Chapter 1
Factoring block Fiedler Companion Matrices*

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Abstract When Fiedler published his "*A note on Companion matrices*" in 2003 in Linear Algebra and its Applications, he could not have foreseen the significance of this elegant factorization of a companion matrix into essentially two-by-two Gaussian transformations, which we will name *(scalar) elementary Fiedler factors*. Since then, researchers extended these results and studied the various resulting linearizations, the stability of Fiedler companion matrices, factorizations of block companion matrices, Fiedler pencils, and even looked at extensions to non-monomial bases. In this chapter, we introduce a new way to factor block Fiedler companion matrices into the product of scalar elementary Fiedler factors. We use this theory to prove that, e.g., a block (Fiedler) companion matrix can always be written as the product of several scalar (Fiedler) companion matrices. We demonstrate that this factorization in terms of elementary Fiedler factors can be used to construct new linearizations. Some linearizations have notable properties, such as low band-width, or allow for factoring the coefficient matrices into unitary-plus-low-rank matrices. Moreover,

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we will provide bounds on the low-rank parts of the resulting unitary-plus-low-rank decomposition. To present these results in an easy-to-understand manner we rely on the flow-graph representation for Fiedler matrices recently proposed by Del Corso and Poloni in Linear Algebra and its Applications, 2017.

1.1 Introduction

It is well known that, given a monic polynomial $p(z) = z^d + a_{d-1}z^{d-1} + \cdots + a_0$, we can build a *(column) companion matrix*[2] that has the roots of $p(z)$ as eigenvalues and whose entries are just 1, 0, and the coefficients of $p(z)$:

$$\Gamma_p := \begin{bmatrix} & & & & -a_0 \\ 1 & & & & -a_1 \\ & 1 & & & -a_2 \\ & & \ddots & & \vdots \\ & & & 1 & -a_{d-1} \end{bmatrix}. \tag{1.1}$$

We remark that constructing a companion matrix is *operation-free*: no arithmetic operations are needed to get Γ_p from p. The pencil $zI - \Gamma_p$ is an example of a *linearization* for $p(z)$. A formal definition of a linearization is the following.

Definition 1. Let $p(z)$ be a $k \times k$ degree d matrix polynomial. Then, the pencil $A - zB$ is a linearization of $p(z)$ if there exist two unimodular matrices $E(z), F(z)$ (i.e., matrix polynomials with non-zero constant determinant) such that $I_{k(d-1)} \oplus p(z) = E(z)(A - zB)F(z)$.

In the above setting, when $B = I$, we say that A is a *companion matrix*[3]. In the rest of the paper, we will never deal with the matrices $E(z)$ and $F(z)$ directly. For us, it is sufficient to know that the column companion matrix identifies a linearization, and that any matrix similar to it still leads to a linearization (see, for instance, [24]).

The fact that a linearization is operation-free can be formalized as follows:

Definition 2. A companion matrix C of a polynomial $p(z) = a_0 + a_1 z + \cdots + z^d$ is called *operation-free* if each of the elements in C is either 0, 1, or one of the scalars a_j (possibly with a minus sign). Similarly, for a block companion matrix linearizing a matrix polynomial, we say that it is *operation-free* if its entries are either $0, 1$, or one entry in the coefficients of the matrix polynomial (possibly with a minus sign).

In 2003 Fiedler showed that Γ_p in (1.1) can be factored as the product of d *(scalar) elementary Fiedler factors* which are equal to the identity matrix with the

[2] We typically abbreviate column companion matrix and omit the word column, unless we want to emphasize it.

[3] Often the term companion matrix indicates a matrix obtained from the coefficients of the polynomial without performing arithmetic operations. Here we have not added this constraint into the definition but — as we will discuss later — all the matrices obtained in our framework satisfy it.

only exception of a 1×1 or 2×2 diagonal block [27]. This factorization has a remarkable consequence: the product of these factors in *any order* provides still a linearization for $p(z)$, since its characteristic polynomial remains $p(z)$. Companion matrices resulting from permuting the elementary Fiedler factors are named *Fiedler companion matrices*.

This theory has then been extended to matrix polynomials, by operating blockwise, and to more general constructions than just permutations of the original factors, which led to Fiedler pencils with repetitions [17,36], generalized Fiedler pencils [2,18] and generalized Fiedler pencils with repetitions [20]. These ideas sparked the interest of the numerical linear algebra community: several researchers have tried to find novel linearizations in this class with good numerical properties [26,28], or which preserve particular structures [21,23,26,33].

The construction of Fiedler companion matrices is connected with permutations of $\{0,\dots,d-1\}$. In this framework, the development of explicit constructions for palindromic, even-odd, and block-symmetric linearizations in the Fiedler class is investigated in [20,21,23,26]. At the same time, several authors have investigated vector spaces of linearizations with particular structures [29,30], and linearizations with good numerical properties [13,19,35]. Recently, a new graph-based classification of Fiedler pencils has been recently introduced by Poloni and Del Corso [31], and has been used to count the number of Fiedler pencils inside several classes, as well as to describe common results in a simpler way.

The aim of this paper is to extend the theory proposed in [31] by introducing manipulations that operate *inside* the blocks and factor them, but at the same time remain *operation-free*, which is a key property of Fiedler-like pencils.

We show that these tools can be used to construct new factorizations of block Fiedler companion matrices. In particular, we prove that (under reasonable assumptions on the constant coefficient) any block Fiedler companion matrix of a monic $k \times k$ matrix polynomial can be factored into k Fiedler companion matrices of *scalar* polynomials. This approach extends a similar factorization for column companion matrices by Aurentz, Mach, Robol, Vandebril, and Watkins [4]. The graph-based representation makes it easy to visualize the unitary-plus-low-rank structure of (block) Fiedler companion matrices; it also provides upper bounds on the rank in the low-rank correction of the unitary-plus-low-rank matrix.

Aurentz, et al. [4] developed a fast method to compute eigenvalues of matrix polynomials by factoring the column block companion matrix into scalar companion matrices and then solving a product eigenvalue problem exploiting the unitary-plus-rank-1 structure of the factors. As we will show in Theorem 8, some block Fiedler companion matrices can be similarly factored as a product of row and column companion matrices (appropriately padded with identities). This makes the algorithm in [4] applicable to devise a fast solver; in fact, this idea is exploited in [3] to describe an algorithm for computing the eigenvalues of unitary-plus-rank-k matrices. As an alternative, after a preliminary reduction of the matrix to Hessenberg form we can employ the algorithm proposed in [8] for generic unitary-plus-rank-k matrices.

As a final application, to illustrate the power of the new representation, we show by an example that these techniques can easily produce novel companion matrices such as thin band or factorizations in which symmetry of the original problem is reflected.

Throughout the article we adopt the following notation: I_n denotes the identity matrix of size $n \times n$; its subscript is dropped whenever its size is clear from the context; e_j denotes the j-th vector of the canonical basis of $\mathbb{C}^n$; and Z denotes the downshift matrix, having ones on the first subdiagonal and zeros elsewhere.

1.2 Fiedler graphs and (block) Fiedler companion matrices

As mentioned in the introduction, Fiedler [27] showed that the column companion matrix Γ_p in (1.1) can be factored as $\Gamma_p = F_0 F_1 \cdots F_{d-1}$, where the F_i, named *elementary Fiedler factors*, are matrices which are equal to the identity except for a diagonal block of size at most 2×2. More precisely,

$$F_0 = F_0(a_0) = (-a_0) \oplus I_{d-1}, \qquad F_i = F_i(a_i) = I_{i-1} \oplus \begin{bmatrix} 0 & 1 \\ 1 & -a_i \end{bmatrix} \oplus I_{d-i-1}. \quad (1.2)$$

We omit the parameters in parentheses if they are clear from context.

The key result due to Fiedler [27] is that any permutation of the factors in the above factorization $C = F_{\sigma(0)} \cdots F_{\sigma(d-1)}$, where σ is a permutation of $\{0, \ldots, d-1\}$, is still a companion matrix for $p(z)$. We call linearizations obtained in this way *Fiedler linearizations* (and the associated matrices *Fiedler companion matrices*). Throughout the paper, we use the letter Γ (with various subscripts) to denote a column (or, occasionally, row) companion matrix, possibly padded with identities, i.e., $I_{h_1} \oplus \Gamma_p \oplus I_{h_2}$, and the letter C to denote Fiedler companion matrices. We will heavily rely on the flow-graph representation established by Poloni and Del Corso [31], so we introduce it immediately.

