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Revising Typical Beliefs: One Revision to Rule Them All

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Abstract

Propositional Typicality Logic (PTL) extends propositional logic with a connective • expressing the most typical (alias normal or conventional) situations in which a given sentence holds. As such, it generalises e.g. preferential logics that formalise reasoning with conditionals such as "birds typically fly". In this paper we study the revision of sets of PTL sentences. We first show why it is necessary to extend the PTL language with a possibility operator and then define the revision of PTL sentences syntactically and characterise it semantically. We show that this allows us to represent a wide variety of existing revision methods, such as propositional revision and revision of epistemic states. Furthermore, we provide several examples showing why our approach is innovative. In more detail, we study the revision of a set of conditionals under preferential closure and the addition and contraction of possible worlds from an epistemic state.

1 Introduction

Recent years have seen an increased interest in formalisms for studying typicality and related concepts. Conditional statements about normality or typicality, such as expressions of the form "typically, if A then B" have been studied for almost half a century (Kraus, Lehmann, and Magidor 1990) but have picked up renewed interest in the last years (Booth et al. 2019; Sezgin and Kern-Isberner 2021a; Booth and Varzinczak 2021), e.g. involving stronger or alternative language constructs. For example, so-called *weak conditionals* expressing that "typically, if A then B" does *not* hold have been studied in (Sezgin and Kern-Isberner 2021a), while Propositional Typicality Logic (PTL) (Booth, Meyer, and Varzinczak 2012; Booth et al. 2019) proposes a connective • that formalizes typicality directly.

Belief revision, which provides a formal theory of the dynamics of beliefs, has been generalized from the propositional context to the context of conditional statements about typicality (Darwiche and Pearl 1997) under the name of *iterated belief revision*. A major insight is that revising a conditional knowledge base corresponds, on the semantical level, to revising an epistemic state (i.e. a total preorder over possible worlds) (Katsuno and Mendelzon 1991). For other language constructs involving typicality, belief revision has not been studied. This constitutes a major gap in the literature, as these language constructs are meant to formalize the same dynamic domains as conditionals.

Contribution: In this paper, we fill this gap by providing a comprehensive study of belief revision in propositional typicality logic. This has the benefit of providing a very broad and expressive framework for the dynamics of typical beliefs, which we demonstrate by providing several representation results. In more detail, the main formal contributions of the paper can be understood as follows. Propositional typicality logic has semantics in terms of epistemic states or rankings over possible worlds. In line with the canonical semantics of propositional belief revision, belief revision of PTL-knowledge bases can be seen semantically as considering a total preorder over these rankings, where revision is then defined by selecting the minimal rankings that satisfy the PTL-formula by which one revises. A major challenge is to account for *representability*, which means that the selection of minimal rankings has to be representable by a PTL-knowledge base. We show that using only the typicality-connective, this is not warranted. To solve this problem, we introduce the *possibility*-connective. We show that this allows us to express additional constructs from the language such as weak conditionals. I.e. it is fruitful independently of the study of revision. Using this extended language, we define and characterise revision of PTL-knowledge bases, and show that it can capture existing approaches to revision such as propositional and iterated revision, and opens up new ground, such as studying the contraction or enlargement of the set of possible worlds in an epistemic state.

Outline of the paper: In section 2 we provide the necessary preliminaries on propositional logic, propositional belief revision, and PTL. In section 3 we study the expressivity of PTL, define the possibility operator, and show how it extends the expressivity of PTL. In Section 4 we define and semantically characterize the revision of PTL-knowledge bases, and in Section 5 we give examples of specific types of revision of PTL-knowledge bases. In Section 6 we conclude in view of related work.

2 Preliminaries

We briefly recap some salient notions on propositional logic, belief revision and propositional typicality logic.

2.1 Propositional Logic

For a non-empty finite set At of *atoms* let $\mathcal{L}(\mathsf{At})$ be the corresponding *Propositional Language* (PL) constructed using the usual connectives $\wedge$, $\vee$, $\neg$, $\rightarrow$ and $\leftrightarrow$. A (classical) *interpretation* (or *possible world*) v w.r.t. $\mathcal{L}(\mathsf{At})$ is a function $v\colon \mathsf{At} \longrightarrow \{\mathsf{T},\mathsf{F}\}$. Let $\mathcal{U}(\mathsf{At})$ denote the set of all interpretations for At. We simply write $\mathcal{L}$ and $\mathcal{U}$ if At is clear from context. An interpretation v *satisfies* (or is a *model* of) an atom $a \in \mathsf{At}$, denoted $v \models a$, iff $v(a) = \mathsf{T}$. The satisfaction relation $\models$ is extended to formulas as usual. As an abbreviation we sometimes identify an interpretation w with its *complete conjunction*, i. e., if $a_1,\ldots,a_n \in \mathsf{At}$ are those atoms that are assigned T by v and $a_{n+1},\ldots,a_m \in \mathsf{At}$ are those propositions that are assigned F by v we identify v by $a_1\ldots a_n\overline{a_{n+1}}\ldots\overline{a_m}$ (or any permutation of this). For example, the interpretation v_1 on $\{a,b,c\}$ with $v_1(a) = v_1(c) = \mathsf{T}$ and $v_1(b) = \mathsf{F}$ is abbreviated by $a\bar{b}c$. For a non-empty finite set $\Phi \subseteq \mathcal{L}(\mathsf{At})$ we also define $v \models \Phi$ iff $v \models \phi$ for every $\phi \in \Phi$. Define the set of models $[\![X]\!]_{\mathsf{PL}} = \{w \in \mathcal{U}(\mathsf{At}) \mid w \models X\}$ for every formula or set of formulas X. A formula ϕ_1 *entails* formula ϕ_2, denoted $\phi_1 \models_{\mathsf{PL}} \phi_2$, if $[\![\phi_1]\!]_{\mathsf{PL}} \subseteq [\![\phi_2]\!]_{\mathsf{PL}}$ (and similarly for sets of formulas), and $Cn_{\mathsf{PL}}(\Phi) = \{\phi \mid \phi \models_{\mathsf{PL}} \phi\}$.

2.2 Revising Propositional Formulas

We now recall the so-called AGM-approach to belief revision (Alchourrón, Gärdenfors, and Makinson 1985) as reformulated for propositional formulas by (Katsuno and Mendelzon 1991). The following postulates for *revision operators* $\star\colon \mathcal{L}(\mathsf{At}) \times \mathcal{L}(\mathsf{At}) \longrightarrow \mathcal{L}(\mathsf{At})$ are formulated:

- (R1) $\phi \star \psi \models_{\mathsf{PL}} \psi$
- (R2) If $\phi \wedge \psi$ is satisfiable, then $\phi \star \psi \leftrightarrow \psi \wedge \phi$
- (R3) If ψ is satisfiable, then so is $\phi \star \psi$
- (R4) If $\emptyset \models_{\mathsf{PL}} \phi_1 \leftrightarrow \phi_2$ and $\emptyset \models_{\mathsf{PL}} \psi_1 \leftrightarrow \psi_2$, $\emptyset \models_{\mathsf{PL}} \phi_1 \star \psi_1 \leftrightarrow \phi_2 \star \psi_2$
- (R5) $(\phi \star \psi) \wedge \mu \models_{\mathsf{PL}} \phi \star (\psi \wedge \mu)$
- (R6) If $(\phi \star \psi) \wedge \mu$ is satisfiable, then $\phi \star (\psi \wedge \mu) \models_{\mathsf{PL}} (\phi \star \psi) \wedge \mu$

An important result is the semantic characterisation of such a belief revision operator. For such a characterisation, we define a function $f\colon \mathcal{L}(\mathsf{At}) \longrightarrow \wp(\mathcal{U}(\mathsf{At}) \times \mathcal{U}(\mathsf{At}))$ assigning to each $\phi \in \mathcal{L}$ a preorder $\preceq_\phi$ over $\mathcal{U}(\mathsf{At})$.

Definition 1 ((Katsuno and Mendelzon 1991))**.** A function $f : \mathcal{L}(\mathsf{At}) \rightarrow \wp(\mathcal{U}(\mathsf{At}) \times \mathcal{U}(\mathsf{At}))$ assigning preorders $\preceq_\phi$ over $\mathcal{U}(\mathsf{At})$ to every formula $\phi \in \mathcal{L}(\mathsf{At})$ is *faithful* iff:

1. for every $\phi \in \mathcal{L}(\mathsf{At})$, if $w, w' \in [\![\phi]\!]_{\mathsf{PL}}$ then $w \not\prec_\phi w'$,
2. for every $\phi \in \mathcal{L}(\mathsf{At})$, if $w \in [\![\phi]\!]_{\mathsf{PL}}$ and $w' \notin [\![\phi]\!]_{\mathsf{PL}}$ then $w \preceq_\phi w'$,
3. for every $\phi, \phi' \in \mathcal{L}(\mathsf{At})$, if $[\![\phi]\!]_{\mathsf{PL}} = [\![\phi']\!]_{\mathsf{PL}}$ then $\preceq_\phi = \preceq_{\phi'}$.

Katsuno and Mendelzon (1991) provided the following representation theorem for an AGM revision operator $\star$.

Theorem 1 ((Katsuno and Mendelzon 1991))**.** An operator $\star\colon \mathcal{L}(\mathsf{At}) \times \mathcal{L}(\mathsf{At}) \longrightarrow \mathcal{L}(\mathsf{At})$ is a revision operator iff there exists a faithful mapping $f^\star\colon \mathcal{L}(\mathsf{At}) \longrightarrow \wp(\mathcal{U}(\mathsf{At}) \times \mathcal{U}(\mathsf{At}))$ that maps each formula $\phi \in \mathcal{L}(\mathsf{At})$ to a total preorder s.t.:

$$[\![\phi \star \psi]\!]_{\mathsf{PL}} = \min_{f^\star(\phi)}([\![\psi]\!]_{\mathsf{PL}}) \tag{1}$$

2.3 Ranked Interpretations and PTL

We recall the necessary preliminaries on propositional typicality logic and its semantics (Booth, Meyer, and Varzinczak 2012; Booth et al. 2019; Booth et al. 2015). A *ranked interpretation* $\mathcal{R}$ is a tuple $\langle \mathcal{V}, \preceq \rangle$ where $\mathcal{V} \subseteq \mathcal{U}(\mathsf{At})$ and $\preceq \subseteq \mathcal{V} \times \mathcal{V}$ is a *modular order* on $\mathcal{V}$.[1] Given $\mathcal{R} = \langle \mathcal{V}, \preceq \rangle$, we will refer sometimes to $\mathcal{V}$ as $\mathcal{V}_{\mathcal{R}}$. Now, the language PTL, denoted $\mathcal{L}^\bullet$, is recursively defined as follows:

$$A ::= \mathsf{At} \mid \neg A \mid A \wedge A \mid \top \mid \bot \mid \bullet A$$

A finite set of PTL-formulas $\mathcal{K}$ is called a *Knowledge Base* (KB). A KB $\mathcal{K} \subseteq \mathcal{L}^\bullet$ is a *Conditional KB* (CKB) if each element of $\mathcal{K}$ is of the form $\bullet\phi \rightarrow \psi$, for $\phi, \psi \in \mathcal{L}$.[2]

Satisfaction is defined inductively as follows (given a ranked interpretation $\mathcal{R} = \langle \mathcal{V}, \preceq \rangle$ and $v \in \mathcal{V}_{\mathcal{R}}$:

- for $a \in \mathsf{At}$, $\mathcal{R}, v \models a$ iff $v(a) = \mathsf{T}$,
- $\mathcal{R}, v \models \bullet A$ iff $\mathcal{R}, v \models A$ and $\nexists\, v'$ s.t. $v' \prec v$ and $v' \models A$.
- $\mathcal{R}, v \models \neg A$ iff $\mathcal{R}, v \not\models A$
- $\mathcal{R}, v \models A \vee B$ iff $\mathcal{R}, v \models A$ or $\mathcal{R}, v \models B$.
- $\mathcal{R}, v \models A \wedge B$ iff $\mathcal{R}, v \models A$ and $\mathcal{R}, v \models B$.

