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Special Collection: 250 Years of Brownian Motion

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ABSTRACT

Inspired by the success of dimensional analysis in continuum mechanics and nondimensional numbers such as Péclet, Schmidt, Lewis, and Prandtl, we introduce the study of the Brownian motion nondimensional parameter $\mathbb{M} = \mathit{conservativeforce}/\mathit{nonconservativeforce}$. In particular, we nondimensionalise the Langevin equation for a Brownian particle of mass m moving in a fluid of viscosity γ with diffusivity coefficient σ under the action of a conservative force in the form of a harmonic oscillator of constant k. We identify all the possible time-scales $\mathbb{T}$, and the associated length-scales $\mathbb{L}$, on the basis of the nondimensional parameter $\mathbb{M} = m\,k\,\gamma^{-2}$. Through this nondimensionalisation we study the small-parameter limits with respect to $\mathbb{M}^{\alpha}$ and we identify five different regimes due to α, in opposition to the three identified by the standard analysis on the basis of the time-scales:$T_1 = m/\gamma$, $T_2 = \sqrt{m/k}$, and $T_3 = \gamma/k$. Because of the ratio-type definition of $\mathbb{M}$ the leading force is established by the sign of α. For very short time-scales ($\alpha > 0$), the particle is trapped at its initial condition. For short times ($\alpha = 0$), the particle has the same dynamics of a free Brownian particle. That is, the action of the conservative force is negligible. For intermediate time ($-1 < \alpha < 0$), the particle diffuses as a Wiener process, that is as a free Brownian particle at large elapsed times. The over-damped timescale ($\alpha = -1$) is a critical timescale where the dynamics are given by the Smoluchowski–Kramers approximation. For long times ($\alpha < -1$), the particle is trapped at the bottom of the potential well.

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I. INTRODUCTION

The Langevin equation is the most physically based mathematical model of the Brownian motion. In this respect, it is worthy to remind that it took indeed a quite large amount of time to achieve it: since the original observation of the phenomenon in 1827—and the official announcement in 1828—by Brown[1] to its explanation in 1905 by Einstein[2] and to its mathematical formulation in 1906 by von Smoluchowski[3] and in 1908 by Langevin.[4,5] The story of the understanding of the Brownian motion experimentally ended definitively after one hundred years with Nobel prizes in Chemistry to Zsigmondy (1925) and Svedberg (1926) and in Physics to Perrin (1926). Historically, such path was not linear but filled of contradictory experimental measurements[6,7] and slowed down by a lack of proper mathematical tools that were worked out only in the 70s' of the XIX century by Boltzmann[8] within the framework of the kinetic theory of gases.[7] As a matter of fact, the philosophical concepts behind the Brownian motion and its mathematical description go much further, then, the original physical phenomenon in itself as proved by the pioneering formulation (five years before Einstein) in 1900 by Bachelier[9] in his thesis with application in finance, whose foundational role was not recognized neither by his thesis director Poincaré.[10] While nowadays, the concept of Brownian motion as a practice for including random fluctuations is very well recognized as a successful modeling approach and it is employed to describe a wide range of phenomena, from thermal fluctuations of microscopic objects,[11] fluctuations in satellite orbits,[12] the evolution of stock prices,[13] and to biological systems.[14] More technically, noisy phenomena are often modeled by stochastic differential equations (SDEs) in the same fashion of the Langevin equation, namely by a deterministic ordinary differential equation with an additional random term.[15] Many review papers about the Brownian motion are available in the literature both technical and historical.[6,7,10,16,17]

The Langevin equation originally allows for describing the motion of a particle immersed in a fluid within the phase-space by taking into account the whole dynamics: that is by including both random perturbative forcing and friction in terms of the thermal bath and of the fluid viscosity, respectively. Forcing and friction are linked in agreement with the conservation laws through the fluctuation-dissipation relation.[18,19] This description provides—in a whole—the full phenomenology of the particle's trajectory embodied by the short-time correlated motion displaying ballistic movement and by the uncorrelated large-time motion displaying diffusion. This last diffusive regime at large elapsed times is sometimes roughly associated with the Brownian motion itself by compressing the rich phenomenology of the actual Brownian motion to the paradigm: Gaussian distribution of particle displacement, linear growth in time of displacement variance, constant diffusion coefficient and diffusion equation.

