Proceedings of the IASS Annual Symposium 2023
Integration of Design and Fabrication
10–14 July 2023, Melbourne, Australia
Y.M. Xie, J. Burry, T.U. Lee and J. Ma (eds.)

FreeGrid: a benchmark on design and optimisation of free-edge gridshells

Luca BRUNO[a,*], Paolo CIGNONI[b], Stefano GABRIELE[c], Ernesto GRANDE[d], Maura IMBIMBO[e], Francesco LACCONE[b], Francesco MARMO[f], Elena MELE[f], Lorenzo RAFFAELE[a], Valentina TOMEI[e], Fiammetta VENUTI[a]

[a,*] Politecnico di Torino,
Viale Mattioli 39, 10126 Torino, Italy
freegrid@ctanet.it

[b] Institute of Information Science and Technologies, National Research Council of Italy, Italy
[c] Università degli studi Roma Tre, Italy
[d] Università degli studi Guglielmo Marconi, Italy
[e] Università degli studi di Cassino e del Lazio Meridionale, Italy
[f] Università degli studi di Napoli Federico II, Italy

Abstract

FreeGrid is meant to offer a common benchmark to test and compare different approaches to the design and optimization of steel gridshells, from man-based heuristic design to AI-based one. FreeGrid sets three design baseline problems: a barrel vault, a paraboloidal dome, and a hyperbolic paraboloid, having their spring line partially not constrained (free-edge) and subjected to symmetric and asymmetric load conditions. Participants are called to modify the baseline gridshell(s) in order to improve their structural performances, buildability, and sustainability, all three of them weighted in a single, bulk quantitative performance metric. Participants shall comply with a limited number of design constraints, while any other design solution is allowed. Baseline setups, performance metrics and design constraints will be fully detailed in technical specifications made publicly available. The full data of the baseline structures will be offered to participants according to an Open Data policy, together with postprocessing utilities intended to align the procedure to obtain the performance metrics. The FreeGrid benchmark will be launched within the IASS Symposium 2023 in Melbourne.

Keywords: freeGrid, benchmark, steel gridshell, free-edge, conceptual design, optimisation

1. Introduction

FreeGrid is a benchmark on design and optimization of gridshells (Fig. 1) conceived by the Authors of the present paper as members of its Steering Committee, advised by an international Scientific Committee. FreeGrid moves from seven general problem statements, and towards the related goals, as briefly listed in the following.

i. The activity in Structural Engineering in general, and about gridshell as well, is increasingly polarized between mechanical modelling and analysis versus design practice and applications [1]. FreeGrid is intended to bridge these two approaches. It proposes a genuine design problem, and targets the rigorous quantitative comparison of the solutions and their performances.

ii. The design and optimization of gridshells as form-resistant discrete structures is intrinsically a multidisciplinary activity jointly carried out by experts in mathematics, computer graphics, mechanics, structural engineering, architecture, designers and builders. FreeGrid aims at gathering competences and input from all the above fields around a common design problem.

iii. Artificial Intelligence (AI) is among the most recent generalized trends in contemporary science, and in structural engineering as well [2]. FreeGrid is meant to offer a common benchmark to comparatively test the readiness of different approaches to the conceptual design of form-resistant structures, from the classical man-based heuristic design, to the optimization approach in its different forms (e.g. gradient-based, topological, genetic-algorithm-based), to the AI-assisted design methods (e.g. artificial neural networks, machine learning).

iv. Horizontal spring line and/or perfect infinitely rigid constraints only seldom occur in built gridshells. Usually and more and more frequently, gridshells include edges free of constraints eventually bounded by stiffening members [3]. Conversely, the structural performances of free-edge gridshells and their design proposals are scarcely treated in scientific systematic studies. FreeGrid aims at filling the gap between the design practice and the scientific literature in this field by adopting free-edge gridshells as case studies.

v. Design is a holistic activity accounting for multiple goals [4]. FreeGrid calls for the holistic improvement of structural, buildability, and sustainability performances. Multiple, even if not necessarily exhaustive, design goals and related performance metrics are defined.

vi. Benchmarking best practices require that *studies and related results are reproducible* [5]. FreeGrid sets mandatory requirements to participants intended to secure the full description of (a) the geometrical and mechanical features of the design solutions, (b) the methods adopted for design/optimization/assessment, (c) the obtained results. FreeGrid adopts an Open Data policy applied to preprocessing and postprocessing files offered by the organizers and/or required to participants.

vii. The *fair comparative evaluation* of contributions in benchmarks on structural design and optimization is recognised as a challenging issue [5]. FreeGrid precisely and analytically adopts objective, purely quantitative performance metrics that do not need a posteriori assessment.

free grid

Figure 1. Logo of the FreeGrid benchmark

The present paper articulately describes in what follows the way in which FreeGrid plans to achieve the above goals. For a better understanding, the terminology used in the present paper and in the benchmark is summarized in Table 1.