We define $[\![A]\!]^{\mathcal{R}} = \{v \in \mathcal{V}_{\mathcal{R}} \mid \mathcal{R}, v \models A\}$. Note that $[\![\bullet A]\!]^{\mathcal{R}} := \min_{\preceq}[\![A]\!]^{\mathcal{R}}$. $A \in \mathcal{L}^\bullet$ is *satisfiable* in $\mathcal{R}$ iff $[\![A]\!]^{\mathcal{R}} \neq \emptyset$. A formula A is true in $\mathcal{R}$, denoted $\mathcal{R} \models A$, if $\mathcal{R}, v \models A$ for every $v \in \mathcal{V}_{\mathcal{R}}$, i.e. $[\![\phi]\!]^{\mathcal{R}} = \mathcal{V}_{\mathcal{R}}$. For a KB $\mathcal{K}$, $[\![\mathcal{K}]\!] = \{\mathcal{R} \mid \mathcal{R} \models \bigwedge \mathcal{K}\}$. We say $\mathcal{K} \models_0 A$ iff $[\![\mathcal{K}]\!] \subseteq [\![A]\!]$. $Cn_0(\mathcal{K}) = \{A \mid \mathcal{K} \models_0 A\}$. $\mathcal{K}_1 \leftrightarrow_0 \mathcal{K}_2$ iff $A \in Cn_0(\{B\})$ and $B \in Cn_0(\{A\})$. We write $A \preceq_{\mathcal{R}} B$ (for $A, B \in \mathcal{L}^\bullet$ and $\mathcal{R} \in \mathfrak{R}$) if for every $w, w' \in \mathcal{V}_{\mathcal{R}}$ s.t. $\mathcal{R}, w \models \bullet A$ and $\mathcal{R}, w' \models \bullet B$, $w \preceq_{\mathcal{R}} w'$. $\mathfrak{R}$ is the set of all rankings.

There exists a normal form for PTL-formulas, which has among its benefits that arbitrary nesting of $\bullet$ can be reduced to a single level of occurences $\bullet$.

Definition 2 ((Booth et al. 2019))**.** A formula $A \in \mathcal{L}^\bullet$ is *in normal form* iff it is of the form $(\bigwedge_{i \leq t} \bullet\theta_i) \rightarrow (\phi \vee \bigvee_{i \leq s} \bullet\psi_i)$ (with $t, s \in \mathbb{N}$), where $\theta_1, \ldots, \theta_t, \phi, \psi_1, \ldots, \psi_s \in \mathcal{L}$.

Proposition 1 ((Booth et al. 2019))**.** For every $A \in \mathcal{L}^\bullet$, there is some $X \subseteq \mathcal{L}^\bullet$ in normal form s.t. $[\![A]\!] = [\![X]\!]$.

Example 1 (Running example)**.** Consider the canonical penguin example: $\mathcal{K}_p = \{p \rightarrow b, \bullet p \rightarrow \neg f, \bullet b \rightarrow f\}$. Then $[\![\mathcal{K}_p]\!]$ contains, among others, the following rankings:

$$\mathcal{R}_1 : \quad \overline{p}\overline{b}\overline{f}, \overline{p}\overline{b}f, b\overline{p}f \prec b\overline{p}\overline{f}, bp\overline{f} \prec pbf$$
$$\mathcal{R}_2 : \quad \overline{p}\overline{b}\overline{f} \prec \overline{p}\overline{b}f, \overline{b}p\overline{f} \prec b\overline{p}\overline{f}, bp\overline{f} \prec pbf$$
$$\mathcal{R}_3 : \quad \overline{p}\overline{b}\overline{f} \prec \overline{p}\overline{b}f \prec \overline{b}p\overline{f} \prec b\overline{p}\overline{f}, bp\overline{f} \prec pbf$$

To get an idea of the additional expressivity allowed by PTL, we can consider an extended penguin example, where we additionally require that all penguins are typical penguins.

[1] An order $\preceq$ over $\mathcal{R}$ is modular if it admits a ranking, i.e. there is some $\kappa : \mathcal{V} \longrightarrow \mathbb{N}$ s.t. $v_1 \preceq v_2$ iff $\kappa(v_1) \leq \kappa(v_2)$.

[2] To keep our results as general as possible, we allow for arbitrary nesting of the $\bullet$ operator. However, results on normal forms show that nesting is in principle not needed (see Proposition 1).

This is done by extending the KB as $\mathcal{K}_{pt} = \mathcal{K}_p \cup \{p \rightarrow \bullet p\}$. We see that this KB has a strictly smaller set of models. For example, $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3 \notin [\![\mathcal{K}_{p'}]\!]$. To see why, notice that e.g. $\mathcal{R}_2, pbf \not\models \bullet p$, and thus $\mathcal{R}_2, pbf \not\models p \rightarrow \bullet p$. On the other hand, an example of a model of $\mathcal{K}_{pt}$ is:

$$\mathcal{R}_4 : \quad \overline{p}\overline{b}\overline{f}, \overline{p}\overline{b}f, \overline{b}p\overline{f} \prec b\overline{p}\overline{f}, bp\overline{f}$$

□

3 The Expressivity of PTL

Before studying revision of PTL-formulas, we first make some observations on the expressivity of PTL. PTL-formulas are given a semantics in terms of ranked interpretations. It's logical to ask what are the expressive capabilities of PTL: i.e. which sets of ranked interpretations can be expressed by PTL-formulas? It turns out that one cannot express *every* set of rankings using a set of PTL-formulas:

Proposition 2. There is $\{\mathcal{R}_1, \ldots, \mathcal{R}_n\} \subseteq \mathfrak{R}$ s.t. there exists no KB $\mathcal{K} \subseteq \mathcal{L}^\bullet$ for which $[\![\mathcal{K}]\!] = \{\mathcal{R}_1, \ldots, \mathcal{R}_n\}$.

Proof. We show this for $\{\mathcal{R}_1, \mathcal{R}_2\}$ with $\mathcal{R}_1 : p \prec \overline{p}$ and $\mathcal{R}_2 : \overline{p} \prec p$. We now show that for any $\mathcal{K} \subseteq \mathcal{L}^\bullet$ for which $[\![\mathcal{K}]\!] \supseteq \{\mathcal{R}_1, \mathcal{R}_2\}$, $\mathcal{R} : p, \overline{p} \in [\![\mathcal{K}]\!]$. In view of Proposition 1, it suffices to show this for a set $\mathcal{K} \subseteq \mathcal{L}^\bullet$ in normal form.

Consider $\mathcal{K} \subseteq \mathcal{L}^\bullet$ for which $[\![\mathcal{K}]\!] \supseteq \{\mathcal{R}_1, \mathcal{R}_2\}$ and consider some $\bigwedge_{i \leq t} \bullet\theta_i \rightarrow (\phi \vee \bigvee_{i \leq s} \bullet\psi_i) \in \mathcal{K}$. We show that $\mathcal{R}, p \models \bigwedge_{i \leq t} \bullet\theta_i \rightarrow (\phi \vee \bigvee_{i \leq s} \bullet\psi_i)$ (the case for $\overline{p}$ is similar). Suppose for this that $\mathcal{R}, p \models \bigwedge_{i \leq t} \bullet\theta_i$, i.e. $\mathcal{R}, p \models \theta_i$ for $i \leq t$, which implies $\mathcal{R}_1, p \models \theta_i$ for $i \leq t$ (since $\theta_i \in \mathcal{L}$ for $i \leq t$). Since $p \in \min_{\preceq_{\mathcal{R}_1}}(\mathcal{V}_{\mathcal{R}_1})$, this implies $\mathcal{R}_1, p \models \bigwedge_{i \leq t} \bullet\theta_i$. Thus, since $\mathcal{R}_1 \in [\![\mathcal{K}]\!]$, $\mathcal{R}_1, p \models (\phi \vee \bigvee_{i \leq s} \bullet\psi_i)$. If $\mathcal{R}_1, p \models \phi$, then (since $\phi \in \mathcal{L}$), $\mathcal{R}, p \models \phi$ and we are done. Otherwise, suppose $\mathcal{R}_1, p \models \bigvee_{i \leq s} \bullet\psi_i$, i.e. $\mathcal{R}_1, p \models \psi_i$ for some $i \leq s$. Thus, since $\psi_i \in \mathcal{L}$, $\mathcal{R}, p \models \psi_i$ for some $i \leq s$. Since $p \in \min_{\preceq_{\mathcal{R}}}(\mathcal{V}_{\mathcal{R}})$, $\mathcal{R}, p \models \bigvee_{i \leq s} \bullet\psi_i$, which concludes. □

Intuitively, PTL formulas cannot express whether something is possible in a ranked model (e.g. that $\neg \bullet p \vee \neg \bullet \neg p$ is possible in $\mathcal{R}_1$ and $\mathcal{R}_2$ in the proof above). To overcome this limitation, we introduce the $\Diamond$ operator interpreted as the universal accessibility relation. That is, for a PTL-formula A, we impose

- $\mathcal{R}, v \models \Diamond A$ iff there is a $w \in \mathcal{V}_{\mathcal{R}}$ s.t. $\mathcal{R}, w \models A$;
- $\mathcal{R} \models \Diamond A$ iff $\mathcal{R}, v \models \Diamond A$ for every $v \in \mathcal{V}_{\mathcal{R}}$.

Remark 1. Note that if $\mathcal{R}, v \models \Diamond A$ for some $v \in \mathcal{V}_{\mathcal{R}}$, then $\mathcal{R}, v \models \Diamond A$ for every $v \in \mathcal{V}_{\mathcal{R}}$, motivating the second definition that is equivalent to $\mathcal{R}, v \models \Diamond A$ for some $v \in \mathcal{V}_{\mathcal{R}}$.

Formally, we extend PTL ($\mathcal{L}^\bullet$) to $\mathcal{L}^\bullet_\Diamond$ as follows ($a \in \mathsf{At}$):

$$A ::= B \mid \Diamond A \mid \neg A \mid A \wedge A\ ,$$

where B is a PTL formula, that is:

$$B ::= a \mid \neg B \mid B \wedge B \mid \top \mid \bot \mid \bullet B\ .$$

Notice that, for simplicity, we do not allow nesting or composition of $\Diamond$ with the typicality operator. We also introduce the classical modal symbol '$\Box$' as an abbreviation of '$\neg\Diamond\neg$'.

Deciding whether a model satisfies a $\Diamond$-formula reduces to deciding whether the model satisfies a PTL formula. In fact, the following holds:

$$\mathcal{R} \models \Diamond A \text{ iff } \mathcal{R} \not\models \neg A$$

or, equivalently, $\mathcal{R} \models \Diamond A$ iff $\mathcal{R} \not\models \bullet A \rightarrow \bot$.

Please note that the $\Diamond$-operator allows us to express some interesting new constructs. For example, we can express a 'possible exceptionally' operator ($\triangledown$) and a 'weak conditional' ($\Rrightarrow$) (Sezgin and Kern-Isberner 2021b):

- $\mathcal{R} \models \triangledown A$ iff $\mathcal{R} \models \Diamond(A \wedge \neg\bullet A)$,

where $\triangledown A$ is read as 'an exceptional A is possible'. The intuition is that $\triangledown A$ expresses that A occurs as non-typical in the ranking, i.e. there is at least one world that satisfies A and it is a non-typical A-world. To the best of our knowledge, such an operator has not been studied in the literature so far.

