SOME EXPLICIT SOLUTIONS FOR NON-LINEAR ELASTIC SOLIDS

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If instead ψ belongs to an interval $[\psi_1, \psi_2]$, different in the two cases, then the solutions involve arising of inelastic strains, which as ψ decreases spread from the inner radius toward the outer one. The solution is unique for both stress and displacement. Subsequently, we deal with a problem having cylindrical symmetry, in which an indefinite hollow cylinder is subjected to its own weight and two radial pressures acting on its inner and outer boundaries and linearly varying along a generatrix. In this case a relatively simple solution can be attained only when $\nu = 0$. As a last problem, we consider and solve a version of the circular ring case, where boundary conditions are chosen in such a way as to generate stress and displacement fields devoid of polar symmetry; this inevitably yields more complex analytical expressions for them.

All the problems except the first are solved by setting $\sigma = 0$ in the constitutive equation, a choice which allows for a less tedious exposition of the results. On the other hand, extending the solution to the case in which $\sigma > 0$ is natural on a purely analytical basis as well.

II. PRELIMINARY CONSIDERATIONS

The constitutive equation of so-called *masonry-like* materials has been studied by a number of authors [see, for example, Del Piero, 1989]. Here we shall recall its main properties and generalise it to the case in which the material can withstand normal positive stresses, providing that they do not exceed an assigned value σ.

Before beginning, we recall that if $\mathbf{A}$ and $\mathbf{B}$ are two tensors, that is two linear applications of $\mathcal{V}$ in $\mathcal{V}$, where $\mathcal{V}$ is a three-dimensional linear space, $\mathbf{A}$ is positive (or, respectively, negative) semi-definite if $\mathbf{u} \cdot \mathbf{A}\,\mathbf{u} \ge 0$ ($\mathbf{u} \cdot \mathbf{A}\,\mathbf{u} \le 0$) for each $\mathbf{u} \in \mathcal{V}$: in this case we shall write $\mathbf{A} \ge 0$ ($\mathbf{A} \le 0$). Analogously, if an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of $\mathcal{V}$ is chosen, we shall write $\mathbf{u} \le 0$ ($\mathbf{u} \ge 0$), with $\mathbf{u} \in \mathcal{V}$, if the components of $\mathbf{u}$ turn out to be non positive (non negative). Moreover, with $\mathbf{A} \cdot \mathbf{B} = \mathrm{tr}\,(\mathbf{A}\,\mathbf{B}^T)$, where $\mathbf{B}^T$ is the transpose of $\mathbf{B}$, and $\mathrm{tr}\,(\mathbf{A})$ is the trace of $\mathbf{A}$, we shall indicate, as usual, the scalar product of $\mathbf{A}$ and $\mathbf{B}$. Finally, we recall that if $\mathbf{A}$ and $\mathbf{B}$ are symmetric, with $\mathbf{A} \ge 0$ and $\mathbf{B} \le 0$, then $\mathbf{A} \cdot \mathbf{B} \le 0$.

To begin with, we assume that the strain tensor $\mathbf{E}$ is the sum of an *elastic* part $\mathbf{E}^e$ and of an *inelastic* one $\mathbf{E}^a$, with $\mathbf{E}^a$ positive semi-definite:

$$\mathbf{E} = \mathbf{E}^e + \mathbf{E}^a\,; \quad \mathbf{E}^a \ge 0\,. \tag{1}$$

In addition, the stress tensor $\mathbf{T}$ is assumed to depend linearly and isotropically on $\mathbf{E}^e$:

$$\mathbf{T} = 2\,G\,\mathbf{E}^e + \frac{\nu\,E}{(1+\nu)(1-2\nu)}\,\mathrm{tr}(\mathbf{E}^e)\,\mathbf{I}\,, \tag{2}$$

where E, ν and $G = \frac{E}{2(1+\nu)}$ are the Young's modulus, the Poisson ratio and the modulus of elasticity in shear, respectively. Finally, given a non-negative number σ, the assumption is made that tensor $\mathbf{T} - \sigma\mathbf{I}$, with $\mathbf{I}$ the identity tensor, is negative semi-definite and orthogonal to $\mathbf{E}^a$:

$$\mathbf{T} - \sigma\mathbf{I} \leq 0 \; ; \quad (\mathbf{T} - \sigma\mathbf{I}) \cdot \mathbf{E}^a = 0 \, . \tag{3}$$

It is a simple matter to prove that as a consequence of (1) and (3), tensors $\mathbf{T}$ and $\mathbf{E}^a$ are coaxial. In fact, due to the symmetry of $\mathbf{T}$ and $\mathbf{E}^a$, there exist two bases of $\mathcal{V}$, say $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ and $\{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\}$, such that

$$\mathbf{T} = \sum_{i=1}^{3} t_i \, \mathbf{f}_i \otimes \mathbf{f}_i \, , \quad \mathbf{E}^a = \sum_{i=1}^{3} a_i \, \mathbf{g}_i \otimes \mathbf{g}_i \, ,$$

where t_1, t_2 and t_3 and a_1, a_2 and a_3 are respectively the eigenvalues of $\mathbf{T}$ and $\mathbf{E}^a$, satisfying the inequalities

$$t_i \leq \sigma \, , \; a_i \geq 0 \, , \; i = 1, 2, 3. \tag{4}$$

Thus, it follows from $(3)_2$ and (4) that

$$(t_j - \sigma) \, a_i \, \mathbf{f}_j \cdot \mathbf{g}_i = 0 \, , \; i, j = 1,2,3,$$

and, consequently,

$$(\mathbf{T} - \sigma \mathbf{I}) \, \mathbf{E}^a = \sum_{i,\, j=1}^{3} (t_j - \sigma) a_i \, (\mathbf{f}_j \cdot \mathbf{g}_i) \, (\mathbf{f}_j \otimes \mathbf{g}_i) = \mathbf{0}.$$

In a similar way we can prove that $\mathbf{E}^a (\mathbf{T} - \sigma\mathbf{I}) = \mathbf{0}$: therefore, $\mathbf{T}$ and $\mathbf{E}^a$ commute and as they are symmetric, they must necessarily be coaxial. If E and ν satisfy the inequalities $E > 0$, $-1 < \nu < 1/2$, relation (2) can be inverted, yielding

$$\mathbf{E}^e = \frac{1}{2G} \mathbf{T} - \frac{\nu}{E} \, \mathrm{tr}(\mathbf{T}) \, \mathbf{I} \, ,$$

and equations $(1)_1$ and (2) result to be equivalent to the following,

$$\mathbf{E} - \mathbf{E}^a = \frac{1+\nu}{E} \mathbf{S} - \frac{\nu}{E} \, \mathrm{tr} \, (\mathbf{S}) \, \mathbf{I} + \frac{\sigma \, (1 - 2\nu)}{E} \mathbf{I} \, , \tag{5}$$

where we set $\mathbf{S} = \mathbf{T} - \sigma \mathbf{I}$. Now, let $\{e_1, e_2, e_3\}$, $\{a_1, a_2, a_3\}$, $\{s_1, s_2, s_3\}$ be the eigenvalues of $\mathbf{E}$, $\mathbf{E}^a$ and $\mathbf{S}$. Since $\mathbf{E}$, $\mathbf{E}^a$ and $\mathbf{S}$ are coaxial, the constitutive relations $(1)_2$, (3) and (5) can be expressed in terms of their eigenvalues, leading to the system

$$\begin{aligned} \tilde{\mathbf{e}} &= \mathbf{D}\,\mathbf{s} + \mathbf{a}, \\ \mathbf{s} &\leq \mathbf{0}, \\ \mathbf{a} &\geq \mathbf{0}, \\ \mathbf{s} \cdot \mathbf{a} &= \mathbf{0}, \end{aligned} \tag{6}$$

in which the symmetric tensor $\mathbf{D}$, positive definite, and the vectors $\tilde{\mathbf{e}}$, $\mathbf{a}$ and $\mathbf{s}$ have the following components:

$$\mathbf{D} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix}, \tilde{\mathbf{e}} = \left[e_1 - \frac{\sigma\,(1-2\nu)}{E}, e_2 - \frac{\sigma\,(1-2\nu)}{E}, e_3 - \frac{\sigma\,(1-2\nu)}{E}\right],$$

$$\mathbf{a} = [a_1, a_2, a_3] \quad , \quad \mathbf{s} = [s_1, s_2, s_3] .$$

System (6) defines a linear complementarity problem, for which it can be proven that, not only does the solution exist, but that it is unique as well [Giannessi, 1982]. The foregoing shows that relations (1)-(3) associate one and only one stress $\mathbf{T}$ to each strain $\mathbf{E}$ and therefore a unique inelastic strain $\mathbf{E}^a$ and a unique elastic strain $\mathbf{E}^e$: as such they provide the constitutive equation of a non-linear elastic material able to withstand tensile stresses less than or equal to an assigned value σ.