Table 1. Main terminology adopted in the paper

entity	**geometry**	**Structure / construction**	**mathematical model**	**Subscript, total number**
0D	vertex	joint	node	j, N
1D	edge / line / arc	member	element	i, M
2D single entity	face	panel	---	f, F
2D overall entity	mesh	grid	discretization	

2. Baseline gridshells

FreeGrid considers three types of gridshells (Fig. 2): a barrel vault, a paraboloidal dome, and a hyperbolic paraboloid, with simple, double gaussian positive, and double gaussian negative curvature, respectively. By 'Background Gridshells' we refer to gridshells fully hinged along their boundaries, while by 'Design Baseline Gridshells' (DBGs) we refer to gridshells having their spring line partially not constrained along what is called a 'free-edge' [3]. Although imperfections play a role in gridshell design and optimization [6], for the sake of clarity DBGs are free from any kind of imperfections induced by constraints, load conditions, mechanical properties, or geometrical features.

2.1. Geometrical setups

The main geometrical features of the DBGs are shown in Figure 2.

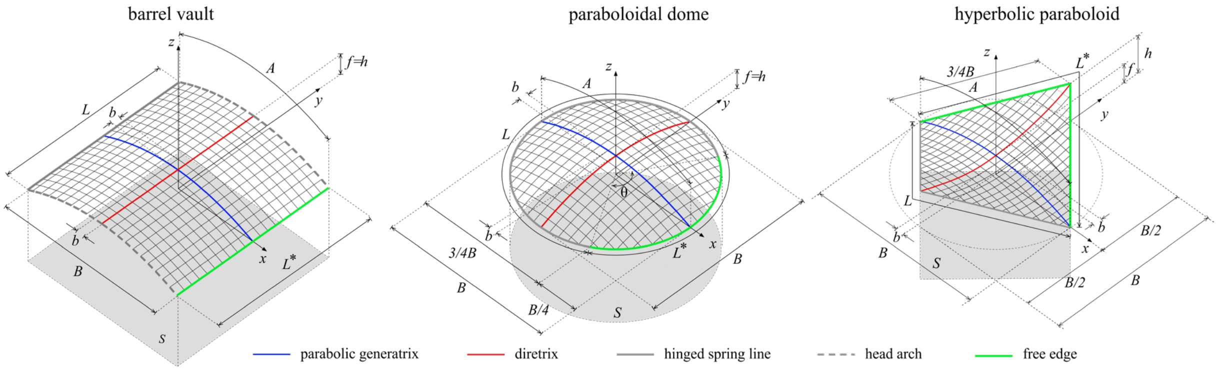
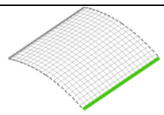
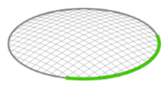
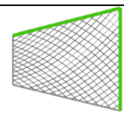


Figure 2. FreeGrid DBGs: barrel vault, paraboloidal dome, hyperbolic paraboloid

All the DBGs share the same parabolic generatrix, whose equation is included in Table 2.

Table 2. Geometrical features of the DBGs

			
Generatrix equation	$z = -\frac{x^2}{2B} + f,\quad D = \left\{-\frac{B}{2} \le x \le \frac{B}{2}\right\}$ $A = \frac{B}{4}\sqrt{5} + B \cdot ln\left(\frac{1+\sqrt{5}}{2}\right)$		
Directrix equation	$z = f,$ $D = \left\{-\frac{B}{2} \le y \le \frac{B}{2}, x = 0\right\}$	$z = -\frac{y^2}{2B} + f,$ $D = \left\{-\frac{B}{2} \le y \le \frac{B}{2}, x = 0\right\}$	$z = \frac{y^2}{2B} + f,$ $D = \left\{-\frac{B}{2} \le y \le \frac{B}{2}, x = 0\right\}$
f	$B/8$		
h	$B/8$	$B/8$	$B/4$
L	A	$\frac{2}{3}\pi B$	$3/2B$
L^*	L	$L/2$	L
S	BL	$\pi B^2/4$	$\pi B^2/2$
b	$A/20$		