This paradigmatic rough Brownian motion describes solely the particle's position by neglecting its velocity and it is also observed—in discrete time and space—in the evolution of standard random walks and—in continuous time and space—in the evolution of the Wiener process.[20] The identification of the Brownian motion as described in the phase-space by the Langevin equation with the Wiener process referred to the particle's position is sometimes a source of misunderstanding between physicists and mathematicians. As a matter of fact, the reduction of the Brownian motion to the solely diffusive regime is so frequent that the corresponding asymptotic Langevin equation, namely the over-damped Langevin equation, calls for a characterization of the full Langevin equation as the under-damped Langevin equation.

Here, we consider the under-damped Langevin equation with a conservative force $F_c(x)$. Specifically, we consider, without loss of generality, the case of a harmonic oscillator. In this framework, the over-damped limit is equivalent to the small-mass limit (or large-viscosity limit) known as the Smoluchowski–Kramers approximation.[21,22] We remind that a rigorous small-mass limit for Langevin-like dynamics differs from the historical-naïve approach since—under certain constraints—a noise-induced drift term emerges.[23–26] However, there are many other limits that can be taken. Thus, we present a framework to analyze *all* such limits under a unifying nondimensional parameter. We are still interested in the asymptotic dynamics in a small-parameter limit, but when such parameter is not the particle's mass. By using the Buckingham π-theorem[27] first we derive the many possible nondimensionalisations of the Langevin equation and later we discuss the corresponding asymptotic behaviors. In spite of the fact that we consider the typical case of a harmonic potential, surprisingly, this study seems to be still new in the literature.

As a matter of fact, the novelty contained in our study goes further than such nondimensionalisation. However, dimensional analysis is an inestimable tool for understanding fluid dynamics and, in general, continuum mechanics. Nondimensional numbers play indeed a key-role in theoretical studies in these fields because they establish universal relationships that allow for unveiling specific dynamical regimes together with their governing laws that actually are not derivable from first principles. The empirical powerfulness of nondimensional numbers lays on the fact that they balance and select among the terms contributing to the process. The prototypical nondimensional number is the Reynolds number. However, within the continuum mechanics framework, transport and diffusion are described and explained, and finally understood, in terms of many other nondimensional numbers: Péclet, Schmidt, Lewis, and Prandtl numbers. At the best of our knowledge, such kind of successful approach was never adopted for a related topic as Brownian motion is. In particular, it was never systematized for studying the rich phenomenology behind a Langevin-type motion of a test-particle. Here, we introduce the nondimensional parameter defined by the ratio

$$\mathbb{M} = \frac{\text{conservative force}}{\text{nonconservative force}}.$$

The effectiveness of this novel formulation allows for studying the whole rich phenomenology in a unitary framework in opposition to the existing approach.

We finally refer the reader to classical treatises in statistical physics for more basic—as well as more high technicalities—that we avoid to report in this paper.[28–31]

The outline of the paper is as follows. In Sec. II, we give a short analysis on three prototypical time-scales: the relaxation time of the velocity, the period of oscillations due to the harmonic oscillator, and the relaxation time for the position. We show that rescaling the Langevin equation according to these three different time-scales results in three different systems. In Sec. III, first, the nondimensional analysis is performed by using the Buckingham π-theorem and all possible time-scales are found, later, the small-parameter limits are analyzed for those different time-scales which corresponds to five cases. This is done by using a heuristic analysis in Sec. III B. Then in Sec. III C, we give results using a direct solution of the SDE as well as a brief argument using the Kolmogorov forward and backward equations (infinitesimal generators). In Sec. IV, we give the conclusions.

II. PRELIMINARIES AND ILLUSTRATIVE EXAMPLES

The Langevin equation essentially is a stochastic formulation of the Newton's second law for a spherical particle of mass m and radius a in a fluid medium with viscosity ν, Stokes' friction $\gamma = 6\pi\nu a$, and random fluctuations. All this reads

$$m\ddot{x}_t = -\gamma\dot{x}_t + \sqrt{2\sigma}\eta_t, \tag{1}$$

for certain initial conditions $x_0 = y$ and $\dot{x}_0 = u$ where x_t, $\dot{x}_t$, and $\ddot{x}_t$ stand for particle's position, velocity and acceleration at time t, η_t is white noise and $\sqrt{2\sigma}$ is the amplitude of the noise and σ is the diffusivity coefficient. From this starting point, the Langevin equation has been well studied through the literature.[32,33] Defining $\dot{x}_t = v_t$ and rewriting Langevin Eq. (1) within the more rigorous setting of SDEs, we have the system