The constitutive equation defined by relations (1)-(3) exhibit a property useful in the presence of plane strain fields. Let us suppose, for example, that the direction $\mathbf{f}_3$ is a principal one and that the corresponding principal strain e_3 is nil:

$$e_3 = \mathbf{f}_3 \cdot \mathbf{E}\,\mathbf{f}_3 = 0 .$$

Let us tentatively assume, furthermore, that the inelastic strain a_3 along $\mathbf{f}_3$ is greater than zero. Then, the following equations,

$$t_3 - \sigma = \frac{E}{1+\nu}(e_3 - a_3) + \frac{\nu E}{(1+\nu)(1-2\nu)}[e_1 + e_2 - (a_1 + a_2 + a_3)] - \sigma = 0 ,$$

must hold, or equivalently,

$$a_3 = \frac{(1+\nu)(1-2\nu)}{E\,(1-\nu)}\left[\frac{\nu E}{(1+\nu)(1-2\nu)}(e_1 + e_2 - a_1 - a_2) - \sigma\right].$$

If $\nu = 0$, from the previous equation it at once follows that $a_3 < 0$, contradicting one of the initial assumptions. For $\nu > 0$, as occurs in the applications of interest to us here, the same equation imposes that

$$e_1 + e_2 - a_1 - a_2 > \frac{(1+\nu)(1-2\nu)}{\nu E}\sigma . \qquad (7)$$

On the other hand, it must also hold that

$$t_1 - \sigma = \frac{E}{1+\nu}(e_1 - a_1) + \frac{\nu E}{(1+\nu)(1-2\nu)}[e_1 + e_2 - (a_1 + a_2 + a_3)] - \sigma \le 0 ,$$

$$t_2 - \sigma = \frac{E}{1+\nu}(e_2 - a_2) + \frac{\nu E}{(1+\nu)(1-2\nu)}[e_1 + e_2 - (a_1 + a_2 + a_3)] - \sigma \le 0 .$$

It can be easily verified that the previous relations taken together yield the following inequality:

$$e_1 + e_2 - a_1 - a_2 \le \frac{2(1-2\nu)}{E}\sigma . \qquad (8)$$

Then, simple comparison of (7) and (8) implies that ν satisfies the inequality $2\nu^2 - 3\nu + 1 < 0$ which is incompatible with the constraint $\nu < 1/2$. Thus, $a_3 = 0$ must be true. In conclusion, if $\nu \ge 0$ and $\mathbf{f}_3$ is a principal direction along which the principal strain is nil, then we have:

$$t_3 = \nu\,(t_1 + t_2) .$$

Let us now turn our attention to a body Ω, made of an elastic material with a low tensile strength, in equilibrium under the action of body forces $\mathbf{b}$ and surface forces $\mathbf{f}$ assigned on the boundary $\partial\Omega$ of Ω. We shall state that a piecewise C^2 displacement field $\mathbf{u}$, a strain field $\mathbf{E}$ and a stress field $\mathbf{T}$ provide a *regular* solution to the equilibrium problem, if in Ω they satisfy the constitutive equation (1)-(3), the strain-displacement relation

$$\mathbf{E} = \frac{1}{2}\left[\text{grad}\,\mathbf{u} + (\text{grad}\,\mathbf{u})^T\right],$$

the equilibrium equation

$$\text{div}\,\mathbf{T} + \mathbf{b} = \mathbf{0} ,$$

and, furthermore, if they satisfy the boundary condition

$$\mathbf{Tn} = \mathbf{f}$$

on $\partial\Omega$, where $\mathbf{n}$ is the outward unit normal to $\partial\Omega$. Let us suppose now that $(\mathbf{u}_1, \mathbf{E}_1, \mathbf{T}_1)$ and $(\mathbf{u}_2, \mathbf{E}_2, \mathbf{T}_2)$ are two distinct regular solutions to the boundary problem, and that $(\bar{\mathbf{u}}, \bar{\mathbf{E}}, \bar{\mathbf{T}})$, with

$$\bar{\mathbf{u}} = \mathbf{u}_1 - \mathbf{u}_2 \ , \ \bar{\mathbf{E}} = \mathbf{E}_1 - \mathbf{E}_2 \ , \ \bar{\mathbf{T}} = \mathbf{T}_1 - \mathbf{T}_2$$

is the difference solution. It can be immediately verified that this latter solves the corresponding equilibrium homogeneous problem, that is in which $\mathbf{b} = \mathbf{0}$ and $\mathbf{f} = \mathbf{0}$. Thus, taking into account the hypothesis of regularity, a simple application of the theorem of virtual work proves that

$$\int_\Omega \bar{\mathbf{T}} \cdot \bar{\mathbf{E}} = 0 \, .$$

On the other hand,

$$\bar{\mathbf{E}} = \bar{\mathbf{E}}^e + \mathbf{E}_1^a - \mathbf{E}_2^a ,$$

where $\mathbf{E}_1^a$ e $\mathbf{E}_2^a$ are the inelastic strain fields corresponding to $\mathbf{E}_1$ and $\mathbf{E}_2$. For (3) we then have

$$\int_\Omega \bar{\mathbf{T}} \cdot \bar{\mathbf{E}}^e \, dV = \int_\Omega \bar{\mathbf{T}} \cdot (\mathbf{E}_2^a - \mathbf{E}_1^a) \, dV = \int_\Omega [\mathbf{T}_1 - \sigma \mathbf{I} - (\mathbf{T}_2 - \sigma \mathbf{I})] \cdot (\mathbf{E}_2^a - \mathbf{E}_1^a) \, dV \, .$$

The last integral is negative or nil; thus $\bar{\mathbf{T}} = \mathbf{0}$ and, consequently, $\mathbf{T}_1 = \mathbf{T}_2$. It has thus been shown that, if the constitutive equation is that described by relations (1)-(3), for every equilibrium problem there exists at most one regular stress field, that is piecewise C^1, satisfying all the field equations and the boundary conditions.

III. THE CIRCULAR RING

An indefinite cylindrical body, whose cross section is a circular ring Ω, made of an isotropic elastic material, is subjected to a plane strain condition as a consequence of two pressures p_e and p_i acting uniformly on the outer and inner surfaces. We choose a cylindrical reference system $\{O, \rho, \theta, z\}$ with the origin at the center of the ring and the z-axis orthogonal to its plane (Figure 1). Because of the symmetry, the only non-zero displacement component is the radial one, $u(\rho)$. If the material is linearly elastic, the principal stresses are

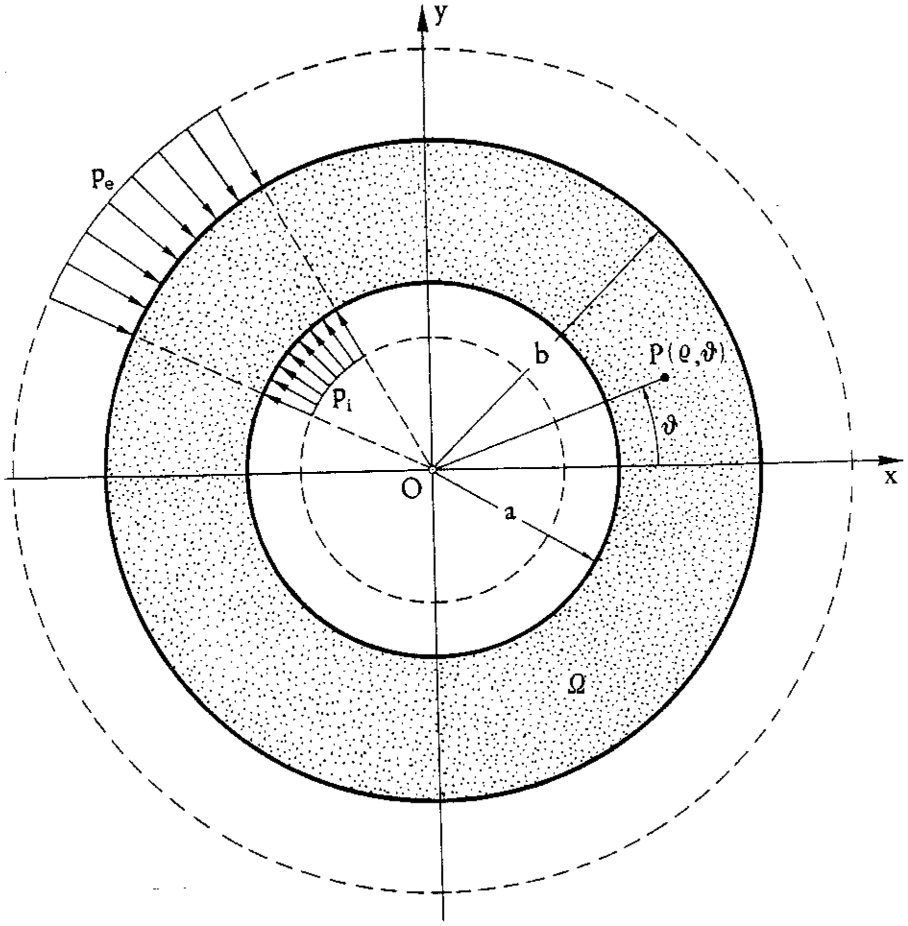

Figure 1. The circular ring.