The generatrix span B =30 m is adopted as reference length, and f =B/8 is its rise. The horizontal plane $z = 0$ is referenced in the following as 'horizontal reference plane'. The generatrix arc length A accommodates 20 edges with constant length b. The directrix is split with the same step b. Therefore, all DBGs are discrete translational surfaces, whose resulting mesh is homogeneous, and made by planar square faces, except for boundary ones that are obtained by intersection with the horizontal reference plane. The other main geometrical features of the DBGs are: $\boldsymbol{h}$ is the maximum height above the horizontal reference plane; the lengths $\boldsymbol{L}$ and $\boldsymbol{L}^*$ of the spring line and free-edge respectively refer to their continuous counterpart; the surface extent $\boldsymbol{S}$ corresponds to the area encircled by the projection of the continuous spring line, free-edge and head arches, if any, on the horizontal reference plane.

2.2. Structural setups

2.2.1. Structural members

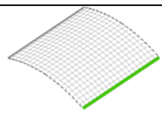
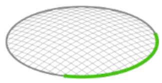
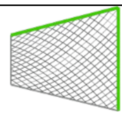
All the structural members of the DBGs are made of steel with a bilinear elastic-perfect plastic constitutive law. Table 3 summarizes the adopted material properties.

Table 3. Material properties of the structural members in the DBGs

Steel grade	density	Young's modulus	Poisson's ratio	yield strength
S355	ρ = 7850 kg/m^3	E = 2.1e+5 MPa	ν = 0.3	f_y = 355 MPa

Structural members have circular hollow cross-section and are not subjected to initial prestressing in all the DBGs. The structural members of a single DBG have the same cross section. The cross-section dimensions differ among the DBGs (Table 4), in order to ensure their homogeneous mechanical performances.

Table 4. Cross-section dimensions of the structural members in the DBGs

			
Section type (type, diameter, thickness [mm])	O/139.7/14.2	O/101.6/10	O/101.6/10
Area [mm^2]	5596	2876	2876
Inertia [mm^4]	11157936	3052611	3052611

2.2.2. Structural constraints

External constraints at the structural joints along the spring lines L are perfect hinges, except for the head arches of the barrel vault, along which only z- and x-wise joint displacements are constrained in order to avoid non-linear stiffening induced by the y-wise members. All the internal structural joints are rigid. The structural joints along the free-edge length L^* are not constrained.

2.2.3. Load Conditions

The design solutions shall be evaluated with reference to two Load Conditions (LC$_k$, k =1:2). Both LCs are simplified and exploratory, intended to assess structural performances under ideal and controlled working conditions. Nevertheless, their moduli have the same order of magnitude of standardized design loads on gridshells. They are sketched in Figure 3 in terms of piecewise uniform loads for the sake of clarity.

The first Load Condition (LC_1) cumulates the distributed self-weight of the structural members and the point loads $Q_{1,j} = (q_1 + q_2)s_j$ applied to all the structural joints, where s_j is the projection on the horizontal plane of the tributary area of the j-th structural joint, q_1 is the uniform distributed load that mimics the permanent weight-like load of a glass cladding, and q_2 is the uniform distributed load that mimics the live snow-like load.

The second Load Condition (LC_2) mimics not only the effects of non-uniform vertical loads, but also the ones of wind-induced loads or other horizontal loads. LC_2 cumulates the distributed self-weight of the structural members, the point loads $Q_{2,1,j} = q_1 s_j$, where q_1 is the uniform distributed load as described above, and the point loads $Q_{2,2,j} = q_2 s_j$, where q_2 is the piecewise uniform distributed load as described above, and it is applied on the surface with $x \geq 0$ (barrel vault and parabolic dome) and $y \geq 0$ (hyperbolic paraboloid).

Structural performances at Ultimate Limit States (ULS) are evaluated by setting $q_1 = 600\ \mathrm{N/m^2}$ and $q_2 = 1200\ \mathrm{N/m^2}$, while the ones at Serviceability Limit States (SLS) are assessed under $q_1 = 400\ \mathrm{N/m^2}$ and $q_2 = 800\ \mathrm{N/m^2}$.