- $\mathcal{R} \models A \Rrightarrow B$ iff $\mathcal{R} \models \Diamond(\bullet A \wedge B)$,

where $A \Rrightarrow B$ is read as 'If A then possibly B'. Moreover, weak conditionals allow us also to express that normal defeasible conditionals do not hold, since

- $\mathcal{R} \models A \Rrightarrow B$ iff $\mathcal{R} \not\models \bullet A \rightarrow \neg B$,

that is not expressible in PTL.

Remark 2. The notion of a weak conditional was introduced and analysed in (Sezgin and Kern-Isberner 2021b), but their definition slightly differs w.r.t. ours since, in our formalism, it would correspond to

- $\mathcal{R} \models A \Rrightarrow B$ iff $\mathcal{R} \models \Diamond(\bullet A \wedge B)$ or $\mathcal{R} \models \neg A$

Example 2 (Running example cont.)**.** We have $\mathcal{R}_1, w \models \Diamond(p \wedge b)$, as there is a $p \wedge b$ world (namely $pb\overline{f}$). We also have $\mathcal{R}_1 \models f \Rrightarrow \neg b$, i.e. there are typical flyers that are not birds. Finally, $\mathcal{R}_1 \models \triangledown p$ as $\mathcal{R}_1, pbf \models p \wedge \neg \bullet p$: i.e. there are exceptional penguins in $\mathcal{R}_1$. □

Using $\Diamond$ and, consequently, $\Rrightarrow$, we can show that any ranking can be represented syntactically. That is,

Proposition 3. Let $\mathcal{R}$ be any ranked interpretation. There is a formula $k_{\mathcal{R}} \in \mathcal{L}^\bullet_\Diamond$ s.t. $[\![k_{\mathcal{R}}]\!] = \{\mathcal{R}\}$.

Proof. We start introducing some notation. Let $\mathcal{R} = \langle \mathcal{V}, \preceq \rangle$ be any ranked interpretation. We can specify *layers* in it, representing the ranks. That is, $\mathcal{V}$ can be partitioned into sets $\{L_0, \ldots, L_n\}$ representing the ranking of the worlds:

$$L_0 = \min_{\preceq}(\mathcal{V})$$
$$L_{i+1} = \min_{\preceq}(\mathcal{V} \setminus (\textstyle\bigcup_{j \leq i} L_j)).$$

Being $\mathcal{V}$ finite, we will end up with a finite n s.t. $\mathcal{V} = L_0 \cup \ldots \cup L_n$. Let us define $\mathtt{Form}(L_i) = \bigvee\{w \mid w \in L_i\}$.

Given a model $\mathcal{R}$ partitioned into $\{L_0, \ldots, L_n\}$, let us define $k_{\mathcal{R}}$ as the conjunction of the following $\mathcal{L}^\bullet$-formulas:

$\bullet\top \rightarrow \mathtt{Form}(L_0)$;
$\bullet(\bigwedge_{j \leq i} \neg\mathtt{Form}(L_j)) \rightarrow \mathtt{Form}(L_{i+1})$, for $0 \leq i < n$;
$\bullet(\neg\mathtt{Form}(L_0) \wedge \ldots \wedge \neg\mathtt{Form}(L_n)) \rightarrow \bot$;

and the following $\mathcal{L}^\bullet_\Diamond$-formulas:

- $\bullet\top \Rrightarrow w$, for all $w \in L_0$;[3]
- $\bullet(\bigwedge_{j\leq i} \neg\texttt{Form}(L_i)) \Rrightarrow w$, for all $w \in L_{i+1}$ and $i < n$.

We need to prove that $[\![k_{\mathcal{R}}]\!] = \{\mathcal{R}\}$. It is immediate to see that $\{\mathcal{R}\} \in [\![k_{\mathcal{R}}]\!]$. Next, we prove that $\mathcal{R}$ is the only model of $k_{\mathcal{R}}$. We are going to use the formalisation of $\mathcal{R}$ as a set of layers of worlds $\{L_0, \dots, L_n\}$, and we prove that any change in the layers implies that some of the conjuncts composing $k_{\mathcal{R}}$ is not satisfied anymore. Consider a ranked model $\mathcal{R}' = \{L'_0, \dots, L'_n\}$ that differs w.r.t. $\mathcal{R}'$, and let i be the lowest number s.t. $L'_i \neq L_i$. We have three possibilities:

$L_i \subset L'_i$. Then, there w s.t. $w \in L'_i$ and $w \notin L_i$. Then the formula $\bullet(\bigwedge_{j<i} \neg\texttt{Form}(L_j)) \to \texttt{Form}(L_i)$ is not satisfied anymore, since w would be one of the interpretations satisfying $\bullet(\bigwedge_{j<i} \neg\texttt{Form}(L_j))$, but not $\texttt{Form}(L_i)$ (if $i = 0$, the formula is $\bullet\top \to \texttt{Form}(L_0)$).

$L'_i \subset L_i$. Then, there is w s.t. $w \in L_i$ and $w \notin L'_i$. Then this case can be proved as the case before.

L'_i, L_i incomparable. then, there are w s.t. $w \in L'_i$ and $w \notin L_i$, and w' s.t. $w' \in L_i$ and $w' \notin L'_i$. In both cases, we may proceed as the cases before.

This concludes the proof. $\square$

Proposition 3 can be generalised to any set of ranked interpretations.

Proposition 4. Let $\{\mathcal{R}_1, \dots, \mathcal{R}_n\}$ be any finite set of ranked interpretations. There is a formula $k_{\mathcal{R}_1,\dots,\mathcal{R}_n} \in \mathcal{L}^{\bullet}_{\Diamond}$ $[\![k_{\mathcal{R}_1,\dots,\mathcal{R}_n}]\!] = \{\mathcal{R}_1, \dots, \mathcal{R}_n\}$.

Proof. Let $k_{\mathcal{R}_1,\dots,\mathcal{R}_n} = \bigvee_{1\leq i\leq n}\{\Box k_{\mathcal{R}_i}\}$.. We need to prove that $[\![\bigvee_{1\leq i\leq n}\{\Box k_{\mathcal{R}_i}\}]\!] = \{\mathcal{R}_1, \dots, \mathcal{R}_n\}$.

We start by showing that, for any $\mathcal{R}, \mathcal{R}' \in \mathfrak{R}$, and $w \in \mathcal{U}$, $\mathcal{R}, w \models \Box k_{\mathcal{R}'}$ iff $\mathcal{R} = \mathcal{R}'$ and, obviously, $w \in \mathcal{V}_{\mathcal{R}'}$. Since $\Box A$ corresponds to $\neg\Diamond\neg A$, for any $\mathcal{R}$ and $w \in \mathcal{V}_{\mathcal{R}}$, $\mathcal{R}, w \models \Box A$ iff $\mathcal{R}, v \models A$ for every $v \in \mathcal{V}_{\mathcal{R}}$. Hence we have that $\mathcal{R}, w \models \Box k_{\mathcal{R}'}$ iff $\mathcal{R}, v \models k_{\mathcal{R}'}$ for every $v \in \mathcal{V}_{\mathcal{R}}$. By Proposition 3 we know that the only interpretation satisfying such a condition for $k_{\mathcal{R}'}$ is $\mathcal{R}'$ itself and the worlds in $\mathcal{V}_{\mathcal{R}'}$. Hence $\mathcal{R}, w \models \Box k_{\mathcal{R}'}$ iff $\mathcal{R} = \mathcal{R}'$ and $w \in \mathcal{V}_{\mathcal{R}'}$.

We can generalise, proving that, for any set $\{\mathcal{R}_1, \dots, \mathcal{R}_n\}$, any interpretation $\mathcal{R}$ and any world $w \in \mathcal{V}_{\mathcal{R}}$, $\mathcal{R}, w \models k_{\mathcal{R}_1,\dots,\mathcal{R}_n}$ iff $\mathcal{R} \in \{\mathcal{R}_1, \dots, \mathcal{R}_n\}$. By Proposition 3, for any $\mathcal{R}_i$ ($1 \leq i \leq n$) and any $w \in \mathcal{V}_{\mathcal{R}_i}$, $\mathcal{R}_i, v \models \Box k_{\mathcal{R}_i}$, and consequently $\mathcal{R}_i, v \models k_{\mathcal{R}_1,\dots,\mathcal{R}_n}$. In the other direction, again by Proposition 3, if $\mathcal{R} \notin \{\mathcal{R}_1, \dots, \mathcal{R}_n\}$, then for every i ($0 \leq i \leq n$), there is some world $w \in \mathcal{V}_{\mathcal{R}}$ s.t. $\mathcal{R}, w \not\models k_{\mathcal{R}_i}$. That implies that for every $w \in \mathcal{V}_{\mathcal{R}}$, $\mathcal{R}, w \not\models \Box k_{\mathcal{R}_i}$ for every i ($0 \leq i \leq n$). That is, $\mathcal{R} \not\models \Box k_{\mathcal{R}_i}$ for every i ($0 \leq i \leq n$), and consequently $\mathcal{R} \not\models \bigvee_{1\leq i\leq n}\{\Box k_{\mathcal{R}_i}\}$, which concludes. $\square$

Proposition 4 proves that $\mathcal{L}^{\bullet}_{\Diamond}$ is sufficiently expressive to represent every set of ranked interpretation.

[3]Notice that '$\bullet\top \Rrightarrow$' is read as '$(\bullet\top) \Rrightarrow$' and not as '$\bullet(\top \Rrightarrow)$'.

4 Revision of PTL-formulas

Next, we address the problem of PTL-knowledge base revision. We consider the revision of PTL-belief sets, i.e. sets $\mathcal{K} \subseteq \mathcal{L}^{\bullet}$ s.t. $Cn_0(\mathcal{K}) = \mathcal{K}$, by PTL-formulas, which results in a revised PTL-belief set. That is,

Definition 3. A *PTL-revision operator* is an operator

$$\circ : \wp(\mathcal{L}^{\bullet}_{\diamond}) \times \mathcal{L}^{\bullet}_{\diamond} \longrightarrow \wp(\mathcal{L}^{\bullet}_{\diamond}).$$

Example 3 (Running example cont.)**.** Suppose that one discovers a colony of penguins where penguins fly. This is clearly the exception, but it is a situation which was not handled in the KB $\mathcal{K}_{pt}$. Thus, one has to revise the knowledge base with the formula $\Diamond\neg\bullet p \wedge f$ which allows for non-typical, flying penguins: that is, $\mathcal{K}_{pt} \circ (\Diamond\neg\bullet p \wedge f)$.

If we want to enforce that typical non-birds are typical non-flyers, we can carry out the revision $\mathcal{K}_{pt} \circ (\bullet\neg b \to \bullet\neg f)$. Notice that this will give rise to a different result than revising by $\mathcal{K}_{pt} \circ \bullet\neg b \to \neg f$ as the latter allows for a ranking where $pb\overline{f} \prec \overline{p}\overline{b}\overline{f} \prec \overline{p}\overline{b}f$, but the former does not.

The generality of our approach also captures the revision of a mixed knowledge base consisting of 'strong' and weak conditionals. Consider e.g. $\mathcal{K}_{pw} = \mathcal{K}_p \cup \{\neg f \Rrightarrow \neg p\}$, which ensures that typical flyers can be non-penguins. Our approach allows to revise this kind of conditional knowledge base by complex PTL-formulas, which includes conjunctions of weak and strong conditionals. $\square$

As we will see in Section 5, PTL-revision will capture:

- propositional revision as $\mathcal{K}_{\mathcal{R}} \circ (\bullet\top \to \phi)$ or $\mathcal{K}_{\mathcal{R}} \circ \phi$, where ϕ is a propositional formula;
- conditional revision as $\mathcal{K}_{\mathcal{R}} \circ (\bullet\phi \to \psi)$, where ϕ and ψ are propositional formulas;
- contraction and expansion of the worlds considered possible; and
- revision of PTL-knowledge bases by PTL-formulas.

