MASONRY-LIKE SOLIDS IN THE PRESENCE OF THERMAL EXPANSION: AN EXPLICIT SOLUTION

Internal Report C95-26

July 1995

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Abstract.
In this paper the constitutive equation of masonry-like materials is generalised in order to account for thermal dilatation. Subsequently, the explicit solution to the equilibrium problem of a circular ring subjected to two uniform radial pressures p_1 and p_2, acting respectively on the inner and outer boundary and a temperature distribution depending linearly on the radius, is calculated.

Sommario.
In questo lavoro si generalizza l'equazione costitutiva dei materiali non resistenti a trazione per tenere conto delle dilatazioni termiche. Successivamente si calcola la soluzione esplicita del problema di equilibrio di una corona circolare costituita da un materiale non resistente a trazione, soggetta a due pressioni radiali uniformi, p_1 e p_2, agenti rispettivamente sul bordo interno e sul bordo esterno e a una distribuzione di temperatura variabile linearmente col raggio.

Key words: Masonry-like material, thermal expansion, equilibrium problem.

1. Introduction

In many applications it is necessary to model the behaviour of masonry-like solids in the presence of thermal dilatation. For example, molten metal production processes, in particular integrated steel manufacturing, require refractory coatings able to withstand the thermomechanical actions produced by high-temperature fluids. Analysis of these coverings is usually carried out by considering the refractory materials to be linear elastic, though they are actually non-resistant to traction. Results obtained by applying such a constitutive model are generally characterised by considerable tensile stress and are thus quite unrealistic. Further applications are presented in [1], where the Pozzuoli's volcanic caldera is studied and in [2] where the influence of the temperature on the stress field in a masonry arch is analysed.

In Section 2 a constitutive equation which can model the behaviour of some refractory materials is presented; to this end the constitutive equation of isotropic masonry-like materials [3] is generalised in order to account for thermal dilatation. Specifically, we suppose that the total strain minus the thermal expansion is the sum of an elastic strain, on which the stress, negative semi-definite, depends isotropically and linearly and of an inelastic part, positive semi-definite and orthogonal to the stress. We thus obtain a non-linear elastic material conforming to a masonry-like material when there is no temperature change. A more detailed description of this generalised constitutive equation is given in [4], where the temperature-dependence of the material constants is taken into account.

In Section 3 we then consider a circular ring made of an elastic material characterised by the constitutive equation introduced in Section 2. The circular ring is subjected to a plane stress brought about by two uniform radial pressures acting on the inner and outer boundary and a temperature distribution varying linearly with the radius. Once the Poisson's ratio, the Young's modulus and the thermal expansion coefficient of the material are fixed, we determine the explicit solution to the equilibrium problem of the circular ring and we study the dependence of this solution on the inner and outer temperatures and pressures. We verify that if they satisfy certain inequalities involving the inner and outer radii and the material's constants, then the elastic solution given in [5] is negative semi-definite and therefore represents precisely the solution for the material under consideration. On the contrary, if these inequalities are not satisfied, the elastic solution is characterised by a tensile circumferential stress which may arise on both the inner and outer boundary of the ring. In this case, by following a procedure similar to that used in [6] for a circular ring at constant temperature, the equilibrated stress field, negative semi-definite, is calculated, and the corresponding fractures determined.

2. The constitutive equation

In this section a constitutive equation able to model the behaviour of some refractory materials is presented; in particular, the constitutive equation of isotropic masonry-like materials [3] is generalised in order to account for the presence of thermal dilatation. The dependence of the material constants on temperature is omitted here because it is irrelevant to the application examined in the next section; a detailed description of the more general constitutive equation is presented in [4].

In the following, t is the temperature change with respect to a reference temperature, say t_e, and ν, E and α denote the Poisson's ratio, the Young's modulus and the thermal expansion coefficient, respectively, which, in this case, are temperature-independent.

Let us indicate as Sym^+ and Sym^- the subsets of the linear space Lin of second order tensors constituted by symmetric positive semi-definite and symmetric negative semi-definite tensors, respectively. The inner product of two tensors **A** and **B** of Lin is $\mathbf{A}\cdot\mathbf{B} = \mathrm{tr}(\mathbf{A}^T\mathbf{B})$, where $\mathbf{A}^T$ is the transpose of **A** and tr(**A**) is the trace of **A**.

Let **E** and $\alpha\, t\, \mathbf{I}$ be the infinitesimal strain and thermal expansion tensor, respectively; we suppose that the tensor $\mathbf{E} - \alpha\, t\, \mathbf{I}$ is the sum of an elastic part $\mathbf{E}^e$ and of a positive semi-definite inelastic part $\mathbf{E}^a$:

$$\text{(2.1)} \qquad \mathbf{E} - \alpha\, t\, \mathbf{I} = \mathbf{E}^e + \mathbf{E}^a , \quad \mathbf{E}^a \in \mathrm{Sym}^+,$$

and that the stress tensor **T** depend linearly and isotropically on $\mathbf{E}^e$:

$$\text{(2.2)} \qquad \mathbf{T} = \frac{E}{1+\nu}\{\mathbf{E}^e + \frac{\nu}{1-2\nu}\,\mathrm{tr}\,(\mathbf{E}^e)\,\mathbf{I}\}.$$

Moreover, we suppose that **T** is negative semi-definite and orthogonal to the inelastic strain:

(2.3)$_1$ $\quad T \in \mathrm{Sym}^-$,

(2.3)$_2$ $\quad \mathbf{T} \cdot \mathbf{E}^a = 0.$

$\mathbf{E}^a$ is sometimes called the fracture strain because the body can be expected to crack in the regions where $\mathbf{E}^a$ is different from zero. (2.1)-(2.3) is the constitutive equation of a non-linear elastic material which, when $t = 0$, reduces to the classical constitutive equation of masonry-like materials described in [3].

By a procedure similar to that used in [7], taking into account that $\alpha\, t\, \mathbf{I}$ is a spherical tensor, it can be proven that tensors $\mathbf{E}$, $\mathbf{T}$, $\mathbf{E}^a$ and $\mathbf{E}^e$ are coaxial, and that the constitutive equation (2.1)-(2.3) has a unique solution. The coaxiality of $\mathbf{E}$, $\mathbf{T}$, $\mathbf{E}^a$ and $\mathbf{E}^e$ allows one to write the constitutive equation (2.1)-(2.3) with respect to the basis $\mathbf{g}_1$, $\mathbf{g}_2$, $\mathbf{g}_3$ of the eigenvectors of $\mathbf{E}$, and thus, given the material's constants, to explicitly calculate the stress and inelastic strain as functions of $\mathbf{E}$ [4].

In view of the application to be described in Section 3, we limit our attention to a plane stress. Let us suppose that the eigenvalue $t_3 = \mathbf{g}_3 \cdot \mathbf{T}\mathbf{g}_3$ of $\mathbf{T}$ is nil; if e_1, e_2, e_3 and a_1, a_2, a_3 are the eigenvalues of $\mathbf{E}$ and $\mathbf{E}^a$, respectively, from (2.2) we obtain

$$e_3 - \alpha\, t - a_3 = \frac{\nu}{1-\nu}\,(a_1 + a_2 - e_1 - e_2 + 2\alpha\, t). \tag{2.4}$$

Moreover, since by virtue of (2.3)$_2$, a_3 is arbitrary, it can be assumed to be equal to zero. By taking (2.4) into account, relation (2.2) can be written in terms of the eigenvalues t_1 and t_2 of $\mathbf{T}$ and the eigenvalues e_1, e_2, a_2 and a_3

$$\text{(2.5)}\qquad \begin{aligned} t_1 &= \frac{E}{1-\nu^2}\,\{e_1 - \alpha\, t - a_1 + \nu\,(e_2 - \alpha\, t - a_2)\}, \\ t_2 &= \frac{E}{1-\nu^2}\,\{e_2 - \alpha\, t - a_2 + \nu\,(e_1 - \alpha\, t - a_1)\}. \end{aligned}$$

3. The circular ring

The circular ring Ω shown in Figure 1, made of a non-linear elastic material with constitutive equation (2.1)-(2.3) and having inner radius a and outer radius b, is subjected to a plane stress as a consequence of two uniform radial pressures p_1 and p_2 acting, respectively, on the inner and outer boundary and a temperature distribution varying linearly with the radius ρ:

$$t(\rho) = -\,\frac{t_1 - t_2}{b - a}\,\rho + \frac{b t_1 - a t_2}{b - a}, \tag{3.1}$$

where t_1 and t_2 are the temperatures of the inner and outer boundary, respectively.

