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LARGE AMPLITUDE WHIRLS OF ROTORS

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SYNOPSIS Large amplitude vibrations are considered in shafts rotating on lubricated bearings, for a certain class of lubricant response; the behaviour of the shaft is described in different ranges of speed of rotation. It is shown finally that the analysis applies partly also to more general cases of lubricant response (e. g., short bearings with cavitating oil film).

1. INTRODUCTION

The aim of this paper (as that of paper in Ref. 1) is a qualitative appreciation of processes leading to unstable behaviour of shafts rotating on lubricated bearings. The analysis of Ref.1 is pursued to cover the study of large amplitude vibrations (see Sect.5 of Ref.1, in particular). Because of the difficulty of the matter, the analysis is restricted to the case of rotors with constant cross-section, running on bearings whose response belongs to a certain class specified below. Notation is standard and, in particular, consistent with the usage of Ref.1 : the radial and transverse components of the lubricant force $\underline{F}$ acting on a rotating journal will be called F_e, F_n with

$$F_e = H\, f_e(a, a', \psi, \psi') \ , \quad F_n = H\, f_n(a, a', \psi, \psi'), \tag{1.1}$$

$$H = Rb^3 \omega u\, c^{-2} \ ,$$

where

R,b,c are radius, width, radial clearance of the bearing,

ω is the rotors speed (in radians per unit time) and

u is the viscosity of the lubricant;

f_e, f_n are non-dimensional functions of journals eccentricity ratio a (true eccentricity ca), of the derivative a' of a with respect to non-dimensional time $\tau = \omega t$, of the attitude angle ψ of the journal evaluated with reference to a fixed direction and of the derivative ψ'.

We will consider the case where the response of the lubricant film on the journal is such that, when $a \sim 1$,

$$f_e = -h_e \frac{a'}{(1-a)^{\alpha}} \left\{ 1 + o\left[(1-a)^{\delta}\right]\right\}, \tag{1.2}$$

$$f_n = h_n \frac{1-2\psi'}{(1-a)^{\alpha-1}} \left\{ 1 + o\left[(1-a)^{\delta}\right]\right\};$$

here h_e, h_n, δ are positive constants and α is within the open interval (1,2); $o(\epsilon)$ indicates of course a quantity which tends to zero more rapidly than ϵ : $\lim\ o(\epsilon)/\epsilon = 0$.

Expressions (1.2) are suggested by well-known cases (for instance, in the so-called Sommerfeld case $\alpha = 3/2$, $\delta = 1$); they summarize properties which are essential for the validity of the analysis which follows. The analysis itself goes along lines similar to those of Ref.2 and 3, but is more general and much simpler, thus facilitating insight. Its first step consists in the remark that there exist paths of the journals centre along which, when a is nearly 1 ,

$$f_e \simeq -\varkappa \ , \quad f_n \simeq \beta\tau , \tag{1.3}$$

here $\varkappa$ and β are constants, the first one positive ; later $K = \varkappa H c^{-1}$ will be interpreted as an equivalent rigidity of the oil film. The paths in question are given by the asymptotic formulae

$$a = 1 - \left[\frac{h_e}{(\alpha-1)\varkappa\tau}\right]^{\frac{1}{\alpha-1}} \tag{1.4}$$

and

$$\psi' = \gamma \ , \qquad \text{with } \gamma = \frac{1}{2} - \frac{h_e \beta}{2\, h_n(\alpha-1)\varkappa} . \tag{1.5}$$

It is easy to check by substitution in (1.2) that, with the choice (1.4) for a and (1.5) for ψ' , relations (1.3) are satisfied to within terms of order $o(\tau^{\frac{\delta}{1-\alpha}})$.

2. SHAFT DYNAMICS

The indefinite equations of rotor whirl around a state of uniform rotation with speed ω are

$$\rho A L^4 \omega^2 \frac{\partial^2 \underline{u}}{\partial \tau^2} + \frac{\partial^2}{\partial \zeta^2}\left(EI \frac{\partial^2 \underline{u}}{\partial \zeta^2}\right) = \underline{0} \tag{2.1}$$

if we choose the z-axis (of unit vector $\underline{k}$) through the points occupied by the centres of the journals in the steady state, put $z = L\zeta$, call $c\underline{u}(\zeta,\tau)$ the dispacement vector, ρA the mass per unit length, EI the flexural rigidity and L the length of the shaft.