The *elementary flow graph* associated to each elementary Fiedler factor is the graph shown in Figure 1.1.

Fig. 1.1 Elementary flow-graphs corresponding to $F_i(a)$ (for $i > 0$) and to $F_0(a)$.

To construct the *Fiedler graph* associated to a product of Fiedler elementary factors $P = F_{i_1}(a_1) \cdots F_{i_k}(a_k)$, each of size $d \times d$, we first draw d horizontal lines labelled with the integers $1, \ldots, d$ (with 1 at the top); this label is called the *height* of a line. Then, we stack horizontally (in the same order in which they appear in P) the graphs corresponding to the elementary factors. These must be properly aligned vertically so that F_{i_j} touches the lines at heights i_j and $i_j + 1$ (or $i_j + 1$ only if $i_j = 0$).

Straight horizontal edges are drawn in grey to distinguish them from the edges of the flow graph.

A Fiedler graph is a representation of multiplying a row vector v, having components v_i, with P. This vector-matrix multiplication can be seen as a composition of additions and multiplications by scalars, and the flow graph depicts these operations, as follows. We imagine that for each i the entry v_i of a row vector v enters the graph from the left along the edge at height i, and moves towards right. A scalar traveling from left to right through an edge with a box carrying the label a, is multiplied by a before it proceeds; an element following a straight edge (with no box) is left unchanged; a scalar arriving at a node with two outgoing edges is duplicated; and finally when two edges meet the corresponding values are added. If one carries on this process, the result at the right end of the graph are the entries of vP, with the jth entry appearing at height j.

Example 1. Consider the simple case in which $d = 2$, and $P = F_0(a_0)F_1(a_1)$ The flow graph associated to P is shown in Figure 1.2.

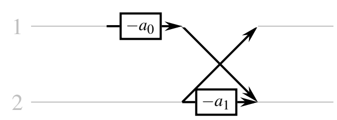


Fig. 1.2 Flow graph for $P = F_0(a_0)F_1(a_1)$.

The element v_1 enters from the left at the first row, hits $-a_0$ resulting in a multiplication $-v_1a_0$ and then moves down to the second row. The element v_2 enters at the second row and is duplicated. Its first clone moves to the top row, its second clone gets multiplied with $-a_1$ and then also ends up in the second row. Since both $-v_1a_0$ and $-v_2a_1$ end up in the bottom row, we add them together. As a result we obtain $[v_2, \ -v_2a_1 - v_1a_0]$ at the right end of the graph, which is exactly $[v_1, \ v_2]P$.

The power of this framework lies in the fact that representing a matrix by a Fiedler graph enables us to easily draw conclusions on the structure of the matrix and its associated elementary Fiedler factors. For instance, to determine the content of the (i, j)-th entry of a product of Fiedler factors it is sufficient to inspect the paths on the graph that start from the left at height i and end on the right at height j.

Consider for instance the column companion matrix (1.1) of degree $d = 4$. The associated Fiedler graph is depicted in Figure 1.3. Indeed, entering the graph from the left on row $i > 1$ yields two elements: one on column $i-1$ (the edge pointing one row up), and the other is $-a_{i-1}$, which follows a descending path until the very last column. This implies that

$$e_i^T \Gamma_p = \left[0_{i-2} \ 1 \ 0_{d-i} \ -a_{i-1}\right], \qquad i > 1,$$

which is exactly the i-th row of Γ_p. The case $i = 1$ produces the row vector

$$e_1^T \Gamma_p = \left[0_{d-1} \ -a_0\right]$$

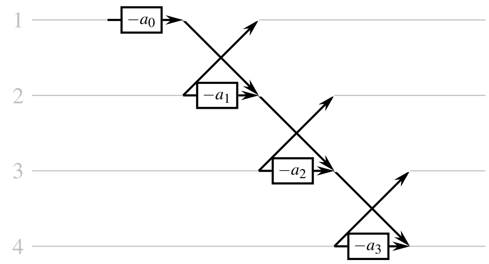


Fig. 1.3 Fiedler graph associated to a column companion matrix of a degree 4 monic polynomial.

Some Fiedler companion matrices are simpler, some are more complex. For instance, we have already considered $F_0 \cdots F_{d-1}$, which is the usual column companion matrix. The transpose of this matrix is $F_{d-1} \cdots F_0$ (all the Fiedler factors are symmetric), which is a Fiedler companion matrix with all the coefficients on the last row:

$$\Gamma_p^T = F_{d-1} \cdots F_0 = \begin{bmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ -a_0 & -a_1 & \dots & -a_{d-1} \end{bmatrix}. \tag{1.3}$$

We refer to (1.3) as a *row companion* matrix[4].

The flow graphs help us to visualize the structure of the factorization in elementary Fiedler factors. For example, the Fiedler companion matrix with $\sigma = (0,1,3,2)$ is associated with the graph in Figure 1.4. Note that the order of the elementary flow graphs coincides with the order of the elementary Fiedler factors.

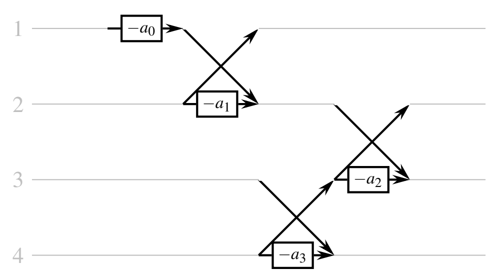


Fig. 1.4 Fiedler graph associated to the matrix $F = F_0F_1F_3F_2$.

Since F_i and F_j commute whenever $|i-j| > 1$, *different* permutations σ may correspond to the *same* matrix. For example, in Figure 1.4, F_1 and F_3 commute, so $F_0F_1F_3F_2 = F_0F_3F_1F_2$. In terms of graphs, we can 'compress' a Fiedler graph by

[4] This is a variation on the usual construction of a row companion matrix having the elements a_i in its first row.

drawing the elements F_i and F_j one on top of the other whenever $|i-j|>1$, or swap them; these are operations that do not alter the topological structure of the graph nor the height of each edge.

For example, we can draw the graph in Figure 1.4 in a compact way as in Figure 1.5. Moreover, we can immediately read off the following equivalences

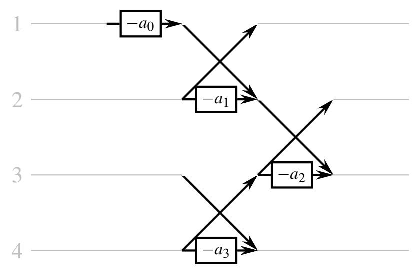


Fig. 1.5 A more compact representation of the graph in Figure 1.4.

$F_0F_3F_1F_2 = F_0F_1F_3F_2 = F_3F_0F_1F_2$, since the factor F_3 is free to slide to the left of the diagram.

If we allow for repositioning factors in this way, two Fiedler companion matrices coincide if and only if their graphs do (see [31] for a detailed analysis of this characterization).

Remark 1. There are a number of different 'standard forms' [31, 36], i.e., canonical ways to order the factors in a Fiedler product or draw the corresponding graph. In this paper, we do not follow any of them in particular. Rather, when drawing the graph associated to a Fiedler companion matrix C, we try to draw them so that the elements form a connected twisted line. (In practice, this can be obtained by drawing first the elementary factor F_{d-1} at the bottom of the graph, and then $F_{d-2}, F_{d-3}, \dots$ each immediately at the left or right of the last drawn element.) This choice gives a better visual interpretation of some of our results; See for instance the discussion after Theorem 6.