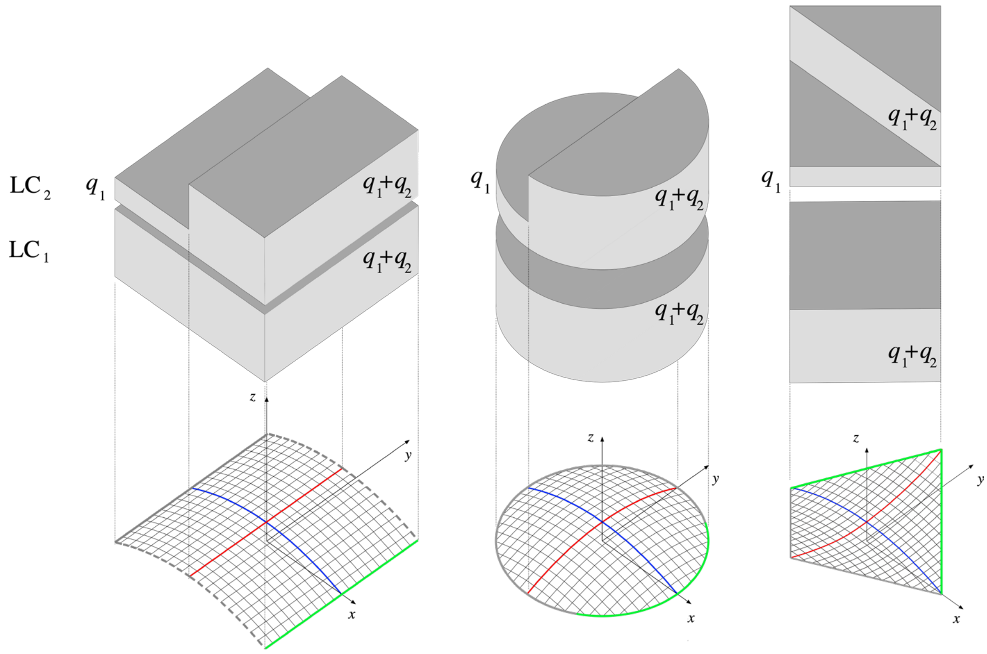


Figure 3. FreeGrid Load Conditions

3. Design goals and performance metrics

Participants to the benchmark are called to modify the above DBG(s), and conceive the Design Solution Gridshell(s) (DSG) in order to achieve seven selected Design Goals (DGs) in a genuine holistic perspective.

The whole performance assessment of each DSG develops in three conceptual steps (Fig. 4). First, each DG is expressed through a quantitative metric to be increased (↑) or decreased (↓), and made dimensionless with respect to the corresponding metric of the DBG (subscript 0). Second, DGs are grouped into three performance categories, covering the structural response (subscript *s*), the buildability (subscript *b*) and the sustainability (subscript *su*) of the DSG(s). Correspondingly, the single DG metrics are clustered into partial performance metrics P_k ($k = s, b, su$). Finally, a bulk performance metric P is obtained as linear combination of the above partial ones.

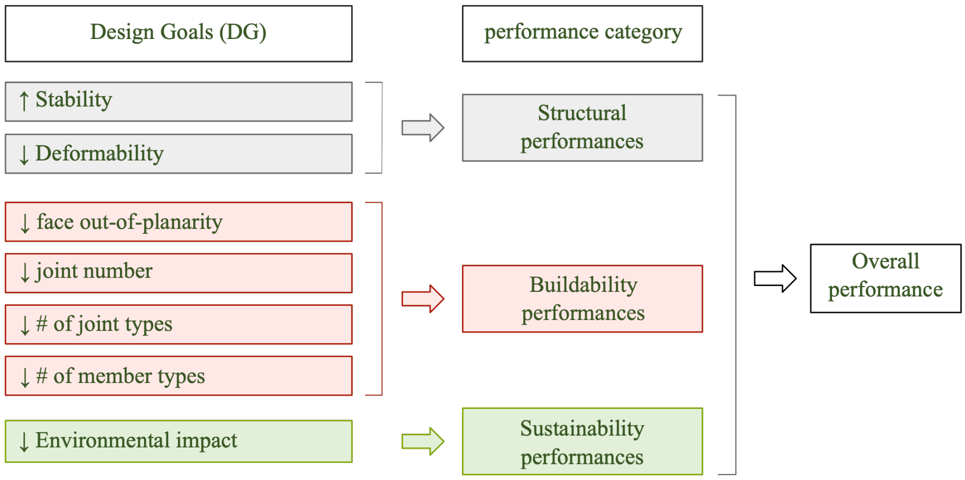


Figure 4. Performance assessment framework

3.1. Structural goals and metrics

Gridshells are form-resistant structures, and the design and optimization of their shape traditionally mainly focus on their mechanical performances, e.g. [7][8][9]. As a result, they are as efficient as exposed to instability and deformability issues [10] at ULS and SLS, respectively.