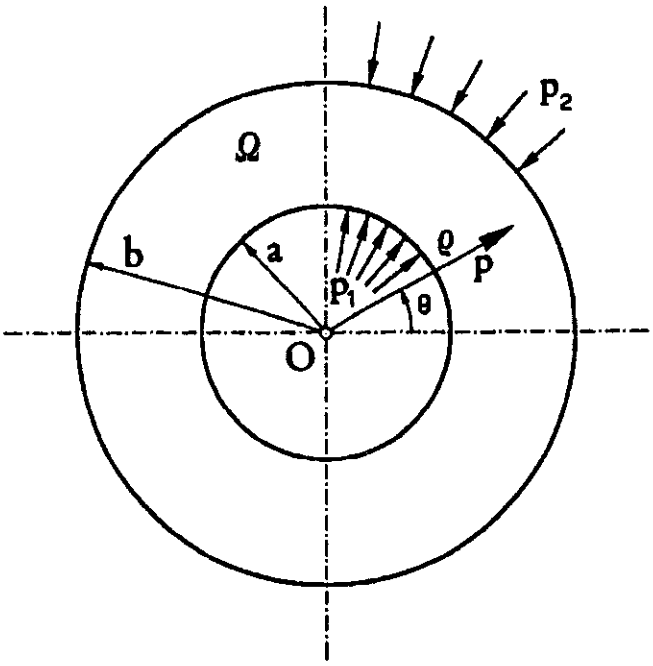

Figure 1. The circular ring.

The problem has been studied in [6], without considering the temperature. In particular, it has been proven that once p_1 is fixed, if p_2 satisfies the inequality $p_2 > \frac{a^2 + b^2}{2b^2} p_1$, then the circular ring is entirely compressed; on the contrary, if p_2 belongs to the interval $\left[\frac{a}{b} p_1, \frac{a^2 + b^2}{2b^2} p_1\right]$, the circular ring is cracked in a region which starts at the inner boundary and varies as p_2 varies. In particular, the crack region is characterised by a transition radius ρ_0 which coincides with a for $p_2 = \frac{a^2 + b^2}{2b^2} p_1$, and increases as p_2 decreases, until the radius b is reached when $p_2 = \frac{a}{b} p_1$.

In this paper we aim to study how in the presence of a linear temperature distribution the crack region varies as $\alpha(t_1 - t_2)$ increases from zero. In particular, we shall prove that once pressures p_1 and p_2 are fixed, there exist ψ_1 and ψ_2, such that if $\alpha(t_1 - t_2)$ belongs to the interval $[\psi_1, \psi_2]$, then the circular ring is compressed; on the contrary, if $\alpha(t_1 - t_2)$ is outside this interval, fractures are present. More precisely, there exists φ_1 such that if $p_2 \geq \varphi_1 p_1$, then the crack region behaves roughly like as in Figure 2; in this picture the amplitude of the region where cracks are present (dashed region) is drawn as function of $\alpha(t_1 - t_2)$.

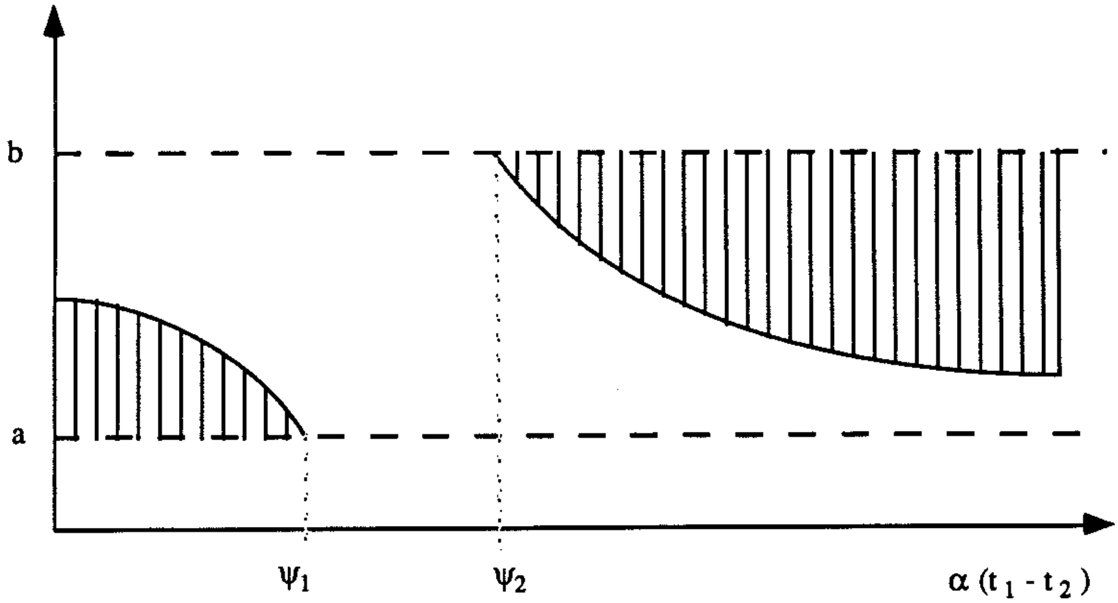

Figure 2. Behaviour of the cracked region when $\alpha(t_1 - t_2)$ varies.

Having chosen a polar reference system $\{O, \rho, \theta\}$ whose origin coincides with the centre of the ring, let us consider the stress field **T** having principal components

$$\sigma_\rho(\rho) = \frac{a^2b^2(p_2 - p_1)}{b^2 - a^2}\frac{1}{\rho^2} + \frac{p_1a^2 - p_2b^2}{b^2 - a^2} + \frac{\alpha(t_1 - t_2)E}{3\rho^2(b^2 - a^2)}\left[(a + b)\rho^3 - (a^2 + b^2 + ab)\rho^2 + a^2b^2\right], \qquad \rho \in [a, b], \tag{3.2}$$

$$\sigma_\theta(\rho) = -\frac{a^2b^2(p_2 - p_1)}{b^2 - a^2}\frac{1}{\rho^2} + \frac{p_1a^2 - p_2b^2}{b^2 - a^2} + \frac{\alpha(t_1 - t_2)E}{3\rho^2(b^2 - a^2)}\left[2(a + b)\rho^3 - (a^2 + b^2 + ab)\rho^2 - a^2b^2\right], \qquad \rho \in [a, b]. \tag{3.3}$$

It is well known that stress field (3.2)-(3.3) is the solution to the equilibrium problem of a circular ring composed of a homogeneous isotropic linear elastic material with temperature distribution (3.1) and subjected to pressures p_1 and p_2 [5].

First of all, we intend to determine the conditions which p_1, p_2, t_1, t_2 and α must satisfy so that the stress field with components (3.2)-(3.3) is negative semi-definite and is therefore precisely the solution to the problem. Subsequently, we shall consider the situations in which these conditions are not verified; in such cases we shall determine the solution, starting from the stress field (3.1)-(3.2). To this end, we have chosen to fix the radial pressures and study the sign of the linear elastic solution as $\alpha(t_1 - t_2)$ varies.

Let us now define the quantities

$$(3.4) \qquad \psi_0 = \frac{3}{E}\,\frac{p_1(a^2+b^2) - 2p_2b^2}{2b^2 - ab - a^2}$$

and

$$(3.5) \qquad \psi_2 = \frac{3}{E}\,\frac{p_2(a^2+b^2) - 2p_1a^2}{b^2 + ab - 2a^2}\,;$$

ψ_2 is positive and the interval $[\psi_0, \psi_2]$ is non empty if we choose pressures p_1 and p_2 satisfying the inequality

$$(3.6) \qquad p_2 \geq \varphi_1 p_1,$$

where

$$(3.7) \qquad \varphi_1 = \frac{b^2 + ab + 4a^2}{4b^2 + ab + a^2}$$

belongs to the interval (0, 1). In the following we shall suppose that condition (3.6) is always verified; in this case ψ_0 and ψ_2 satisfy the inequality

$$(3.8) \qquad \psi_0 \leq \psi_2,$$

thus we can consider the values of $\alpha(t_1 - t_2)$ belonging to the interval $[\psi_0, \psi_2]$ and those outside it. In particular, if $p_2 = \varphi_1 p_1$, then $\psi_0 = \psi_2 = \frac{3}{E}\, p_1 \,\frac{b^2 - a^2}{4b^2 + ab + a^2}$ and the interval $[\psi_0, \psi_2]$ reduces to the point ψ_0.

