DFT TRANSFORMERS FOR THE EVALUATION OF SUCCESSIVE PERIODOGRAMS

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Two DFT transformers for evaluating in real time modified periodograms of partially overlapped segments of a sampled signal are presented. The first transformer works whatever may be the data window for which every segment is multiplied, while the second transformer, which has a reduced complexity, requires that the data window is rectangular.

1. Introduction

In many applications it may be of interest to process a sequence of overlapped segments of a sampled signal, each segment having a length N and the starting points of these segments being Q units apart (Q divides N). Such processing consists in multiplying each segment for a data window and in evaluating the Discrete Fourier Transform (DFT) of the segments so modified [1]. Commonly used data windows are the Hanning window, the Parzen window, the cosine-bell window, and in particular cases the rectangular window. The successive DFTs of the segments so modified (modified periodograms) can be used in order to estimate the power spectrum of the sampled signal [2].

If the successive periodograms are to be evaluated in real time, we must perform one DFT of N elements every time a new set of Q elements arrives. Then a DFT transformer that evaluates P=N/Q DFT coefficients at the arrival of every sample is needed.

In this work the most important results of a general study carried out by the authors [3-5] are reported. More precisely two DFT transformers working in real time are described. Each transformer has a degree of parallelism P in the sense that at every step each computational element of the transformer (multiplier, adder) performs an operation and P DFT coefficients are generated. The first transformer refers to an arbitrary data window, while the second refers to a rectangular data window and it has a very reduced com-

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plexity with respect to the first one. Both the transformers are based on a two-dimensional procedure for evaluating the DFT [6], which is recalled in the next Section.

2. A two-dimensional procedure for evaluating the DFT

Let us consider the set A constituted by N=PQ complex numbers (elements):
$A = \{f_n, \; n=0,1,\dots,N-1\}$.

The DFT of the set A is the other set

$$B = \{F_k = \sum_{n=0}^{N-1} f_n W^{nk}, \quad k=0,1,\dots,N-1\} \qquad (1)$$

where $W=\exp(-2\pi\sqrt{-1}/N)$. If we express subscripts n and k as

$$\begin{cases} n = Qp+q \\ k = s+Pr \end{cases}$$

where $p,s=0,1,\dots,P-1$ and $q,r=0,1,\dots,Q-1$, we obtain for the k-th coefficient F_k:

$$F_k = F_{s+Pr} = \sum_{q=0}^{Q-1} \sum_{p=0}^{P-1} f_{Qp+q} W^{(Qp+q)(s+Pr)} . \qquad (2)$$

By expanding $W^{(Qp+q)(s+Pr)}$ and by observing that $W^{PQpr}=W^{Npr}=1$, we can rearrange (2) in the following way:

$$F_{s+Pr} = \sum_{q=0}^{Q-1} H_{Qs+q} T^{q(r+s/P)} \qquad (3)$$

where

$$H_{Qs+q} = \sum_{p=0}^{P-1} f_{Qp+q} V^{ps} \qquad (4)$$

and $T=W^P$, $V=W^Q$.

The quantities f_{Qp+q}, H_{Qs+q}, F_{s+Pr} can be regarded, respectively, as elements of the matrices:

$Ma = \{f_{p,q} = f_{Qp+q}, \quad p=0,1,\dots,P-1, \; q=0,1,\dots,Q-1\}$

$Mh = \{H_{s,q} = H_{Qs+q}, \quad s=0,1,\dots,P-1, \; q=0,1,\dots,Q-1\}$

$Mb = \{F_{s,r} = F_{s+Pr}, \quad s=0,1,\dots,P-1, \; r=0,1,\dots,Q-1\}$

For every fixed q, the quantity H_{Qs+q} in relation (4) can be regarded as the s-th coefficient of the DFT of the q-th column of Ma. For every fixed s, the quantity F_{s+Pr} in relation (3) can be regarded as the r-th coefficient of another kind of transform of the s-th row of Mb. This transform is called Generalized in Frequency Discrete Fourier Transform with frequency parameter s/P, and it is denoted by $GF^{s/P}$ or simply by GF, where the specification of the parameter is not necessary. The GF is a particular case of the generalized discrete Fourier transform (GFT) introduced in [6].