If we assume that the shaft rests on two identical bearings without overhangs, then the boundary conditions are

$$\frac{\partial^2 \underline{u}}{\partial \zeta^2} = \underline{0} \ , \qquad \text{for } \zeta = 0,1 \ ; \tag{2.2}$$

$$\frac{\partial}{\partial \zeta}\left(\frac{EIc}{HL^3} \frac{\partial^2 \underline{u}}{\partial \zeta^2}\right) = \pm \underline{F}/H \ , \tag{2.3}$$

again for $\zeta = 0$, $\zeta = 1$, but with alternate sign; in (2.3) $\underline{F}$ is the vector of components (F_e, F_n)

with a specified as $|\underline{u}(0,\tau)|$ or $|\underline{u}(1,\tau)|$ and ψ' as $|\underline{u}|^{-2}\,(\frac{\partial \underline{u}}{\partial \tau} \times \underline{u})\cdot \underline{k}$ again for $\zeta = 0,\ 1$.

To keep the notation simple and make developments shorter we consider here only the case of a uniform shaft (A and I constant) and we restrict also our attention to "symmetric" solutions (i.e., solutions such that $\underline{u}(\zeta,\tau)=\underline{u}(1-\zeta,\tau)$), thus needing to consider only boundary conditions at $\zeta = 0$. With a certain caution, however, most results canbe extended to the general case; in particular no difficulty arises, except formal, in considering "antisymmetric" modes.

For later purposes we need to introduce a notation for some special solutions of (2.1), i.e. solutions of the type

$$\underline{u}(\zeta,\tau) = U(\zeta)\ \underline{r}(\gamma\tau) = U(\zeta)\ (\ \underline{i}\cos\gamma\tau + \underline{j}\sin\gamma\tau\ ) \quad , \tag{2.4}$$

where $\underline{i}$, $\underline{j}$ are mutually orthogonal unit vectors, normal to $\underline{k}$. $U(\zeta)$ must satisfy the equation

$$\frac{d^4U}{d\zeta^4} = \frac{\rho A L^4 \omega^2 \gamma^2}{E I}\ U \quad ; \tag{2.5}$$

there are, of course, a great many U. We call $U^{(r,s)}(\zeta;B,C)$ the solution of (2.5) which satisfies the boundary conditions (at $\zeta = 0$) involving the derivatives of order r and s ($r,s = 0,1,2,3$) which are put equal respectively to B and C.

Remember that, if $B^2+C^2 \neq 0$, there is one and only one $U^{(r,s)}(\zeta;B,C)$, provided $\gamma\omega$ differs from all values of a critical sequence $\omega_{2n+1}^{(r,s)}$ ($n = 0,1,2,..$; the odd index is related to our proposal to consider only symmetric solutions); $U^{(r,s)}(\zeta;B,C)$ exists also for $B^2+C^2\neq 0$ and $\gamma\omega = \omega_{2n+1}^{(r,s)}$ provided B bears an appropriate relation to C, but then $U^{(r,s)}(\zeta;B,C)$ is not unique.

$U^{(r,s)}(\zeta;0,0)$ exists only if $\gamma\omega = \omega_{2n+1}^{(r,s)}$ and then we will consider it uniquely defined requiring that $\max U^{(r,s)}(\zeta;0,0) = 1$.