For the sake of simplicity we shall limit our treatment by supposing that the thermal expansion coefficient and temperatures t_1 and t_2 are such that the inequality

$$(3.9) \qquad \alpha(t_1 - t_2) \geq 0$$

is satisfied[1]. Thus, since the quantity ψ_0 is positive when the inequality

$$(3.10) \qquad p_2 < \varphi_2 p_1,$$

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1 Accounting for negative values of $\alpha(t_1 - t_2)$ as well would make both calculation and their description more onerous, without however improving the quality of the solution itself.

is satisfied, where

$$(3.11) \qquad \varphi_2 = \frac{a^2 + b^2}{2b^2}$$

ranges from 0 to 1 and verifies the inequality $\varphi_1 < \varphi_2$, we define a new number ψ_1 as follows: $\psi_1 = \psi_0$, if (3.10) holds , and $\psi_1 = 0$, when (3.10) does not hold.

The next subsections are dedicated to separate treatment of two cases: that in which $\alpha(t_1 - t_2)$ belongs to the interval $[\psi_1, \psi_2]$, and that in which $\alpha(t_1 - t_2)$ is outside this interval.

3.1 $\alpha(t_1 - t_2)$ BELONGS TO THE INTERVAL $[\psi_1, \psi_2]$

In this subsection we prove that, with a fixed pressure p_1 on the inner boundary of Ω, if (3.6) is satisfied and if temperatures t_1 e t_2 are such that $\alpha(t_1 - t_2) \in [\psi_1, \psi_2]$, then the stress **T** with components (3.2)-(3.3) is negative semi-definite.

The stress component σ_ρ in (3.2) is the sum of a stress due to the pressures p_1 and p_2, say $\sigma_\rho^m(\rho) = \frac{a^2b^2(p_2 - p_1)}{b^2 - a^2}\frac{1}{\rho^2} + \frac{p_1a^2 - p_2b^2}{b^2 - a^2}$, and a thermal stress $\sigma_\rho^t(\rho) = \frac{\alpha(t_1 - t_2)E}{3\rho^2(b^2 - a^2)}[(a + b)\rho^3 - (a^2 + b^2 + ab)\rho^2 + a^2b^2]$. Since $\sigma_\rho^m(a) = -p_1$, $\sigma_\rho^m(b) = -p_2$ and $\sigma_\rho^m(\rho)$ is a monotonic function of ρ, we can deduce that $\sigma_\rho^m(\rho)$ is negative in the interval [a, b]. Taking into account both the boundary conditions $\sigma_\rho^t(a) = \sigma_\rho^t(b) = 0$ and assumption (3.9), simple calculations reveal that $\sigma_\rho^t(\rho)$ is non-positive in [a, b] and, finally, that $\sigma_\rho(\rho)$ is non-positive for each $\rho \in [a, b]$.

We now move on to evaluating the sign of the circumferential stress σ_θ given in (3.3). Let us begin by noting that the circumferential stresses on the boundary

$$(3.12) \qquad \sigma_\theta(a) = \frac{p_1(a^2 + b^2) - 2p_2b^2}{b^2 - a^2} + \alpha(t_1 - t_2)\,E\,\frac{ab + a^2 - 2b^2}{3(b^2 - a^2)}$$

and

$$(3.13) \qquad \sigma_\theta(b) = \frac{2p_1a^2 - p_2(a^2 + b^2)}{b^2 - a^2} + \alpha(t_1 - t_2)\,E\,\frac{b^2 + ab - 2a^2}{3(b^2 - a^2)},$$

are both non-positive, since $\alpha(t_1 - t_2)$ belongs to the interval $[\psi_1, \psi_2]$. Moreover, $\sigma_\theta(a) = 0$, if $\alpha(t_1 - t_2) = \psi_1$, and $\sigma_\theta(b) = 0$, if $\alpha(t_1 - t_2) = \psi_2$.

We now aim to prove that σ_θ is non-positive in the internal part of Ω as well. In order to arrive at this result, we consider the third degree polynomial

$$(3.14) \qquad s(\rho) = 2(a + b)\alpha(t_1 - t_2)E\,\rho^3 + [3(p_1a^2 - p_2b^2) - \alpha(t_1 - t_2)E\,(b^2 + ab + a^2)]\rho^2 +$$

$$- 3a^2b^2(p_2 - p_1) - \alpha(t_1 - t_2)E\, a^2b^2 ,$$

whose sign coincides to that of $\sigma_\theta(\rho)$. Polynomial $s(\rho)$ is non-positive in [a, b], in fact $s(\rho)$ has a maximum at $\rho_1 = 0$ and a minimum at $\rho_2 = \dfrac{\alpha E\,(t_1 - t_2)(a^2 + b^2 + ab) - 3(p_1a^2 - p_2b^2)}{3\alpha E\,(t_1 - t_2)(b + a)} > 0$[2]. Three situations can occur: $\rho_2 \in [a, b)$, $\rho_2 \leq a$ and, finally, $b < \rho_2$. In the first case, $s(\rho)$ has a minimum in (a, b); in the second case, $s(\rho)$ is an increasing function in [a, b]; in the third case, $s(\rho)$ is decreasing in [a, b]. These results, taken together with the fact that $s(a) \leq 0$ and $s(b) \leq 0$, are sufficient to guarantee the non-positiveness of $s(\rho)$ in [a, b].

For values of $\alpha(t_1 - t_2)$ greater than ψ_2 or less than ψ_1, the radial stress (3.2) is still non-positive, on the contrary the circumferential stress (3.3) becomes positive. In particular, if we progressively increase the quantity $\alpha(t_1 - t_2)$, examination of (3.13) reveals that, beginning at ψ_2, positive circumferential stresses arise on the outer boundary and spread in the internal part of Ω. Instead, in view of (3.12), the circumferential stress for values of $\alpha(t_1 - t_2)$ less than ψ_1 becomes positive starting at the inner boundary. Therefore, for values of $\alpha(t_1 - t_2)$ outside the interval $[\psi_1, \psi_2]$, the stress field with components (3.2)-(3.3) does not constitute the solution of the equilibrium problem of the circular ring made of a material having constitutive equation (2.1)-(2.3).

3.2 $\alpha(t_1 - t_2)$ IS GREATER THAN ψ_2

In this subsection we consider the values of $\alpha(t_1 - t_2)$ satisfying the inequality $\alpha(t_1 - t_2) \geq \psi_2$ and calculate the solution for a material with constitutive equation (2.1)-(2.3), by starting with the solution (3.2)-(3.3) corresponding to a linear elastic material. The procedure followed in determining the solution is similar to that used in [6] for a circular ring made of a masonry-like material, subjected to pressures p_1 and p_2.

Let us suppose that the circumferential stress vanishes in the circular ring $\Omega_{2r} = \{(\rho, \theta);\ \rho \in [\rho_r, b]\}$, where the radius $\rho_r \in [a, b]$ is unknown. In this region the radial stress must be determined in such a way as to satisfy the equilibrium equation

(3.15) $$\frac{d\sigma_\rho}{d\rho} + \frac{\sigma_\rho}{\rho} = 0.$$

Moreover, by virtue of (2.5), taking into account that the circumferential stress and the circumferential inelastic strain ε^a_θ are nil, we have

(3.16) $$\sigma_\rho = E\,(\varepsilon_\rho - \alpha t),$$

[2] Since $\varphi_1 > a/b$, condition (3.6) implies that $p_2 > (a/b)p_1$ and then the numerator of ρ_2 is positive.

where ε_ρ is the radial strain and $\varepsilon_\rho^e = \varepsilon_\rho - \alpha t$ the elastic strain. Denoting u as the radial displacement, in view of the relation $\varepsilon_\rho = \frac{du}{d\rho}$ and (3.16), (3.15) is equivalent to the differential equation

$$\rho \frac{d^2u}{d\rho^2} - \alpha\rho \frac{dt}{d\rho} + \frac{du}{d\rho} - \alpha t = 0, \tag{3.17}$$

from which we obtain

$$\frac{du}{d\rho} = \alpha t + \frac{c}{\rho} \tag{3.18}$$

and then $\sigma_\rho = E \frac{c}{\rho}$, where the constant c is determined imposing the boundary condition $\sigma_\rho(b) = - p_2$ and it holds that $c = - \frac{bp_2}{E}$. Therefore, the stress components in Ω_{2r} are

$$\begin{aligned} \sigma_\rho(\rho) &= - \frac{b}{\rho} p_2, && \rho \in [\rho_r, b], \\ \sigma_\theta(\rho) &= 0, && \rho \in [\rho_r, b]. \end{aligned} \tag{3.19}$$