By taking into account relations (4) and (3), the following two-dimensional procedure for obtaining the coefficients F_k is deduced:

1) evaluate a DFT of the q-th column of Ma, for q=0,1,...,Q-1, so obtaining a new matrix Mh
2) evaluate a $GF^{s/P}$ of the s-th row of Mh, for s=0,1,...,P-1, so obtaining the final matrix of coefficients.

In the next Section a fast algorithm and a pipeline structure for evaluating a GF of a set H of Q elements are briefly recalled, Q being a power of two.

3. Fast GF computation.

It is well known that a DFT can be evaluated by means of fast algorithms (FFT algorithms). It can be easily verified that also for evaluating the GF there exists a fast algorithm (FGF algorithm) similar to the FFT algorithm based on "decimation in time" [7]. In fact, if H is a set of Q elements, the coefficients G_r, r=0,1,...,Q-1, of the GF^t of H can be expressed as

$$\begin{cases} G_{r'} = X_{r'} + T^{(r'+t)} Y_{r'} \\ G_{r'+Q/2} = X_{r'} - T^{(r'+t)} Y_{r'} \end{cases} \qquad r'=0,1,\dots,Q/2-1$$

where $X_{r'}$ and $Y_{r'}$ are the coefficients of the GF^t of the blocks constituted by the elements of H having even and odd indices, respectively. This relation, if iterated, leads to the FGF algorithm. Observe that at the z-th iteration, $z=1,2,\dots,\log_2 Q$, the exponents of T can be obtained from those relative to the DFT by adding $2^{z-1}t$. The complexity of an FGF algorithm on Q elements is equal to the complexity of an FFT algorithm on Q elements.

In order to compute the GF in a fast manner, a radix-2 pipeline structure similar to that introduced for the DFT [8] can be utilized. More precisely, a radix-2 pipeline structure is able to perform a GF with frequency parameter t_1 of the set whose elements arrive sequentially at its upper input, and a GF with frequency parameter t_2 of the set whose elements arrive sequentially at its lower input. The z-th stage ($z=1,2,\dots,\log_2 Q$ from the output side to the input side) is illustrated in Fig. 1. The commutator control

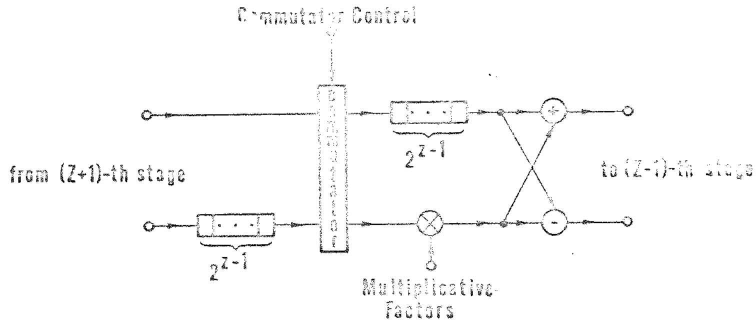

Fig. 1. The z-th stage of a radix-2 pipeline structure

switches every 2^{z-1} steps. The multiplicative factors to be presented at the multiplier can be obtained from those relative to the DFT by adding to the exponent of T the quantity $2^{z-1}t_1$ for Q/2 steps (while the stage works on the GF^{t1}), and the quantity $2^{z-1}t_2$ for the next Q/2 steps (while the stage works on the GF^{t2}). Then we have the two successive ordered sets of multiplicative factors

$$\left\{ T^{(2^{z-1})(s_z+t_1)}, \quad s_z=0,1,\dots,\frac{Q}{2^z}-1 \text{ in bit reversed order} \right\}$$

$$\left\{ T^{(2^{z-1})(s_z+t_2)}, \quad s_z=0,1,\dots,\frac{Q}{2^z}-1 \text{ in bit reversed order} \right\}$$

The first set is repeated 2^{z-1} times, then the second set is repeated 2^{z-1} times, and so on.