With some caution the notation just introduced can be used recursively : for instance $U^{(2,3)}(\zeta;0,-\ U(0))$ is one of the modes of vibration of the shaft on elastic supports of rigidity $K = H\pi/c$ and exists only if $\omega\gamma$ coincides with one of the critical speeds $\omega_{2n+1}^{(K)}$; then we will use for it the notation $U_{2n+1}^{(K)}(\zeta)$. Notice that $U_{2n+1}^{(\infty)}$ is one of $U^{(0,2)}(\zeta;0,0)$, and that

$$\omega_{2n+1}^{(0,2)} = \omega_{2n+1}^{(\infty)} = \frac{(2n+1)^2\pi^2}{L^2}\ \left(\frac{EI}{\rho A}\right)^{1/2},$$

$$U_{2n+1}^{(\infty)} = \sin\,(2n+1)\pi\zeta\ .$$

We need also later the solution of the following problem

$$\frac{d^4V}{d\zeta^4} - \pi^4\left(\frac{\omega\gamma}{\omega_1^{(\infty)}}\right)^2 V = e\ U_1(\zeta) = e\sin\pi\zeta \tag{2.6}$$

$$V(0) = 1 \quad , \quad V''(0) = 0\ . \tag{2.7}$$

For this solution to exist when $e\neq 0$, $\omega\gamma$ must coincide with $\omega_1^{(\infty)}$; then

$$V(\zeta) = \frac{-e}{4\pi^3}\left(\frac{1}{2} - \zeta\right)\cos\pi\zeta + U^{(0,2)}\left(\zeta\ ;\ \frac{e}{8\pi^3} + 1,\ \frac{-e}{8\pi}\right).$$

But we have already remarked that, under the present conditions $U^{(0,2)}$ with non-null parameters does not exist in general; for existence, the parameters must be appropriately chosen. This bears upon e, which simple calculations show must be taken equal to $-4\pi^3$, if solutions of (2.6), (2.7) have to exist, so that, when all conditions are met, the solution is

$$V(\zeta) = \left(\frac{1}{2} - \zeta\right)\cos\pi\zeta + U^{(0,2)}\left(\zeta;\frac{1}{2},\frac{\pi^2}{2}\right); \tag{2.8}$$

or explicitly

$$V(\zeta) = \tilde{V}(\zeta) = d\sin\pi\zeta + \frac{1}{2}\,\frac{1-\cosh\pi}{\sinh\pi}\sinh\pi\zeta + \frac{1}{2}\cosh\pi\zeta + \left(\frac{1}{2} - \zeta\right)\cos\pi\zeta\ ; \tag{2.9}$$

we remark that V contains still the indeterminated parameter d.

There is an interest also in the solution $W(\zeta)$ of the problem (2.6), (2.7) when $e = 0$, but $\omega\gamma \neq \omega_1^{(\infty)}$; then, of course, the solution is

$$W(\zeta) = U^{(0,2)}(\zeta;1,0)\ , \tag{2.10}$$

or, explicitly,

$$W(\zeta) = \tilde{W}(\zeta) = \frac{1-\cos\lambda}{2\sin\lambda}\sin\lambda\zeta + \frac{1-\cosh\lambda}{2\sinh\lambda}\sinh\lambda\zeta + \frac{1}{2}(\cos\lambda\zeta + \cosh\lambda\zeta)\ , \quad \lambda = \pi\left(\frac{\gamma\omega}{\omega_1^{(\infty)}}\right)^{1/2}. \tag{2.11}$$

We examine in the following sections the full dynamic problem; we must remark though that, even within all restrictions accepted so far on the response of the lubricant and on the aspect of the shaft, the non-linear problem obtained by introducing in (2.3) for $\underline{F}$ explicit expressions for the forces generated by the lubricant is still in general a formidable one. However, some definite conclusions canbe reached through an analysis of the asymptotic behaviour under conditions when a is nearly 1 (i.e., when the amplitude of vibration in the bearings is nearly equal to the clearance) and the asymptotic behaviour of $\underline{F}$ is adequately described by formulae (1.2).

3. LARGE AMPLITUDE VIBRATIONS

Under a variety of choices for the radial and transverse component of $\underline{F}$ (for instance, when formulae due to Sommerfeld or to Ocvirk are used, which refer to the case of fully lubricated full cylindrical bearings) a steady rotation of the shaft is found to be linearly unstable. The analysis below tries to follow up such a case of linearly unstable behaviour and assumes that a has reached already a value near 1. Briefly the idea is that of constructing a function $\underline{u}$ which satisfies asymptotically eqn (2.1) and the boundary condition (2.2) and has the following properties : (i) at $\zeta=0$, $|\underline{u}|$ behaves as in (1.4); (ii) the vector $\underline{u}$ rotates with a speed which, at $\zeta=0$, approximates $\omega\psi'$ as given by (1.5). If the construction is successful we need only to specify parameters so that $\frac{\partial^3 \underline{u}}{\partial \zeta^3}$ has radial and transverse components proportional to $-\varkappa$ and $\beta\tau$ respectively, as required by formulae (1.3). It is within the spirit of the whole paper to aim at an asymptotic evaluation valid for large τ.