Before proceeding to show how PTL-revision captures these different kinds of revision, we proceed with an axiomatization of PTL-revision:

Definition 4. Let $\mathcal{K} \cup \{A, B\} \subseteq \mathcal{L}^{\bullet}_{\diamond}$ be given. Then we define the following postulates for $\circ$:

AGM1 $\mathcal{K} \circ A \models_0 A$.
AGM2 $\mathcal{K} \cup \{A\} \not\models_0 \bot$ implies $\mathcal{K} \circ A = Cn_0(\mathcal{K} \cup \{A\})$.
AGM3 If $\{A\} \not\models_0 \bot$ then $\mathcal{K} \circ A \not\models \bot$.
AGM4 If $A \leftrightarrow_0 B$ then $Cn_0(\mathcal{K} \circ A) = Cn_0(\mathcal{K} \circ B)$.
AGM5 $(\mathcal{K} \circ A) \cup \{B\} \models_0 \mathcal{K} \circ (A \wedge B)$.
AGM6 If $(\mathcal{K} \circ A) \cup \{B\} \not\models_0 \bot$ then $\mathcal{K} \circ (A \wedge B) \models_0 (\mathcal{K} \circ A) \wedge B$.

In Section 5 we will see that these postulates reduce to known postulates for propositional and conditional revision when appropriately restricting their scope.

Please note that these postulates are adaptions of the classical AGM-postulates with the following changes: firstly, we allow for revision of PTL-knowledge base by PTL-formulas, and thus extend the language. This means we also had to make a choice w.r.t. entailment. We replaced propositional entailment with the weakest form of PTL-entailment: $\models_0$. These adapted postulates are a minimal requirement on PTL-revision. For example, the success postulate **AGM1** now says: all rankings satisfying the revised knowledge base $\mathcal{K} \circ A$ should satisfy A. We conjecture that stronger requirements, e.g. an adaption of these postulates to a stronger consequence operator such as RC- or LM-entailment (Booth et al. 2015) can be obtained by adding additional postulates.

Remark 3. We remark that the antecedent of **AGM2**, $\mathcal{K} \cup \{A\} \not\models_0 \bot$ is, in general *not* equivalent to $\mathcal{K} \not\models_0 \neg A$. To see this, consider the ranking $\mathcal{R}$ defined by: $pq \prec p\overline{q}$. Now, notice that:

- $\mathcal{R}, pq \not\models \bullet\top \to \neg p$, and
- $\mathcal{R}, p\overline{q} \models \bullet\top \to \neg p$.

Thus, $\mathcal{R} \notin [\![\bullet\top \to \neg p]\!]$ and $\mathcal{R} \notin [\![\neg(\bullet\top \to \neg p)]\!]$. In particular, this means that $\{\mathcal{R}\} \cap [\![\bullet\top \to \neg p]\!] = \emptyset$, i.e. $\mathcal{K}_{\mathcal{R}} \cup \{\bullet\top \to \neg p\} \models_0 \bot$ yet $\mathcal{K}_{\mathcal{R}} \not\models_0 \neg(\bullet\top \to \neg p)$.

From a semantics point of view, we approach $\circ$ by considering mappings from PTL-knowledge bases $\mathcal{K}$ to orders $\preceq_{\mathcal{K}} \subseteq (\mathfrak{R} \times \mathfrak{R})$ over rankings. In more detail, we define the notion of faithful mappings as follows:

Definition 5. A *faithful mapping* for PTL-knowledge bases is a function $f\colon \wp(\mathcal{L}^{\bullet}_{\diamond}) \longrightarrow \wp((\mathfrak{R} \times \mathfrak{R}))$, assigning to every PTL-knowledge base $\mathcal{K} \subseteq \mathcal{L}^{\bullet}_{\diamond}$ a total preorder over rankings $\preceq_{\mathcal{K}} \subseteq \mathfrak{R} \times \mathfrak{R}$ that satisfies the following conditions:

1. if $\mathcal{R}_1, \mathcal{R}_2 \in [\![\mathcal{K}]\!]$ then $\mathcal{R}_1 \preceq_{\mathcal{K}} \mathcal{R}_2$;
2. if $\mathcal{R}_1 \in [\![\mathcal{K}]\!]$ and $\mathcal{R}_2 \notin [\![\mathcal{K}]\!]$ then $\mathcal{R}_1 \prec_{\mathcal{K}} \mathcal{R}_2$;
3. if $\mathcal{K}_1, \mathcal{K}_2 \subseteq \mathcal{L}^{\bullet}$ and $\mathcal{K}_1 \leftrightarrow_0 \mathcal{K}_2$ then $\preceq_{\mathcal{K}_1} = \preceq_{\mathcal{K}_2}$.

In the following, we give several concrete methods to obtain faithful mappings.

A trivial but faithful mapping can be obtained by constructing the two-layer ranking consisting of $[\![\mathcal{K}]\!]$ in the lower-most layer and $\mathfrak{R} \setminus [\![\mathcal{K}]\!]$ in the second layer. It is easy to check that this gives rise to a faithful mapping.

A second method consists in looking at the difference between two rankings by assigning penalties to worlds ranked differently, or a world that is considered by one of the rankings only: i.e. $\mathtt{dist}^1(\mathcal{R}_1, \mathcal{R}_2) = |\{w_1, w_2 \in \mathcal{V}_{\mathcal{R}_1} \cap \mathcal{V}_{\mathcal{R}_2} \mid w_1 \prec_{\mathcal{R}_i} w_2 \text{ and } w_1 \not\prec_{\mathcal{R}_j} w_2 \text{ for } i, j = 1, 2 \text{ and } i \neq j\}| + |(\mathcal{V}_{\mathcal{R}_1} \setminus \mathcal{V}_{\mathcal{R}_2}) \cup (\mathcal{V}_{\mathcal{R}_2} \setminus \mathcal{V}_{\mathcal{R}_1})|$. We then define:

$$\kappa^{\mathcal{K}}_{\mathtt{dist}^1}(\mathcal{R}) = \min_{\mathcal{R}' \in [\![\mathcal{K}]\!]} \left(\mathtt{dist}^1(\mathcal{R}', \mathcal{R})\right)$$

and let $f^{\mathtt{dist}^1}(\mathcal{K})$ be the total preorder $\preceq^{\mathtt{dist}^1}_{\mathcal{K}}$ over $\mathfrak{R}$ s.t. $\mathcal{R}_1 \preceq^{\mathtt{dist}^1}_{\mathcal{K}} \mathcal{R}_2$ iff $\kappa^{\mathcal{K}}_{\mathtt{dist}^1}(\mathcal{R}_1) < \kappa^{\mathcal{K}}_{\mathtt{dist}^1}(\mathcal{R}_2)$. We now show that $f^{\mathtt{dist}^1}$ is a faithful mapping:

Proposition 5. $f^{\mathtt{dist}^1}$ is a faithful mapping.

Proof. We show that the three conditions from Definition 5 hold (where $\mathcal{K}, \mathcal{K}_1, \mathcal{K}_2 \subseteq \mathcal{L}^{\bullet}_{\diamond}$):

1. Suppose that $\mathcal{R}_1, \mathcal{R}_2 \in [\![\mathcal{K}]\!]$. As $\mathtt{dist}^1(\mathcal{R}_1, \mathcal{R}_1) = \mathtt{dist}^1(\mathcal{R}_2, \mathcal{R}_2) = 0$, we see that $\kappa^{\mathcal{K}}_{\mathtt{dist}^1}(\mathcal{R}_1) = \kappa^{\mathcal{K}}_{\mathtt{dist}^1}(\mathcal{R}_2) = 0$ and thus $\mathcal{R}_1 \preceq^{\mathtt{dist}^1}_{\mathcal{K}} \mathcal{R}_2$.
2. Suppose that $\mathcal{R}_1 \in [\![\mathcal{K}]\!]$ and $\mathcal{R}_2 \notin [\![\mathcal{K}]\!]$. Then for every $\mathcal{R} \in [\![\mathcal{K}]\!]$, there is some w_1 s.t. either (2) $w_2 \prec_{\mathcal{R}} w_1$ and $w_2 \not\prec_{\mathcal{R}_2} w_1$, or (3) $w_1 \in (\mathcal{V}_{\mathcal{R}_2} \setminus \mathcal{V}_{\mathcal{R}}) \cup (\mathcal{V}_{\mathcal{R}} \setminus \mathcal{V}_{\mathcal{R}_2})$. Thus, $\mathtt{dist}^1(\mathcal{R}_2, \mathcal{R}) \geq 1$ for every $\mathcal{R} \in [\![\mathcal{K}]\!]$ and thus $0 = \kappa^{\mathcal{K}}_{\mathtt{dist}^1}(\mathcal{R}_1) < \kappa^{\mathcal{K}}_{\mathtt{dist}^1}(\mathcal{R}_2)$.
3. Follows immediately from the fact that $\mathcal{K}_1 \leftrightarrow_0 \mathcal{K}_2$ implies $[\![\mathcal{R}_1]\!] = [\![\mathcal{R}_2]\!]$.

$\square$

Example 4. Consider the KB $\mathcal{K} = \{\bullet\top \to p, \Diamond\neg p\}$. Notice that $[\![\mathcal{K}]\!] = \{p \prec \neg p\}$. We obtain the following faithful mapping $f^{\mathtt{dist}^1}(\mathcal{K})$:

i	1	2	3	4	5
$\mathcal{R}_i$:	$p \prec \overline{p}$	$p, \overline{p}$	$\overline{p} \prec p$	p	$\overline{p}$
$\kappa^{\mathcal{K}}_{\mathtt{dist}^1}(\mathcal{R}_i)$:	0	1	2	1	1

If we would, e.g. revise by $\bullet\top \to \neg p$, we select the $\kappa^{\mathcal{K}}_{\mathtt{dist}^1}$-minimal rankings that satisfy $\bullet\top \to \neg p$:

$$\min_{\preceq^{\mathtt{dist}^1}_{\mathcal{K}}} ([\![\bullet\top \to \neg p]\!]) = \{\mathcal{R}_2, \mathcal{R}_5\}$$

$\square$

More fine-grained approaches are possible as well: Instead of assigning a binary penalty for disagreement between rankings, one can count the distance of ranks assigned to worlds. We assume that every ranking $\mathcal{R}$ is represented as a corresponding mapping $\kappa_{\mathcal{R}} : \mathcal{V}_{\mathcal{R}} \to \mathbb{N}$, where the $\mathcal{R}$-minimal worlds w receive the rank $\kappa_{\mathcal{R}}(w) = 0$. We can then define the overall distance between two rankings as: $\mathtt{dist}^2(\mathcal{R}_1, \mathcal{R}_2) = \sum_{w \in \mathcal{V}_{\mathcal{R}_1} \cap \mathcal{V}_{\mathcal{R}_2}} |\kappa_{\mathcal{R}_1}(w) - \kappa_{\mathcal{R}_2}(w)| + |(\mathcal{V}_{\mathcal{R}_1} \setminus \mathcal{V}_{\mathcal{R}_2}) \cup (\mathcal{V}_{\mathcal{R}_2} \setminus \mathcal{V}_{\mathcal{R}_1})|$ It can be easily verified that the resulting order over rankings is a faithful mapping as well.

More generally, there is a myriad of options available to generate faithful mappings, e.g. by manipulating the penalty for missing worlds ($|(\mathcal{V}_{\mathcal{R}_1} \setminus \mathcal{V}_{\mathcal{R}_2}) \cup (\mathcal{V}_{\mathcal{R}_2} \setminus \mathcal{V}_{\mathcal{R}_1})|$).