$$dx_t = v_t\,dt, \tag{2a}$$

$$m\,dv_t = F_c(x_t)\,dt - \gamma\,v_t\,dt + \sqrt{2\sigma}\,dW_t, \tag{2b}$$

where dW_t are the increments of the Wiener process. Therefore, the motion of the particle is driven by the sum of a conservative force $F_c(x)$ plus the friction force and plus the thermal perturbative forcing. As a matter of fact, when the sum of these three terms is null then the particle is in the over-damped regime, that is the small-mass limit $m \to 0$ (or large-viscosity limit $\gamma \to \infty$) considered in the Smoluchowski–Kramers approximation, and if γ and σ are constant, the solution of (2) converges almost surely[11] to the solution of

$$dx_t = \frac{F_c(x_t)}{\gamma}\, dt + \frac{\sqrt{2\sigma}}{\gamma}\, dW_t. \tag{3}$$

It is worthy to report that, for arbitrary space- or state-dependent γ and σ, the solution of (3) depends on the interpretation of the stochastic term.[23–25]

As matter of fact, the small-mass limit and the analog large-viscosity limit encode the small-time limit with respect to the timescale m/γ. However, this over-damped timescale m/γ is not the unique small-parameter of the problem.[34,35] As illustrative examples for claryfing the spirit of our study, we show that a small parameter arises in different forms when Langevin Eq. (2) is rescaled according to some time-scales of physical interest. Thus, when, for example, the conservative force is the harmonic oscillator, namely,

$$F_c(x) = -k\, x, \tag{4}$$

actually, there are three time-scales: the relaxation time of the velocity (T_1), the period of oscillations due to the harmonic oscillator (T_2) and the relaxation time of the position (T_3),[34,35] i.e.,

$$T_1 = \frac{m}{\gamma}, \quad T_2 = \sqrt{\frac{m}{k}}, \quad T_3 = \frac{\gamma}{k}. \tag{5}$$

By assuming $m \to 0$ for bounded γ and k, and $\gamma \to \infty$ for bounded m and k, we have the inequalities

$$T_1 < T_2 < T_3. \tag{6}$$

In a nutshell, in the small-mass or in the large-viscosity limit, the velocity relaxation time becomes much smaller than the oscillation period, which in turn is much smaller than the position relaxation time. Through this analysis the dimensionless parameter

$$\mathbb{M} = \frac{mk}{\gamma^2} \tag{7}$$

can be guessed, being $\mathbb{M} = (T_1/T_2)^2 = (T_2/T_3)^2$. Parameter $\mathbb{M}$ is the parameter that we consider in our small-parameter limits, since it corresponds with the small-mass (large-viscosity) or over-damped limit.

To ease the analysis, Langevin Eq. (2) equipped with (4) is nondimensionalised. For the nondimensionalisation, we introduce the spatial- and temporal- scales L and T, respectively, and we define the nondimensional variables $X = x/L$, $S = t/T$, and $V = Tv/L$ such that system (2) turns out to be

$$dX_S = V_S\, dS, \tag{8a}$$

$$dV_S = -\frac{kT^2}{m} X_S\, dS - \frac{\gamma T}{m} V_S\, dS + \sqrt{\frac{2\sigma T^3}{m^2 L^2}}\, dW_S. \tag{8b}$$

Note that the Wiener process under a time change has the property $W_t = W_{TS} = \sqrt{T}\, W_S$.

We start from the short timescale, and we set $T = T_1 = m/\gamma$. Therefore, system (8) is re-written as

$$dX_S = V_S\, dS, \tag{9a}$$

$$dV_S = -\mathbb{M} X_S\, dS - V_S\, dS + \sqrt{2}\, dW_S, \tag{9b}$$

where we have also defined the corresponding short length-scale

$$L = L_1 = \sqrt{\frac{\sigma m}{\gamma^3}} = \sqrt{\frac{\sigma}{\gamma^2}}\, T_1^{1/2}. \tag{10}$$

Note that since L_1 is a multiple of $T_1^{1/2}$, this is a small length-scale. Finally, by setting $T = T_1$ the nondimensional parameter $\mathbb{M}$ arises naturally and it appears only in front of the conservative force term.