From (3.18) we obtain the expression for the radial displacement

$$u(\rho) = \int_{\rho_r}^{\rho} \alpha\, t(\rho')\, d\rho' - \frac{bp_2}{E} \ln \rho + d, \qquad \rho \in [\rho_r, b], \tag{3.20}$$

where d is a constant which will be determined subsequently by imposing the continuity of u at $\rho = \rho_r$. By virtue of (2.1), the relation $\varepsilon_\theta = \frac{u}{\rho}$ and (2.5), the circumferential inelastic strain is

$$\varepsilon_\theta^a(\rho) = \varepsilon_\theta(\rho) - \alpha t(\rho) - \varepsilon_\theta^e(\rho) = \frac{u(\rho)}{\rho} - \alpha t(\rho) - \frac{\nu}{E} \frac{bp_2}{\rho}, \qquad \rho \in [\rho_r, b]. \tag{3.21}$$

Let now consider the remaining circular ring $\Omega_{1r} = \{(\rho, \theta); \rho \in [a, \rho_r]\}$, it is subjected to pressures p_1 and $p_r = \frac{bp_2}{\rho_r}$, acting on the inner and outer boundary, respectively, and has the linear temperature distribution $t(\rho) = - \frac{t_1 - t_r}{\rho_r - a} \rho + \frac{\rho_r t_1 - a t_r}{\rho_r - a}$, where t_r is the temperature at ρ_r

which, in view of (3.1), is $t_r = t(\rho_r) = -\dfrac{t_1 - t_2}{b - a}\rho_r + \dfrac{bt_1 - at_2}{b - a}$. On the other hand, by virtue of the results obtained in subsection 3.1, the linear elastic solution in Ω_{1r}

$$\text{(3.22)} \qquad \sigma_\rho(\rho) = \frac{a^2\rho_r^2(p_r - p_1)}{\rho_r^2 - a^2}\,\frac{1}{\rho^2} + \frac{p_1a^2 - p_r\rho_r^2}{\rho_r^2 - a^2} + $$

$$+ \frac{\alpha(t_1 - t_r)E}{3\rho^2(\rho_r^2 - a^2)}\left[(a + \rho_r)\rho^3 - (a^2 + \rho_r^2 + a\rho_r)\rho^2 + a^2\rho_r^2\right], \qquad \rho \in [a, \rho_r],$$

$$\text{(3.23)} \qquad \sigma_\theta(\rho) = -\frac{a^2\rho_r^2(p_r - p_1)}{\rho_r^2 - a^2}\,\frac{1}{\rho^2} + \frac{p_1a^2 - p_r\rho_r^2}{\rho_r^2 - a^2} + $$

$$+ \frac{\alpha(t_1 - t_r)E}{3\rho^2(\rho_r^2 - a^2)}\left[2(a + \rho_r)\rho^3 - (a^2 + \rho_r^2 + a\rho_r)\rho^2 - a^2\rho_r^2\right], \qquad \rho \in [a, \rho_r],$$

is negative semi-definite, under the assumption that the condition

$$\text{(3.24)} \qquad \alpha(t_1 - t_r) = \frac{3}{E}\,\frac{(a^2 + \rho_r^2)p_r - 2a^2p_1}{\rho_r^2 + a\rho_r - 2a^2}$$

holds[3]. (3.24), being equivalent to $\sigma_\theta(\rho_r) = 0$, expresses the continuity of the circumferential stress at ρ_r. Taking into account the expressions of t_r and p_r, from (3.24) it can be deduced that ρ_r is a root of the following fourth-degree polynomial:

$$\text{(3.25)} \qquad q(\rho) = \alpha E\,(t_1 - t_2)\rho^4 - 3[\alpha E\,(t_1 - t_2)a^2 + (b - a)bp_2]\rho^2 +$$

$$+ 2[\alpha E\,(t_1 - t_2)a^3 + 3(b - a)a^2p_1]\rho - 3(b - a)a^2bp_2.$$

At the end of this subsection we will prove that polynomial q has a unique root ρ_r in the interval (a, b].

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3 If $\alpha(t_1 - t_2) \geq 0$, then $\alpha(t_1 - t_r)$ is also non-negative by virtue of the linear dependence of t on ρ and this is sufficient to guarantee the non-positiveness of the radial stress (3.22). The non-positiveness of the circumferential stress (3.23), by virtue of (3.12), (3.8), (3.6), (3.7) and the relation $p_r = \dfrac{b}{\rho_r}p_2$, is due to the inequality

$$\frac{\rho_r^3 + a\rho_r^2 + 4a^2\rho_r}{4\rho_r^2 + a\rho_r + a^2} < \frac{b^3 + ab^2 + 4a^2b}{4b^2 + ab + a^2},$$

which holds for each ρ_r belonging to [a, b].

Now let us calculate the strains and displacement corresponding to the stress field (3.22)-(3.23) in the region Ω_{1r}; here the inelastic strain is nil, therefore

$$(3.26)\quad \varepsilon_\rho(\rho) = \alpha\, t(\rho) + \varepsilon_\rho^e(\rho) = \frac{1}{E}\left[(1+\nu)\frac{a^2\rho_r^2(p_r - p_1)}{\rho_r^2 - a^2}\frac{1}{\rho^2} + (1-\nu)\frac{p_1 a^2 - p_r\rho_r^2}{\rho_r^2 - a^2}\right] +$$

$$+ \alpha\, t(\rho) + \frac{\alpha(t_1 - t_r)}{3(\rho_r^2 - a^2)}[(1-2\nu)(a+\rho_r)\rho +$$

$$- (1-\nu)(a^2 + a\rho_r + \rho_r^2) + (1+\nu)\frac{a^2\rho_r^2}{\rho^2}], \qquad \rho \in [a, \rho_r],$$

$$(3.27)\quad \varepsilon_\theta(\rho) = \alpha\, t(\rho) + \varepsilon_\theta^e(\rho) = \frac{1}{E}\left[-(1+\nu)\frac{a^2\rho_r^2(p_r - p_1)}{\rho_r^2 - a^2}\frac{1}{\rho^2} + (1-\nu)\frac{p_1 a^2 - p_r\rho_r^2}{\rho_r^2 - a^2}\right] +$$

$$+ \alpha\, t(\rho) + \frac{\alpha(t_1 - t_r)}{3(\rho_r^2 - a^2)}[(2-\nu)(a+\rho_r)\rho +$$

$$- (1-\nu)(a^2 + a\rho_r + \rho_r^2) - (1+\nu)\frac{a^2\rho_r^2}{\rho^2}], \qquad \rho \in [a, \rho_r],$$

and the radial displacement is

$$(3.28)\quad u(\rho) = \frac{1}{E}\left[-(1+\nu)\frac{a^2\rho_r^2(p_r - p_1)}{\rho_r^2 - a^2}\frac{1}{\rho} + (1-\nu)\frac{p_1 a^2 - p_r\rho_r^2}{\rho_r^2 - a^2}\right] +$$

$$+ \frac{\alpha(t_1 - t_r)}{3(\rho_r^2 - a^2)}\left[(1-2\nu)(a+\rho_r)\frac{\rho^2}{2} + (\nu - 1)(a^2 + a\rho_r + \rho_r^2)\rho - (1+\nu)a^2\rho_r^2\frac{1}{\rho}\right] +$$

$$+ \frac{\alpha}{\rho_r - a}\left[-(t_1 - t_r)\frac{\rho^2}{2} + (\rho_r t_1 - a t_r)\rho\right], \qquad \rho \in [a, \rho_r].$$

We are now in a position to calculate the constant d contained in the relation (3.20), by imposing the continuity of the radial displacement at $\rho = \rho_r$ and then determine the radial displacement in the region Ω_{2r}.

$$(3.29)\quad u(\rho) = \frac{bp_2}{E}\ln\left(\frac{\rho_r}{\rho}\right) + \frac{\rho_r}{(\rho_r^2 - a^2)E}\,[2a^2p_1 - (\rho_r^2 + a^2)p_r + \nu(\rho_r^2 - a^2)p_r] +$$

$$+\frac{\alpha}{\rho_r - a}\left[-(t_1 - t_r)\frac{\rho^2}{2} + (\rho_r t_1 - a t_r)\rho\right] +$$

$$-\frac{\alpha\rho_r}{6(\rho_r^2 - a^2)}(t_1 - t_r)(\rho_r^2 + a\rho_r + 4a^2), \qquad \rho\in[\rho_r, b].$$