One of the main results of this work is the soundness and completeness of the semantic characterisation of PTL-revision in terms of faithful mappings (Definition 5) w.r.t. the axiomatization of Definition 4.

Proposition 6. Assume that a faithful mapping for PTL-knowledge bases $f : \wp(\mathcal{L}^{\bullet}_{\Diamond}) \longrightarrow \wp((\mathfrak{R} \times \mathfrak{R}))$ is given. Then the revision operator $\circ$ defined by

$$[\![\mathcal{K} \circ A]\!] = \min_{\preceq_{\mathcal{K}}} [\![A]\!]$$

satisfies **AGM1**-**AGM6**.

Proof. **AGM1:** This case is clear as $\min_{\preceq_{\mathcal{K}}}([\![A]\!]) \subseteq [\![A]\!]$.
AGM2: suppose $\mathcal{K} \cup \{A\} \not\models_0 \bot$, i.e. $[\![\mathcal{K}]\!] \cap [\![A]\!] \neq \emptyset$. Then by definition of a faithful ranking, $\min_{\preceq_{\mathcal{K}}}([\![A]\!]) = [\![\mathcal{K}]\!] \cap [\![A]\!]$ and thus $\mathcal{K} \circ A = Cn_0(\mathcal{K} \cup \{A\})$.
AGM3: it is clear that if $[\![A]\!] \neq \emptyset$ then, since $\preceq$ is a total preorder, $\min_{\preceq_{\mathcal{K}}}([\![A]\!]) \neq \emptyset$.
AGM4: this case follows from condition 3 of the definition of a faithful mapping.
AGM5 and AGM6: We show the non-trivial case, where $[\![(\mathcal{K} \circ A) \cup \{B\}]\!] \neq \emptyset$. Consider some $R \in [\![\mathcal{K} \circ A]\!] \cap [\![B]\!]$ and suppose towards a contradiction $R \notin [\![\mathcal{K} \circ (A \wedge B)]\!]$.

Since $R \in [\![A]\!] \cap [\![B]\!]$, there is some $R' \in [\![A]\!] \cap [\![B]\!]$ s.t. $R' \prec_{\mathcal{K}} R$. But this contradicts $R \in \min_{\preceq_{\mathcal{K}}}[\![A]\!]$. Thus, we have shown $[\![\mathcal{K} \circ A]\!] \cap [\![B]\!] \subseteq [\![\mathcal{K} \circ (A \wedge B)]\!]$. The other direction is similar. □

Proposition 7. Consider a revision operator $\circ$ that satisfies **AGM1**-**AGM6**. Then there exists a faithful mapping for PTL-knowledge bases $f : \wp(\mathcal{L}^{\bullet}_{\Diamond}) \longrightarrow \wp((\mathfrak{R} \times \mathfrak{R}))$ s.t.

$$[\![\mathcal{K} \circ A]\!] = \min_{\preceq_{\mathcal{K}}}[\![A]\!].$$

Proof. Assume $\circ$ satisfies **AGM1**-**AGM6**. We define, for an arbitrary PTL-knowledge base $\mathcal{K}$, the preorder $\preceq_{\mathcal{K}}$ as follows: for any $\mathcal{R}_1, \mathcal{R}_2 \in \mathfrak{R}$, $\mathcal{R}_1 \preceq^{\mathcal{K}} \mathcal{R}_2$ if (i) $\mathcal{R}_1 \in [\![\mathcal{K}]\!]$ or (ii) $[\![\mathcal{K} \circ \mathcal{K}_{\{\mathcal{R}_1,\mathcal{R}_2\}}]\!] = \{\mathcal{R}_1\}$ (recall Proposition 3). We now show that this defines a faithful mapping. We first show this defines a total preorder. Notice first that, as $[\![\mathcal{K}_{\{\mathcal{R}_1,\mathcal{R}_2\}}]\!] = \{\mathcal{R}_1, \mathcal{R}_2\}$ with Proposition 3, with **AGM1** and **AGM3**, $\emptyset \neq [\![\mathcal{K} \circ \mathcal{K}_{\{\mathcal{R}_1,\mathcal{R}_2\}}]\!] \subseteq \{\mathcal{R}_1, \mathcal{R}_2\}$. Thus, $\preceq_{\mathcal{K}}$ is total. Letting $\mathcal{R}_1 = \mathcal{R}_2$ gives us reflexivity. We now show transitivity. Assume $\mathcal{R}_1 \preceq_{\mathcal{K}} \mathcal{R}_2$ and $\mathcal{R}_2 \preceq_{\mathcal{K}} \mathcal{R}_3$. We consider three cases: (1) $\mathcal{R}_1 \in [\![\mathcal{K}]\!]$. Then $\mathcal{R}_1 \preceq_{\mathcal{K}} \mathcal{R}_3$ by definition of $\preceq_{\mathcal{K}}$. (2) $\mathcal{R}_1 \notin [\![\mathcal{K}]\!]$ and $\mathcal{R}_2 \in [\![\mathcal{K}]\!]$. But then, since $[\![\mathcal{K} \wedge \mathcal{K}_{\{\mathcal{R}_1,\mathcal{R}_2\}}]\!] = \{\mathcal{R}_2\}$, by **AGM2**, $[\![\mathcal{K} \circ \mathcal{K}_{\{\mathcal{R}_1,\mathcal{R}_2\}}]\!] = \{\mathcal{R}_2\}$. Thus, $\mathcal{R}_1 \not\preceq_{\mathcal{K}} \mathcal{R}_2$, contradiction. (3) $\mathcal{R}_1 \notin [\![\mathcal{K}]\!]$ and $\mathcal{R}_2 \notin [\![\mathcal{K}]\!]$. By **AGM1** and **AGM3**, $\emptyset \neq [\![\mathcal{K} \circ \mathcal{K}_{\{\mathcal{R}_1,\mathcal{R}_2,\mathcal{R}_3\}}]\!] \subseteq \{\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3\}$. We now consider two cases: (3.1) $[\![\mathcal{K} \circ \mathcal{K}_{\{\mathcal{R}_1,\mathcal{R}_2,\mathcal{R}_3\}}]\!] \cap \{\mathcal{R}_1, \mathcal{R}_2\} = \emptyset$. In that case, $[\![\mathcal{K} \circ \mathcal{K}_{\{\mathcal{R}_1,\mathcal{R}_2,\mathcal{R}_3\}}]\!] = \{\mathcal{R}_3\}$. Then by **AGM5** and **AGM6**, we obtain $[\![\mathcal{K} \circ \mathcal{K}_{\{\mathcal{R}_1,\mathcal{R}_2,\mathcal{R}_3\}}]\!] \cap \{\mathcal{R}_2, \mathcal{R}_3\} = [\![\mathcal{K} \circ \mathcal{K}_{\{\mathcal{R}_2,\mathcal{R}_3\}}]\!] = \{\mathcal{R}_3\}$. But this contradicts $\mathcal{R}_2 \preceq_{\mathcal{K}} \mathcal{R}_3$ and $\mathcal{R}_2 \notin [\![\mathcal{K}]\!]$, thus case (3.1) can be ruled out. (3.2) $[\![\mathcal{K} \circ \mathcal{K}_{\{\mathcal{R}_1,\mathcal{R}_2,\mathcal{R}_3\}}]\!] \cap \{\mathcal{R}_1, \mathcal{R}_2\} \neq \emptyset$. Since $\mathcal{R}_1 \preceq_{\mathcal{K}} \mathcal{R}_2$ and $\mathcal{R}_1 \notin [\![\mathcal{K}]\!]$, $\mathcal{R}_1 \in [\![\mathcal{K} \circ \mathcal{K}_{\{\mathcal{R}_1,\mathcal{R}_2\}}]\!]$. Thus, by **AGM5** and **AGM6**, we obtain: $[\![\mathcal{K} \circ \mathcal{K}_{\{\mathcal{R}_1,\mathcal{R}_2,\mathcal{R}_3\}}]\!] \cap \{\mathcal{R}_1, \mathcal{R}_2\} = [\![\mathcal{K} \circ \mathcal{K}_{\mathcal{R}_1,\mathcal{R}_2}]\!]$. Thus $\mathcal{R}_1 \in [\![\mathcal{K} \circ \mathcal{K}_{\{\mathcal{R}_1,\mathcal{R}_2,\mathcal{R}_3\}}]\!] \cap \{\mathcal{R}_1, \mathcal{R}_2\}$. By **AGM5** and **AGM6**, we can obtain $\mathcal{R}_1 \in [\![\mathcal{K} \circ \mathcal{K}_{\{\mathcal{R}_1,\mathcal{R}_3\}}]\!]$. Thus, $\mathcal{R}_1 \preceq_{\mathcal{K}} \mathcal{R}_3$.

We now show the three conditions of Definition 5. Condition 1 follows from the definition of $\preceq_{\mathcal{K}}$. For Condition 2 now $\mathcal{R}_1 \in [\![\mathcal{K}]\!]$ and $\mathcal{R}_2 \notin [\![\mathcal{K}]\!]$. Then by **AGM2**, $[\![\mathcal{K} \circ \mathcal{K}_{\mathcal{R}_1,\mathcal{R}_2}]\!] = \{\mathcal{R}_1\}$ and thus $\mathcal{R}_1 \prec_{\mathcal{K}} \mathcal{R}_2$. The third condition follows from **AGM4**. □

5 Specific Revision Types

In this section, we illustrate the expressive strength of the framework by giving several examples of types of revisions, such as propositional revision and conditional revision, and show how assumptions about the possible worlds making up the rankings can be taken into account.

5.1 Propositional Revision

The first and simplest example of an existing form of revision we can capture is propositional revision. We show that PTL-revision restricted to propositional formulas satisfies all the AGM-postulates. We start with the following observation:

Proposition 8. For $\mathcal{K} \cup \{\phi\} \subseteq \mathcal{L}$, $\mathcal{K} \models \phi$ iff $\mathcal{K} \models_0 \phi$.

Proof. We first show the following Lemma:

Lemma 1. For any $\phi \in \mathcal{L}$, $[\![\phi]\!] = \{\mathcal{R} \in \mathfrak{R} \mid \mathcal{V}_{\mathcal{R}} \subseteq [\![\phi]\!]_{\mathsf{PL}}\}$.

Proof. This follows from the fact that $\mathcal{R} \in [\![\phi]\!]$ iff $\mathcal{R}, v \models \phi$ for every $v \in \mathcal{V}_{\mathcal{R}}$. □

We now show the main claim. By definition, ($\mathcal{K} \models \phi$ iff $\mathcal{K} \models_0 \phi$) iff ($[\![\mathcal{K}]\!] \subseteq [\![\phi]\!]$ iff $[\![\mathcal{K}]\!]_{\mathsf{PL}} \subseteq [\![\mathcal{K}]\!]_{\mathsf{PL}}$). By Lemma 1, $[\![\mathcal{K}]\!] \subseteq [\![\phi]\!]$ iff $[\![\mathcal{K}]\!]_{\mathsf{PL}} \subseteq [\![\phi]\!]_{\mathsf{PL}}$, which concludes the proof. □

In view of the above result, it can be observed that the postulates from Definition 4 reduce to the classical AGM-postulates when looking at propositional belief bases $\mathcal{K}$ and revision formulas A.