$$\sigma_\rho(\rho) = \frac{1}{b^2 - a^2}\left\{a^2 p_i - b^2 p_e + a^2 b^2 (p_e - p_i)\frac{1}{\rho^2}\right\},$$

$$\sigma_\theta(\rho) = \frac{1}{b^2 - a^2}\left\{a^2 p_i - b^2 p_e - a^2 b^2 (p_e - p_i)\frac{1}{\rho^2}\right\},$$

$$\sigma_z(\rho) = \nu\,[\sigma_\rho(\rho) + \sigma_\theta(\rho)] = \frac{2\nu}{b^2 - a^2}\{a^2 p_i - b^2 p_e\},$$

where a and b are the inner and outer radius of the ring, respectively (Lame', 1852). Stresses σ_ρ and σ_θ are monotonic functions of ρ. Moreover, by hypothesis $\sigma_\rho(a)$ and $\sigma_\rho(b)$ are both non positive. On the inner boundary the circumferential stress is less than σ if the condition

$$p_e + \sigma \geq (p_i + \sigma)\frac{1 + \eta^2}{2\eta^2}$$

holds, where $\eta = b/a$. In turn, $\sigma_\theta(b)$ is less than σ if when ψ decreases from $\psi_2 = \frac{1+\eta^2}{2\eta^2}$ to $\psi_1 = \frac{1}{\eta}$, ρ_0 varies correspondingly from a to b. It is now easy to verify that the stress field

$$\sigma_\rho(\rho) = \begin{cases} -\frac{a}{\rho}(p_i + \sigma) + \sigma, & a \le \rho \le \rho_0, \\ -a(p_i + \sigma)\left\{\frac{\rho_0}{2\rho^2} + \frac{1}{2\rho_0}\right\} + \sigma, & \rho_0 < \rho \le b, \end{cases}$$

$$\sigma_\theta(\rho) = \begin{cases} \sigma, & a \le \rho \le \rho_0, \\ a(p_i + \sigma)\left\{\frac{\rho_0}{2\rho^2} - \frac{1}{2\rho_0}\right\} + \sigma, & \rho_0 < \rho \le b, \end{cases} \tag{9}$$

$$\sigma_z(\rho) = \nu\,[\sigma_\rho(\rho) + \sigma_\theta(\rho)], \quad a \le \rho \le b,$$

is equilibrated and it is such that principal stresses are less than or equal to σ.

According to the constitutive equation, in Ω_1

$$\varepsilon_\rho(\rho) = \frac{1+\nu}{E}\left\{(1-\nu)\,\sigma_\rho(\rho) - \nu\,\sigma_\theta(\rho)\right\} =$$

$$= \frac{1+\nu}{E}\left\{(1-2\nu)\,\sigma - (1-\nu)\frac{a\,(p_i+\sigma)}{\rho}\right\}, \quad a \le \rho \le \rho_0,$$

$$\varepsilon_\theta(\rho) = \frac{1+\nu}{E}\left\{(1-\nu)\,\sigma_\theta(\rho) - \nu\,\sigma_\rho(\rho)\right\} + \varepsilon_\theta^a(\rho) = \tag{10}$$

$$= \frac{1+\nu}{E}\left\{(1-2\nu)\,\sigma + \frac{\nu a\,(p_i+\sigma)}{\rho}\right\} + \varepsilon_\theta^a(\rho), \quad a \le \rho \le \rho_0,$$

where $\varepsilon_\theta^a(\rho)$ is a non-negative function of ρ to be determined. In the region Ω_2, where σ_ρ and σ_θ are less than σ, the strain components coincide with the linear elastic ones:

$$\varepsilon_\rho(\rho) = \frac{1+\nu}{E}\left\{(1-2\nu)\left(\sigma - \frac{a\,(p_i+\sigma)}{2\rho_0}\right) - \frac{a\,\rho_0\,(p_i+\sigma)}{2\rho^2}\right\}, \quad \rho < \rho_0 \le b,$$

$$\varepsilon_\theta(\rho) = \frac{1+\nu}{E}\left\{(1-2\nu)\left(\sigma - \frac{a\,(p_i+\sigma)}{2\rho_0}\right) + \frac{a\,\rho_0\,(p_i+\sigma)}{2\rho^2}\right\}, \quad \rho < \rho_0 \le b. \tag{11}$$

Since $u(\rho) = \rho\,\varepsilon_\theta(\rho)$, it is at once evident that

$$u(\rho) = \frac{1+\nu}{E}\left\{(1-2\nu)\left(\sigma - \frac{a(p_i+\sigma)}{2\rho_0}\right)\rho + \frac{a\,\rho_0\,(p_i+\sigma)}{2\rho}\right\}, \quad \rho_0 < \rho \le b. \qquad (12)$$

On the other hand, $u'(\rho) = \varepsilon_\rho(\rho)$; thus

$$u(\rho) = \frac{1+\nu}{E}\left\{(1-2\nu)\sigma\,\rho - (1-\nu)(p_i+\sigma)a\ln\rho\right\} + C_1, \quad a \le \rho \le \rho_0 \qquad (13)$$

where

$$C_1 = \frac{1+\nu}{E}\; a\,(p_i+\sigma)\left\{\nu + (1-\nu)\ln\rho_0\right\}$$

is a constant whose value is determined by imposing the continuity of the radial displacement at $\rho = \rho_0$. Then, the circumferential inelastic strain is

$$\varepsilon^a_\theta(\rho) = \begin{cases} \dfrac{a\,(p_i+\sigma_0)(1-\nu^2)}{E\,\rho}\ln\left(\dfrac{\rho_0}{\rho}\right), & a \le \rho \le \rho_0, \\ 0, & \rho_0 < \rho \le b. \end{cases} \qquad (14)$$

Given that the function $\varepsilon^a_\theta(\rho)$ just determined is always non-negative, the foregoing allows us to conclude that, if the *masonry-like* material has the contitutive equation (1)-(3), and if $\psi_1 \le \psi \le \psi_2$, then a solution for the equilibrium problem is given by the stresses, strains and radial displacement yielded by equations (9)-(14). A result proven in Section II shows that, if we limit ourselves to regular solutions, the solution found is unique in terms of stress. It seems worthwile to point out that, unlike other simple equilibrium problems, in this case, if $\psi \ne \psi_1$ the inelastic strain and radial displacement are also unique. This appears related to the fact that the region Ω_1, in which inelastic strains are different from zero, is included in another region Ω_2, in which the solution coincides with a linear elastic one. We observe furthermore that the radius ρ_0, as well the inelastic strain themselves do not depend separately upon the external pressures; more precisely, for all problems in which the sums $p_i + \sigma$ and $p_e + \sigma$ are the same, they will coincide. When $\psi = \psi_1$ and the whole ring is the seat of circumferential inelastic strains, constant C_1, as well as the inelastic strains themselves and the radial displacement are not univocally determined.

Finally, it is easy to prove that, if $\psi < 1/\eta$, there are no stress fields in equilibrium with the external pressures and satisfying inequalilty $(3)_2$. To this end, let us consider the region of the circular ring between straight lines $\theta = 0$ and $\theta = \theta_0$, with, for instance, $0 < \theta_0 < \pi/2$ (Figure 2). For equilibrium reasons the resultant of the external forces must have a nil component in the $\theta = 0$ direction, that is to say,

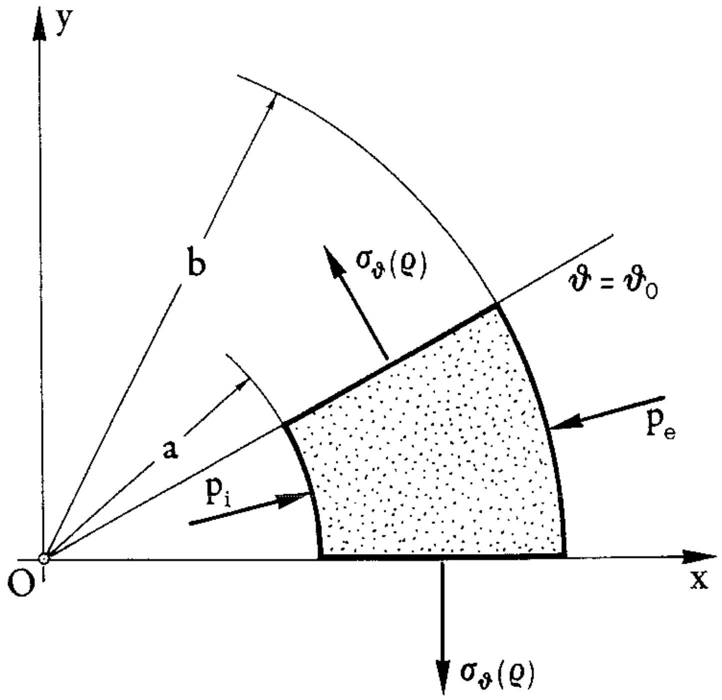

Figure 2. The equilibrium of a portion of the circular ring.