We must distinguish two cases :

$$\omega > 2\omega_1^{(\infty)} \quad , \quad \omega < 2\omega_1^{(\infty)} .$$

In the first case the appropriate asymptotic expression for $\underline{u}$ is as follows

$$\underline{u} = -\frac{2\omega_1^{(\infty)}}{\pi\omega}\left\{1 - \left[\frac{h_e}{(\alpha-1)\varkappa\tau}\right]^{\frac{1}{\alpha-1}}\left[\frac{\alpha-1}{\alpha-2}\right]\right\} \times$$

$$\times\, \tau \sin \pi\zeta\ \underline{k} \times \underline{r} \quad + \tag{3.1}$$

$$+\, \tilde{V}(\zeta)\left\{1 - \left[\frac{h_e}{(\alpha-1)\varkappa\tau}\right]^{\frac{1}{\alpha-1}}\right\}\underline{r} ,$$

where $\tilde{V}(\zeta)$ is given by formula (2.9) and $\underline{r}$ is the rotating unit vector already defined.

Substitution in (2.1) shows that in fact that equation is satisfied apart from terms which are of order $O(\frac{1}{\tau})$; the boundary condition (2.2) is also satisfied, of course, and so are conditions

$$|\underline{u}(0)| = a ,$$
$$|\underline{u}|^{-2}\left(\frac{\partial \underline{u}}{\partial \tau} \times \underline{u}\right)\cdot \underline{k} = \psi' , \tag{3.2}$$

with a and ψ' given by (1.4) and (1.5) respectively, apart from terms which are again of order $O(\frac{1}{\tau})$.

Condition (2.3) remains to be satisfied; in view of the fact that a and ψ' are (approximately) given by (1.4), (1.5) we can take simply

$$\frac{EIc}{HL^3}\left(\frac{\partial^3 \underline{u}}{\partial \zeta^3}\right)_{\zeta=0} = F/H = -\varkappa\underline{r} + \beta\tau\,\underline{k}\times\underline{r} \tag{3.3}$$

But $\frac{\partial^3 \underline{u}}{\partial \zeta^3}$ has the following components

$$-2\frac{\omega_1^{(\infty)}}{\omega}\pi^2\left\{1 - \left[\frac{h_e}{(\alpha-1)\varkappa\tau}\right]^{\frac{1}{\alpha-1}}\left(\frac{\alpha-1}{\alpha-2}\right)\right\}\tau ,$$

$$\left\{1 - \left[\frac{h_e}{(\alpha-1)\varkappa\tau}\right]^{\frac{1}{\alpha-1}}\right\}\pi^3\left(\frac{1-\cosh\pi}{2\sinh\pi} - d\right) .$$

where, we remember, d is the parameter left unspecified in $\tilde{V}$. Therefore condition (2.3) can be satisfied, up to terms of order

$$O\left(\left(\frac{1}{\tau}\right)^{\frac{2-\alpha}{\alpha-1}}\right)$$

by taking

$$d = \frac{1}{2}\,\frac{1-\cosh\pi}{\sinh\pi} + \frac{HL^3\varkappa}{EIc\,\pi^3} , \tag{3.4}$$

$$\beta = 2\,\frac{EIc}{HL^3}\,\frac{\omega_1^{(\infty)}}{\omega}\pi^2 .$$

Hence in this case β is positive; this means $\gamma < 1/2$. But on the other hand $\gamma = \omega_1^{(\infty)}/\omega$, because otherwise $\tilde{V}(\zeta)$ would not exist; hence the solution (3.1) applies for $\omega > 2\omega_1^{(\infty)}$. In conclusion:

1. <u>If $\omega > 2\omega_1^{(\infty)}$, an asymptotic solution of our problem exists where the mode of vibration is approximately the first mode on rigid supports, the speed of the whirl is $\omega_1^{(\infty)}$ independently of the speed of rotation, the amplitude of vibration at the bearings tends to the clearance, the amplitude of vibration at the centre of the shaft increases indefinitely in time and the displacement at the centre is out of phase with the displacement at the bearings by the angle $-\pi/2$.</u>

Let us consider now the alternative case $\omega < 2\omega_1^{(\infty)}$; then the appropriate expression of $\underline{u}$ is

$$\underline{u} = \tilde{W}(\zeta)\left\{1 - \left[\frac{h_e}{(\alpha-1)\varkappa\tau}\right]^{\frac{1}{\alpha-1}}\right\}\underline{r} , \tag{3.5}$$

where $\tilde{W}(\zeta)$ is given by (2.11). Substitution in (2.1) shows that the equation is satisfied to within terms of order $o(1/\tau)$; boundary condition (2.2) is satisfied exactly and so are the relations (3.2).

The boundary condition (2.3) can again be written in the form (3.3) but now with $\beta = 0$. All is required to obtain matching, up to terms $O((1/\tau)^{\frac{1}{\alpha-1}})$, is that $\varkappa$ be chosen as follows

$$\varkappa = \frac{EIc}{2HL}\lambda^3\,\frac{(1-\cos\lambda)\sinh\lambda-(1-\cosh\lambda)\sin\lambda}{\sin\lambda\,\sinh\lambda} ,$$
$$\lambda = \pi\,(\omega/2\omega_1^{(\infty)})^{1/2} . \tag{3.6}$$

It is easy to show that the function of λ in the right-hand side of this formula is positive when λ is in the interval $(0, \pi)$ and increases monotonically in that interval from 0 to $+\infty$; therefore (3.6) is compatible if

$$\omega < 2\omega_1^{(\infty)} .$$

It is also obvious from eqn (3.6) and the explicit expression (2.11) for $\tilde{W}(\zeta)$ that, for $\tau \to \infty$, the path of each point of the shaft approaches a circumference. In particular, when ω is near $2\omega_1^{(\infty)}$ the leading term in (2.11) is the first, for $\zeta \neq 0$; then the amplitude of vibration in the middle of the shaft can be estimated with the formula

$$|\underline{u}(\tfrac{1}{2},\tau)| \sim \tilde{W}(\tfrac{1}{2}) \sim$$

$$\sim \frac{1-\cos\pi\left(\frac{\omega}{2\omega_1^{(\infty)}}\right)^{1/2}}{2\sin\pi\left(\frac{\omega}{2\omega_1^{(\infty)}}\right)^{1/2}}$$

So we conclude that:

2. When $\omega < 2\,\omega_1^{(\infty)}$, an asymptotic solution of our problem exists where the mode of vibration is approximately the first mode on supports of appropriate rigidity increasing from 0 to $+\infty$ as the speed of rotation increases from 0 to $2\,\omega_1^{(\infty)}$; the speed of the whirl is always half the rotational speed; the amplitude of vibration is specified by the condition to be equal to the bearings clearance at the ends (hence the whirl is not a severe one although the amplitude is the larger, the nearer ω is to $2\,\omega_1^{(\infty)}$) : the phase of vibration is approximately the same all along the shaft.

4. EXTENSIONS; FINAL REMARKS

Is it possible to confirm the validity of our results beyond the rather narrow confines of the hypotheses expressed by formulae (1.2) ? the statement 2 underlined in Sect.3 can be easily shown to have wider applicability: in fact, if one follows the analysis again, one can check that it applies simply if α is larger than 1. Thus the case of fully lubricated short cylindrical bearings (sometimes called the Ocvirk case) can be accomodated: in that case $\alpha = 5/2$, $\delta = 1$. But one can go beyond:

Formulae (1.2) need not even be assumed at all, provided only the lubricant forces satisfy (1.3), whenever a and ψ' behave as required by (1.4), (1.5) perhaps within terms of appropriate order, say $o(\tau^{\frac{1}{1-\alpha}})$.