Proposition 9. For $\mathcal{K} \cup \{\phi, \psi\} \subseteq \mathcal{L}$, any revision operator $\circ$ that satisfies **AGM1**-**AGM6** satisfies the following postulates:

1. $\mathcal{K} \circ \phi \models_{\mathsf{PL}} \phi$.
2. $\mathcal{K} \not\models_{\mathsf{PL}} \neg\phi$ implies $Cn_0(\mathcal{K} \circ \phi) \cap \mathcal{L} = Cn_{\mathsf{PL}}(\mathcal{K} \cup \{\phi\})$.
3. If $\{\phi\} \not\models_{\mathsf{PL}} \perp$ then $\perp \notin Cn_0(\mathcal{K} \circ \phi) \cap \mathcal{L}$.
4. If $\models_{\mathsf{PL}} \phi \leftrightarrow \psi$ then $Cn_0(\mathcal{K} \circ \phi) \cap \mathcal{L} = Cn_0(\mathcal{K} \circ \psi) \cap \mathcal{L}$.
5. If $(\mathcal{K} \circ \phi) \cup \{\psi\} \not\models_{\mathsf{PL}} \perp$ then $Cn_0((Cn_0(\mathcal{K} \circ \phi) \cap \mathcal{L}) \cup \{\psi\}) \cap \mathcal{L} = Cn_0(\mathcal{K} \circ \phi \wedge \psi) \cap \mathcal{L})$.

Proof. This follows immediately from Proposition 8. □

It has to be noted that a faithful ranking still allows for the possibility of a propositional belief base being revised by a propositional formula ϕ resulting in a proper PTL-knowledge base:

Example 5. Consider the knowledge base $\mathcal{K} = \{p \rightarrow b\}$. The following ranking is faithful:

$$[\![p \rightarrow b]\!] \prec_{\mathcal{K}} \mathcal{R}_1 \prec_{\mathcal{K}} \dots ,$$

where $\mathcal{R}_1 = \overline{p}\overline{b} \prec p\overline{b}$. If we revise by $p \vee b$, we get, among others, $\mathcal{R}_2 \in [\![\mathcal{K} \circ p \vee b]\!]$ where $\mathcal{R}_2 = pb \prec \overline{p}b$ (as $\mathcal{R}_2 \in [\![\mathcal{K}]\!]$). Notice that, on the purely propositional level, this does not violate any of the AGM-postulates. Likewise, revising by $p \rightarrow b$ will result in the PTL-formula which has as a model the ranking $\mathcal{R}_1$. We thus see that the result of revising a propositional knowledge base with a propositional formula might not be a propositional formula. □

5.2 Contraction and Extension of Possible Worlds

Revision of PTL-knowledge bases, even those that represent a single ranking $\mathcal{R}$, might lead to contraction or extension of the possible worlds $\mathcal{V}_{\mathcal{R}}$, as opposed to a mere reordering of the possible worlds. Intuitively, this means that we do not just revise what is expected, but also what is possible.

Example 6 (Running example cont.)**.** Consider again ranking $\mathcal{R}_1$ from Example 1. In this ranking, flying penguins are considered possible. Thus, $\mathcal{R}_1 \not\models p \rightarrow \neg f$. If one would be certain enough about this to enforce it as strict knowledge, this could be achieved by the revision $\mathcal{K}_p \circ (p \rightarrow \neg f)$. With **AGM1**, we know that for any $\mathcal{R} \in [\![\mathcal{K}_p \circ (p \rightarrow \neg f)]\!]$, $\mathcal{V}_{\mathcal{R}} \subseteq [\![p \rightarrow \neg f]\!]_{\mathsf{PL}}$, i.e. at least some rankings in $[\![\mathcal{K}_p]\!]$, such as $\mathcal{R}_1$ will have the set of possible worlds contracted by e.g. pbf when revising by $(p \rightarrow \neg f)$. □

We have seen that revision might necessitate the contraction (or extension) of the set of possible worlds. But, depending on the context at hand, contraction and extension of possible worlds might be seen as a rather drastic operation (since worlds previously seen as possible are now rejected, respectively worlds previously seen as impossible are now seen as possible). Therefore, we now consider revision operators which avoid changing the set of possible worlds as much as possible. We call this property *Stability of the Universe* (SU), i.e. worlds are only removed or added if this is absolutely necessary:

SU: For any $\mathcal{R}, \mathcal{R}_1, \mathcal{R}_2 \in \mathfrak{R}$ if $\mathcal{V}_{\mathcal{R}_1} = \mathcal{V}_{\mathcal{R}}$ and $\mathcal{V}_{\mathcal{R}_2} \neq \mathcal{V}_{\mathcal{R}}$ then $\mathcal{R}_1 \prec_{\mathcal{R}} \mathcal{R}_2$.

We can generalize this to sets of rankings, e.g. models of PTL-knowledge bases, as follows:

SU: For any $\mathcal{R}_1, \mathcal{R}_2 \in \mathfrak{R}$ if $\mathcal{V}_{\mathcal{R}_1} \in \{\mathcal{V}_{\mathcal{R}} \mid \mathcal{R} \in [\![\mathcal{K}]\!]\}$ and $\mathcal{V}_{\mathcal{R}_2} \notin \{\mathcal{V}_{\mathcal{R}} \mid \mathcal{R} \in [\![\mathcal{K}]\!]\}$ then $\mathcal{R}_1 \prec_{\mathcal{K}} \mathcal{R}_2$.

We next show that axioms **SU** can be expressed syntactically as well.

In order to do this, we first define the formula $\mathtt{worlds}(\mathcal{R})$ as follows:

$$\mathtt{worlds}(\mathcal{R}) \coloneqq \bigwedge_{w \in \mathcal{V}_{\mathcal{R}}} \Diamond w \wedge \bigwedge_{w \notin \mathcal{V}_{\mathcal{R}}} \neg \Diamond w .$$

$\mathtt{worlds}(\mathcal{R})$ encodes which worlds are part of $\mathcal{V}_{\mathcal{R}}$, in the sense that the rankings that satisfy $\mathtt{worlds}(\mathcal{R})$ are exactly the rankings that are rankings over the universe of $\mathcal{R}$:

Proposition 10. For any $\mathcal{R}$, $[\![\mathtt{worlds}(\mathcal{R})]\!] = \{\mathcal{R}' \in \mathfrak{R} \mid \mathcal{V}_{\mathcal{R}'} = \mathcal{V}_{\mathcal{R}}\}$.

Proof. We first show $[\![\mathtt{worlds}(\mathcal{R})]\!] \subseteq \{\mathcal{R}' \in \mathfrak{R} \mid \mathcal{V}_{\mathcal{R}'} = \mathcal{V}_{\mathcal{R}}\}$. Suppose that $\mathcal{R}' \in [\![\mathtt{worlds}(\mathcal{R})]\!]$. We show that $\mathcal{V}_{\mathcal{R}} = \mathcal{V}_{\mathcal{R}'}$. Suppose first that $w \in \mathcal{V}_{\mathcal{R}'}$ and suppose towards a contradiction that $w \notin \mathcal{V}_{\mathcal{R}}$. Then $\mathcal{R}' \models \Diamond w$, contradiction to $\mathcal{R}' \in [\![\mathtt{worlds}(\mathcal{R})]\!]$. Suppose now (again towards a contradiction) that $w \in \mathcal{V}_{\mathcal{R}} \setminus \mathcal{V}_{\mathcal{R}'}$. Then $\mathcal{R}' \not\models \Diamond w$, contradiction to $\mathcal{R}' \in [\![\mathtt{worlds}(\mathcal{R})]\!]$.

The other direction is similar, which concludes the proof. $\square$

As next, we generalise the above encoding to a KB $\mathcal{K} \subseteq \mathcal{L}^{\bullet}$. To this end, we define

$$\mathtt{worlds}(\mathcal{K}) = \bigvee_{\mathcal{R} \in [\![\mathcal{K}]\!]} \mathtt{worlds}(\mathcal{R}) .$$

We can now define the syntactic counterpart **SUR** (*Stability of the Universe for Revision*) of **SU** as follows: for $\mathcal{K} \cup \{A\} \subseteq \mathcal{L}^{\bullet}$

SUR: If $A \wedge \mathtt{worlds}(\mathcal{K}) \not\models_0 \perp$ then $\mathcal{K} \circ A \models \mathtt{worlds}(\mathcal{K})$.

Example 7. A trivial example of a faithful mapping satisfying **SU** is the following three-layer mapping:

$$[\![\mathcal{K}]\!] \prec_{\mathcal{K}} \mathfrak{R}^{\mathbf{SU}}(\mathcal{K}) \prec_{\mathcal{K}} \mathfrak{R} \setminus (\mathfrak{R}^{\mathbf{SU}}(\mathcal{K}) \cup [\![\mathcal{K}]\!]) ,$$

where $\mathfrak{R}^{\mathbf{SU}}(\mathcal{K}) = \{\mathcal{R} \in \mathfrak{R} \setminus [\![\mathcal{K}]\!] \mid \mathcal{V}_{\mathcal{R}} = \mathcal{V}_{\mathcal{R}'}$ for some $\mathcal{R}' \in [\![\mathcal{K}]\!]$. It is easy to observe that this is the minimal faithful mapping satisfying **SU**: any other faithful mapping satisfying **SU** will be a refinement of this mapping. $\square$

The following proposition can be shown.

Proposition 11. Consider a revision operator $\circ$ that satisfies **AGM1**-**AGM6** and **SUR**. Then there exists a faithful mapping for PTL-knowledge bases $f\colon \wp(\mathcal{L}^{\bullet}_{\Diamond}) \longrightarrow \wp(\mathfrak{R} \times \mathfrak{R})$ satisfying **SU** s.t.

$$[\![\mathcal{K} \circ A]\!] = \min_{\preceq_{\mathcal{K}}} [\![A]\!] .$$

Proof. In view of Proposition 7, it suffices to show that **SUR** implies **SU**. Assume thus that **SUR** holds and consider some $\mathcal{R}_1, \mathcal{R}_2 \in \mathfrak{R}$ s.t. $\mathcal{V}_{\mathcal{R}_1} \in \{\mathcal{V}_{\mathcal{R}} \mid \mathcal{R} \in [\![\mathcal{K}]\!]\}$ and $\mathcal{V}_{\mathcal{R}_2} \notin \{\mathcal{V}_{\mathcal{R}} \mid \mathcal{R} \in [\![\mathcal{K}]\!]\}$. **SUR** ensures that $[\![\mathcal{K} \circ A]\!] \subseteq [\![\mathtt{worlds}(\mathcal{K})]\!]$ if $[\![A]\!] \cap [\![\mathtt{worlds}(\mathcal{K})]\!] \neq \emptyset$. Notice that $(\mathcal{K}_{\mathcal{R}_1} \vee \mathcal{K}_{\mathcal{R}_2}) \wedge \mathtt{worlds}(\mathcal{K}) \not\models_0 \perp$ as $\mathcal{K}_{\mathcal{R}_1} \models_0 \mathtt{worlds}(\mathcal{R}_1)$ and $\mathcal{V}_{\mathcal{R}_1} \in \{\mathcal{V}_{\mathcal{R}} \mid \mathcal{R} \in [\![\mathcal{K}]\!]\}$. Thus, $[\![\mathcal{K} \circ \mathcal{K}_{\mathcal{R}_1} \vee \mathcal{K}_{\mathcal{R}_2}]\!] \subseteq [\![\mathtt{worlds}(\mathcal{K})]\!]$. As $[\![\mathcal{K} \circ \mathcal{K}_{\mathcal{R}_1} \vee \mathcal{K}_{\mathcal{R}_2}]\!] = \{\mathcal{R}_1, \mathcal{R}_2\}$, we see that $[\![\mathcal{K} \circ \mathcal{K}_{\mathcal{R}_1} \vee \mathcal{K}_{\mathcal{R}_2}]\!] = \{\mathcal{R}_1\}$ and thus, with the construction method of the proof of Proposition 7 $\mathcal{R}_1 \prec \mathcal{R}_2$. $\square$

Proposition 12. Consider a faithful mapping for PTL-knowledge bases $f\colon \wp(\mathcal{L}^{\bullet}_{\Diamond}) \longrightarrow \wp(\mathfrak{R} \times \mathfrak{R})$ that satisfies **SU**. Then the revision operator $\circ$ defined by

$$[\![\mathcal{K} \circ A]\!] = \min_{\preceq_{\mathcal{K}}} [\![A]\!].$$

satisfies **AGM1**-**AGM6** and **SUR**.