$$- \int_0^{\theta_0} a\, p_i \cos\theta \, d\theta + \int_0^{\theta_0} b\, p_e \cos\theta \, d\theta - \int_a^b \sigma_\theta(\rho) \sin\theta_0 \, d\theta = 0 ,$$

or also

$$b\, p_e = a\, p_i - \int_a^b \sigma_\theta(\rho)\, d\theta \; .$$

Given that $\sigma(\rho) \leq \sigma$,

$$b\, p_e \geq a\, p_i - \sigma(b - a) ,$$

must hold, or equivalently, $\psi \geq 1/\eta$.

Figures 3, 4 and 5 show the behaviour of the radial and circumferential stresses and the radial displacement in the linear case and for two different values of σ. The values $a = 1$ m, $b = 2$ m, $p_e = 0.502$ MPa, $p_i = 1$ MPa, have been chosen in such a way that in the first case ($\sigma = 0.1$ MPa) $\rho_0 = 1.299$ m; in the second ($\sigma = 0$) $\rho_0 = 1.828$ m; moreover, we set $E = 5{,}000$ MPa and $\nu = 0.1$. From Figure 5 it is evident that σ exerts considerable influence, not only on the pattern of circumferential stress, as is obvious, but on the intensity of the radial displacement as well. Moreover, Figure 6 shows the radial displacement of the points belonging to the external surface as a function of ψ with ψ ranging from ψ_1 to ψ_2 ; one of two curves refers to the non-linear case with σ = 0, the other to the linear one.

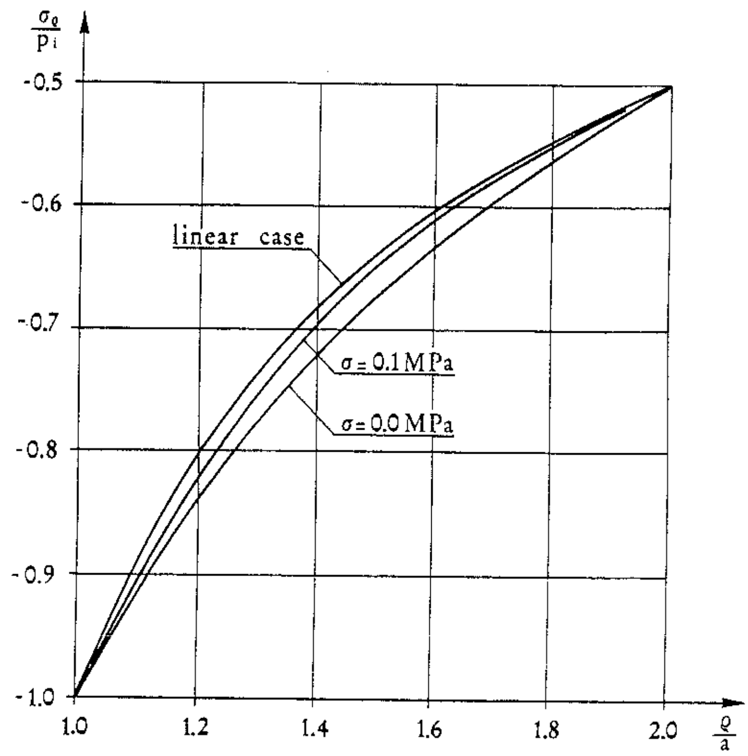

Figure 3. Radial stress $\sigma_\rho(\rho)$.

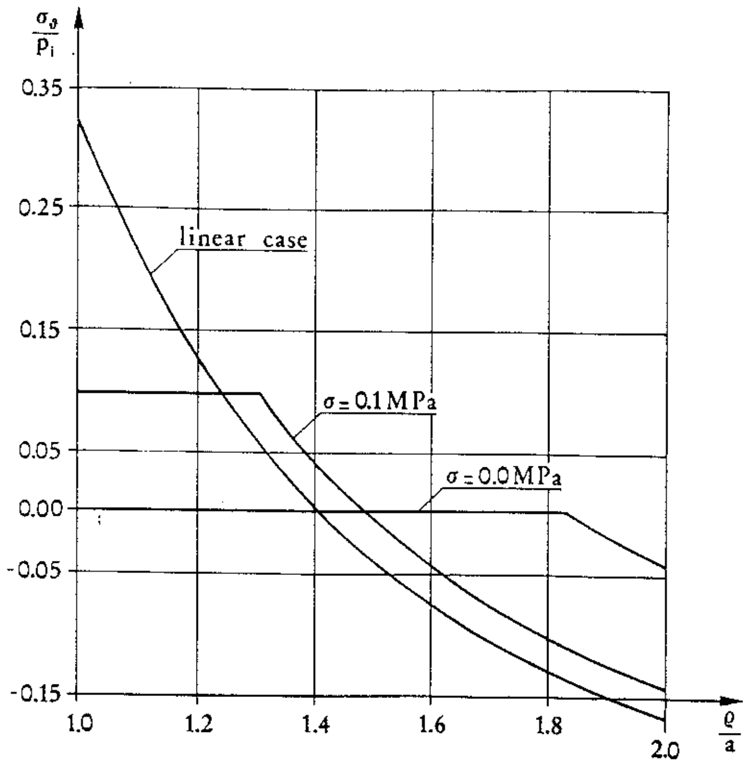

Figure 4. Circumferential stress $\sigma_\theta(\rho)$.

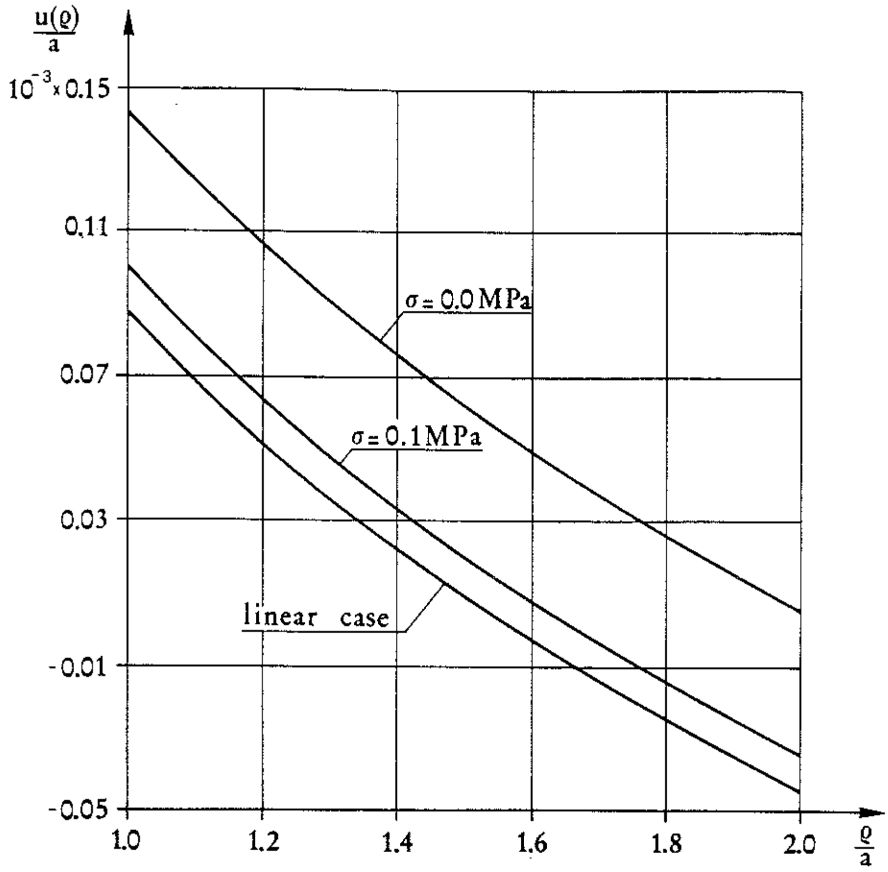

Figure 5. Radial displacement $u(\rho)$.