Let us evaluate the arithmetic and memory requirements of a radix-2 pipeline structure. As the z-th stage needs 1 multiplier, 2 adders, and 2^z memory elements, the total numbers of multipliers, adders, and memory elements are resepctively $\log_2 Q$, $2\log_2 Q$ and $2(Q-1)$.

4. First transformer

Let $\dots,x_{i-2},x_{i-1},x_i,\dots$ be a sampled signal, and let N, P and Q be three powers of two, such that N=PQ. Let $\dots,X^{u-2},X^{u-1},X^u,\dots$ be a sequence of segments, each of N elements, such that

$$X^j = \{x_n^j = x_{jQ+n}, \quad n=0,1,\dots,N-1\} \quad j=\dots,u-2,u-1,u,\dots \qquad (5)$$

Two successive segments X^{j-1} and X^j are said to be staggered of Q samples, in the sense that the last N-Q elements of X^{j-1} coincide with the first N-Q elements of X^j. In fact, from (5) it follows that $x_n^j = x_{jQ+n}$, $x_n^{j-1} = x_{jQ-Q+n}$ and then that

$$x_m^j = x_{Q+m}^{j-1}, \quad m=0,1,\ldots,N-Q-1$$

Note that two successive segments are staggered of 1 element for Q=1, are disjoined for Q=N, and are staggered of 2,4,...,N/2 elements in the intermediate cases.

Let us consider the processing consisting in:

1) multiplying every segment X^j for a data window $R=\{\rho_0,\rho_1,\ldots,\rho_{N-1}\}$, so obtaining a new sequence of segments $\ldots,A^{u-2},A^{u-1},A^u,\ldots$;
2) evaluating the DFT of every segment A^j, so obtaining a final sequence $\ldots,B^{u-2},B^{u-1},B^u,\ldots$.

A transformer that performs the operations involved in the previous points 1) and 2) in real time is depicted in Fig. 2. The transformer is based

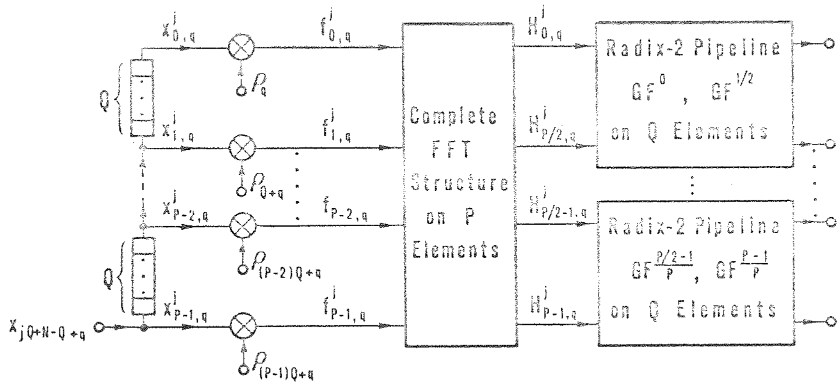

Fig. 2. The first transformer

on the two-dimensional procedure for evaluating a DFT recalled in Section 2, and consists of four main parts.

The first part is constituted by a shift register of N-Q elements having P intermediate outputs. When the samples $x_{jQ+N-Q+q}$, $q=0,1,\ldots,Q-1$, arrive sequentially at the input of the transformer, the P outputs of the first part give sequentially the Q columns of a PxQ matrix

$$Mx^j = \{x_{p,q}^j = x_{jQ+pQ+q}, \quad p=0,1,\ldots,P-1,\ q=0,1,\ldots,Q-1\} \qquad (6)$$

representing X^j properly rearranged.