By virtue of (3.21) and (3.29), the circumferential inelastic strain has the expression

$$\varepsilon_\theta^a(\rho) = \frac{1}{\rho}\ \{\ \frac{1}{2}\rho^2\frac{\alpha(t_1 - t_r)}{\rho_r - a} + \frac{bp_2}{E}\ln(\rho_r/\rho) - \frac{\nu}{E}bp_2 + \tag{3.30}$$

$$+\frac{\rho_r}{(\rho_r^2 - a^2)E}[2a^2p_1 - (\rho_r^2 + a^2)p_r + \nu(\rho_r^2 - a^2)p_r] +$$

$$-\frac{\alpha\rho_r}{6(\rho_r^2 - a^2)}(t_1 - t_r)(\rho_r^2 + a\rho_r + 4a^2)\}, \qquad \rho\in[\rho_r, b].$$

In view of (3.24), it can be immediately verified that $\varepsilon_\theta^a(\rho_r) = 0$, and that ε_θ^a is moreover a positive function of ρ in the interval $(\rho_r, b]$. In order to prove this, it can first be noted that the sign of the circumferential inelastic strain (3.30) coincides with the sign of the function $f(\rho) = \rho\ \varepsilon_\theta^a(\rho)$, which, setting $\bar\rho = \sqrt{\frac{(\rho_r - a)b\,p_2}{\alpha(t_1 - t_r)E}}$, has a maximum at $\rho = -\bar\rho$ and a minimum at $\rho = \bar\rho$. It can then be proven that the minimum point $\bar\rho$ is less than the transition radius ρ_r. In fact, in view of (3.24), the inequality $\bar\rho < \rho_r$ is equivalent to the condition $p_2 > \frac{3a^2\rho_r^2}{b(\rho_r^3 + 3a^2\rho_r - a^3)}p_1$ which is trivial to verify by virtue of the relations $\frac{3a^2\rho_r^2}{b(\rho_r^3 + 3a^2\rho_r - a^3)} < \varphi_1$ and of (3.6). Therefore, $f(\rho)$ is increasing for $\rho \geq \rho_r$ and so, since $f(\rho_r) = 0$, it can be concluded that $f(\rho)$ and, consequently $\varepsilon_\theta^a(\rho)$, are both non negative in $[\rho_r, b]$.

Our aim is now to prove that polynomial q given in (3.25) has a unique root in the interval (a, b]. To this end, let us first note that $q(b) = \alpha(t_1 - t_2)E\,b(b - a)(b^2 + ab - 2a^2) - 3b(b - a)[(a^2 + b^2)p_2 - 2a^2p_1]$ is positive when $\alpha(t_1 - t_2) > \psi_2$ and is nil for $\alpha(t_1 - t_2) = \psi_2$. Moreover, $q(a) = 6(b - a)a^2(ap_1 - bp_2)$ is temperature-independent and negative in view of the inequalities $p_2 \geq \varphi_1 p_1 > \frac{a}{b}p_1$. Therefore, there exist at least one ρ_r belonging to the interval (a, b] such that $q(\rho_r) = 0$.

In order to prove the uniqueness of ρ_r, let us consider the polynomial

$$p(\rho) = \rho^4 - 3\left[a^2 + \frac{(b-a)bp_2}{\alpha E\,(t_1 - t_2)}\right]\rho^2 + 2\left[a^3 + \frac{3(b-a)a^2p_1}{\alpha E\,(t_1 - t_2)}\right]\rho - \frac{3(b-a)a^2bp_2}{\alpha E\,(t_1 - t_2)}$$

having the same roots of q, and then the second derivative of p:

$$p''(\rho) = 12\rho^2 - 6\left[a^2 + \frac{(b-a)bp_2}{\alpha E\,(t_1 - t_2)}\right]$$

which is an increasing function in [a, b]. By setting $\psi_3 = \frac{1}{E}\,\frac{(b-a)bp_2}{a^2}$, it is easy to verify that p''(a) is non-negative if and only if $\alpha(t_1 - t_2) \geq \psi_3$. In this case p is a convex function and has a unique root ρ_r in (a, b); ρ_r may coincide with b, if $\psi_3 < \psi_2$. On the contrary, if $\alpha(t_1 - t_2) < \psi_3$, then $p''(a) < 0$ and further operations are needed. *A priori*, we can distinguish the following cases:

case (i) $\quad p''(b) \geq 0$, then there exists $\rho^* \in (a, b]$ such that $p''(\rho^*) = 0$.

case (ii) $\quad p''(b) < 0$, then there exists $\rho^* > b$ such that $p''(\rho^*) = 0$.

Let us put $\psi_4 = \frac{1}{E}\,\frac{(b-a)bp_2}{2b^2 - a^2}$ and observe that $\psi_4 < \psi_3$ and $\psi_4 < \psi_2$. Firstly, we note that $p''(b) < 0$ if and only if $\alpha(t_1 - t_2) < \psi_4$, a situation which can never occur, given that we have supposed $\alpha(t_1 - t_2) > \psi_2$; case (ii) is therefore excluded. Now we need only examine case (i), which holds when the interval $[\psi_2, \psi_3)$ is non-empty and $\alpha(t_1 - t_2) \in [\psi_2, \psi_3)$. In this instance $p'(\rho)$ has a minimum at ρ^*. Since $p'(a) = \frac{6a(b-a)}{\alpha E\,(t_1 - t_2)}(ap_1 - bp_2) < 0$, because $p_2 > ap_1/b$, p'(b) must be positive, given that if it were negative or nil, then p' would be negative in [a, b] and p decreasing in [a, b], something which is excluded by $p(a) = \frac{6a^2(b-a)}{\alpha E\,(t_1 - t_2)}(ap_1 - bp_2) < 0$ and $p(b) = b(b-a)\left[b^2 + ab - 2a^2 - 3\,\frac{(a^2 + b^2)p_2 - 2a^2p_1}{\alpha E\,(t_1 - t_2)}\right] \geq 0$. Thus, there exists $\rho^{**} \in (a, b)$, such that $p'(\rho^{**}) = 0$. The polynomial p decreases in $[a, \rho^{**})$, has a minimum at ρ^{**}, increases in $[\rho^{**}, b]$, and thus there is a unique $\rho_r \in (\rho^{**}, b]$, such that $p(\rho_r) = 0$.

We have thus proven that with a, b, p_1 and p_2 fixed so that (3.6) holds, for each value of $\alpha(t_1 - t_2) \in [\psi_2, +\infty)$, there is a unique $\rho_r(\alpha(t_1 - t_2)) \in (a, b]$ root of the polynomial p and therefore of the polynomial q.

The explicit value of $\rho_r(\alpha(t_1 - t_2))$ could be determined using the well-known formulas for the calculus of the roots of a fourth-degree polynomial. This calculation is omitted here, however we include an analysis of the dependence of ρ_r on $\alpha(t_1 - t_2)$; more precisely we shall prove that $\rho_r(\alpha(t_1 - t_2))$ is a decreasing function of $\alpha(t_1 - t_2)$ and when $\alpha(t_1 - t_2)$ goes to $+\infty$, then $\rho_r(\alpha(t_1 - t_2))$ goes to a. To this end, let us put $\beta = \alpha(t_1 - t_2)$ and consider the function two variables $w(\beta,\rho) = \beta E\,\rho^4 - 3[\beta E\,a^2 + (b-a)bp_2]\rho^2 + 2[\beta E\,a^3 + 3(b-a)a^2p_1]\rho - 3(b-a)a^2bp_2$. From (3.25) it follows that $w(\beta, \rho_r(\beta)) = 0$ and by virtue of the implicit function's theorem, we have

$$(3.31) \qquad \frac{d\rho_r}{d\beta} = \frac{(-\rho_r^3 + 3a^2\rho_r - 2a^3)\rho_r E}{4\beta E\,\rho_r^3 - 6[\beta E\,a^2 + (b-a)bp_2]\rho_r + 2[\beta E\,a^3 + 3(b-a)a^2p_1]},$$

where the numerator is negative because $\rho_r > a$, and the denominator never vanishes since ρ_r is a root of q having multiplicity equal to 1. Therefore, ρ_r is a monotonic function of β, which result, together with the fact that $a < \rho_r(\beta) \le b$, leads to the conclusion that the limit of $\rho_r(\beta)$ for β going to $+\infty$ exists and is finite. Let us indicate this limit by c and consider the relation

$$(3.32) \qquad \beta = \frac{3(b-a)(b\rho_r^2(\beta)p_2 - 2a^2\rho_r(\beta)p_1 + a^2bp_2)}{E\,\rho_r(\beta)(\rho_r(\beta) - a)(\rho_r^2(\beta) + a\rho_r(\beta) - 2a^2)},$$

obtained by deriving β from $w(\beta, \rho_r(\beta)) = 0$. When β goes to $+\infty$, taking into account that the numerator of (3.32) is positive, we get $c = a$. In particular, $\rho_r(\beta)$ is a decreasing function of β in $[\psi_2, +\infty)$. Figure 3 shows the behaviour of the radius ρ_r as $\alpha(t_1 - t_2) \ge \psi_2$ varies. This result has been obtained by assuming a = 1. m, b = 2. m, $p_1 = 0.1$ MPa, $p_2 = 0.063$ MPa, $E = 5000.$ MPa, and the corresponding value of ψ_2 is $0.17\ 10^{-4}$.

