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Eigenvalue analysis of the Gerchberg method in the 2D band-pass case: Limited-angle data

Analisi agli autovalori del metodo di Gerchberg in casi passa-banda bidimensionali: Dati ad angolo limitato

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As is known, the application of spectrum extrapolation (or *superresolution*) algorithms to bandlimited images with low space-bandwidth product can be very useful to improve the estimation of the physical object that has been reconstructed through the bandlimiting system [1]. This result has been particularly established, both theoretically and experimentally, for the so called Gerchberg method (GM) and some generalizations of it [3,6], which, for a compact-support object function, achieve spectrum extrapolation by means of iterated enforcements of known constraints in the object and in the Fourier domains. The performance and the convergence rate of the GM have been investigated by analyzing the behavior of the eigenpairs of the imaging operator. Indeed, for finite extent objects, the operator related to a linear imaging system is compact, and its eigenfunctions are complete in the object space. In the cases of interest, the discrete and denumerably infinite eigenvalues are all contained in the open interval between zero and one. By expanding the object and the data image on the eigenfunction basis, the generic GM estimate of the object can be expressed by a noniterative formula, from which the properties of the solutions and the convergence rate of the GM for a particular operator can be inferred if its eigenpairs are known. The eigenfunctions of a linear space- and band-limited operator are related to the generalized prolate spheroidal wavefunctions [8]. Their shapes and their eigenvalues have been derived theoretically for 1D low-pass systems and for a few other cases, such as 2D low-pass systems with circular [8] or square [2] pupils, and for a special 1D band-pass operator [5]. The possibilities of spectrum extrapolation for some of these cases were investigated, among others, by Bertero and Pike [2] and Gori and Wabnitz [4]. The interest of an eigenvalue analysis for 2D band-pass cases is motivated from the growing importance of some coherent imaging systems, whose transfer functions are inherently band-pass. To my knowledge, there is no theoretical study on generalized prolate spheroidal functions in 2D band-pass cases so far; some results from numerical calculations with annular pupils are reported in [7]. The analysis of these results has led to confirm general results found in the low-pass cases and to find some peculiarities for band-pass systems. These results were also tested with true experimental images and numerical simulations.

In this presentation, I further generalize the imaging system assumed, by admitting that its passband can also be just a limited-angle sector of an annulus in the Fourier plane, and show some numerically calculated generalized prolate spheroidals for these cases.

Neglecting the presence of the measurement noise, let us suppose we have the following imaging equation:

$$I(x,y) = BTF(x,y)$$

where $I(x,y)$ is the image, obtained from a compact support object $F(x,y)$ through the cascade of the linear operators $\mathcal{B}$ and $\mathcal{T}$. $\mathcal{B}$ is a linear space-invariant band-pass operator, whose passband is

$$B = \{(s_x, s_y) \text{ s.t. } \rho_{min} \leq \sqrt{{s_x}^2 + {s_y}^2} \leq \rho_{max} \text{ and } \mid \arctan \frac{s_y}{s_x} \mid \leq \theta_{max}\} \tag{2}$$

that is, an annular sector in the Fourier plane, and $\mathcal{T}$ is a space limiting operator, whose support domain is

$$D = \{(x,y) \text{ s.t. } \sqrt{x^2 + y^2} \leq r_{max}\} \tag{3}$$

that is, a circle in the object plane. If the support of F is contained in D, the operator $\mathcal{T}$ has no effect in (1), however, its presence is justified to enable us to consider both F and I as belonging to the Hilbert space $L^2(\Re^2)$ of the square-integrable functions in $\Re^2$. It can be shown (see *e.g.* [4]) that the estimate of $F(x,y)$ at the n-th iteration of GM can be expressed as:

$$F_n(x,y) = \sum_{i=1}^{\infty} f_i \; \{1 - [1 - \mu_i]^n\} \phi_i(x,y) \tag{4}$$

where μ_i is the i-th eigenvalue of $\mathcal{BT}$, $\phi_i(x,y)$ is the related eigenfunction, and f_i is the i-th component of $F(x,y)$ on the eigenfunction basis. For the properties of the eigenvalues recalled above, $F_n(x,y)$ approaches $F(x,y)$ as n approaches infinity. This is how the GM achieves infinite resolution in the noiseless case. Practical resolution limits for this method are set by both the noise level and the magnitude of the eigenvalues: very small eigenvalues, apart from noise effects, would require a very long iteration to correctly recover the corresponding object components. This also means that this version of the algorithm is normally very slow. The number of significant eigenfunctions for spectrum extrapolation for a given noise level is denoted as number of degrees of freedom, and, compared with the number of eigenvalues very close to 1, gives a measure of the extrapolation actually achievable.

I will show the behavior of the dominant eigenvalues for the domains specified in (2) and (3), for several values of the parameters there appearing, r_{max}, ρ_{min}, ρ_{max}, and θ_{max}, and some numerically calculated eigenfunctions in the same cases. The general features found for the full-angle case have been so far confirmed, however, as expected, the number of degrees of freedom decreases with the angular extent of the Fourier domain. This means that the extrapolation possibilities are lower in the cases examined here than in the full-angle cases. Furthermore, and this could also be expected, the resolution of the estimated object remains strongly anisotropic for low values of θ_{max}.

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