The second part of the transformer is constituted by P multipliers, whose outputs give sequentially the Q columns of a PxQ matrix

$$Ma^j = \{f^j_{p,q} = x^j_{p,q}\ \rho_{pQ+q}\ ,\quad p=0,1,\dots,P-1,\ q=0,1,\dots,Q-1\}$$

representing A^j properly rearranged.

The third part of the transformer is constituted by a structure (complete FFT structure) that implements the flow-graph of an FFT algorithm on P elements. It executes sequentially the DFT of the columns of Ma^j and generates sequentially the columns of a new PxQ matrix

$$Mh^j = \{H^j_{s,q}\ ,\quad s=0,1,\dots,P-1,\ q=0,1,\dots,Q-1\}\ .$$

The fourth part of the transformer is constituted by P/2 radix-2 pipeline structures. Each structure receives sequentially pairs of elements belonging to two rows of Mh^j, and performs successively the GFs (with pertinent parameters) of two sets of Q elements each. Then the P outputs of the transformer give at every step P coefficients of the DFT of Ma^j.

Let us evaluate the arithmetic and memory requirements of the transformer described in this Section. The first and the second part require, respectively, N-Q memory elements and P multipliers. The third part, i.e., the complete FFT structure on P elements, needs $(P/2)\log_2 P$ multipliers and $P\log_2 P$ adders. Observe that some multipliers have as multiplicative factors 1 or $-\sqrt{-1}$ and they can be replaced by simple combinational circuits. Their number $\psi(P)$ can be determined by observing that in decimation in time algorithm, 1) a DFT of P elements is expressed in terms of two DFTs of P/2 elements and 2) one multiplication for 1 and one multiplication for $-\sqrt{-1}$ are introduced, among other multiplications and additions. Moreover, for a DFT of two elements, the unique multiplication required is for 1. Then $\psi(P)$ is the unique solution of the equation $\psi(P)=2\psi(P/2)+2$, with the constraint $\psi(2)=1$. Such a solution is $\psi(P)=(3/2)P-2$. Therefore, the number of effective multipliers for a complete FFT structure on P elements is $(P/2)\log_2(P/8)+2$. Since the fourth part is constituted by P/2 radix-2 pipeline structures, it requires $(P/2)\log_2 Q$ multipliers, $P\log_2 Q$ adders and N-P memory elements.

Thus, the numbers $MU^I(N,P)$, $AD^I(N,P)$, $ME^I(N,P)$ of multipliers, adders and memory elements, respectively, required by the transformer are:

$$\Big\{\ MU^I(N,P) = \frac{P}{2}\log_2 N - \frac{P}{2} + 2$$

$$\begin{cases} AD^{I}(N,P) = P \log_2 N \\ ME^{I}(N,P) = 2N - \frac{N}{P} - P \end{cases}$$

5. Second transformer

Let us consider the particular case in which the data window is rectangular, i.e. $\rho_n=1$ for $n=0,1,\dots,N-1$. In this hypothesis a second transformer can be obtained by properly simplifying the portion of the first transformer preceeding the radix-2 pipeline structures.

An obvious simplification consists in eliminating the P multipliers at the inputs of the complete FFT structure (see Fig. 3a).

More consistent simplifications can be obtained by taking into account the fact that the segments $A^j=X^j$, $j=\dots,u-2,u-1,u,\dots$ are partially overlapped. Let us apply the decimation in time algorithm for the computation of the DFT of the q-th column of $Ma^j=Mx^j$. It results that

$$\begin{cases} H^{j}_{s',q} = K[2]^{j}_{s',q} + V^{s'} H[2]^{j}_{s',q} \\ H^{j}_{s'+P/2,q} = K[2]^{j}_{s',q} - V^{s'} H[2]^{j}_{s',q} \end{cases} \tag{7}$$