We now generalize this construction to monic matrix polynomials. Given a degree-d matrix polynomial with $k \times k$ coefficients

$$P(z) = Iz^d + A_{d-1}z^{d-1} + \cdots + A_0 \in \mathbb{C}^{k\times k}[z],$$

we can factor its column companion matrix as

$$\Gamma_P = \begin{bmatrix} 0 & 0 & \cdots & 0 & -A_0 \\ I_k & 0 & & 0 & -A_1 \\ & I_k & & 0 & -A_2 \\ & & \ddots & \vdots & \vdots \\ & & & I_k & -A_{d-1} \end{bmatrix} = \mathscr{F}_0\mathscr{F}_1\cdots\mathscr{F}_{d-1},$$

where all $\mathscr{F}_i$ are *block elementary Fiedler factors*, that is

$$\mathscr{F}_0 = \mathscr{F}_0(A_0) = (-A_0) \oplus I_{k(d-1)}, \qquad \mathscr{F}_i = \mathscr{F}_i(A_i) = I_{(i-1)k} \oplus \begin{bmatrix} 0 & I \\ I & -A_i \end{bmatrix} \oplus I_{(d-i-1)k},$$

for all $i = 0, \ldots, d-1$. Again, each permutation of these factors gives a *(block) Fiedler companion matrix*. We can construct graphs associated to their products in the same way; the entities we operate on are now matrices instead of scalar entries. For instance, for $A \in \mathbb{C}^{k\times k}$, the active part of a block elementary Fiedler factor, which is the diagonal block differing from the identity, can be represented as in Figure 1.6. All the results of this section concerning the reordering the block elementary

$\mathscr{F}_i(A) =$  $\mathscr{F}_0(A) =$.

Fig. 1.6 Graph representing the active part of the block elementary Fiedler factor $\mathscr{F}_i(A)$, for $i > 0$ and of $\mathscr{F}_0(A)$.

Fiedler factors remain valid also in the block case. In particular, block Fiedler flow graphs represent the multiplication of a 'block row vector' $[V_1, V_2, \ldots, V_d] \in \mathbb{C}^{k\times kd}$ by a product of block Fiedler matrices.

This construction can be thought of as the "blocked" version of the one we had in the scalar case: we treat the blocks as atomic elements, which we cannot inspect nor separate. However, it does not have to be that way, and in the next section we will explore the idea of splitting these blocks into smaller pieces.

In particular, we will show in Section 1.3.1 how each of the block elementary Fiedler factors can be decomposed as a product of (scalar) elementary Fiedler factors. So we have a coarse (block) level factorization and graph; and a fine (entry) level factorization and corresponding graph.

1.3 Factoring elementary block Fiedler factors

We discuss the block Fiedler factors $\mathscr{F}_i$ for $i > 0$ and $i = 0$ in different subsections because they require different treatments.

1.3.1 Block factors $\mathscr{F}_i$, for $i > 0$

To ease readability and avoid additional notation we will use, in this section, $\mathscr{F}(A)$ to denote the active part (the one different from the identity) of an arbitrary block Fiedler factor $\mathscr{F}_i(A)$, for $i > 0$. In particular, we have

$$\mathscr{F}(A) := \begin{bmatrix} 0_{k\times k} & I_k \\ I_k & -A \end{bmatrix} \in \mathbb{C}^{2k\times 2k}. \tag{1.4}$$

Consider the graph of a single Fiedler factor $\mathscr{F}(A)$ given in the left of Figure 1.6. This graph represents the multiplication of $\mathscr{F}(A)$ by a block row vector $[V_1,V_2]$, so the two horizontal levels in the graph correspond to the blocks $1:k$ and $k+1:2k$ (in Fortran/Matlab notation).

We show that it can be converted into a more fine-grained graph in which each line represents a single index in $\{1,2,\ldots,2k\}$. We call this construction a *scalar-level graph* (as opposed to the *block-level* graph appearing in Figure 1.6).

We first show the result of this construction using flow graphs, to get a feeling of what we are trying to build.

Example 2. Let $k=3$, and $A=(a_{ij})$, with $1\le i,j\le 3$. In this case, the elementary block factor $\mathscr{F}(A)$ in Figure 1.6 has size 6×6. A scalar-level graph associated to $\mathscr{F}(A)$ is depicted in Figure 1.7.

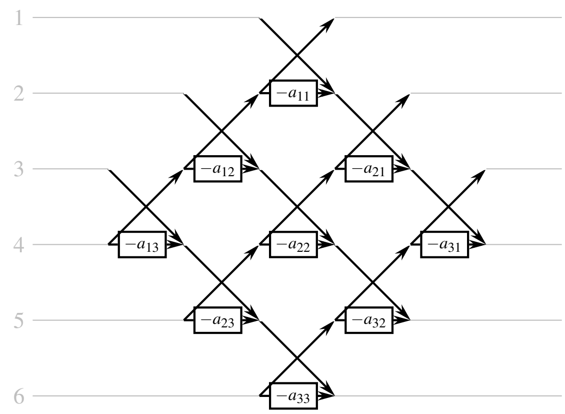


Fig. 1.7 Scalar-level graph associated to $\mathscr{F}(A)$ where A is a 3×3 matrix with entries a_{ij}.

Theorem 1. *Let $\mathscr{F}(A)\in\mathbb{C}^{2k\times 2k}$ be a block elementary Fiedler factor as defined by equation* (1.4)*. Then, $\mathscr{F}(A)$ can be factored into k^2 scalar elementary Fiedler factors associated to the elements of the matrix $A=(a_{ij})$, as follows*

$$\mathscr{F}(A) = \Gamma_k\Gamma_{k-1}\cdots\Gamma_1,$$
$$\Gamma_j = F_j(a_{1j})F_{j+1}(a_{2j})\cdots F_{j+k-1}(a_{kj}), \quad j=1,2,\ldots,k.$$

Proof. From a linear algebra viewpoint, the proof can be obtained simply multiplying the various factors together. Alternatively, one can construct the $2k\times 2k$ analogue of Figure 1.7, and follow the edges of the graph to check the value of each matrix element. $\square$

Remark 2. Each Γ_j is a (scalar) column companion matrix padded with identities, i.e., it has the form $I_{j-1} \oplus \Gamma_{a_j} \oplus I_{k-j}$, where Γ_{a_j} is a particular column companion matrix of size $k+1$. Indeed,

$$\Gamma_j = \begin{bmatrix} I_{j-1} & & \\ & Z + a_j e_{k+1}^T & \\ & & I_{k-j} \end{bmatrix}, \qquad a_j := \begin{bmatrix} 1 \\ -Ae_j \end{bmatrix},$$

where Z is the downshift matrix, with ones on the first subdiagonals and zero elsewhere. In the following we call *column (resp. row) companion matrices* also matrices with this form, ignoring the additional leading and trailing identities.

We could have proved that $\mathscr{F}(A)$ is the product of k column companion matrices also by inspecting the associated scalar-level graph. Indeed, for simplicity let us restrict ourselves to the running example in Figure 1.7 of size 6×6. We replot the graph in Figure 1.7 inserting gaps between some elements. Hence, the graph is the

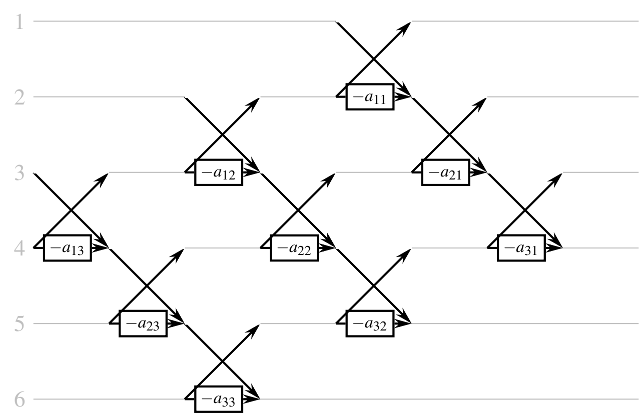


Fig. 1.8 Replot of the graph in Figure 1.7 with gaps between descending diagonal lines. This reveals the factorization into companion matrices.

concatenation of three sequences of factors that can be arranged in a descending diagonal line each. These correspond precisely to $\Gamma_3, \Gamma_2, \Gamma_1$; indeed, any descending line of diagonal factors forms a column companion matrix (padded with identities).