Proof. In view of Proposition 6, it suffices to show that **SU** implies **SUR**. Suppose $A \wedge \mathtt{worlds}(K) \not\models_0 \perp$, i.e. $[\![A]\!] \cap [\![\mathtt{worlds}(\mathcal{K})]\!] \neq \emptyset$, which on its turn means that (†) there are some $\mathcal{R} \in [\![\mathcal{K}]\!]$ and $\mathcal{R}_{\mathcal{A}} \in [\![A]\!]$ s.t. $\mathcal{V}_{\mathcal{R}} = \mathcal{V}_{\mathcal{R}_{\mathcal{A}}}$. By **SU**, for any $\mathcal{R}^1_A, \mathcal{R}^2_A \in [\![A]\!]$ s.t. $\mathcal{V}_{\mathcal{R}^1_A} \in \{\mathcal{V}_{\mathcal{R}} \mid \mathcal{R} \in [\![\mathcal{K}]\!]\}$ and $\mathcal{V}_{\mathcal{R}^2_A} \notin \{\mathcal{V}_{\mathcal{R}} \mid \mathcal{R} \in [\![\mathcal{K}]\!]\}$, $\mathcal{R}^1_A \prec_{\mathcal{K}} \mathcal{R}^2_A$. This means (with (†)) that $\min_{\prec_{\mathcal{K}}} [\![A]\!] \in \{\mathcal{V}_{\mathcal{R}} \mid \mathcal{R} \in [\![\mathcal{K}]\!]\}$ and thus $\mathcal{K} \circ A \models_0 \mathtt{worlds}(\mathcal{K})$. $\square$

Example 8 (Running example cont.)**.** Suppose we are interested in revising $\mathcal{R}_1$ from Example 1 with $\bullet\neg b \rightarrow \bullet\neg f$. In that case, there exists a ranking that satisfies $\bullet\neg b \rightarrow \bullet\neg f$ where no worlds have to be added or removed:

$$\mathcal{R}_5 : \quad \overline{p}\overline{b}\overline{f} \prec \overline{p}\overline{b}f, b\overline{p}f \prec b\overline{p}\overline{f}, bp\overline{f} \prec pbf .$$

Thus, if we use a revision operator that satisfies **SUR**, we are guaranteed to have $\mathcal{K}_{\mathcal{R}_1} \circ \bullet\neg b \rightarrow \bullet\neg f \models \mathtt{worlds}(\mathcal{K}_{\mathcal{R}_1})$. On the other hand, as we saw in Example 6, revising with $p \rightarrow \neg f$ will result in a contraction of the possible worlds of the result of the revision. $\square$

5.3 Revision of Epistemic States

Iterated belief revision extends propositional revision based on the observation that propositional revision does not impose any requirements on the revision of the revised states. On a semantic level, iterated revision corresponds to revising rankings over possible worlds. The canonical work on iterated revision (Darwiche and Pearl 1997), proposes a set of postulates that iterated revision is expected to satisfy.

In our setting, we also deal with revision of epistemic states in the sense that both the starting point and the result of a revision are a set of rankings. I.e., we are in the setting of revising epistemic states.

We next adapt the postulates from Darwiche and Pearl (1997) and characterize them semantically. As a special case, by setting the knowledge base to $\mathcal{K}_{\mathcal{R}}$, we obtain the revision of an epistemic state $\mathcal{R}$ by a proposition, which corresponds to the original setting of Darwiche and Pearl (1997). We notice that, unlike the case of propositional revision, this is not equivalent to iterative revision of PTL-knowledge bases. Indeed, iterative revision of PTL-knowledge bases would mean, in our setting, revising an order over $\mathfrak{R}$ into another order over $\mathfrak{R}$. We leave such considerations for future work.

The following postulates for revision were given in (Darwiche and Pearl 1997), which we adapt to the setting of PTL-revision.

Definition 6. Let $\mathcal{K} \subseteq \mathcal{L}^\bullet_\Diamond$ and $A, B, C \in \mathcal{L}^\bullet$ s.t. $[\![\mathcal{K}]\!] \subseteq [\![\Diamond B]\!]$.[4] Then we define the following postulates for belief revision operator $\circ$:

DP1$\circ$: If $\emptyset \models_0 B \to A$ then $\mathcal{K} \models_0 \bullet B \to C$ iff $\mathcal{K} \circ (\bullet\top \to A) \models \bullet B \to C$.

DP2$\circ$: If $\emptyset \models_0 B \to \neg A$ then $\mathcal{K} \models_0 \bullet B \to C$ iff $\mathcal{K} \circ (\bullet\top \to A) \models \bullet B \to C$.

DP3$\circ$: If $\mathcal{K} \models_0 \bullet B \to A$ then $\mathcal{K} \circ (\bullet\top \to A) \models_0 \bullet B \to A$.

DP4$\circ$: If $\mathcal{K} \models_0 \bullet B \to \neg A$ then $\mathcal{K} \circ (\bullet\top \to A) \models_0 \bullet B \to \neg A$.

Let us explain the intuition behind them: for **DP1**$\circ$, if $B \to A$ is true according to all rankings, i.e. every B-world is also an A-world,[5] then what is the case in typical B-worlds according to $\mathcal{K}$ should be exactly what is the case in typical B-worlds according to $\mathcal{K} \circ (\bullet\top \to A)$. In other words, the relative structure of B-worlds (which are a subset of the A-worlds) should not be changed by revising by $\bullet\top \to A$. Notice that we do not talk about $\mathcal{K} \circ A$. This would be a much stronger case, since then, already when $A \in \mathcal{L}$, every ranking in $[\![\mathcal{K} \circ A]\!]$ can only contain A-worlds. Combining this with the ideas behind the DP-postulates would result in simply taking $[\![\mathcal{K} \circ A]\!] = \{\mathcal{R} \cap [\![A]\!]^{\mathcal{R}} \mid \mathcal{R} \in [\![\mathcal{K}]\!]\}$, which is arguably a very contrived case. Likewise, **DP2**$\circ$ requires that the relative structure of B-worlds is preserved when revising by $\bullet\top \to A$ in case all B-worlds are $\neg A$-worlds. **DP3**$\circ$ require that if typically B implies A, revising with $\bullet\top \to A$ should not influence this derivation. Similarly for **DP4**$\circ$. Notice that we require that for no $\mathcal{R} \in [\![\mathcal{K}]\!]$, $\mathcal{R}, w \not\models \bullet B$ for any $w \in \mathcal{V}_{\mathcal{R}}$, i.e. B is a formula that holds in at least one world in every model of $\mathcal{K}$. This excludes the trivial case where B is false and, thus, $\bullet B \to C$ is true in models of $\mathcal{K}$.

We can characterize the postulates from Definition 6 semantically using the following properties on faithful mappings (again adapted from (Darwiche and Pearl 1997))

[4]In other words, we assume that B is possible in the ranking under consideration. This is necessary to ensure that the semantic requirements lead to satisfaction the postulates defined here.

[5]It is easy to see that $\emptyset \models_0 B \to A$ iff for every ranking $\mathcal{R}$, for every $w \in \mathcal{V}_{\mathcal{R}}, \mathcal{R}, w \models B \to A$. This is in contrast to $\{B\} \models_0 A$.

DP1sem: For every $B_1, B_2 \in \mathcal{L}^\bullet$ s.t. $\emptyset \models_0 B_i \to A$ (for $i = 1, 2$), it holds that: (for every $\mathcal{R} \in [\![\mathcal{K}]\!]$, $B_1 \prec_{\mathcal{R}} B_2$) iff (for every $\mathcal{R} \in \min_{\preceq_{\mathcal{K}}} [\![\bullet\top \to A]\!]$, $B_1 \prec_{\mathcal{R}} B_2$).

DP2sem: For every $B_1, B_2 \in \mathcal{L}^\bullet$ s.t. $\emptyset \models_0 B_i \to \neg A$ (for $i = 1, 2$), it holds that: (for every $\mathcal{R} \in [\![\mathcal{K}]\!]$, $B_1 \prec_{\mathcal{R}} B_2$) iff (for every $\mathcal{R} \in \min_{\preceq_{\mathcal{K}}} [\![\bullet\top \to A]\!]$, $B_1 \prec_{\mathcal{R}} B_2$).

DP3sem: For every $B \in \mathcal{L}^\bullet$ it holds that: if (for every $\mathcal{R} \in [\![\mathcal{K}]\!]$, $B \wedge A \prec_{\mathcal{R}} B \wedge \neg A$) then (for every $\mathcal{R} \in \min_{\preceq_{\mathcal{K}}} [\![\bullet\top \to A]\!]$, $B \wedge A \prec_{\mathcal{R}} B \wedge \neg A$).

DP4sem: For every $B \in \mathcal{L}^\bullet$ it holds that: if (for every $\mathcal{R} \in [\![\mathcal{K}]\!]$, $B \wedge \neg A \prec_{\mathcal{R}} B \wedge A$) then (for every $\mathcal{R} \in \min_{\preceq_{\mathcal{K}}} [\![\bullet\top \to A]\!]$, $B \wedge \neg A \prec_{\mathcal{R}} B \wedge A$).

These postulates semantically ensure that any conditional beliefs are preserved after revision as much as possible. So, for example, **DP1sem** requires that if all original epistemic states agree on the relative plausibility of B_1 and B_1, and B_1 and B_1 are both subsumed by A, then this relation is preserved after revising the epistemic states by $\bullet\top \to A$.

Remark 4. It is not hard to see that the above conditions **DP1**$\circ$-**DP4**$\circ$ and **DP1sem**-**DP4sem** reduce, when restricted to $\mathcal{K}_{\mathcal{R}}$ for a ranking $\mathcal{R}$ and propositional formulas, to the well-known Darwiche-Pearl postulates on faithful rankings from (Darwiche and Pearl 1997).

Proposition 13. Consider a faithful mapping for a PTL-knowledge base $f \colon \wp(\mathcal{L}^\bullet) \longrightarrow \wp(\mathfrak{R} \times \mathfrak{R})$ that satisfies **DP1sem**-**DP4sem**. Then the revision operator $\circ$ defined by $[\![\mathcal{K} \circ A]\!] = \min_{\preceq_{\mathcal{K}}} [\![\bullet\top \to A]\!]$ satisfies **DP1**$\circ$-**DP4**$\circ$.