where $s'=0,1,\dots,P/2-1$, and

$$\begin{cases} K[2]^{j}_{s',q} = \sum_{p'=0}^{P/2-1} x^{j}_{2p',q} (V^2)^{p's'} \\ H[2]^{j}_{s',q} = \sum_{p'=0}^{P/2-1} x^{j}_{2p'+1,q} (V^2)^{p's'} \end{cases}$$

are the coefficients of the DFTs of the two sets constituted, respectively, by the elements of the q-th column of Mx^j having even and odd indices. Since from (6) it results that

$$x^{j}_{2p',q} = x^{j-1}_{2p'+1,q}$$

we have that

$$K[2]^{j}_{s',q} = H[2]^{j-1}_{s',q} \tag{8}$$

Then from (7) and (8) it results that

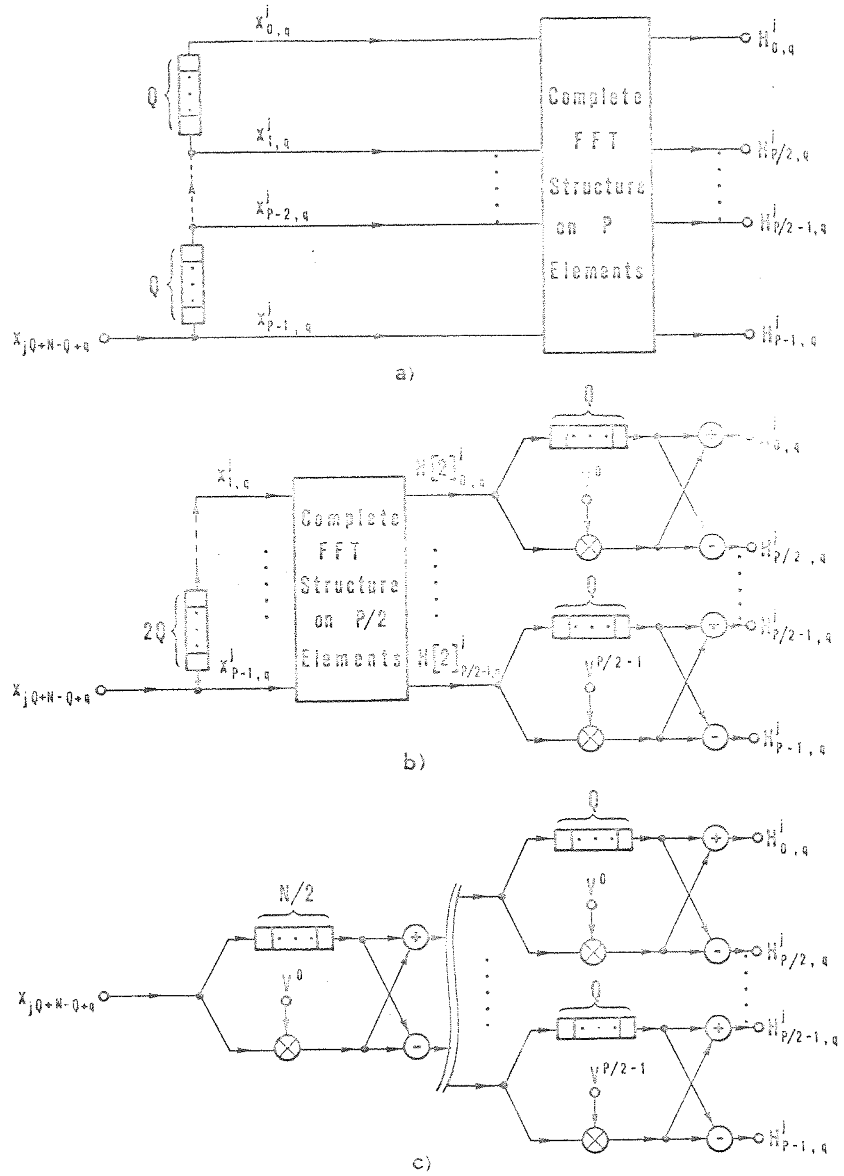

Fig. 3. Successive simplifications that lead to the first portion of the second transformer

$$\begin{cases} H^{j}_{s',q} = H[2]^{j-1}_{s',q} + V^{s'}\, H[2]^{j}_{s',q} \\ H^{j}_{s'+P/2,q} = H[2]^{j-1}_{s',q} - V^{s'}\, H[2]^{j}_{s',q} \end{cases}$$