Example 3. Written explicitly, the factors that compose the 6×6 matrix $\mathscr{F}(A)$ of our running example are

$$\mathscr{F}(A) = \left[\begin{array}{ccc|ccc} 0&0&0 & 1 & 0 & 0 \\ 0&0&0 & 0 & 1 & 0 \\ 0&0&0 & 0 & 0 & 1 \\ \hline 1&0&0 & -a_{11} & -a_{12} & -a_{13} \\ 0&1&0 & -a_{21} & -a_{22} & -a_{23} \\ 0&0&1 & -a_{31} & -a_{32} & -a_{33} \end{array}\right] = \begin{bmatrix} 0 & I \\ I & -A \end{bmatrix} = \Gamma_3 \Gamma_2 \Gamma_1$$

$$= \left[\begin{array}{cc|cccc} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ \hline 0&0&0&0&0&1 \\ 0&0&1&0&0&-a_{13} \\ 0&0&0&1&0&-a_{23} \\ 0&0&0&0&1&-a_{33} \end{array}\right] \left[\begin{array}{c|cccc|c} 1&0&0&0&0&0 \\ \hline 0&0&0&0&1&0 \\ 0&1&0&0&-a_{12}&0 \\ 0&0&1&0&-a_{22}&0 \\ 0&0&0&1&-a_{32}&0 \\ \hline 0&0&0&0&0&1 \end{array}\right] \left[\begin{array}{cccc|cc} 0&0&0&1&0&0 \\ 1&0&0&-a_{11}&0&0 \\ 0&1&0&-a_{21}&0&0 \\ 0&0&1&-a_{31}&0&0 \\ \hline 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{array}\right].$$

Remark 3. We can also factor $\mathscr{F}(A)$ into row-companion matrices. If we group the elements in Figure 1.7 following ascending diagonals, we obtain an analogous factorization, shown in Figure 1.9, into row companion matrices, each containing entries from one row of A. This new decomposition can be identified from the graph in Figure 1.9. This result is immediately obtained by applying Theorem 1 to $\mathscr{F}(A)^T$.

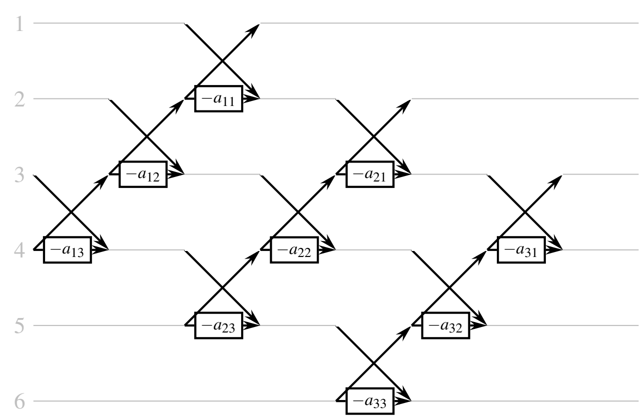


Fig. 1.9 Replot of the graph in Figure 1.7 adding gaps between ascending diagonal lines.

Remark 4. Poloni and Del Corso [31], only consider elementary blocks of the form $\mathscr{F}_i = \mathscr{F}_i(A_i)$, where A_i is a coefficient of the matrix polynomial $P(z) = Iz^d + A_{d-1}z^{d-1} + \cdots + zA_1 + A_0$. Here, we allow for a more general case, where two elementary blocks $F_i(a)$ and $F_i(b)$ with the same index i can have different parameters $a \neq b$.

1.3.2 The block Fiedler factor $\mathscr{F}_0$

In this section we provide a factorization for the block $\mathscr{F}_0$ into the product of scalar companion matrices. Note that the active part of the elementary Fiedler factor $\mathscr{F}_0$ is confined to the first k rows instead of $2k$ rows like the factors $\mathscr{F}_i$ for $i > 0$.

Since our goal is to build linearizations of matrix polynomials, we can perform a preliminary transformation that does not alter the spectrum. If there exist two invertible matrices E, G, such that $EP(\lambda)G = Q(\lambda)$, then the matrix polynomials $P(\lambda)$ and $Q(\lambda)$ are said to be *strictly equivalent* [24]. When this happens, their spectra (both finite and infinite) coincide. If the matrices E and G are also unitary, then the condition number of their eigenvalues also matches[5], hence we need not worry about instabilities resulting from using this factorization.

In particular, we can choose an orthogonal (resp. unitary) matrix E and let $G = E^T \Pi$ (resp. $G = E^H \Pi$), where Π is the counter-identity matrix, so that the monic matrix polynomial $P(\lambda)$ is transformed into a monic polynomial $Q(\lambda)$ with A_0 lower anti-triangular (i.e., $(A_0)_{i,j} = 0$ whenever $i + j \leq k$). This unitary matrices can be obtained by computing the Schur form of $A_0 = QTQ^T$, and then setting $E = Q^T$. For these reasons, we may assume that A_0 is lower anti-triangular.

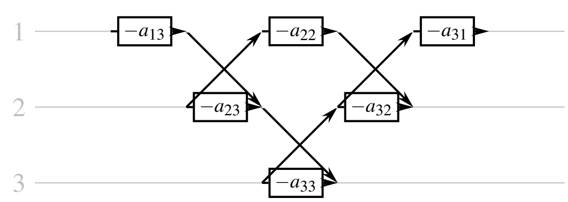
Theorem 2. *Let $A \in \mathbb{C}^{k \times k}$ be a lower anti-triangular matrix. Then, $\mathscr{F}_0(A)$ can be factored as the product of $\frac{k(k+1)}{2}$ scalar elementary Fiedler factors as follows*

$$\mathscr{F}_0(A) = \Gamma_k \Gamma_{k-1} \cdots \Gamma_1,$$
$$\Gamma_j = F_0(a_{k-j+1,j}) F_1(a_{k-j+2,j}) \cdots F_{j-1}(a_{k,j}).$$

Moreover, each Γ_j is a scalar column companion matrix (padded with identities).

The proof is analogous to the one of Theorem 1. Again, we can consider the flow graph associated with $\mathscr{F}_0$.

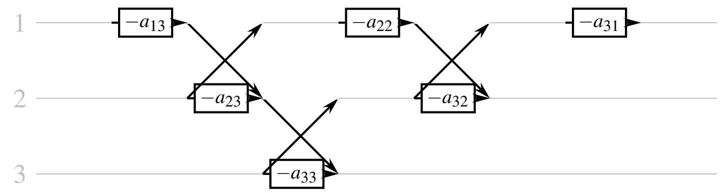
Example 4. Consider again $k = 3$. Then the flow-graph associated with $\mathscr{F}_0$ is



Separating the elementary factors into three descending diagonals we get the decompositions into three column companion matrices.

---

[5] Here, by condition number we mean the unhomogeneous absolute or relative condition number defined in [34] (see also [1] where several definitions are compared). It is easy to verify that substituting the change of basis in the formula for the unhomogeneous condition number in [34] does not change the result.



The explicit matrices are

$$\begin{aligned}\mathscr{F}_0 &= \begin{bmatrix} 0 & 0 & -a_{13} \\ 0 & -a_{22} & -a_{23} \\ -a_{31} & -a_{32} & -a_{33} \end{bmatrix} = \Gamma_3 \Gamma_2 \Gamma_1 \\ &= \begin{bmatrix} 0 & 0 & -a_{13} \\ 1 & 0 & -a_{23} \\ 0 & 1 & -a_{33} \end{bmatrix} \left[\begin{array}{cc|c} 0 & -a_{22} & 0 \\ 1 & -a_{32} & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c|cc} -a_{31} & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right].\end{aligned}$$

We can adapt this decomposition to work with lower triangular matrices, but the result is more complicated.

Theorem 3. *Let $A \in \mathbb{C}^{k\times k}$ be a lower triangular matrix. Then, $\mathscr{F}_0(A)$ can be factored as the product of k^2 scalar elementary Fiedler factors as follows*

$$\mathscr{F}_0(A) = \Gamma_1 \Gamma_2 \cdots \Gamma_k, \tag{1.5a}$$
$$\Gamma_j = F_0(a_{j,j}) F_1(a_{j+1,j}) \cdots F_{k-j}(a_{k,j}) F_{k-j+1}(0) F_{k-j+2}(0) \cdots F_{k-1}(0).$$

Moreover, each Γ_j is a scalar column companion matrix (padded with identities).

Again, we can prove this factorization either algebraically or by following the edges along the associated Fiedler graph, which is shown in Figure 1.10.

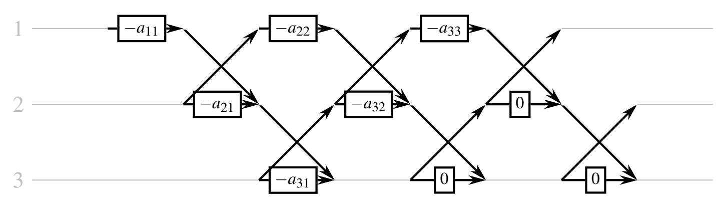


Fig. 1.10 Fiedler graph associated to $\mathscr{F}_0(A)$ where A is a lower triangular 3×3 matrix.