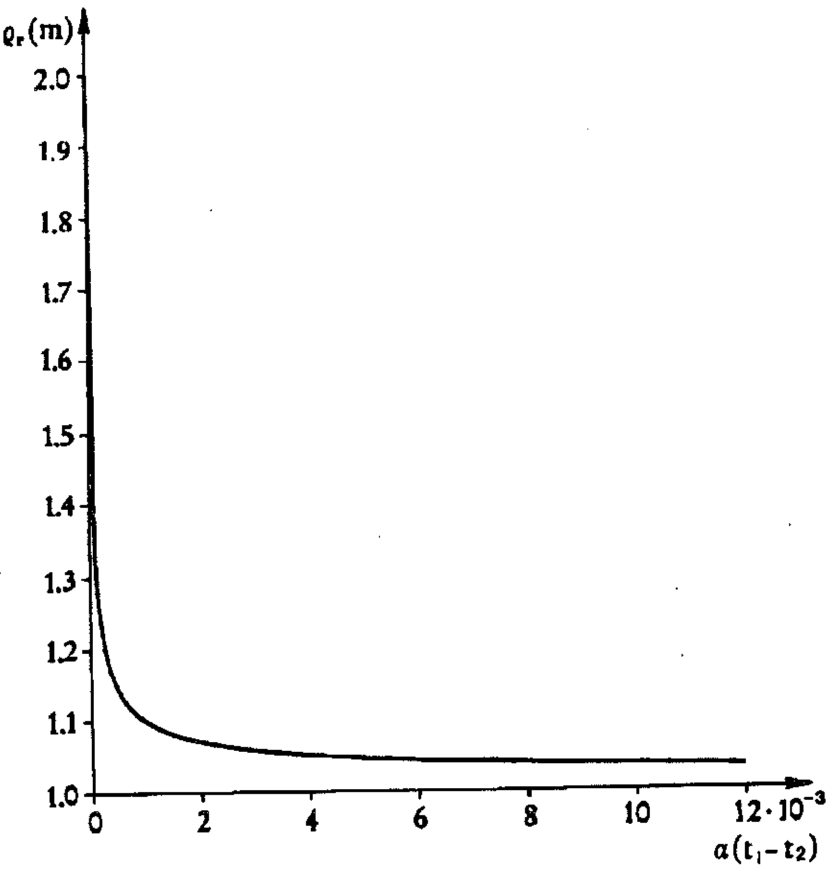

Figure 3. Radius ρ_r vs. $\alpha(t_1 - t_2)$.

3.3 $\alpha(t_1 - t_2)$ BELONGS TO THE INTERVAL $[0, \psi_1]$

In this subsection we shall consider values of $\alpha(t_1 - t_2)$ belonging to the interval $[0, \psi_1]$. First of all, we notice that if $\psi_1 = 0$, then the interval reduces to the point 0. On the other hand, if $\psi_1 > 0$, that is, if pressures p_1 and p_2 satisfy the inequality (3.10), then the interval $[0, \psi_1]$ is non-empty. Given p_1, let us choose $p_2 \in [\varphi_1 p_1, \varphi_2 p_2)$ and fix the temperatures t_1 and t_2 such that $\alpha(t_1 - t_2) \in [0, \psi_1]$. As already noted in Subsection 3.1, the linear elastic solution (3.2)-(3.3) is not negative semi-definite because it is characterised by positive circumferential stresses emerging from the inner boundary of the circular ring. It is possible to determine a negative semi-definite stress field by following the procedure used in Subsection 3.2. Also in this case we suppose that in the circular ring $\Omega_{1s} = \{(\rho, \theta); \rho \in [a, \rho_s]\}$, where ρ_s is unknown, the stress components are

$$\sigma_\rho(\rho) = -\frac{a}{\rho} p_1, \qquad \rho \in [a, \rho_s]$$

(3.33)

$$\sigma_\theta(\rho) = 0, \qquad \rho \in [a, \rho_s].$$

In view of (2.1) and $(2.3)_2$, we have $\varepsilon_\rho^a = 0$, $\varepsilon_\rho^e = \varepsilon_\rho - \alpha\, t$ and the radial displacement is

$$u(\rho) = \int_a^\rho \alpha\, t(\rho')\, d\rho' - \frac{a p_1}{E} \ln \rho + d, \qquad \rho \in [a, \rho_s] \tag{3.34}$$

where the constant d is determined by imposing the continuity of u at $\rho = \rho_s$. The circumferential inelastic strain is

$$\varepsilon_\theta^a(\rho) = \varepsilon_\theta(\rho) - \alpha t(\rho) - \varepsilon_\theta^e(\rho) = \frac{u(\rho)}{\rho} - \alpha t(\rho) - \frac{\nu}{E}\frac{a p_1}{\rho}, \qquad \rho \in [a, \rho_s] \tag{3.35}$$

where u is supplied by (3.34).

The remaining circular ring $\Omega_{2s} = \{(\rho, \theta); \rho \in [\rho_s, b]\}$ is subjected to pressures $p_s = \frac{a p_1}{\rho_s}$ and p_2 acting on the inner and outer boundary, respectively, and experiences the temperature distribution $t(\rho)$ varying linearly from $t_s = -\frac{t_1 - t_2}{b - a}\rho_s + \frac{b t_1 - a t_2}{b - a}$ for $\rho = \rho_s$, to t_2 for $\rho = b$. In Ω_{2s} the linear elastic solution

$$\sigma_\rho(\rho) = \frac{b^2 \rho_s^2 (p_2 - p_s)}{b^2 - \rho_s^2}\frac{1}{\rho^2} + \frac{p_s \rho_s^2 - p_2 b^2}{b^2 - \rho_s^2} + \tag{3.36}$$

$$+ \frac{\alpha(t_s - t_2)E}{3\rho^2(b^2 - \rho_s^2)} [(b + \rho_s)\rho^3 - (b^2 + \rho_s^2 + b\rho_s)\rho^2 + b^2\rho_{sl}^2], \quad \rho \in [\rho_s, b].$$

(3.37) $$\sigma_\theta(\rho) = - \frac{b^2\rho_s^2(p_2 - p_s)}{b^2 - \rho_s^2} \frac{1}{\rho^2} + \frac{p_s\rho_s^2 - p_2 b^2}{b^2 - \rho_s^2} +$$

$$+ \frac{\alpha(t_s - t_2)E}{3\rho^2(b^2 - \rho_s^2)} [2(b + \rho_s)\rho^3 - (b^2 + \rho_s^2 + b\rho_s)\rho^2 - b^2\rho_s^2], \quad \rho \in [\rho_s, b].$$

is negative semi-definite, if the condition [4]

(3.38) $$\alpha(t_s - t_2) = \frac{3}{E} \frac{(b^2 + \rho_s^2)p_s - 2b^2 p_2}{2b^2 - \rho_s^2 - b\rho_s};$$

is satisfied. (3.38) is equivalent to the continuity of the circumferential stress at ρ_s. Taking into account the expressions of t_s and p_s, it follows that (3.38) is satisfied if and only if ρ_s is a root of the following fourth-degree polynomial:

(3.39) $$h(\rho) = \alpha E\,(t_1 - t_2)\rho^4 - 3[\alpha E\,(t_1 - t_2)b^2 + (b - a)ap_1]\rho^2 +$$

$$+ 2[\alpha E\,(t_1 - t_2)b^3 + 3(b - a)b^2 p_2]\rho - 3(b - a)b^2 a p_1 .$$

We begin by noting that $h(a) = \alpha(t_1 - t_2)E\, a(b - a)(2b^2 - ab - a^2) - 3a(b - a)[(a^2 + b^2)p_1 - 2b^2 p2]$ is negative for $\alpha(t_1 - t_2) < \psi_1$ and nil for $\alpha(t_1 - t_2) = \psi_1$. Moreover, $h(b) = 6(b - a)b^2 (bp_2 - ap_1)$ is temperature-independent and positive, by virtue of the inequality $p_2 \geq \varphi_1 p_1 > \frac{a}{b} p_1$. Therefore, there exists at least one ρ_s in the interval [a, b), such that $h(\rho_s) = 0$. By a procedure similar to that used in Subsection 3.2 for the polynomial q, it is possible to prove that $h(\rho)$ has a unique real root ρ_s in [a, b]. The proof is omitted here for the sake of brevity. Therefore, with a, b, p_1 and p_2 fixed in a such a way that both (3.6) and (3.10) hold, for each value of $\alpha(t_1 - t_2) \in [0, \psi_1]$, there is a unique root $\rho_s(\alpha(t_1 - t_2)) \in [a, b)$ of the polynomial h. $\rho_s(\alpha(t_1 - t_2))$ is a decreasing function of $\alpha(t_1 - t_2)$; in particular, when $\alpha(t_1 - t_2) = \psi_1$, $\rho_s = a$ and when $\alpha(t_1 - t_2) = 0$, the fourth degree polynomial h reduces to the second degree polynomial

(3.40) $$n(\rho) = - 3(b - a)(ap_1\rho^2 - 2b^2 p_2 \rho + b^2 a p_1) ,$$

---

[4] If $\alpha(t_1 - t_2) \geq 0$, then $\alpha(t_s - t_2)$ is also non-negative by virtue of the linear dependence of t on ρ and this is sufficient to gurantee the non-positiveness of the radial stress (3.36). The non-positiveness of the circumferential stress (3.37) can be proven with arguments similar to those used in the footnote 3 for the case $\alpha(t_1 - t_2) > \psi_2$.

a result already obtained in [6]. In this case the explicit value of the radius ρ_s can be easily calculated and it holds that $\rho_s = b\,\dfrac{b\,p_2 - \sqrt{b^2p_2^2 - a^2p_1^2}}{a\,p_1}$. Figure 4 shows the behaviour of ρ_s as a function of $\alpha(t_1 - t_2)$ in the interval $[0, \psi_1]$. This result has been obtained using the following parameter values: a = 1. m, b = 2. m, p_1 = 1. MPa, p_2 = 0.536 MPa, E = 5000. MPa, the corresponding value of ψ_1 is 0.85 10^{-4}.

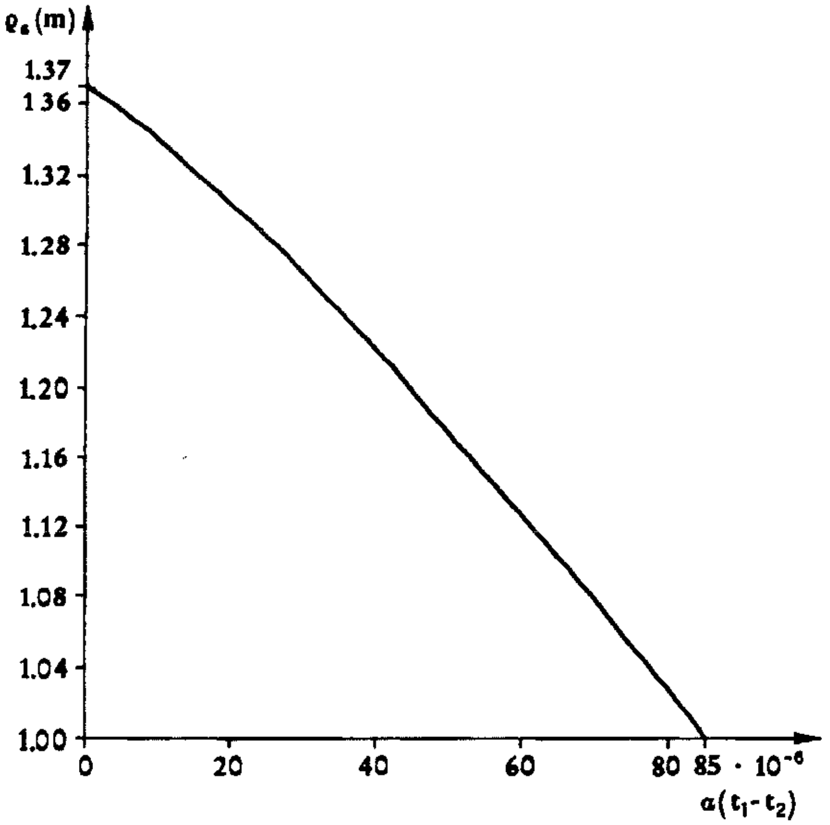

Figure 4. Radius ρ_s vs. $\alpha(t_1 - t_2)$.

The radial displacement corresponding to the stress field (3.36)-(3.37) in the region Ω_{2s} where the inelastic strain is nil is thus

$$(3.41)\quad u(\rho) = \frac{1}{E}\left[-(1+\nu)\frac{b^2\rho_1^2(p_2 - p_s)}{b^2 - \rho_s^2}\frac{1}{\rho} + (1-\nu)\frac{p_s\rho_s^2 - p_2b^2}{b^2 - \rho_s^2}\right] +$$

$$+\frac{\alpha(t_s - t_2)}{3(b^2 - \rho_s^2)}\left[(1-2\nu)(b+\rho_s)\frac{\rho^2}{2} + (\nu - 1)(b^2 + b\rho_s + \rho_s^2)\rho - (1+\nu)b^2\rho_s^2\frac{1}{\rho}\right] +$$

$$+\frac{\alpha}{b-\rho_s}\left[-(t_s-t_2)\frac{\rho^2}{2}+(bt_s-\rho_s t_2)\rho\right], \qquad \rho\in[\rho_s, b].$$

With this we are now in a position to calculate constant d contained in (3.34) by imposing the continuity of the radial displacement at $\rho = \rho_s$ and then determining the radial displacement in Ω_{1s},

$$u(\rho) = \frac{ap_1}{E}\ln\left(\frac{\rho_s}{\rho}\right) + \frac{\rho_s}{(b^2-\rho_s^2)E}\left[(\rho_s^2+b^2)p_s - 2b^2p_2 + \nu(b^2-\rho_s{}^2)p_s\right] + \tag{3.42}$$

$$+\frac{\alpha}{b-\rho_s}\left[-(t_s-t_2)\frac{\rho^2}{2}+(bt_s-\rho_s t_2)\rho\right] +$$

$$-\frac{\alpha\rho_s}{6(b^2-\rho_s^2)}(t_s-t_2)(\rho_s^2+b\rho_s+4b^2), \qquad \rho\in[a, \rho_s].$$

In view of (3.35) and (3.42), the circumferential inelastic strain is

$$\varepsilon_\theta^a(\rho) = \frac{1}{\rho}\ \{\ \frac{1}{2}\rho^2\frac{\alpha(t_s-t_2)}{b-\rho_s} + \frac{ap_1}{E}\ln(\rho_s/\rho) - \frac{\nu}{E}ap_1 + \tag{3.43}$$

$$+\frac{\rho_s}{(b^2-\rho_s^2)E}\left[(\rho_s^2+b^2)p_s - 2b^2p_2 + \nu(b^2-\rho_s{}^2)p_s\right] +$$

$$-\frac{\alpha\rho_s}{6(b^2-\rho_s^2)}(t_s-t_2)(\rho_s^2+b\rho_s+4b^2)\}, \qquad \rho\in[a, \rho_s].$$

From (3.38), it follows immediately that $\varepsilon_\theta^a(\rho_s) = 0$. Moreover, ε_θ^a is positive in the interval $[a, \rho_s)$. In proof of this, we first note that the sign of the circumferential inelastic strain (3.43) coincides with the sign of the function $g(\rho) = \rho\ \varepsilon_\theta^a(\rho)$, which by setting $\tilde{\rho} = \sqrt{\frac{(b-\rho_s)a\,p_1}{\alpha(t_s-t_2)E}}$, has a minimum at $\rho = \tilde{\rho}$. The non-negativeness of g in $[a, \rho_s)$ follows from the inequality $\tilde{\rho} > \rho_s$, which in turn derives from the condition $p_2 > \frac{a(\rho_s^3+3b^2\rho_s-b^3)}{3\,b^2\rho_s^2}\,p_1$, which is trivial to verify through the inequalities $\frac{a(\rho_s^3+3b^2\rho_s-b^3)}{3\,b^2\rho_s^2} < \varphi_1$ and (3.6).