Therefore the portion of the transformer in Fig. 3a can be simplified as shown in Fig. 3b, where the Q memory elements having as input $H[2]^{j}_{s',q}$ store the quantities [$^{j}_{s',q-1}$, ... $H[2]^{j}_{s',0}$, $H[2]^{j-1}_{s',Q-1}$, ..., $H[2]^{j-1}_{s',q}$ previously computed. This reasoning can be iterated $\log_2 P$ times, so obtaining for the first portion of the second transformer the structure shown in Fig. ...

Let us evaluate the arithmetic and the memory requirements of the second transformer. The first portion (see Fig. 3c) requires $P/2+P/4+...+1=P-1$ multipliers, $2(P-1)$ adders and $(N/2)\log_2 P$ memory elements. On the other hand, observe that in every one of the $\log_2 P$ stages there are two multipliers having as multiplicative factors 1 and $-\sqrt{-1}$, respectively, with the exception of the first stage in which the unique multiplier has the multiplicative factor equal to 1. Therefore, the number of effective multipliers of the first portion of the transformer is $P-2\log_2 P$. By taking into account the arithmetic and memory requirements of the P/2 radix-2 pipeline structures, it results that the numbers $MU^{II}(N,P)$, $AD^{II}(N,P)$, $ME^{II}(N,P)$ of multipliers, adders and memory elements, respectively, required by the second transformer are

$$\begin{cases} MU^{II}(N,P) = \frac{P}{2}\log_2 N - (\frac{P}{2}+2)\log_2 P + P \\ AD^{II}(N,P) = P\log_2 N - P\log_2 \frac{P}{4} - 2 \\ ME^{II}(N,P) = \frac{N}{2}\log_2 4P - P \end{cases}$$

6. Conclusions

Two DFT transformers with degree of parallelism P, that evaluate successive DFTs of sets of N elements staggered of Q, are described. Also the radix-P pipelines presented in [9] can be used for the same goal, since they have a degree of parallelism P. On the other hand, observe that these last transformers require that N is a power of P, while the transformers described in this paper work on the hypothesis that P divides N.

Observe that, for every fixed N, the complexity of the two presented transformers increases with the increasing of P, and then with the decreasing of the number of samples that separate two successive segments to be transformed.

Let us now compare the two presented transformers. Such a comparison is consistent only on the hypothesis that also the first transformer works with a rectangular data window (i.e., that the P multipliers at the input the complete FFT structure are eliminated). In this case the differences in arithmetic and memory requirements between the first and the second transformer are

$$
\begin{cases}
\Delta MU(N,P) = \frac{P}{2}\log_2\frac{P}{32} + 2\log_2 P + 2 \\
\Delta AD(N,P) = P\log_2\frac{P}{4} + 2 \\
\Delta ME(N,P) = -\frac{N}{2}\log_2\frac{P}{4} - \frac{N}{P}
\end{cases}
$$

For $P=2$ the two transformers coincide. For $P=4$ and $P=8$ the second transformer requires less adders and the same number of multipliers with respect to the first one, while for $P \geq 16$ both the number of adders and the number of multipliers is inferior in the second transformer. As a counterpart, the second transformer requires more memory elements.

From the previous reasonings it follows that, with the increasing of P, it increases both the complexity of the transformers and the gain in arithmetic requirements of the second transformer with respect to the first one.

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