Stability (ULS). The adopted DG quantitative metric to be increased is the critical Load Factor $\widehat{LF}$, with reference to both LC_1 and LC_2 defined in subsect. 2.2.3. $\widehat{LF}$ accounts for global, local and member instability, and/or member plasticization [11][12][13]. The adopted lower limit value is $\widehat{LF}_l = 1$.

Deformability (SLS). The adopted DG quantitative metric to be decreased is the modulus of the maximum joint vertical displacement $|\hat{\delta}_z|$ over the whole gridshell, under LC_1 and LC_2 load conditions. The adopted upper limit value is $\hat{\delta}_{z,l} = B/200$.

The structural performance metric is averaged over the Load Conditions LC_k (k =1:2) and is defined as

$$P_s = \frac{\sum_{k=1}^{2} \frac{\widehat{LF}_k}{\widehat{LF}_{k,0}} \Big/ \frac{|\hat{\delta}_{z,k}|}{|\hat{\delta}_{z,k,0}|}}{2}. \tag{1}$$

Under the setup conditions specified in Sect. 2, the background gridshells fulfil the selected structural requirements at both ULS and SLS, while the DBGs do not.

3.2. Buildability goals and metrics

To the authors' best knowledge, buildability performances (also known as 'fabrication-aware design') are not thoroughly and unanimously defined in literature, in spite of their paramount design role and recent excellent proposals [14][15][16]. As such, the selected DGs, the related metrics and the resulting partial metric are intended to be an intentionally not all-encompassing, although rigorous, buildability performance model. In particular, the selected buildability DGs only refer to the geometry of 2D gridshell panels coincident with the face, 1D line-like structural members coincident with edges, and 0D joints at vertices. Hence, in a truly conceptual design perspective, the adopted DGs do not include issues related to the panel-face offset and to the joint 3D manufacturing (e.g. kinks at joints, edge offset [15]).

Face out-of-planarity. Face planarity is the most widespread and traditionally considered construction constraint for double curvature gridshells (e.g. [17][18]). Indeed, planar faces accommodate cheaper flat panels that are significantly less expensive than moulded or cold bent doubly curved panels. The adopted

DG quantitative metric $\overline{\Delta}$ to be decreased is the average, computed over the whole gridshell, of the modulus of the mean distances of the face vertices from the best fitting plane divided by the face semiperimeter [19][20].

Joint number. The number of structural joints largely affects the overall cost and buildability [21]. The adopted DG quantitative metric to be decreased is the cardinality of the structural joints $\#(N)$.

Uniformity of structural joints. The DG is aimed at shortening the joints chart, but also affecting the gridshell aesthetics. In what follows the joint type results from its valence v and the relative angles θ between converging members [22]. The adopted DG quantitative metric to be decreased is the cardinality of the joint type $\#(J)$.

Uniformity of structural members. The DG is aimed at shortening the members chart, but also affecting the gridshell aesthetics. In what follows the member type results from its length and cross section. The corresponding DG quantitative metrics to be decreased are the standard deviation of members length $\tilde{l}$ [14][23] and the cardinality of the members cross-sections $\#(C)$.

The buildability performance metric results from the average of the DG metrics as

$$P_b = \frac{1}{\frac{1}{5}\left[\frac{1+\bar{\Delta}}{1+\bar{\Delta}_0} + \frac{\#(N)}{\#(N_0)} + \frac{\#(J)}{\#(J_0)} + \frac{1+\tilde{l}}{1+\tilde{l}_0} + \frac{\#(C)}{\#(C_0)}\right]}. \quad (2)$$

3.3. Sustainability goals and metrics

Weight reduction is the traditional, widespread and largely emphasized design objective for structures in general, and for lightweight ones in particular, gridshells included, e.g. [24]. In a Life Cycle Assessment perspective, steel weight reduction is rephrased in the reduction of the gridshell embodied carbon that depends on the steel weight and grade, and on the type of sections of the structural members [25][26]. The adopted corresponding DG quantitative metric to be decreased is $W = \sum_{i=1}^{M} g_i \cdot l_i \cdot \alpha_i$, where the summation over the *M* structural members includes the weight per unit length g, the length l of the i-th member, and a correction coefficient α that depends on the steel grade and the type of member cross section, and that is normalized with respect to hollow sections made of S235.