The additional blocks with zeros are in fact permutations necessary for positioning each element correctly. Even though this factorization is still operation-free, meaning that there are no arithmetic operations involving the a_{ij}, we see that this is only because the trailing elementary Fiedler factors have 0. Indeed, if one replaces the zeros appearing in Figure 1.10 with different quantities, the resulting product requires arithmetic operations. This is an instance of a more general result, linked to *operation-free* linearizations, as in Definition 2.

Theorem 4 ([31, 36]). *Consider the product $\mathscr{P} = M_1 M_2 \cdots M_\ell$, where for each $k = 1, 2, \ldots, \ell$ the factor M_k is an elementary Fiedler factor $F_{i_k}(a_{j_k})$. Then, $\mathscr{P}$ is operation-free for each choice of the scalars (or matrices) a_{j_k} if and only if between every pair of factors $M_{i_k}, M_{i_{k'}}$ with the same index $i_k = i_{k'} = i$ there is a factor with index $i+1$. In terms of diagrams, this means that between every two factors at height i there must appear a factor at height $i+1$.*

Again this theorem holds for the block version as well. While all the other products of Fiedler factors that we consider in this paper are operation-free *a priori* because of this theorem, the one in (1.5) does not satisfy this criterion. It is only operation-free because of the zeros.

Remark 5. It is impossible to find an operation-free factorization of $\mathscr{F}_0(A)$ for an unstructured A. Indeed, if there existed a factorization $\mathscr{F}_0(A) = M_1 M_2 \cdots M_{k^2}$, where each M_i is a scalar elementary Fiedler factor, then by writing $\mathscr{F}_0(A)^{-1} = M_{k^2}^{-1} \cdots M_2^{-1} M_1^{-1}$ one could solve any linear system $Ax = b$ in $O(k^2)$ flops, which is known to be impossible [32].

1.4 Factoring block companion matrices

In this section, we use the previous results to show that any block Fiedler companion matrix can be factored as $\mathscr{C} = C_1 C_2 \cdots C_k$, where each C_j is a scalar Fiedler companion. This generalizes the results for block column companion matrices from Aurentz et al. [4].

These factorizations have a nice property: they are low-rank perturbations of unitary matrices. This allows the design of fast algorithms for computing the eigenvalues of $\mathscr{C}$, by working on the factored form [4].

This novel factorization allows to build even more linearizations. When all factors C_j are invertible, all the cyclic permutations of the factors provide again linearizations for the same matrix polynomial, since they are all similar.

For column block companion matrices we have the following (see also Aurentz, et al. [4]).

Theorem 5. *Let $P(z) \in \mathbb{C}[z]^{k \times k}$ be a monic matrix polynomial of degree d with lower anti-triangular constant term A_0. Then, the associated block column companion matrix can be factored as a product of k scalar companion matrices of size $dk \times dk$.*

A formal proof will follow as a special case of Theorem 6. Here we only point out that this factorization is again easy to detect using the graph representation.

Example 5. Let $d = k = 3$ and $P(z) = Iz^3 + A_2 z^2 + A_1 z + A_0$, with A_0 lower anti-triangular. Then, using the scalar-level factorizations of each Fiedler block, the column companion matrix of $P(z)$ links to the flow graph in Figure 1.11.

It is easy to decompose this graph as the product of the three factors drawn in the figure in different colors. Moreover, each of these three factors is a column

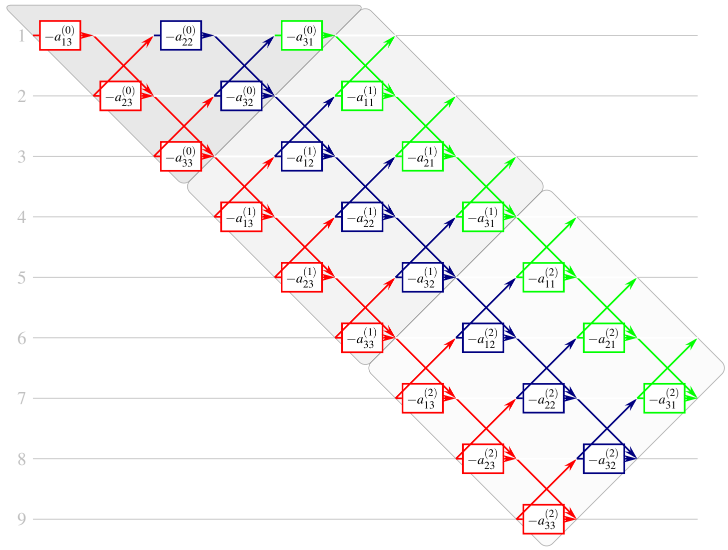


Fig. 1.11 Graph of the block column companion matrix associated to the monic matrix polynomial $P(z)$. The constant coefficient A is lower anti-triangular. To simplify the notation we used $a_{ij}^{(i)}$ to denote the entries of the matrices A_i.

companion matrix[6] constructed from a polynomial whose coefficients are a column of $[A_0^T \;\; A_1^T \;\; A_2^T]^T$.

This construction can be generalized to a generic block Fiedler companion. To prove the theorem formally, we need some additional definitions [22, 31] and a lemma which is a variation of [16, Proposition 2.12].

Definition 3. Let $P = F_{i_1} F_{i_2} \cdots F_{i_\ell}$ be a product of ℓ Fiedler elementary factors. For each $i = 0, \ldots, d-1$, the *layer* $\mathscr{L}_{i:i+1}(P)$ is the sequence formed by the factors of the form $F_i(a)$ and $F_{i+1}(b)$, for any values of a, b, taken in the order in which they appear in P.

Definition 4. Let $C = F_{\sigma(0)} F_{\sigma(1)} \cdots F_{\sigma(d-1)}$ be a Fiedler companion matrix, where σ is a permutation of $\{0, 1, \ldots, d-1\}$. We say that C has

- a *consecution* at i, $0 \le i \le d-2$, if $\mathscr{L}_{i:i+1}(C) = (F_i, F_{i+1})$;
- an *inversion* at i, $0 \le i \le d-2$, if $\mathscr{L}_{i:i+1}(C) = (F_{i+1}, F_i)$.

For instance, the Fiedler companion matrix whose associated graph is depicted in Figure 1.4 has two consecutions at 0 and 1, and an inversion at 2. Note that in the flow graph a consecution corresponds to the subgraph of F_i being to the left of the subgraph of F_{i+1}, and *vice versa* for an inversion. The definition extends readily to the block case.

[6] Similarly one could factor it into dk row companion matrices linked to polynomials of degree 3.

The layers of a factorization in elementary Fiedler factors uniquely define the resulting product as stated in the next lemma.

Lemma 1. *Let F and G be two products of (scalar or block) elementary Fiedler factors of size $d \times d$. If $\mathscr{L}_{i:i+1}(F) = \mathscr{L}_{i:i+1}(G)$ for all $i = 0, \dots, d-2$, then the two products can be reordered one into the other by only swapping commuting factors, and hence $F = G$ (as matrices).*

Proof. See [16, Proposition 2.12]. □

Theorem 6. *Let $\mathscr{C} = \mathscr{F}_{\sigma(0)} \cdots \mathscr{F}_{\sigma(d-1)}$ be a block Fiedler companion matrix of the monic matrix polynomial $P(z) = Iz^d + A_{d-1}z^{d-1} + \cdots + A_1 z + A_0 \in \mathbb{C}[z]^{k \times k}$, with the matrix A_0 lower anti-triangular. Then, $\mathscr{C} = C_1 C_2 \cdots C_k$, where each of the matrices C_j is a scalar Fiedler companion matrix.*

Proof. In the following, we use the notation $a_{ij}^{(i)}$ to denote the (i, j) entry of A_k.