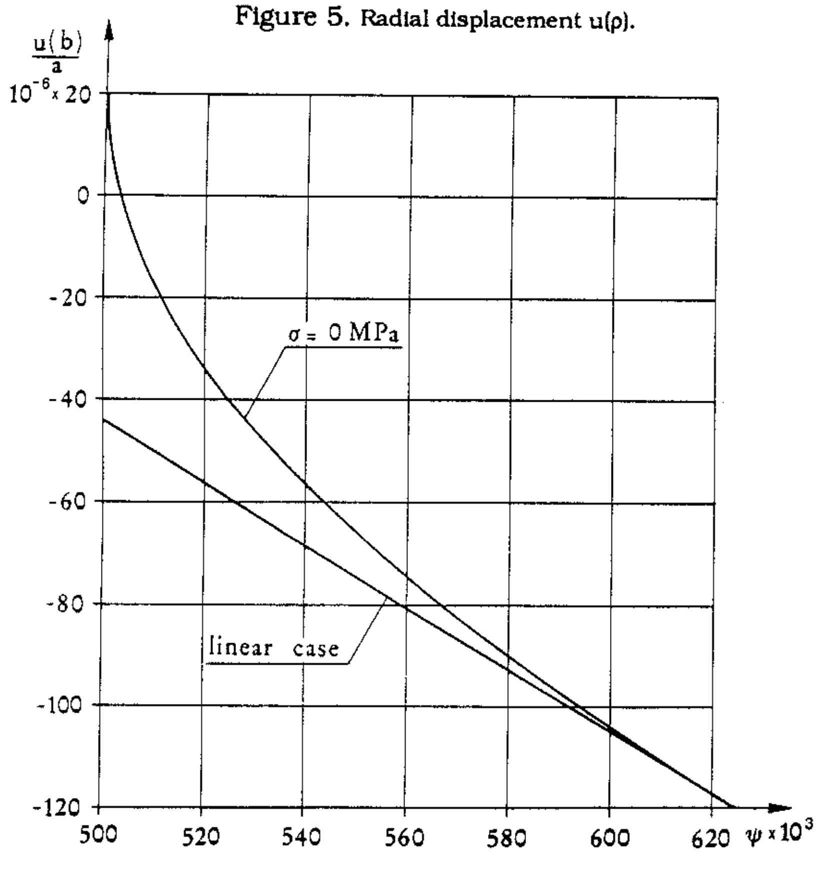

Figure 6. Radial displacement $u(b)$ as a function of ψ.

It is quite interesting to deal with the limit case in which the ring's external radius becomes infinite. It is easy to show that, if $\psi > \frac{1}{2}$, then the solution coincides with the linear elastic one. On the contrary, if $\psi < \frac{1}{2}$, calculations, here ometted for the sake of brevity, lead to the following expression for the radial displacement:

$$u(\rho) = \begin{cases} a\, p_i \dfrac{1+\nu}{E}\left\{(1-\nu)\ln\left[\dfrac{\rho_0}{\rho}\right] + \nu\right\}, & a \le \rho \le \rho_0, \\ -\, p_e \dfrac{1+\nu}{E}\left\{(1-2\nu)\,\rho - \dfrac{\rho_0^2}{\rho}\right\}, & \rho > \rho_0. \end{cases}$$

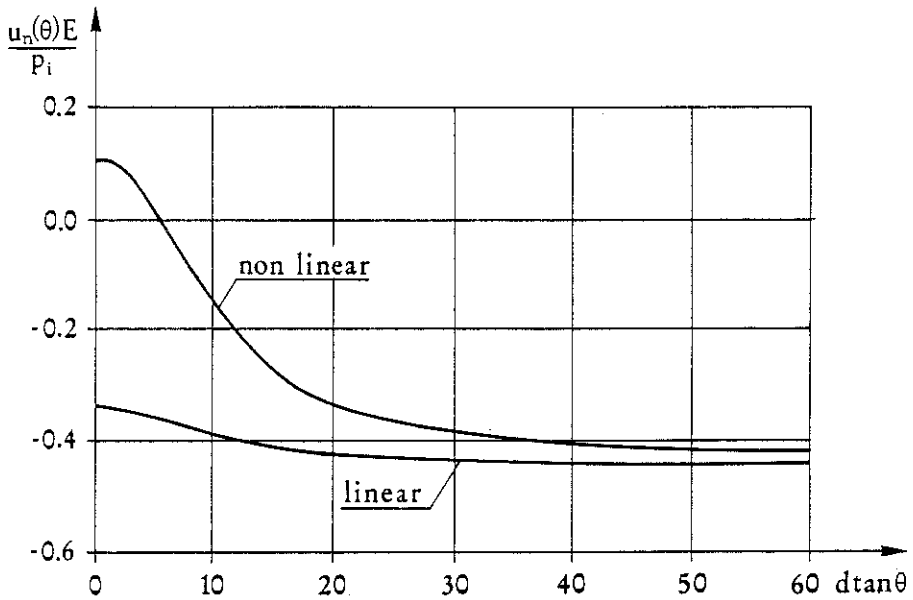

Figure 7. Displacement u_n vs. d tan(θ) for a linear elastic and a *masonry-like* material.

In the foregoing equations $\rho_0 = \frac{a}{2\psi}$ is the value of the radius of the circumference separating the inner region, which is the seat of the circumferential inelastic strain

$$\varepsilon_\theta^a(\rho) = a\, p_i \frac{1-\nu^2}{E} \frac{1}{\rho} \ln\left(\frac{\rho_0}{\rho}\right),$$

from the other, where it is nil. Setting $\psi < \frac{1}{2}$, let r be a straight line at distance $d > \rho_0$ from the centre of the ring. The points belonging to the line undergo displacements whose component perpendicular to the line is

$$u_{n0}(\theta) = -\psi\, p_i \frac{1+\nu}{E}\left\{(1-2\nu)\, d - \frac{a^2}{4d\psi^2}\cos^2(\theta)\right\};$$

if the material is instead linear elastic, this component is

$$u_{n1}(\theta) = p_i \frac{1+\nu}{E}\left\{(2\nu-1)d\,\psi - \frac{a^2}{d}(\psi-1)\cos^2(\theta)\right\}.$$

Figure 7 shows, as an example, the behaviour of $u_{n0}(\theta)$ and $u_{n1}(\theta)$ as functions of the distance $d\tan(\theta)$. In the figure $a = 1$ m, $d = 10$ m, $\psi = 0.05$, $E = 5{,}000$ MPa and $\nu = 0.1$. As it can be seen, switching from the linear elastic case to the non-linear one causes the displacement field to undergo remarkable changes in both intensity and shape.

IV. THE HOLLOW SPHERE

Let us consider a hollow sphere Ω_S, with inner radius a and outer radius b, which we will suppose for the moment to be made of a linear elastic material. The hollow sphere is subjected to two uniform radial pressures: a pressure p_e acting on the external boundary and a pressure p_i acting on the internal boundary. Let $\{O, \rho, \theta, \phi\}$ be a spherical reference system, with origin O coinciding with the centre of the sphere. For symmetry reasons, the only displacement component different from zero is the radial one $u(\rho)$; moreover, the non-zero stress components are the following [Lame', 1852]:

$$\sigma_\rho(\rho) = \frac{a^3 b^3 (p_e - p_i)}{b^3 - a^3}\,\frac{1}{\rho^3} + \frac{a^3 p_i - b^3 p_e}{b^3 - a^3},$$

$$\sigma_\theta(\rho) = \sigma_\phi(\rho) = -\frac{a^3 b^3 (p_e - p_i)}{2\left(b^3 - a^3\right)}\,\frac{1}{\rho^3} + \frac{a^3 p_i - b^3 p_e}{b^3 - a^3}.$$

As for the circular ring in the previous Section, it is easy to verify that the principal stresses are negative in Ω_S if the inequality $\sigma_\theta(a) < 0$ is satisfied, namely if

$$\psi > \frac{2+\eta^3}{3\eta^3},$$

where we have again set $\eta = b/a$ and $\psi = p_e/p_i$, respectively. If the inequality does not hold, tractions arise on the inner boundary and, when ψ decreases, spread to the interior and reach the outer boundary for $\psi = \dfrac{3}{1+2\eta^2}$.