The sustainability performance metric reads as

$$P_{su} = \frac{1}{\frac{W}{W_0}}. \quad (3)$$

3.4. Bulk performance metric

A bulk performance metric P results from the three partial performance metrics above

$$P = \gamma_s \cdot P_s + \gamma_b \cdot P_b + \gamma_{su} \cdot P_{su} \quad (4)$$

where $\gamma_s, \gamma_b, \gamma_{su}$ are partial weighting factors constrained by $\gamma_s + \gamma_b + \gamma_{su} = 1$, so that $P_0 = 1$. In the FreeGrid benchmark, the partial weighting factors are uniformly set equal to $\gamma_s = \gamma_b = \gamma_{su} = 1/3$ in order to offer to the participants a common term of reference. In a broader perspective, designers are in charge to discuss the overall performance of DSG(s) by setting other values of the partial weighting factors.

Being P a sortable performance metric, the ranking of the DSG issued from the benchmark for each DBG does not depend to a subjective and *a posteriori* assessment.

4. Design constraints

DSGs shall fulfil the following Geometrical (GC) and Mechanical (MC) Constraints. They are listed in the following by making general reference to a design solution with generic shape and structural features:

GC.1. the single-layer gridshell structural type cannot be changed, i.e. the mesh is 2-manifold, that is non-manifold vertices and edges are not permitted [27];
GC.2. the position, shape and length of the continuous spring line and of the head arches (for barrel vault only) cannot be modified;
GC.3. the gridshell spans along the x and y directions B_x and B_y, respectively, shall not be shorter than 30 m, being B_x and B_y generally defined as the maximum span free of external constraints;
GC.4. the extent of the projection of the overall gridshell surface on the horizontal reference plane shall be no smaller than $S - 0.05S$;
GC.5. the rise shall be kept equal to f=B/8, being f generally defined as the distance between the horizontal reference plane and the horizontal tangent plane to the shell surface having the minimum height;
GC.6. the height h shall be not longer than B /4, being h generally defined as the distance between the horizontal reference plane and the horizontal plane passing through the shell vertex having the maximum height;
GC.7. geometrical vertices and structural joints cannot lie below the horizontal reference plane;
MC.1. along the spring line L, x-, y- and z-displacements of all the structural joints resulting from mesh generation shall be externally constrained (perfect hinges);
MC.2. along the head arches (barrel vault), x- and z-displacements of all the structural joints resulting from mesh generation shall be externally constrained;
MC.3. additional structural external constraints are not allowed anywhere;
MC.4. the structural members shall have commercial cross sections;
MC.5. the structural material shall be steel.

Whichever design parameter can be varied if not explicitly excluded above, for instance geometry (e.g. overall shape of the gridshell, node position, grid density and topology), number and properties of the structural members (cross section, length), number and type of the structural joints, steel grade, prestressing magnitude, et cetera.

5. Design approaches and performance assessment methods

Whatever approach to design and optimization is welcome, for instance man-based heuristic design, AI-based design (e.g. neural networks, machine learning), gradient-based or topological optimization, physical scale model-based, models based on continuous analogy, et cetera.

Any kind of structural models can be used by the participants during the design/optimization (e.g. Reduced Order Models, surrogate models, simplified models such as linear static or buckling analysis).

Final structural performance assessment of the retained DSG(s) shall be carried out by a Geometrically and Materially Nonlinear Analysis (GMNA).

6. Requirements for participants

The method adopted for the performance assessment of the retained DSG(s) shall be well documented in order to guarantee the *reproducibility* of the results, and the *comparability* among them.

In order to reduce the model uncertainties, the following specifications about the Finite Element Model and the numerical solver used in GMNA are given:

- Finite Elements shall be 3D beam elements based on Timoshenko model, with cubic shape function;
- The Distributed Plasticity (DP) approach [28] shall be adopted, with nonlinear behaviour modelled along the element and over the element cross section;

- each structural member should be preferably discretised by 4 Finite Elements of equal length, in order to account for member buckling;
- the load control path-following procedure shall be applied within the analysis. The load step magnitude shall be set equal to 1/1000 the magnitude of the load condition LC_i;
- the iterative convergence shall be accomplished at each load step by means of the standard Newton-Raphson method, with a tolerance set equal to 5e-3 in terms of weighted residuals of the variables.