For all $i = 0, 1, \dots, d-2$ and $j = 1, 2, \dots, k$, we use j' as a shorthand for $k - j + 1$; let the matrix $M_{i,j}$ be defined as

$$M_{i,j} = F_{ki}(a_{j,j'}^{(i)}) F_{ki+1}(a_{j+1,j'}^{(i)}) \cdots F_{ki+j'-1}(a_{k,j'}^{(i)}) \\ F_{ki+j'}(a_{1,j'}^{(i+1)}) F_{ki+j'+1}(a_{2,j'}^{(i+1)}) \cdots F_{ki+k-1}(a_{j-1,j'}^{(i+1)}) \quad (1.6)$$

if $\mathscr{C}$ has a (block) consecution at i, or

$$M_{i,j} = F_{ki+k-1}(a_{j,j'-1}^{(i+1)}) F_{ki+k-2}(a_{j,j'-2}^{(i+1)}) \cdots F_{ki+j}(a_{j,1}^{(i+1)}) \\ F_{ki+j-1}(a_{j,k}^{(i)}) F_{ki+j-2}(a_{j,k-1}^{(i)}) \cdots F_{ki}(a_{j,j'}^{(i)}) \quad (1.7)$$

if $\mathscr{C}$ has a (block) inversion at i.

When $i = d-1$, we can take either of (1.6) or (1.7) as $M_{i,j}$, omitting all terms containing entries of $a^{(d)}$. (Hence, in particular, one can find two different factorizations for $\mathscr{C}$.)

We will prove that $\mathscr{C} = C_1 C_2 \cdots C_k$, where

$$C_j = (M_{\sigma(0),j} M_{\sigma(1),j} \cdots M_{\sigma(d-1),j}). \quad (1.8)$$

Each of the C_j contains exactly one factor of the form $F_0, F_1, \dots, F_{dk-j}$, hence it is a scalar Fiedler companion matrix, linked to a polynomial whose coefficients are elements out of the original block coefficients.

To show that $\mathscr{C} = C_1 C_2 \cdots C_k$, we rely on Lemma 1. Note that $\mathscr{C} = \mathscr{F}_{\sigma(0)} \mathscr{F}_{\sigma(1)} \cdots \mathscr{F}_{\sigma(d-1)}$ can be seen as a product of scalar elementary Fiedler factors of size $kd \times kd$ using the factorizations in Theorems 1 and 2. Relying on Lemma 1, we simply have to verify that its layers coincide with those of $C_1 C_2 \cdots C_k$. Indeed, let $0 \le i < d$, and $1 \le \ell \le k$; if $\mathscr{C}$ has a block consecution at i then

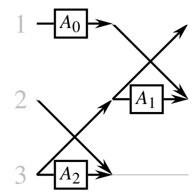
$$\mathscr{L}_{ki+\ell-1:ki+\ell}(\mathscr{C}) = \mathscr{L}_{ki+\ell-1:ki+\ell}(\mathscr{F}_i\mathscr{F}_{i+1}) =$$
$$\Big(F_{ki+\ell-1}(a^{(i)}_{\ell,k}), F_{ki+\ell}(a^{(i)}_{\ell+1,k}), F_{ki+\ell-1}(a^{(i)}_{\ell+1,k-1}), F_{ki+\ell}(a^{(i)}_{\ell+2,k-1}), \dots, F_{ki+\ell-1}(a^{(i)}_{k,\ell}),$$
$$F_{ki+\ell}(a^{(i+1)}_{1,\ell}), F_{ki+\ell-1}(a^{(i+1)}_{1,\ell-1}), F_{ki+\ell}(a^{(i+1)}_{2,\ell-1}), F_{ki+\ell-1}(a^{(i+1)}_{2,\ell-2}), \dots, F_{ki+\ell}(a^{(i+1)}_{\ell,1})\Big)$$
$$= \mathscr{L}_{ki+\ell-1:ki+\ell}(M_{i,1}M_{i+1,1}M_{i,2}M_{i+1,2}\cdots M_{i,k}M_{i+1,k}) = \mathscr{L}_{ki+\ell-1:ki+\ell}(C_1C_2\cdots C_k).$$

Similarly, if $\mathscr{C}$ has an inversion in i then

$$\mathscr{L}_{ki+\ell-1:ki+\ell}(\mathscr{C}) = \mathscr{L}_{ki+\ell-1:ki+\ell}(\mathscr{F}_{i+1}\mathscr{F}_i) =$$
$$\Big(F_{ki+\ell}(a^{(i+1)}_{1,\ell}), F_{ki+\ell-1}(a^{(i+1)}_{1,\ell-1}), F_{ki+\ell}(a^{(i+1)}_{2,\ell-1}), F_{ki+\ell-1}(a^{(i+1)}_{2,\ell-2}), \dots, F_{ki+\ell}(a^{(i+1)}_{\ell,1}),$$
$$F_{ki+\ell-1}(a^{(i)}_{\ell,k}), F_{ki+\ell}(a^{(i)}_{\ell+1,k}), F_{ki+\ell-1}(a^{(i)}_{\ell+1,k-1}), F_{ki+\ell}(a^{(i)}_{\ell+2,k-1}), \dots, F_{ki+\ell-1}(a^{(i)}_{k,\ell})\Big)$$
$$= \mathscr{L}_{ki+\ell-1:ki+\ell}(M_{i+1,1}M_{i,1}M_{i+1,2}M_{i,2}\cdots M_{i+1,k}M_{i,k}) = \mathscr{L}_{ki+\ell-1:ki+\ell}(C_1C_2\cdots C_k). \qquad \square$$

This tedious algebraic proof hides a simple structure that is revealed by the associated graphs: the scalar elementary Fiedler factors appearing in the graph of $\mathscr{C}$ can be split into k twisted lines that run diagonally, parallel one to the other. Moreover it is interesting to remark that all resulting Fiedler companion matrices have the same structure, with consecutions and inversions in the same positions. This is illustrated clearly in the next example.

Example 6. Consider the block Fiedler companion matrix of the matrix polynomial $P(z) = Iz^3 + A_2z^2 + A_1z + A_0$, with $d = k = 3$, defined as $\mathscr{C} = \mathscr{F}_2\mathscr{F}_0\mathscr{F}_1$. Its block-level graph is:



and has a consecution at block level 0 and an inversion at block level 1. Its scalar-level diagram is presented in Figure 1.12. The elements belonging to the three factors C_1, C_2, C_3 are drawn in three different colors. Formally, we have

$$\begin{aligned}
M_{0,1} &= F_0(a^{(0)}_{13})F_1(a^{(0)}_{23})F_2(a^{(0)}_{33}), & M_{1,1} &= F_5(a^{(2)}_{12})F_4(a^{(2)}_{11})F_3(a^{(1)}_{13}),\\
M_{0,2} &= F_0(a^{(0)}_{22})F_1(a^{(0)}_{32})F_2(a^{(1)}_{12}), & M_{1,2} &= F_5(a^{(2)}_{21})F_4(a^{(1)}_{23})F_3(a^{(1)}_{22}),\\
M_{0,3} &= F_0(a^{(0)}_{31})F_1(a^{(1)}_{11})F_2(a^{(1)}_{21}), & M_{1,3} &= F_5(a^{(1)}_{33})F_4(a^{(1)}_{32})F_3(a^{(1)}_{31})
\end{aligned}$$

and finally, using (1.7) we have

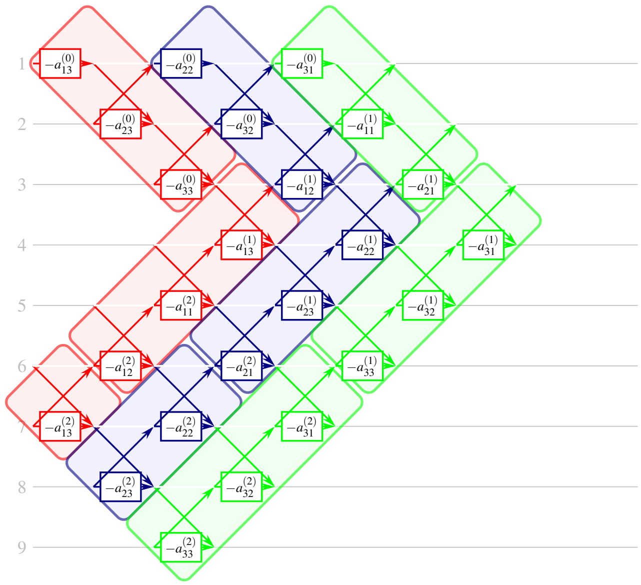


Fig. 1.12 The scalar-level diagram associated to Example 6. Each diagonal segment in a box represents a term $M_{i,j}$. We use the notation $a_{ij}^{(k)}$ to denote the entry in position (i,j) of A_k.

$$M_{2,1} = F_6(a_{13}^{(2)}), \quad M_{2,2} = F_7(a_{23}^{(2)})F_6(a_{22}^{(2)}), \quad M_{2,3} = F_8(a_{33}^{(2)})F_7(a_{32}^{(2)})F_6(a_{31}^{(2)}).$$

In accordance with (1.8) we get $C_1 = M_{2,1}M_{0,1}M_{1,1}, C_2 = M_{2,2}M_{0,2}M_{1,2}$ and $C_3 = M_{2,3}M_{0,3}M_{1,3}$.