If the elastic material cannot withstand tensile stresses and its constitutive equation is described by relations (1)-(3), where $\sigma = 0$, in looking for the solution we may suppose that the spherical region $\Omega_{S1} = \{(\rho, \theta, \phi), \rho \in [a, \rho_1]\}$, where $\rho_1 \in [a, b]$, is subjected to the equilibrated stress field

$$\sigma_\rho(\rho) = -\frac{a^2 p_i}{\rho^2} ,$$

$$\sigma_\theta(\rho) = \sigma_\phi(\rho) = 0 . \qquad (15)$$

As a consequence, the remaining spherical region Ω_{S2} is subjected to external pressure p_e and to internal pressure $p_1 = \dfrac{a^2 p_i}{\rho_1^2}$. On the other hand, for continuity reasons, equalities $\sigma_\theta(\rho_1^+) = \sigma_\phi(\rho_1^+) = 0$ must hold. Then, in view of (15), ratio $x = \rho_1/a$ is a solution to the cubic equation

$$x^3 + a_1(\psi)\, x^2 + a_3 = 0 ,$$

where $a_1(\psi) = -\dfrac{3\eta^2\psi}{2}$, $a_3 = \dfrac{\eta^3}{2}$. As is well-known, the roots of the equation are

$$x_1 = u_1 - \frac{p}{3u_1} - \frac{a_1}{3}, \quad x_2 = u_1\varepsilon_1 - \frac{p\,\varepsilon_2}{3u_1} - \frac{a_1}{3}, \quad x_3 = u_1\varepsilon_2 - \frac{p\,\varepsilon_1}{3u_1} - \frac{a_1}{3},$$

where $\varepsilon_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $\varepsilon_2 = \frac{1}{2} - \frac{\sqrt{3}}{2}i$ are the cubic complex roots of unity, and, moreover,

$$u_1 = \left\{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right\}^{1/3} ,$$

with

$$p = -\frac{a_1^3}{3} \;, q = -\frac{2a_1^3 + 27a_3}{27} ;$$

furthermore, u_1 is the only one of the three cubic roots such that $-\frac{\pi}{3} \le \arctan\left\{\dfrac{Im(u_1)}{Re(u_1)}\right\} \le \frac{\pi}{3}$. Other inequalities, omitted here for the sake of brevity, allow us to prove that the only real root belonging to the interval $[1, \eta]$ is x_3. Thus,

$$\rho_1 = a\left\{u_1 \varepsilon_2 - \frac{p\,\varepsilon_1}{3u_1} - \frac{a_1}{3}\right\}. \tag{16}$$

As ψ varies from $\psi_{s2} = \dfrac{2+\eta^3}{3\eta^3}$ to $\psi_{s1} = \dfrac{1}{\eta^2}$, radius ρ_1 varies correspondingly from a to b. Thus, for $\psi \in [\psi_{s1}, \psi_{s2}]$, stress field

$$\sigma_\rho(\rho) = \begin{cases} -\dfrac{a^2 p_i}{\rho^2}, & a \le \rho \le \rho_1, \\[2ex] \dfrac{\rho_1^3 b^3 (p_e - p_1)}{b^3 - \rho_1^3}\dfrac{1}{\rho^3} + \dfrac{\rho_1^3 p_1 - b^3 p_e}{b^3 - \rho_1^3}, & \rho_1 < \rho \le b, \end{cases}$$

$$\sigma_\theta(\rho) = \sigma_\phi(\rho) = \begin{cases} 0, & a \le \rho \le \rho_1, \\[2ex] -\dfrac{\rho_1^3 b^3 (p_e - p_1)}{2\left(b^3 - \rho_1^3\right)}\dfrac{1}{\rho^3} + \dfrac{\rho_1^3 p_1 - b^3 p_e}{b^3 - \rho_1^3}, & \rho_1 < \rho \le b, \end{cases} \tag{17}$$

is negative semi-definite in Ω_S. Then, according to the constitutive equation, the strain field is the following:

$$\varepsilon_\rho(\rho) = \begin{cases} -\dfrac{1}{E}\dfrac{a^2 p_i}{\rho^2}, & a \le \rho \le \rho_1, \\[2ex] \dfrac{p_1}{3E}\left\{2\nu - 1 - \dfrac{2\rho_1^3}{\rho^3}(1+\nu)\right\}, & \rho_1 < \rho \le b; \end{cases}$$

$$\varepsilon_\theta(\rho) = \varepsilon_\phi(\rho) = \begin{cases} \dfrac{\nu}{E}\dfrac{a^2 p_i}{\rho^2} + \varepsilon_\theta^a(\rho), & a \le \rho \le \rho_1, \\[2ex] \dfrac{p_1}{3E}\left\{2\nu - 1 + \dfrac{\rho_1^3}{\rho^3}(1+\nu)\right\}, & \rho_1 < \rho \le b. \end{cases} \tag{18}$$

In equation (18) the inelastic strain $\varepsilon_\theta^a(\rho)$ is an unknown non-negative function to be determined. In turn, radial displacement $u(\rho)$ has the following expression:

$$u(\rho) = \begin{cases} \frac{1}{E}\frac{a^2 p_l}{\rho} + C_l, & a \le \rho \le \rho_1, \\ -\frac{p_l}{3E}\left\{(1-2\nu)\rho - \frac{\rho_l^3}{\rho^2}(1+\nu)\right\}, & \rho_l < \rho \le b. \end{cases} \tag{19}$$

Since $u(\rho_1^-) = u(\rho_1^+)$, the value of C_1 is

$$C_l = -(1-\nu)\frac{a^2 p_l}{E\rho_l}. \tag{20}$$

On the other hand, as $u(\rho) = \rho\,\varepsilon_\theta(\rho)$, the inelastic circumferential strain

$$\varepsilon_\theta^a(\rho) = \varepsilon_\theta(\rho) - \varepsilon_\theta^e(\rho) = \frac{a^2 p_l}{E}(1-\nu)\left\{\frac{1}{\rho^2} - \frac{1}{\rho_l\rho}\right\} \tag{21}$$

results to be non-negative in Ω_{s1}, and is equal to zero only for $\rho = \rho_1$. This last result proves that, if $\psi \in [\psi_{s1}, \psi_{s2}]$, the solution to the boundary problem is described by equations (15)-(21). Once again in this case the inelastic strain and the radial displacement are unique.

V. THE WELL

Let us now suppose that the cylindrical indefinite body, whose cross section is the circular ring Ω described in Section III, belongs solely to the half-space $z \le 0$ (Figure 8), and is subjected to its own weight and the pressures $\gamma|z|$ e $\tau|z|$, varying linearly with z and acting on its inner and outer surfaces, respectively. The loading condition provide a rough representation of that occurring in a very deep well filled inside up to the free surface and externally subjected to the pressure of the surrounding soil. Due to the existing cylindrical symmetry, the only non-zero displacement components are the radial one $u(\rho,z)$ and the axial one $w(\rho,z)$. If the material is linearly elastic, it can be immediatley shown that the stress field has the components

$$\sigma_\rho(\rho, z) = -\frac{z}{b^2 - a^2}\left\{a^2\gamma - b^2\tau + \frac{a^2 b^2}{\rho^2}(\tau - \gamma)\right\},$$

$$\sigma_\theta(\rho, z) = -\frac{z}{b^2 - a^2}\left\{a^2\gamma - b^2\tau - \frac{a^2 b^2}{\rho^2}(\tau - \gamma)\right\},$$

$$\sigma_z(\rho, z) = p\,z, \quad \tau_{\rho z} = \tau_{\rho\theta} = \tau_{\theta z} = 0,$$

where p is the material's specific weight. Stress σ_z is non-positive; moreover, a direct calculation reveals that the principal stresses σ_ρ and σ_θ are non-positive, providing that

$$\omega \geq \frac{1+\eta^2}{2\eta^2}$$

where $\omega = \frac{\tau}{\gamma}$. Let us now consider the case in which the previous inequality is not satisfied and the non-linear elastic material has constitutive equation (1)-(3), where, for simplicity's sake, we set $\sigma = 0$. Following a line of reasoning quite similar to that in Section III, it can be easily shown that for values of ω belonging to the intaerval $[\omega_1, \omega_2]$, with $\omega_1 = \frac{1}{\eta}$, $\omega_2 = \frac{1+\eta^2}{2\eta^2}$, the stress field

$$\sigma_\rho(\rho, z) = \begin{cases} \frac{a\gamma}{\rho} z , & a \leq \rho \leq \rho_0, \\ \frac{a\gamma}{2}\left\{\frac{\rho_0}{\rho^2}+\frac{1}{\rho_0}\right\} z , & \rho_0 < \rho \leq b , \end{cases}$$

$$\sigma_\theta(\rho, z) = \begin{cases} 0, & a \leq \rho \leq \rho_0, \\ \frac{a\gamma}{2}\left\{-\frac{\rho_0}{\rho^2}+\frac{1}{\rho_0}\right\} z , & \rho_0 < \rho \leq b , \end{cases} \tag{22}$$

$$\sigma_z(\rho, z) = p\, z , \quad \tau_{\rho z} = \tau_{\rho\theta} = \tau_{\theta z} = 0 ,$$

is equilibrated and negative semi-definite. In equations (22)

$$\rho_0 = a\,\eta\,\left\{\eta\omega - \sqrt{\eta^2\omega^2 - 1}\right\} \tag{23}$$