To pursue reproducibility and comparability, data about DSG, design approaches, performance assessment tools, and results shall be provided by the participants as Open Data. The complete set of requested data is detailed in the FreeGrid Technical Specifications.

In order to guarantee *comparability*, participants are strongly advised to precisely specify for each proposed design solution each single Design Goal metric, partial performance metrics, and bulk performance metric.

In order to allow *reproducibility*, participants are warmly encouraged to precisely specify for each proposed design solution: (i) *data on DSG*, i.e. geometrical and structural features; (ii) *data on models*, i.e. adopted design approach (e.g. heuristic approach, optimization algorithm(s), AI technique), name and version of the Finite Element code(s) and details about the Finite Element Model used for performance assessment; (iii) *main output* of the structural performance assessment.

Open Access automatic tools will be made available in order to assist participants in checking the fulfilment of the design constraints during the conceptual design, and in the calculation of buildability performance metrics during the final assessment.

The above information shall be delivered by participants in the form of data uploaded on the FreeGrid web site, technical report/white paper sent to the FreeGrid Steering Committee, or published scientific paper.

7. Outlook and conclusion

The FreeGrid benchmark is launched at the IASS 2023 Annual Symposium in Melbourne. The benchmark technical specifications are made available at the FreeGrid website (https://sites.google.com/view/freegrid). A position paper will discuss in deep the mechanical behaviour of the background and design baseline gridshells, in order to provide to participants stimuli for design solutions.

In itinere special sessions will be possibly organized in the following years at international conferences. The benchmark first milestone is tentatively set within the IASS Annual Symposium 2026, when the FreeGrid ArcelorMittal Steligence Awards will be remitted according to given specific regulation to the young author(s) of the most performing design solution for each type of DGS, together with an additional award remitted to the author(s) who will deal with all the three DGS and obtain the highest performances in average.

Acknowledgements

The Authors gratefully acknowledge the support of the Italian Council for Steel Structures (CTA, https://www.collegiotecniciacciaio.it), the umbrella of the International Association for Shell and Spatial Structures (IASS, https://iass-structures.org), and the partnership with ArcelorMittal Steligence (https://steligence.arcelormittal.com). The Authors thank Riccardo Zanon and Marina D'Antimo (ArcelorMittal Steligence) for the stimulating insight and fruitful discussion about steel gridshell sustainability. The Authors warmly thank Yi Min ‘Mike’ Xie and Jane Burry, Co-Chairs of the IASS annual Symposium 2023, and its Organizing Committee for hosting the launch of the benchmark within the event.

References

[1] I. Paya-Zaforteza and M.E.M. Garlock, “Structural Engineering Heroes and Their Inspirational Journey”, *Structural Engineering International*, vol. 31(4), pp. 584-597, 2021.

[2] H. Salehi and R. Burgueño, “Emerging artificial intelligence methods in structural engineering”, *Engineering Structures*, vol. 171, pp. 170-189, 2018.

[3] F. Venuti and L. Bruno, “Influence of in-plane and out-of-plane stiffness on the stability of free-edge gridshells: A parametric analysis”, *Thin-Walled Structures*, vol. 131, pp. 755–768, 2018.

[4] R.Q. Feng, L. Zhang and J.M. Ge, “Multi-objective morphology optimization of free-form cable-braced grid shells”, *International Journal of Steel Structures*, vol.15(3), pp. 681-691, 2015.

[5] O. Sigmund, “On benchmarking and good scientific practise in topology optimization”, *Structural and Multidisciplinary Optimization*, vol. 65(11), pp. 315, 2022.

[6] V. Tomei, E. Grande and M. Imbimbo, “Influence of geometric imperfections on the efficacy of optimization approaches for grid-shells”, *Engineering Structures*, vol. 228, pp. 111502, 2021.

[7] E. Vouga, M. Höbinger, J. Wallner and H. Pottmann, “Design of self-supporting surfaces”, *ACM Transactions on Graphics*, vol. 31(4), art. no. 87, 2012.

[8] L. Mourad, J. Bleyer, R. Mesnil, J. Nseir, K. Sab and W. Raphael, “Topology optimization of load-bearing capacity”, *Structural and Multidisciplinary Optimization*, vol. 64(3), pp. 1367 – 1383, 2021.