Note that the factorization is not unique since we can additionally incorporate the terms $F_7(a_{23}^{(2)})F_8(a_{33}^{(2)})$ in C_1, thereby defining the matrix $M_{2,1}$ as in (1.6) rather than (1.7). In that case also $M_{2,2}$ and $M_{2,3}$ should be defined in accordance with (1.6).

The corresponding matrices are

$$\mathscr{C} = \begin{bmatrix} 0 & A_0 & 0 \\ 0 & 0 & I \\ I & A_1 & A_2 \end{bmatrix} = C_1C_2C_3,$$

with

$$C_1 = \begin{bmatrix} & & & -a_{13}^{(0)} & & & & & \\ 1 & & & -a_{23}^{(0)} & & & & & \\ & 1 & & -a_{33}^{(0)} & & & & & \\ & & & & 1 & & & & \\ & & & & & 1 & & & \\ & & & & & & 1 & & \\ & & 1 & -a_{13}^{(1)} & -a_{11}^{(2)} & -a_{12}^{(2)} & -a_{13}^{(2)} & & \\ & & & & & & & 1 & \\ & & & & & & & & 1 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} & & & -a_{22}^{(0)} & & & & & \\ 1 & & & -a_{32}^{(0)} & & & & & \\ & 1 & & -a_{12}^{(1)} & & & & & \\ & & & & 1 & & & & \\ & & & & & 1 & & & \\ & & & & & & 1 & & \\ & & & & & & & 1 & \\ & & 1 & -a_{22}^{(1)} & -a_{23}^{(1)} & -a_{21}^{(2)} & -a_{22}^{(2)} & -a_{23}^{(2)} & \\ & & & & & & & & 1 \end{bmatrix},$$

$$C_3 = \begin{bmatrix} & & & -a_{31}^{(0)} & & & & & \\ 1 & & & -a_{11}^{(1)} & & & & & \\ & 1 & & -a_{21}^{(1)} & & & & & \\ & & & & 1 & & & & \\ & & & & & 1 & & & \\ & & & & & & 1 & & \\ & & & & & & & 1 & \\ & & & & & & & & 1 \\ & & 1 & -a_{31}^{(1)} & -a_{32}^{(1)} & -a_{33}^{(1)} & -a_{31}^{(2)} & -a_{32}^{(2)} & -a_{33}^{(2)} \end{bmatrix}.$$

1.5 Unitary-plus-low-rank structure

Unitary (or orthogonal) plus low rank matrices appear frequently in fast algorithms for polynomial root-finding [4–7, 9–12, 14, 15]. To the best of our knowledge, these techniques have been applied to row and column companion matrices (either block or scalar ones), but never to general Fiedler linearizations, even though recent advances [8] point towards novel algorithms for eigenvalue computations of unitary-plus-low-rank matrices.

In this section we show that (block) Fiedler companion matrices are unitary-plus-low-rank, and that an upper bound on the rank of the correction is easily determined from their structure. This result is not new, as it is already present in a very similar form in [25, Section 6]; however, we report an alternative proof making use of the Fiedler graphs.

Lemma 2. *Let A_1 be unitary plus rank t_1, and A_2 be unitary plus rank t_2. Then, A_1A_2 is unitary plus rank (at most) $t_1 + t_2$.*

Proof. It is sufficient to write $A_i = Q_i + u_i v_i^T$, with Q_i unitary and $u_i, v_i \in \mathbb{C}^{n \times t_i}$, for $i = 1, 2$. Then,

$$A_1A_2 = (Q_1 + u_1v_1^T)A_2 = Q_1A_2 + u_1v_1^TA_2 = Q_1Q_2 + Q_1u_2v_2^T + u_1v_1^TA_2. \quad \square$$

We introduce the concept of *segment decomposition* of a Fiedler companion matrix, which groups together elementary Fiedler factors with consecutive indices.

Definition 5. Let $C = F_{\sigma(0)}F_{\sigma(1)} \cdots F_{\sigma(d-1)}$ be a scalar Fiedler companion matrix. We say that C has t *segments* (or, equivalently, that its graph has t segments) if t is the minimal positive integer such that $C = \Gamma_1 \cdots \Gamma_t$, for a certain set of indices i_j satisfying

$$\Gamma_j = F_{\sigma(i_j)} \cdots F_{\sigma(i_{j+1}-1)}, \qquad 0 = i_1 < i_2 < \cdots < i_{t+1} = d,$$

and such that the integers $\sigma(i_j), \ldots, \sigma(i_{j+1} - 1)$ are consecutive (either in increasing or decreasing order).

Note that each Γ_j is either a column or row companion matrix possibly padded with identities. Segments are easily identified in the Fiedler graph, as they corresponds to sequences of diagonally aligned elementary graphs. For instance, Figure 1.13 depicts, on the left, the graph of the Fiedler companion matrix of $C = F_0F_1F_5F_4F_2F_6F_7F_3$. Swapping commuting blocks we can rearrange the elementary Fiedler factors as follows $C = (F_0F_1F_2)(F_5F_4F_3)(F_6F_7)$ identifying the 3 column and row companion matrices and hence the 3 segments. Similarly the graph on the right has 2 segments.

Remark 6. The paper [22] defines a sequence of integers called the *consecution-inversion structure sequence* (CISS) of a Fiedler companion. The number of segments can be deduced from the CISS: namely, it is the length of CISS (excluding a leading or trailing zero if there is one) minus the number of distinct pairs of consecutive 1's appearing in it.

Theorem 7. *Let C be a scalar Fiedler companion matrix with t segments. Then, C is unitary plus rank (at most) t.*

Proof. If C has t segments, then by definition $C = \Gamma_1\Gamma_2 \cdots \Gamma_t$, where each Γ_j is either a column or a row companion matrix (possibly padded with identities). In fact, if $\Gamma_j = F_{\sigma(i_j)} \cdots F_{\sigma(i_{j+1}-1)}$ and the integers $\sigma(i_j), \ldots, \sigma(i_{j+1} - 1)$ are consecutive in increasing order we obtain a column companion matrix; if instead they are consecutive in decreasing order we obtain a row companion matrix. Each row or column companion matrix is unitary plus rank 1 (since it is sufficient to alter the last row

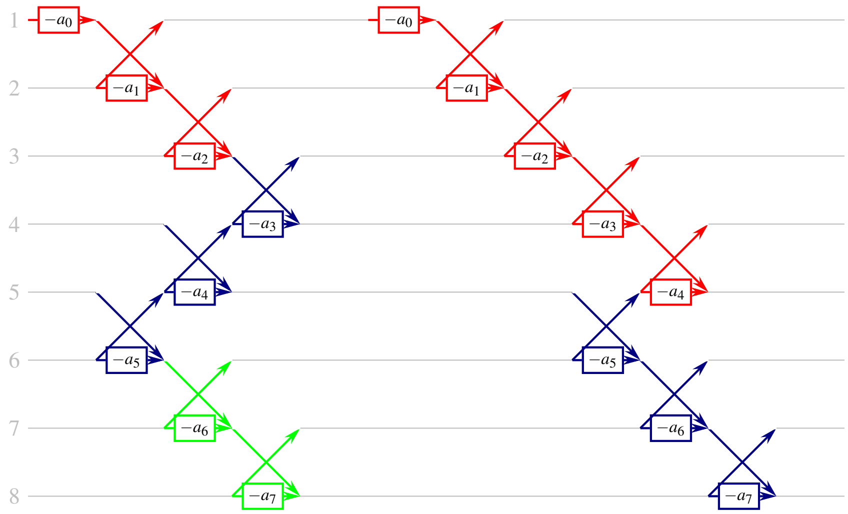


Fig. 1.13 Two graphs associated to scalar Fiedler companion matrices. The example on the left is composed of 3 segments, while the one on the right of only 2.

or column to turn it into the unitary cyclic shift matrix $Z + e_1 e_n^T$). Hence C is the product of t unitary-plus-rank-1 matrices, which is unitary plus rank (at most) t by Lemma 2. □

The above result can be used to prove another interesting fact.