4. Conclusion

The analysis performed in Section 3 shows that when a linear temperature distribution is present in the circular ring, fractures can arise, not only from the inner boundary, as occurs when the temperature is uniform [6], but from the outer boundary as well. Figure 5 depicts the behaviour of the cracked region as $\alpha(t_1 - t_2)$ varies from 0, for $p_2 \in (\varphi_1 p_1, \varphi_2 p_1)$. In particular, for $\alpha(t_1 - t_2) = 0$ the cracked region is delineated by radius $\rho_s = b \frac{b\, p_2 - \sqrt{b^2 p_2^2 - a^2 p_1^2}}{a\, p_1}$ (Figure 5 a); for values of $\alpha(t_1 - t_2)$ belonging to the interval $(0, \psi_1)$, the radius ρ_s, root of the polynomial (3.39), decreases until it coincides with the inner radius a of Ω for $\alpha(t_1 - t_2) = \psi_1$ (Figure 5 b). When $\alpha(t_1 - t_2)$ belongs to the interval (ψ_1, ψ_2), the circular ring is entirely compressed and then the inelastic strain is nil (Figure 5 c). Finally, for $\alpha(t_1 - t_2) \geq \psi_2$ (Figure 5 d), the cracked region is determined by the radius ρ_r, root of the polynomial 3.25; ρ_r equals b for $\alpha(t_1 - t_2) = \psi_2$ and decreases as $\alpha(t_1 - t_2)$ increases. Therefore the radial extension $b - \rho_r$ of the cracked region increases, spreading towards the interior of Ω.

For $p_2 > \varphi_2 p_1$, ψ_1 is equal to 0 and the behaviour of the crack region is that depicted in Figure 6; in this case, the inelastic strain is different from zero only starting at the outer boundary.

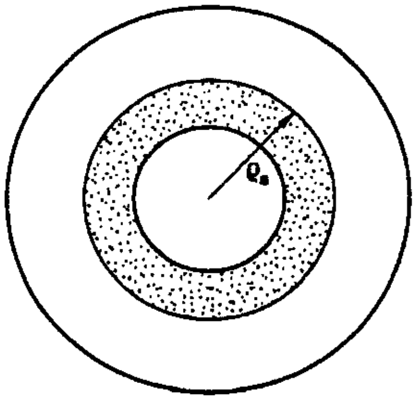

Figure 5 a. $\alpha(t_1 - t_2) = 0$.

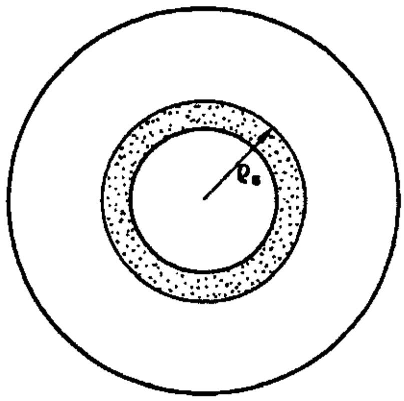

Figure 5 b. $\alpha(t_1 - t_2) \in (0, \psi_1)$.

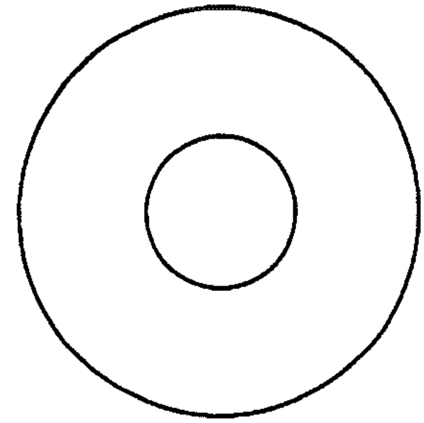

Figure 5 c. $\alpha(t_1 - t_2) \in [\psi_1, \psi_2]$.

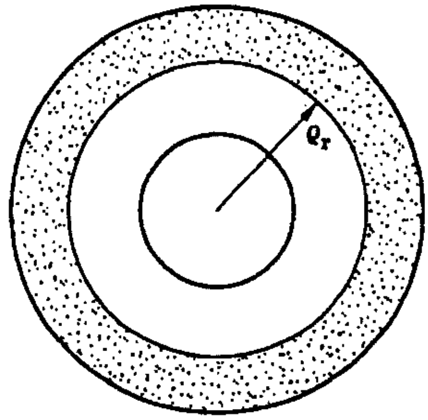

Figure 5 d. $\alpha(t_1 - t_2) > \psi_2$.

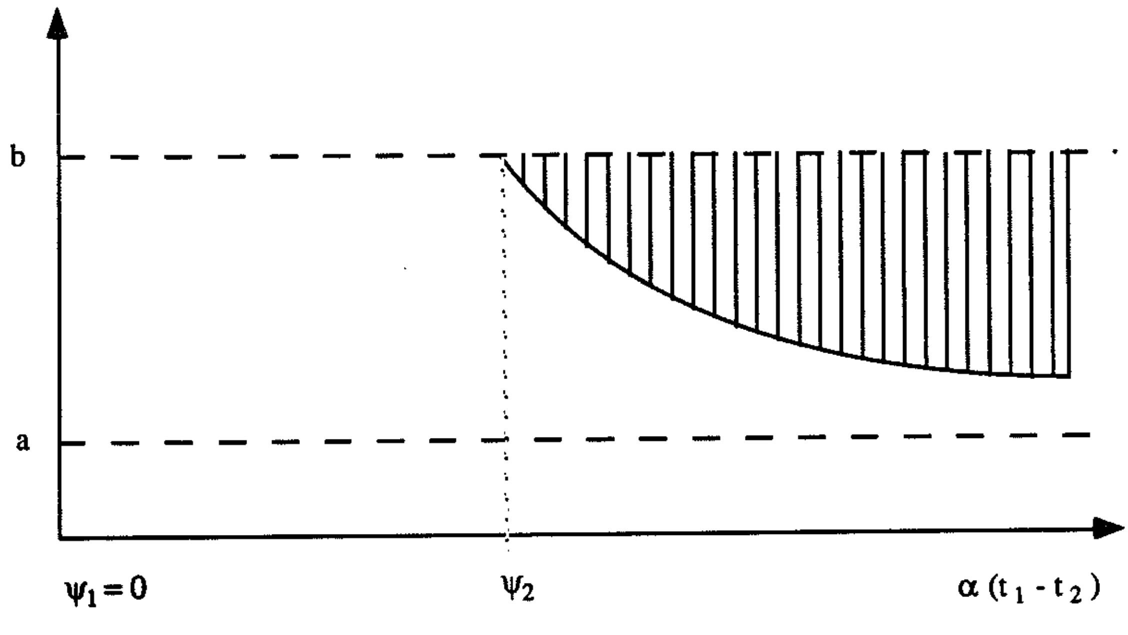

Figure 6. Behaviour of the cracked region for $p_2 > \varphi_2 p_1$ when $\alpha(t_1 - t_2)$ varies.

References

1. M. Como, M. Lembo, "Sulla modellazione termomeccanica delle attività di una caldera: deformazioni e fratturazioni allo stato eruttivo ed applicazioni al bradisismo flegreo". Atti del Convegno Nazionale Meccanica dei Materiali e delle Strutture in ricordo di Riccardo Baldacci e Michele Capurso, Roma 25-27 ottobre 1989.

2. C. Guidi, "Influenza della temperatura sulle costruzioni murarie". Atti R. Accad. delle Scienze di Torino, (1906).
3. G. Del Piero, "Constitutive equations and compatibility conditions of the externalk loads for linear elastic masonry-like materials". Meccanica **24**, 3 pp. 150-162 (1989).
4. M. Lucchesi, C. Padovani, G. Pasquinelli, "A numerical method for solving the equilibrium problem of masonry-like solids subjected to thermal loads". In preparation (1995).
5. S. P. Timoshenko and J. N. Goodier, Theory of elasticity. McGraw-Hill Kogakusha, Ltd (1970).
6. S. Bennati and C. Padovani, "Soluzioni esplicite per problemi di equilibrio di solidi non resistenti a trazione". Atti del XI Congresso Nazionale dell'Associazione Italiana di Meccanica Teorica e Applicata, Meccanica dei Solidi e delle Strutture, pp. 73-78, Trento 28 settembre-2 ottobre 1992.
7. M. Lucchesi, C. Padovani and A. Pagni, "A numerical method for solving equilibrium problems of masonry-like solids". Meccanica, 29 pp. 175-193 (1994).