is the radius of the cylindrical surface Γ_0 which separetes the body into two regions, an external one, for which the solution coincides with a linear elastic one, and an internal one, where the circumferential stress is nil and the solution is non-linear. Trivial calculations, omitted here for brevity's sake, show that the following strains correspond to the previous stress components:

$$\varepsilon_\rho(\rho, z) = \begin{cases} \dfrac{1}{E}\left(-\nu p + \dfrac{\gamma a}{\rho}\right) z\ , & a \le \rho \le \rho_0\ , \\[2ex] \left\{\dfrac{\gamma a}{4G}\left(\dfrac{\rho_0}{\rho^2} + \dfrac{1}{\rho_0}\right) - \dfrac{\nu}{E}\left(p + \dfrac{\gamma a}{\rho_0}\right)\right\} z\ , & \rho_0 < \rho \le b\ ; \end{cases} \qquad (24)_1$$

$$\varepsilon_\theta(\rho, z) = \begin{cases} -\dfrac{\nu}{E}\left(p + \dfrac{\gamma a}{\rho}\right) z + \varepsilon_\theta^a(\rho, z)\ , & a \le \rho \le \rho_0\ , \\[2ex] \left\{\dfrac{\gamma a}{4G}\left(-\dfrac{\rho_0}{\rho^2} + \dfrac{1}{\rho_0}\right) - \dfrac{\nu}{E}\left(p + \dfrac{\gamma a}{\rho_0}\right)\right\} z\ , & \rho_0 < \rho \le b\ ; \end{cases} \qquad (24)_2$$

$$\varepsilon_z(\rho, z) = \begin{cases} \dfrac{z}{E}\left(p - \dfrac{\nu \gamma a}{\rho}\right)\ , & a \le \rho \le \rho_0\ , \\[2ex] \left\{\dfrac{p}{2G} - \dfrac{\nu}{E}\left(p + \dfrac{\gamma a}{\rho_0}\right)\right\} z\ , & \rho_0 < \rho \le b\ . \end{cases} \qquad (24)_3$$

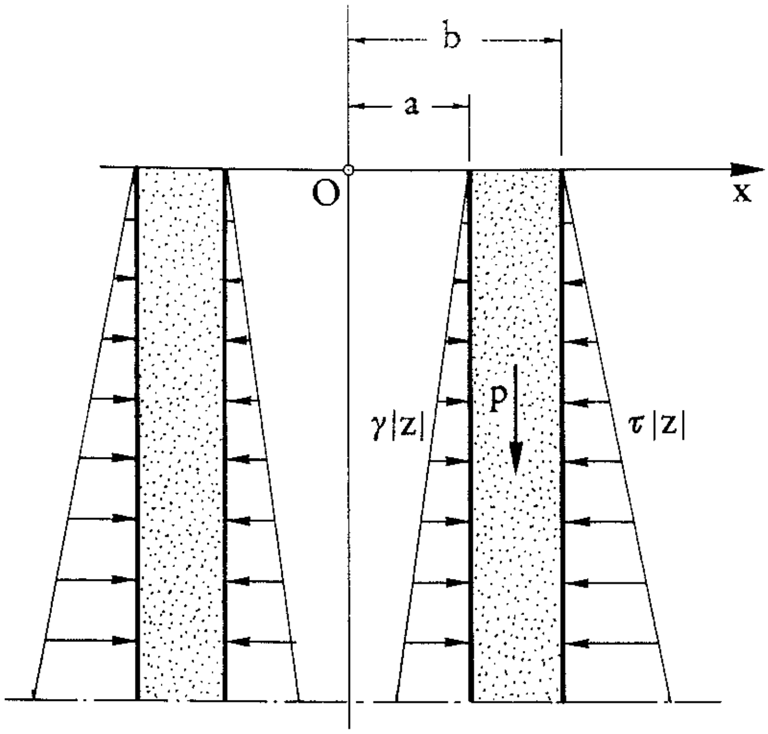

Figure 8. The well.

In $(24)_2$ $\varepsilon_\theta^a(\rho, z)$ is a non-negative function to be determined. On the other hand, since $u = \rho\, \varepsilon_\theta$ and $u_{,\rho} = \varepsilon_\rho$, by taking the continuity of the radial displacements at $\rho = \rho_0$ into account, we have:

$$u(\rho, z) = \begin{cases} \frac{z}{E}\left(a\gamma \ln\left(\frac{\rho}{\rho_0}\right) - \nu\,(a\gamma + p\rho)\right), & a \le \rho \le \rho_0, \\ \left\{\frac{\gamma a}{4G}\left(\frac{\rho}{\rho_0} - \frac{\rho_0}{\rho}\right) - \frac{\nu}{E}\left(p + \frac{\gamma a}{\rho_0}\right)\rho\right\}z, & \rho_0 < \rho \le b, \end{cases} \tag{25}$$

$$\varepsilon_\theta^a(\rho, z) = -\frac{a\gamma}{E}\ln\left(\frac{\rho_0}{\rho}\right)z, \quad a \le \rho \le \rho_0. \tag{26}$$

The displcement component $w(\rho,z)$ remains to be determined. However, $w_{,z} = \varepsilon_z$, and, furthermore, $\tau_{\rho z} = 0$: so, it must hold that $u_{,z} + w_{,\rho} = 0$. The foregoing equations imposes that

$$w(\rho, z) = \begin{cases} \frac{1}{2E}\left(p - \frac{\nu\gamma a}{\rho}\right)z^2 + g(\rho), & a \le \rho \le \rho_0, \\ \left\{\frac{p}{2G} - \frac{\nu}{E}\left(p + \frac{\gamma a}{\rho_0}\right)\right\}\frac{z^2}{2} + \gamma a\left\{\frac{\rho_0}{4G}\ln\rho + \left(\frac{\nu}{E} - \frac{1}{4G}\right)\frac{\rho^2}{2\rho_0}\right\}, & \rho_0 < \rho \le b \end{cases} \tag{27}$$

with $g(\rho)$ verifying the differential equation

$$g'(\rho) = -\frac{\nu a\gamma z^2}{2E\rho^2} - \frac{1}{E}\left[a\gamma\ln\left(\frac{\rho}{\rho_0}\right) - \nu\,(a\gamma + p\rho)\right]. \tag{28}$$

The last expression clearly shows that the solution built up to this point looses all meaning if ν is different from zero: indeed, if this is the case, equation (28) can never be satisfied and no regular displacement field generate strain field determined above. Vice versa, if $\nu = 0$, by accounting for the continuity of $w(\rho,z)$ at $\rho = \rho_0$, it is easily proved that for $a \le \rho \le \rho_0$,

$$w(\rho, z) = \frac{p\,z^2}{2E} + \frac{a\gamma}{E}\left[1 + \ln\left(\frac{\rho_0}{\rho}\right)\right]\rho + \frac{a\gamma\rho_0}{4E}\,[2\ln\rho_0 - 5], \tag{29}$$

and the non-linear elastic solution is described by equations (22)-(27) and (29). Such a result underscore how an *admissible* stress field , that is equilibrated and negative semi-definite, will often have no corresponding regular displacement field compatible with the constitutive equation.

VI. THE CIRCULAR HALF-RING

The previous solutions are all characterized by one form of symmetry or other, be it spherical, cylindrical or polar. In the following we deal with a problem lacking any of the above symmetries.

Let us consider an indefinite cylindrical body made of elastic isotropic material whose cross section is a circular half-ring Ω_p of inner radius a and external radius b. The body is referred to a cylindrical reference system {O, ρ, θ, z} (Figure 9) and subjected to a plane strain field, with pressures

$$\hat{p}_e(\theta) = p_e - \chi \frac{\sin\theta}{b}, \quad \hat{p}_i(\theta) = p_i - \chi \frac{\sin\theta}{a}, \tag{30}$$

acting on the external and internal boundaries, respectively. In equations (30) p_i e p_e are positive constants and χ is a parameter belonging to interval $(-\infty, ap_i]$; moreover, we suppose the existence of surface forces on the rectilinear portions AB and CD of $\partial\Omega_p$, perpendicular to the boundary and of such intensity as to insure the equilibrium of the half-ring.