[9] F. Laccone, L. Malomo, M.Froli, P. Cignoni, N. Pietroni, “Automatic Design of Cable-Tensioned Glass Shells”, *Computer Graphics Forum*, vol. 39.1, pp. 260-273, 2020.

[10] D. Tonelli, N. Pietroni, E. Puppo, M. Froli, P. Cignoni, G. Amendola and R. Scopigno, “Stability of Statics Aware Voronoi Grid-Shells”, *Engineering Structures*, vol. 116, pp. 70-82, 2016.

[11] T. Bulenda and J. Knippers, “Stability of grid shells”, *Computers and Structures*, vol. 79(12), pp. 1161–74, 2001.

[12] M. Mohammadi, K. Abedi and N. Taghizadieh, Stability analysis of single-layer barrel vault space structures. *International Journal of Space Structures*, vol. 7(4), pp. 203-218, 2012.

[13] J. Yan, F. Qin, Z. Cao, F. Fan and Y. Mo, Mechanism of coupled instability of single-layer reticulated domes, *Engineering Structures*, vol. 114, pp. 158-170, 2016.

[14] R. Mesnil, C. Douthe, O. Bavarel, B. Léger, J.F. Caron, “Isogonal moulding surfaces: A family of shapes for high node congruence in free-form structures”, *Automation in Construction*, Vol. 59, pp. 38-47, 2015.

[15] X. Tellier, C. Douthe, L. Hauswirth, O. Baverel, “Caravel meshes: A new geometrical strategy to rationalize curved envelopes”, *Structures,* vol. 28, pp. 1210-1228, 2020.

[16] V. Tomei, M. Imbimbo and E. Mele, “Optimization of structural patterns for tall buildings: the case of diagrid”, *Engineering Structures*, vol. 171, pp. 280-297, 2018.

[17] J. Schlaich and H. Schober, “Glass roof for the Hippo House at the Berlin Zoo”. *Structural Engineering International*, vol. 7(4), pp. 252–254, 1997.

[18] R. Mesnil, C. Douthe, C. Richter and O. Bavarel, “Fabrication-aware shape parametrisation for the structural optimisation of shell structures”, *Engineering Structures*, vol. 176, pp. 569 – 5841, 2018.

[19] C. Tang, X. Sun, A. Gomes, J. Wallner, and H. Pottmann, “Form-finding with polyhedral meshes made simple”, *ACM Transactions on Graphics*, vol. 33(4), pp. 70:1–70:9, 2014.

[20] N. Pietroni, D. Tonelli, E. Puppo, M. Froli, R. Scopigno and P. Cignoni, “Statics Aware Grid Shells”, *Computer Graphics Forum*, vol. 34(2), pp. 627-641, 2015.

[21] H. Seifi, A.R. Javan, S. Xu, Y. Zhao, Y.M. Xie, "Design optimization and additive manufacturing of nodes in gridshell structures", *Engineering Structures*, vol. 160, pp. 161-170, 2018.

[22] Y. Liu, T.U. Lee, A. Koronaki, N. Pietroni and Y.M. Xie, "Reducing the number of different nodes in space frame structures through clustering and optimization", *Engineering Structures*, vol. 284 (116016), 2023.

[23] F. Otto, "*IL10: Gitterschalen - Grid Shelss*", Institut für Leichte Flächentragwerke, Stuttgart, 1974.

[24] M. Kilian, D. Pellis, J. Wallner and H. Pottmann, "Material-minimizing forms and structures", *ACM Transactions on Graphics*, vol. 36 (6), art. no. a173, 2017.

[25] O. P. Gibbons, J. J. Orr, C. Archer-Jones, W. Arnold, D. Green, "*How to calculate embodied carbon*" (2nd ed.), The Institution of Structural Engineers, London, UK, 2022.

[26] R. Stroetmann, "High Strength Steel for Improvement of Sustainability", in *Proceedings of Eurosteel 2011*, Budapest, Hungary, August 31 - September 2, 2011.

[27] L.W. Tu, "*An Introduction to Manifolds"* (2nd ed.), New York: Springer, 2011.

[28] F.F. Taucer, E. Spacone, F.C. Filippou, "*A Fiber Beam-Column Element for Seismic Response Analysis of Reinforced Concrete Structures*", Report No. UCB/EERC-91/17, Earthquake Engineering Research Center College of Engineering University of California, Berkeley, 1991.