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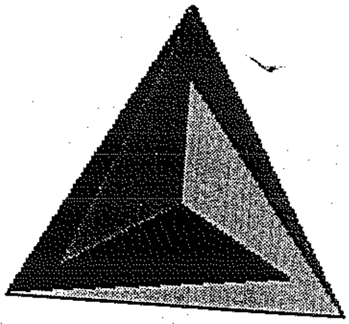

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Power Diagram Depth Sorting

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1 Introduction

The problem of depth sorting has received considerable attention in the literature: a lot of research deals with the two dimensional version of the problem. A good overview of the depth sorting problem is contained in the De Berg's PhD Thesis [2]. Beside its theorethic importance, the depth ordering finds many useful real applications, from visibility problems in GIS [4] to the Direct Volume Visualization of unstructured meshes [5, 6].

The main contribution of this paper is the proposal of a new technique for depth-sorting cell complexes that belongs to the class of projective complexes. Given a cell complex Γ in $\mathbb{E}^d$, the approach is based on the preliminary construction of the convex complex Γ^* in $\mathbb{E}^{d+1}$ corresponding to Γ and on its representation as a power diagram. This approach exhibits a $O(n \log n)$ runtime complexity to sort a complex and requires only linear storage.

1.1 Definitions

Obstruction relation The obstruction relation $\prec_p$ (usually called *infront/behind* relation) for a pair of non self-intersecting cells γ_1, γ_2 with respect to a viewpoint p, defines a visibility/depth order of a set of objects; that is sequence of such objects such that, if object A obstructs object B when seen from p, then A preceeds B in the sequence.

A cell complex Γ is called *acyclic* with respect to a given viewpoint p if and only if relation $\prec_p$ defines a partial order over the cells of Γ. In this case, it is possible to order the cells of Γ either front-to-back or back-to-front with respect to the viewpoint.

Cell complexes in $\mathbb{E}^d$ that can be obtained as the orthogonal projection of the lower part of the boundary of a convex polytope in $\mathbb{E}^{d+1}$ are called *regular cell complexes* [3]. These complexes are also known as *projective complexes* [1].

An important result due to Edelsbrunner [3] for regular cell complexes is that they are acyclic. Delaunay simplicial complexes are acyclic with respect to any viewpoint [3], and it assures that a volume dataset represented by a Delaunay complex can always be sorted and correctly visualized.

In the following we will use the following concepts and definitions about power diagrams, a complete introduction about power diagrams can be found in [1]. Let S be a finite set of spheres in $\mathbb{E}^d$, we denote the *power diagram* of S with PD(S), the power of a sphere s w.r.t. a point p with $pow(p, s)$ and the chordale between two spheres s, t with $chor(s, t)$.

1.2 Power diagrams and convex polyhedra

Let $\mathbb{E}^{d+1}$ be spanned by the coordinate axes $x_1, \cdots, x_{d+1}$ and let h_0 denote the hyperplane $x_{d+1} = 0$. The following result, that relates PD in $\mathbb{E}^d$ and convex polyhedra in $\mathbb{E}^{d+1}$, is presented in [1]:

Theorem 1. *(Aurenhammer [1]) For any (d+1)-polyhedron P, which can be expressed as the intersection of upper halfspaces, there exists an affinely equivalent power diagram in h_0, and viceversa.*

This theorem is based on the transform Π that maps a sphere $s \subset h_0$ with center z and radius r

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into a hyperplane $\Pi(s)$ in $\mathbb{E}^{d+1}$ and viceversa.

An interesting property of this transform is that, if s and t are two non-concentric spheres in h_0, then $chor(s,t)$ is the vertical projection of $\Pi(s) \cap \Pi(t)$ onto h_0. Therefore, given a polyhedron P, it is possibile to obtain a set of spheres S whose PD is equal to the projection of P onto h_0.

The theorem 1 refers only to polyhedra formed by the intersection of non-vertical upper halfspaces, thus unbounded infinite polyhedra whose projection onto h_0 is a partition of the whole plane. With the following theorem, we extend this relation in order to be able to manage bounded polyhedra such as the lower part of a convex hull.

Theorem 2. *Let P be the lower part of a convex polyedron bounded by simplicial facets; there exists a power diagrams $PD(S)$ of a set of spheres S whose cells are superset of the projection of the facets of P onto h_0.*

Using this theorem and some other lemmas we can prove that the obstruction relation betweens cells of a projective simplical complex $\sigma \prec_p \tau$ agrees with the power of the spheres s_σ, s_τ corresponding to each cell $pow(p, s_\sigma) < pow(p, s_\tau)$. This result suggests a technique for depth sorting a simplicial complex Γ in $\mathbb{E}^d$ that is the projection of the lower part of a convex polyhedron in $\mathbb{E}^{d+1}$ Γ^*. Infact, given the viewpoint p, it is sufficient to sort the d-cells σ_i of the complex according to $pow(p, s_{\sigma_i})$, where s_{σ_i} is the sphere obtained by the transformation Π from the plane h_i affine to the facet σ_i^* of Γ^*. Hereafter we will refer this approach to depth sorting as *Power Diagram Depth Sorting* or PDD sorting.

Using some corollaries of the above theorem we are able to show that this approach can also be used to sort scattered triangles in space with a complexity of $O(n \log n)$, with just a linear overhead in storage (the center and radius for each sphere).

The most interesting aspect of this approach is the *clearness* of the structure needed for the sorting once the corresponding polyhedron Γ^* has been found: for each d-simplex it is only necessary to store the corresponding power circle.

1.3 Lifting a simplicial complex

In the common cases we have just a simplicial complex Γ in $\mathbb{E}^d$, but not the complex Γ^* such that Γ is the vertical projection of Γ^*. We denote the problem of finding Γ^*, if there exists one, as the *lifting problem*. In some special cases it is simple to find the convex polyhedron, for example if Γ is a Delaunay simplicial complex we can exploit the well known correspondence with convex hull in $\mathbb{E}^{d+1}$ to find Γ^*. We show that the lifting problem can be formulated as a linear programming problem. The lifted complex is convex if and only if the dihedral angle between any two lifted $(d-1)$-adjacent simplex is convex. This condition can be expressed by a linear inequality and a solution of the resulting system can be found used the simplex algorithm.

Some experiments in two dimensions showed that the we can *lift* a complex of one thousand triangles in less than a minute on a small personal computer using a public domain LP solver. This preliminary result seems to show that the lifting of a complex is a reasonable preprocessing step for the subsequent PDD Sorting.

References

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