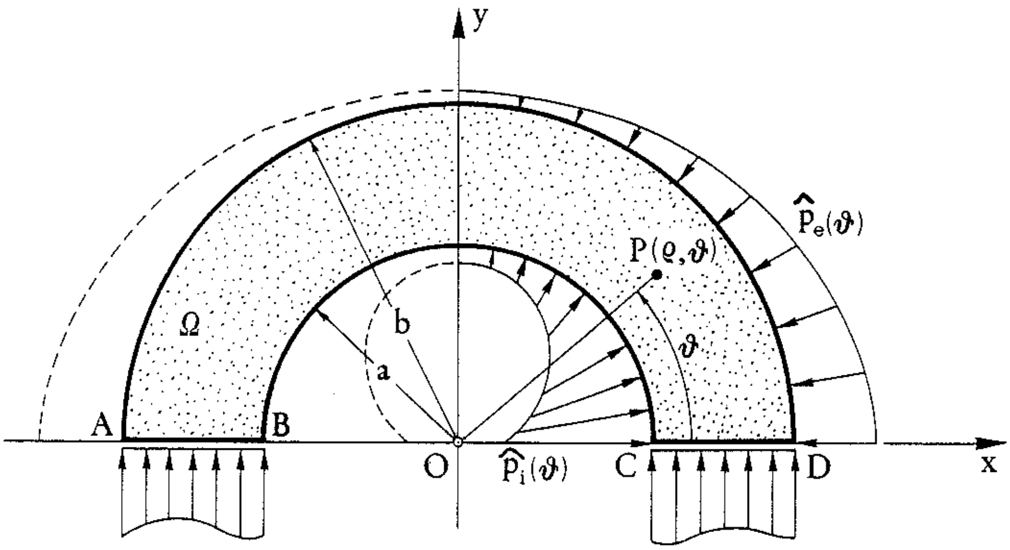

Figure 9. The circular half-ring.

If the elastic material is linear, it can be immediately verified that the principal stresses

$$\sigma_\rho(\rho,\theta) = \frac{1}{b^2-a^2}\left\{a^2 p_i - b^2 p_e + \frac{a^2 b^2}{\rho^2}(p_e - p_i)\right\} + \chi\frac{\sin\theta}{\rho},$$

$$\sigma_\theta(\rho,\theta) = \frac{1}{b^2-a^2}\left\{a^2 p_i - b^2 p_e - \frac{a^2 b^2}{\rho^2}(p_e - p_i)\right\},$$

$$\sigma_z(\rho,\theta) = \nu\,[\sigma_\rho(\rho,\theta) + \sigma_\theta(\rho,\theta)] = \frac{2\nu}{b^2-a^2}\{a^2 p_i - b^2 p_e)\} + \nu\chi\frac{\sin\theta}{\rho},$$

correspond to one of the infinite solutions to the equilibrium problem, in which normal stresses are non-positive providing that the ratio $\psi = p_e/p_i$ is greater than $\frac{1+\eta^2}{2\eta^2}$, once again with $\eta = b/a$. On the other hand, it is a trivial matter to prove that, if $\frac{1}{\eta} \leq \psi \leq \frac{1+\eta^2}{2\eta^2}$, the stress field

$$\sigma_\rho(\rho,\theta) = \begin{cases} -\frac{a}{\rho}p_i + \chi\frac{\sin\theta}{\rho}, & a \leq \rho \leq \rho_0, \\ -a p_i\left\{\frac{\rho_0}{2\rho^2} + \frac{1}{2\rho_0}\right\} + \chi\frac{\sin\theta}{\rho}, & \rho_0 < \rho \leq b, \end{cases}$$

$$\sigma_\theta(\rho,\theta) = \begin{cases} 0, & a \leq \rho \leq \rho_0, \\ a p_i\left\{\frac{\rho_0}{2\rho^2} - \frac{1}{2\rho_0}\right\}, & \rho_0 < \rho \leq b, \end{cases} \tag{31}$$

$$\sigma_z(\rho,\theta) = \nu\,[\sigma_\rho(\rho,\theta) + \sigma_\theta(\rho,\theta)], \quad a \leq \rho \leq b,$$

in which $\rho_0 = a\eta\left\{\eta\psi - \sqrt{\eta^2\psi^2 - 1}\right\}$ does not depend on θ, is equilibrated and negative semi-definite in Ω_p. It is therefore reasonable to ask if the above field has associated strain and displacement fields which taken together can solve the problem when the elastic material is non-resistant to tensile stresses.

We begin by observing that the stress field separates the body into two half-rings, Ω_{p1} and Ω_{p2}, the first of which has internal radius a and external radius ρ_0. In Ω_{p2} the inelastic strains are nil and it is an easy matter to show that the strains and the radial and circumferential displacements are, respectively,

$$\varepsilon_\rho(\rho,\theta) = \frac{1+\nu}{E}\left\{-\frac{a\,p_i}{2\rho_0}(1-2\nu) - \frac{a\,p_i\,\rho_0}{2\rho^2} + (1-\nu)\chi\frac{\sin\theta}{\rho}\right\},$$

$$\varepsilon_\theta(\rho, \theta) = \frac{1+\nu}{E}\left\{-\frac{a\, p_l}{2\rho_0}(1-2\nu) + \frac{a\, p_l\, \rho_0}{2\rho^2} - \nu\chi\frac{\sin\theta}{\rho}\right\},$$

$$u(\rho, \theta) = \frac{1+\nu}{E}\left\{\frac{a\, p_l}{2}\left[\frac{\rho_0}{\rho} - (1-2\nu)\frac{\rho}{\rho_0}\right] + (1-\nu)\chi\frac{\sin\theta}{\rho}\ln\rho + \right.$$

$$\left. - \frac{1-2\nu}{2}\chi\left[\sin\theta + (\theta - \frac{\pi}{2})\cos\theta\right]\right\},$$

$$v(\rho, \theta) = \chi\frac{1+\nu}{E}\left\{[\nu + (1-\nu)\ln\rho]\cos\theta - \frac{(1-2\nu)}{2}\sin\theta\,(\frac{\pi}{2} - \theta)\right\}.$$

In Ω_{pl}

$$\varepsilon_\rho(\rho, \theta) = \varepsilon_\rho^e(\rho, \theta) = -\frac{(1-\nu^2)(a\, p_l - \chi\sin\theta)}{E\,\rho}.$$

Moreover, since $\varepsilon_\rho = u_{,\rho}$, we have

$$u(\rho, \theta) = -\frac{(1-\nu^2)(a\, p_l - \chi\sin\theta)\ln\rho}{E} + C_1(\theta),$$

where

$$C_1(\theta) = \frac{1+\nu}{E}\left\{a\, p_l\,(\nu + (1-\nu)\ln\rho_0) - \frac{1-2\nu}{2}\chi\left[\sin\theta + (\theta - \frac{\pi}{2})\cos\theta\right]\right\}$$

is a function determined by imposing the continuity of $u(\rho, \theta)$ at $\rho = \rho_0$. In turn,

$$\varepsilon_\theta(\rho, \theta) = \varepsilon_\theta^e(\rho, \theta) + \varepsilon_\theta^a(\rho, \theta) = \frac{\nu(1+\nu)(a\, p_l - \chi\sin\theta)}{E\,\rho} + \varepsilon_\theta^a(\rho, \theta),$$

where $\varepsilon_\theta^a(\rho, \theta)$ is again a non-negative function, for the moment unknown. Finally, it can be shown that by accounting for the continuity of $v(\rho, \theta)$ at $\rho = \rho_0$, then

$$v(\rho, \theta) = \chi\frac{1+\nu}{E}\left\{[\nu + (1-\nu)\ln\rho]\cos\theta - \frac{(1-2\nu)}{2}\sin\theta\,(\frac{\pi}{2} - \theta)\right\}, \quad a \le \rho \le \rho_0$$

must hold, and, consequently,

$$\varepsilon_\theta^a(\rho, \theta) = \frac{(1-\nu^2)\, a\, p_l \ln\left(\frac{\rho_0}{\rho}\right)}{E\,\rho}.$$

Thus, in this case as well, if $\frac{1}{\eta} \leq \psi \leq \frac{1+\eta^2}{2\eta^2}$, stress field (31) may be completed by a strain field and a piecewise C^2 displacement field which together provide a solution to the equilibrium problem when Ω_p is made of a material with constitutive equation (1)-(3) in which σ has again been set equal to zero, for the sake of simplicity.

VII. CONCLUSIONS

In order to conclude, some remarks are proper. The foregoing explicit solutions have been obtained by assuming that the behavior of the material might be described by an elastic constitutive equation with a non-linear stress-strain law. Such a model is only roughly representative of the complex behavior of materials with low tensile strength such as stone and masonry. In particular, if $\sigma > 0$, it is assumed that the material is able to withstand a tensile stress equal to its tensile strength without incurring fractures. Moreover, its bounded compressive strength is also neglected. Although it is not difficult to account for this latter by appropriately modifying the constitutive equation [see Lucchesi *et al.*, 1995], the objection arises that a load-history dependent material cannot be described by a purely elastic model. However, it is worthwhile noting that the constitutive equation accounts for an essential feature of the response of masonry and stone structures, that is to say their non linearity (see Figures 4 and 6). This is perhaps the reason why, despite its simplicity, the model often delivers useful indications to understand them [Lucchesi *et al.*, 1994b]. Finally, it is to be observed that, although inelastic strains are present in extended region of the solid, the displacement fields are regular, that is C^2 piecewise.

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