One such case is quoted in Sect.4 of Ref. 4 ; it is the case of very short bearings with cavitation. The case is particularly interesting because it is the only one for which explicit formulae for the lubricant forces are known, when the hypothesis of full lubrication is abandoned (a hypothesis, needless to say, which fails usually in practice). The situation is complex, naturally, because a new variable is involved : the angle θ which specifies the attitude of the lubricated portion of the clearance (from θ to $\theta + \varkappa$, measuring

$$\theta = \arctan \frac{2a'}{(1 - 2\psi')\,a}$$

from the radius which points towards the maximum of the clearance).

Then one has

$$f_e = \left(\frac{1}{2} - \psi'\right) a\, g_1(a,\theta) - 2a'\, g_2(a,\theta) \;,$$

$$f_n = \left(\frac{1}{2} - \psi'\right) a\, g_3(a,\theta) - 2a'\, g_1(a,\theta) \;,$$

where g_1, g_2, g_3 have complicated expressions in terms of the variables a and θ (see formulae (4.3) of Ref.4) . Now again the appropriate exponent in (1.4) is obtained by taking $\alpha = 5/2$: but terms of higher order must be specified in formula (1.5)

$$\psi' \simeq 1/2 - G\,\tau^{-\epsilon} \;, \qquad \epsilon > 0 \;.$$

The choice of ϵ influences the phase of the attitude angle of the cavitating zone during vibration. However, in any case, f_e tends to a constant different from zero and f_n tends to zero; the attendand compatibility conditions with equations (2.1), (2.2), (2.3) lead again, for $\omega < 2\,\omega_1^{(\infty)}$, to statement 2.

Another interesting case is that of a heavy shaft, when it must be inquired if trajectories of the journals exist such that along them

$$f_e \simeq -\varkappa - w\cos\psi \;,$$

$$f_n \simeq w\sin\psi \;,$$

where w is the ratio W/H , with W load on the bearing :

$$W = \frac{1}{2}\,\rho\, g\, A\, L \;.$$

Under hypotheses (1.2) these trajectories can be found

$$a = 1 - \left(\frac{h_e}{(\alpha-1)\varkappa\tau}\right)^{\frac{1}{\alpha-1}} \left[1 - \frac{2w\sin(\tau/2)}{(\alpha-1)\varkappa\tau}\right] ,$$

$$\psi' = 1/2 - \frac{h_e}{2h_n}\,\frac{w\sin(\tau/2)}{(\alpha-1)\varkappa\tau} \;.$$

Because of the properties of the sine-integral function, ψ grows in time as $\tau/2$, apart from terms of order $O(1/\tau)$. All we need in our analysis is to change formula (3.5) into

$$u = \tilde{W}(\xi)\left\{1 - \left(\frac{h_e}{(\alpha-1)\varkappa\tau}\right)^{\frac{1}{\alpha-1}}\left[1 - \frac{2w\sin(\tau/2)}{(\alpha-1)\varkappa\tau}\right]\right\}(\underline{i}\cos\psi + \underline{j}\sin\psi) \;,$$

to prove again statement 2.

The generalization of statement 1 is not so easy; it needs be qualified anyway. The statement underlined above in this Section is still applicable but the restriction $\alpha < 2$ remains heavy. Fomula (3.1) must be modified considerably to cover new cases, such as those we have just mentioned, and the analysis becomes altogether unpleasantly complex.

REFERENCES

1. G.CAPRIZ, Self-excited vibrations of rotors. Dynamics of Rotors. Springer, Berlin, 1975, p. 128-141.
2. G.CAPRIZ, On the vibrations of shafts rotating on lubricated bearings. Ann.Mat.Pura Appl., 1960, vol.50, p. 223-248.
3. A.LARATTA, Oil whirl of rotors. Meccanica,1969, vol.4, p. 1-13.
4. G.CAPRIZ, A.LARATTA, Vibrations of shafts rotating on lubricated bearings. Proc. Wordl Congress on the theory of machines and mechanisms, Newcastle upon Tyne, 1975, p. 513-516.