Theorem 8. *Let $\mathscr{C}$ be a block Fiedler companion matrix with a block-level graph composed of t segments. Then, $\mathscr{C}$ is unitary plus rank (at most) kt.*

Proof. The result follows by generalizing the proof of Theorem 7 to block Fiedler companion matrices, noticing that each block Fiedler companion matrix is unitary plus rank k. □

Remark 7. Given a Fiedler companion matrix $\mathscr{C}$ with t segments and its factorization $\mathscr{C} = C_1 C_2 \cdots C_k$ obtained through Theorem 6, each C_j has the same number of segments, but it may happen that this number is larger than t. An example is given by the block version of the pencil on the right of Figure 1.13, i.e., $\mathscr{C} = \mathscr{F}_5 \mathscr{F}_6 \mathscr{F}_7 \mathscr{F}_0 \mathscr{F}_1 \mathscr{F}_2 \mathscr{F}_3 \mathscr{F}_4$. Indeed, each of its scalar Fiedler companion factors C_j has 3 segments rather than 2.

Remark 8. Theorem 8 shows that we can apply to $\mathscr{C}$ structured methods for fast eigenvalues computation, provided that the number of segments in the graph associated with the Fiedler companion matrix is limited.

In addition, it gives an explicit factorization of $\mathscr{C}$ into unitary plus rank 1 matrices, therefore providing all the tools required to develop a fast method similar to the one presented in [4] for column block companion matrices.

1.6 A thin-banded linearization

Another interesting use of scalar-level factorizations of block Fiedler companion matrices is to construct new companion matrices by rearranging factors. We present an example using the flow graphs.

Example 7. We consider the matrix polynomial $P(z) = Iz^4 + A_3z^3 + A_2z^2 + A_1z + A_0$, with $d = 4, k = 3$. Assume for simplicity that A is already anti-triangular. The graph associated to its column companion matrix Γ is shown in Figure 1.14.

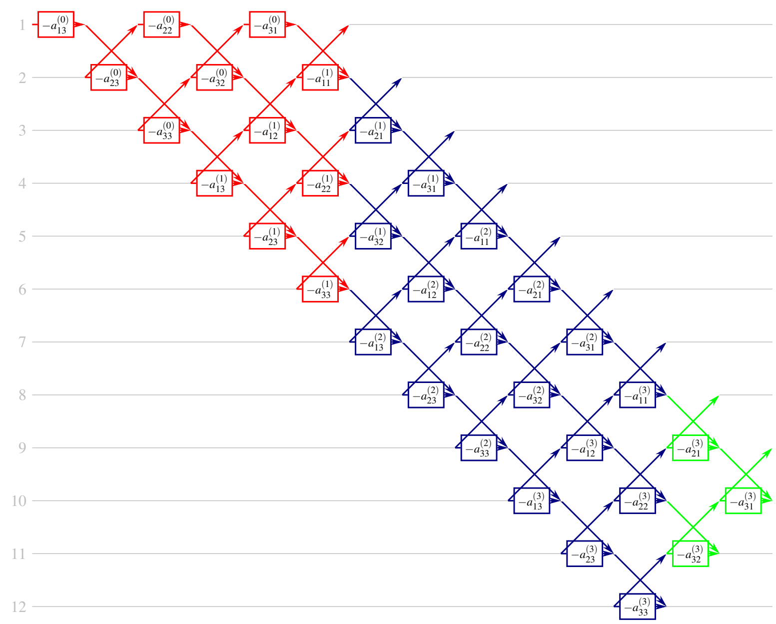


Fig. 1.14 The flow graph of the column companion matrix in Example 7.

We can factor this matrix as the product of three factors $\Gamma = RST$, which we have drawn in different colors in Figure 1.14. Note that R and T commute, and that S, T are invertible, being products of nonsingular Fiedler factors. Hence, RST is similar to $TRS = RTS$, which is in turn similar to TSR. This proves that $\mathscr{C} = TRS$ is also a companion matrix for $P(z)$. The graph of $\mathscr{C}$ is depicted in Figure 1.15.

Note that $\mathscr{C}$ is not a Fiedler companion matrix, as it cannot be obtained by permuting block-level factors; "breaking the blocks" is required to construct it. This construction can be generalized to arbitrary d and k, and it has a couple of nice features.

- $\mathscr{C}$ is a banded matrix. Since we have drawn its diagram inside six columns in Figure 1.15, there is no path from the left to the right of the diagram that moves up or down more than 5 times; this means that $\mathscr{C}_{i,j} = 0$ whenever $|j - i| \geq 6$.

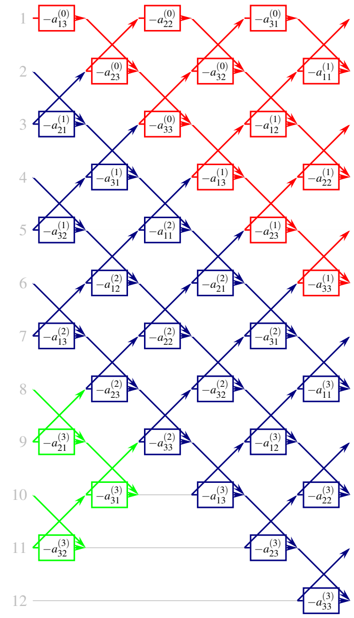


Fig. 1.15 The graph of $\mathscr{C}$ in Example 7.

Generalizing this construction to arbitrary k and d, one gets $\mathscr{C}_{i,j} = 0$ whenever $|j-i| \geq 2k$. Finding low-bandwidth linearizations and companion matrices has attracted quite some interest in the past: for instance, [2, 27] present a (block) pentadiagonal companion matrix (which can also be expressed as a block tridiagonal linearizing pencil). The new companion matrix $\mathscr{C}$ has the same bandwidth as this classical example.

- Whenever the coefficients of $P(z)$ are symmetric matrices, we can factor $\mathscr{C}$ into the product of two symmetric matrices $\mathscr{C} = \mathscr{C}_1\mathscr{C}_2$: it is sufficient to take $\mathscr{C}_1$ as the product of all factors appearing in the first five columns of Figure 1.15, and $\mathscr{C}_2$ as the product of all factors appearing in the sixth and last one, i.e., $\mathscr{C}_2 = F_1(a_{11}^{(1)})F_3(a_{22}^{(1)})F_5(a_{33}^{(1)})F_7(a_{11}^{(3)})F_9(a_{22}^{(3)})F_{11}(a_{33}^{(3)})$. This means that we can construct a symmetric pencil $\mathscr{C}_1 - \mathscr{C}_2^{-1}z$ which is a linearization of $P(z)$. (Note that $\mathscr{C}_2^{-1}$ is operation-free.) Finding symmetric linearizations for symmetric matrix polynomials is another problem that has attracted research interest in the past [16, 31].

Remark 9. We remark that it is not clear that thin-banded linearizations provide practical advantages in numerical computation. Commonly used eigenvalue algorithms (namely, QZ and QR) cannot exploit this structure, unless the matrix at hand is also symmetric (or the pencil is symmetric/positive definite).

1.7 Conclusions

We have presented an extension of the graph-based approach by Poloni and Del Corso [31] that allows to produce scalar-level factorizations of block Fiedler companion matrices.

We have shown that this framework can be used for several purposes, such as identifying new factorizations of products of Fiedler matrices, revealing their structures (such as the unitary-plus-rank-t structure), and combining them to build new linearizations.

Once the reader is familiar with reading the Fiedler graphs, there are many more factorizations that could be devised. Every time a diagonal line of factors appears in a graph, it can be transformed into a factorization that involves row or column companion matrices.

The presented approach allows a more general and particularly easy manipulation of these linearizations. It might lead to the development of efficient algorithms for the computation of eigenvalues of matrix polynomials using the product form with the unitary-plus-rank-1 structure.

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