Proof. Consider some $\mathcal{K} \subseteq \mathcal{L}^\bullet$ s.t. there is no $\mathcal{R} \in [\![\mathcal{K}]\!]$ s.t. $\mathcal{R}, w \not\models \bullet B$ for any $w \in \mathcal{V}_{\mathcal{R}}$.
DP1$\circ$: For the $\Rightarrow$-direction, suppose $\emptyset \models_0 B \to A$ and $\mathcal{K} \models \bullet B \to C$. The latter means that for every $\mathcal{R} \in [\![\mathcal{K}]\!]$, $B \wedge C \prec_{\mathcal{R}} B \wedge \neg C$ or for no $w \in \mathcal{R}$, $\mathcal{R}, w \models B$. The latter is excluded by assumption. By **DP1sem**, for every $\mathcal{R} \in \min_{\preceq_{\mathcal{K}}} [\![\bullet\top \to A]\!]$, $B \wedge C \prec_{\mathcal{R}} B \wedge \neg C$. Thus, since $[\![\mathcal{K} \circ (\bullet\top \to A]\!]) = \min_{\preceq_{\mathcal{K}}} [\![\bullet\top \to A]\!]$, $\mathcal{K} \circ (\bullet\top \to A) \models_0 \bullet B \to C$. The $\Leftarrow$-direction is similar.
DP2$\circ$: similar to **DP1**$\circ$.
DP3$\circ$: suppose $\mathcal{K} \models_0 \bullet B \to A$, i.e. for every $\mathcal{R} \in [\![\mathcal{K}]\!]$, $B \wedge A \prec_{\mathcal{R}} B \wedge \neg A$ or for every $w \in \mathcal{V}_{\mathcal{R}}, \mathcal{R}, w \not\models \bullet B$. The latter case is excluded by assumption. Thus, by **DP3sem**, $B \wedge A \prec_{\mathcal{R}} B \wedge \neg A$ for every $\mathcal{R} \in \min_{\preceq_{\mathcal{K}}} [\![\bullet\top \to A]\!]$, which implies $\mathcal{K} \circ (\bullet\top \to A) \models_0 \bullet B \to A$.
DP4$\circ$: suppose $\mathcal{K} \models_0 \bullet B \to \neg A$, i.e. for every $\mathcal{R} \in [\![\mathcal{K}]\!]$, $B \wedge \neg A \prec_{\mathcal{R}} B \wedge A$ or for every $w \in \mathcal{V}_{\mathcal{R}}, \mathcal{R}, w \not\models \bullet B$. The latter case is excluded by assumption. Thus, with **DP4sem**, $B \wedge \neg A \prec_{\mathcal{R}} B \wedge A$ for every $\mathcal{R} \in \min_{\preceq_{\mathcal{K}}} [\![\bullet\top \to A]\!]$, which implies $\mathcal{K} \circ (\bullet\top \to A) \models_0 \bullet B \to \neg A$. $\square$

Proposition 14. Consider a revision operator $\circ$ that satisfies **AGM1**-**AGM6** and **SUR**. Then there exists a faithful mapping for PTL-knowledge bases $f \colon \wp(\mathcal{L}^\bullet) \longrightarrow \wp(\mathfrak{R} \times \mathfrak{R})$ that satisfies **SU** s.t.

$$[\![\mathcal{K} \circ A]\!] = \min_{\preceq_{\mathcal{K}}} [\![A]\!].$$

Proof. **DP1sem**: For the $\Rightarrow$-direction, suppose $B_1, B_2 \in \mathcal{L}^\bullet$ are given s.t. $\emptyset \models_0 B_i \to A$ for $i = 1, 2$ and for every

$\mathcal{R} \in [\![R]\!]$, $B_1 \prec_{\mathcal{R}} B_2$. This means that $\mathcal{K} \models_0 \bullet(B_1 \vee B_2) \to \neg B_2$. By **DP1**$\circ$, this means that $\mathcal{K} \circ (\bullet\top \to A) \models_0 \bullet(B_1 \vee B_2) \to \neg B_2$ and thus for every $R \in [\![\mathcal{K} \circ (\bullet\top \to A)]\!] = \min_{\preceq_{\mathcal{K}}}[\![\bullet\top \to A]\!]$, $B_1 \prec_{\mathcal{R}} B_2$ or $\mathcal{R}, w \models B$. By **SUR** (and Proposition 12) we can exclude the latter case, and thus we have shown that for every $\mathcal{R} \in \min_{\preceq_{\mathcal{K}}}[\![\bullet\top \to A]\!]$, $B_1 \prec_{\mathcal{R}} B_2$. For the $\Leftarrow$-direction, suppose $B_1, B_2 \in \mathcal{L}^\bullet$ are given s.t. $\emptyset \models_0 B_i \to A$ for $i = 1, 2$ and for every $\mathcal{R} \in \min_{\preceq_{\mathcal{K}}}[\![\bullet\top \to A]\!]$, $B_1 \prec_{\mathcal{R}} B_2$. Then $\mathcal{K} \circ (\bullet\top \to A) \models_0 \bullet(B_1 \vee B_2) \to \neg B_2$ and thus, by **DP1**sem, $\mathcal{K} \models_0 \bullet(B_1 \vee B_2) \to \neg B_2$. Thus, for every $\mathcal{R} \in [\![\mathcal{K}]\!]$, $B_1 \prec_{\mathcal{R}} B_2$ or for no $w \in \mathcal{R}$, $\mathcal{R}, w \models B$. By **SUR** (and Proposition 12), the latter case can be excluded. Thus, we have shown that for every $\mathcal{R} \in [\![\mathcal{K}]\!]$, $B_1 \prec_{\mathcal{R}} B_2$.

DP3sem: Suppose that for every $B \in \mathcal{L}^\bullet$, it holds that for every $\mathcal{R} \in [\![\mathcal{K}]\!]$, $B \wedge A \prec_{\mathcal{R}} B \wedge \neg A$. Thus, $\mathcal{K} \models_0 \bullet B \to A$ and, with **DP3**$\circ$, $\mathcal{K} \circ (\bullet\top \to A) \models_0 \bullet B \to A$. This means that for every $\mathcal{R} \in [\![\mathcal{K} \circ (\bullet\top \to A)]\!] = \min_{\preceq_{\mathcal{K}}}([\![\bullet\top \to A]\!])$, $B \wedge A \prec_{\mathcal{R}} B \wedge \neg A$ or $\mathcal{R}, w \models B$. By **SUR** (and Proposition 12), the latter case can be excluded. Thus, we have shown that for every $\mathcal{R} \in \min_{\preceq_{\mathcal{K}}}([\![\bullet\top \to A]\!])$, $B \wedge A \prec_{\mathcal{R}} B \wedge \neg A$. **DP4**sem and **DP2**sem are similar to **DP3**sem respectively **DP1**sem. ☐

Example 9. We now give an example of a constructive revision operator, based on *c-revisions* (Kern-Isberner 2001). We assume that every ranking $\mathcal{R}$ is represented as a corresponding mapping $\kappa_{\mathcal{R}} : \mathcal{V}_{\mathcal{R}} \to \mathbb{N}$, where the $\mathcal{R}$-minimal worlds w receive the rank $\kappa_{\mathcal{R}}(w) = 0$. The rank of a formula is determined by the $\mathcal{R}$-minimal world validating that formula: $\kappa_{\mathcal{R}}(A) = \min_{\mathcal{R}, w \models A}(\kappa_{\mathcal{R}}(w))$. We then define

$$\kappa_{\mathcal{R}\star(\bullet\top\to A)}(w) = \begin{cases} \kappa_{\mathcal{R}}(w) - \kappa_{\mathcal{R}}(A) & \text{if } w \models A \\ \kappa_{\mathcal{R}}(w) & \\ + \max\{0, -\kappa_{\mathcal{R}}(\neg A) + 1\} & \text{if } w \models \neg A \end{cases}$$

Notice that every $\kappa_{\mathcal{R}\star(\bullet\top\to A)}$ determines a ranking $\mathcal{R} \star (\bullet\top \to A)$. We can now define the revision of a PTL-belief set by a conditional as:

$$[\![\mathcal{K} \circ (\bullet\top \to A)]\!] = \{\mathcal{R} \star (\bullet\top \to A) \mid \mathcal{R} \in [\![\mathcal{K}]\!]\}\ .$$

This gives rise to (a restriction of) a faithful mapping.

Generalizing this to a constructive method that allows to determine a set of rankings that satisfies an arbitrary PTL-formula is far from being straightforward. Indeed, consider the ranking formula $\neg p$ and suppose we want to revise the formula by $\bullet p$. This cannot be achieved by a mere reordering of the worlds in the rankings in $[\![\neg p]\!]$, but requires adding and removing possible worlds. Defining constructive methods for doing this is left for future work. ☐

Example 10 (Running example cont.)**.** Suppose we go on an Atlantic cruise, which means that birds become far less typical than non-birds (i.e. we'll see more fish than birds). This can be modelled by revising $\mathcal{K}_p$ (Example 1) with $\bullet\top \to \neg b$. For simplicity, we restrict attention to $\mathcal{R}_1$ from Example 1, i.e. we carry out the revision $\mathcal{K}_{\mathcal{R}_1} \circ (\bullet\top \to \neg b)$ which results, using to the revision scheme defined in Example 9 in the following ranking:

$$\mathcal{R}_6 : \quad \overline{p}\overline{b}\overline{f}, \overline{p}\overline{b}f \prec b\overline{p}f \prec b\overline{p}\overline{f}, bp\overline{f} \prec pbf$$

Intuitively, the b-worlds are shifted down one layer whereas all $\overline{b}$-worlds stay on the lowermost layer. We see that any information about birds is still derivable in view of $\mathbf{DP2}\circ$ and the fact that $\emptyset \models_0 b \to \neg\neg b$, e.g. $\mathcal{R}_6 \models \bullet b \to f$. Likewise, as in $\mathcal{R}_1$ penguins are typically birds ($\mathcal{R}_1 \models \bullet p \to b$), we also see that revising with $\bullet\top \to \neg b$ does not change this inference: $\mathcal{R}_6 \models \bullet p \to b$. ☐

5.4 Revision under Preferential Closure

Eventually, we show how one can straightforwardly define and characterise the revision of a conditional KB under preferential closure (Kraus, Lehmann, and Magidor 1990). At first, recall that given a conditional KB $\mathcal{K}$, a conditional $\bullet A \to B$ is in the preferential closure of $\mathcal{K}$ if $[\![\bullet A \to B]\!] \supseteq [\![\mathcal{K}]\!]$. This means that we can capture revision by a knowledge base under preferential closure without having to assume any additional postulates, as the postulates from Definition 4 already refer to $\models_0$, which reduces to preferential entailment when restricting attention to conditionals. It should be noted, though, that the resulting revision formula might not be characterisable by a set of conditionals. To ensure this, one needs to characterise the sets of rankings that can be represented by a set of conditionals. This is an open question, left for future work.

Example 11 (Running example cont.)**.** The preferential closure of $\mathcal{K}_p$ from Example 1 is given by $[\![\mathcal{K}_p]\!]$ and thus we have already considered revision of the preferential closure in the examples above. As an additional example, suppose we revise by $\bullet\neg b \to \neg f$, we obtain, in view of $\mathbf{AGM2}$ and $\mathcal{K}_p \cup \{\bullet\neg p \to \neg f\} \not\models \bot$, the set $[\![\mathcal{K}_p \cup \{\bullet\neg p \to \neg f\}]\!]$, i.e. the preferential closure of $\mathcal{K}_p \cup \{\bullet\neg p \to \neg f\}$.

6 Conclusion, in View of Related Work

The PTL framework (Booth, Meyer, and Varzinczak 2012; Booth, Meyer, and Varzinczak 2013) is, to the best of our knowledge, the first attempt to introduce a typicality operator into propositional logic without imposing any constraint in its use in the language. A logic that is close to PTL is the non-monotonic Description Logic $\mathcal{ALC}+\mathbf{T}$ (Giordano et al. 2009; Giordano et al. 2015), where the Description Logic $\mathcal{ALC}$ is enriched with a modal operator $\mathbf{T}$ that is semantically similar to our $\bullet$-operator, but its use in the language is constrained in such a way that a propositional version of their logic would correspond to a conditional logic, making it less expressive than PTL.

Several works (Girard 2008; Souza, Vieira, and Moreira 2021) give an account of belief revision in *preference logic* or similar systems which are similar to our results. There are several differences to our work, however. For example, these works consider revision and belief change as an operator in the object language, and sometimes do not allow for revision by modal formulas. Furthermore, despite the conceptual similarities, the exact formal relation between PTL and preference logic remains to be investigated.

Future work includes applying our results to the revision of PTL-knowledge bases under the closure operators proposed by Booth et al. (2019) and a deeper investigation into the extension of the language which we proposed here.

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