For the intermediate timescale, we set $T = T_2 = \sqrt{m/k}$. Therefore, system (8) is re-written as

$$dX_S = V_S\, dS, \tag{11a}$$

$$\mathbb{M}^{1/2}\, dV_S = -\mathbb{M}^{1/2} X_S\, dS - V_S\, dS + \sqrt{2}\, dW_S, \tag{11b}$$

where in analogy with (10), we have also defined the corresponding intermediate length-scale

$$L = L_2 = \sqrt{\frac{\sigma m^{1/2}}{k^{1/2}\gamma^2}} = \sqrt{\frac{\sigma}{\gamma^2}}\, T_2^{1/2}. \tag{12}$$

Finally, by setting $T = T_2$ the nondimensional parameter $\mathbb{M}^{1/2}$ arises naturally and it appears in front of the inertial and the conservative force terms.

For the long timescale, we set $T = T_3 = \gamma/k$. Therefore, system (8) is re-written as

$$dX_S = V_S\, dS, \tag{13a}$$

$$\mathbb{M}\, dV_S = -X_S\, dS - V_S\, dS + \sqrt{2}\, dW_S, \tag{13b}$$

where in analogy with (10) and (12), we have also defined the corresponding large length-scale

$$L = L_3 = \sqrt{\frac{\sigma}{\gamma k}} = \sqrt{\frac{\sigma}{\gamma^2}}\, T_3^{1/2}. \tag{14}$$

Finally, by setting $T = T_3$ the nondimensional parameter $\mathbb{M}$ arises naturally and it appears in front of the inertial term only. This last is the classical over-damped limit equation.

Thus, the three different scalings lead to three different resulting equations and seemingly three different dynamics. Hence, the question we study in the Sec. III is: can these time-scales be unified under a single theory? In the following, we systematically identify all possible time-scalings and derive their dynamics.

III. SMALL-PARAMETER LIMITS

A. Nondimensional analysis

We perform a systematic nondimensional analysis of Langevin Eq. (2) where all time-scales, and associated length-scales, are identified by using the Buckingham π-theorem.[27] As in the Sec. II, we consider the case with a harmonic oscillator (4). To apply the Buckingham π-theorem, we remind that the physical dimensions of the involved parameters are

$$[m] = \text{mass}, \tag{15a}$$

$$[\gamma] = \text{mass} \times \text{time}^{-1}, \tag{15b}$$

$$[k] = \text{mass} \times \text{time}^{-2}, \tag{15c}$$

$$[\sigma] = \text{mass}^2 \times \text{length}^2 \times \text{time}^{-3}, \tag{15d}$$

where $[\cdot]$ denotes the physical dimension of the parameter. First, the dimensionless combinations are found. This is done by solving the equation

$$\left[m^a \gamma^b k^c \sigma^d\right] = \text{unitless}. \tag{16}$$

By using the dimensions from (15), formula (16) becomes

$$\left[m^a\gamma^b k^c\sigma^d\right] = \text{mass}^{a+b+c+2d} \times \text{time}^{-b-2c-3d} \times \text{length}^{2d}. \tag{17}$$

A dimensionless constant is obtained if all the exponents are zero. This corresponds to the matrix equation

$$\begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{18}$$

The null space of this matrix is $[a, b, c, d] = c[1, -2, 1, 0]$. This corresponds to the dimensionless parameter $\mathbb{M}^c$ where $\mathbb{M}$ is defined in (7). Looking for the timescale, formula (17) is set dimensionally equal to time. This corresponds to the matrix equation

$$\begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \tag{19}$$

Particular solutions to this problem have the form $[1+c, -1-2c, c, 0]$. All solutions to the system can be written as

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1+c \\ -1-2c \\ c \\ 0 \end{pmatrix} + \theta\begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \quad \theta \in \mathbb{R}, \tag{20}$$

which provides the timescale

$$\mathbb{T} = m^{1+\alpha}\gamma^{-1-2\alpha}k^\alpha = \frac{m}{\gamma}\mathbb{M}^\alpha, \quad c + \theta = \alpha \in \mathbb{R}, \tag{21}$$

where $\mathbb{M}$ is defined in (7). The same solution can be get by setting $c = 0$ without losing any generality. Then, we characterize all possible time-scales by $\mathbb{T}$ and derive the limits for all cases of α. Because of the ratio-type definition of $\mathbb{M}$ and of the small-parameter limit, the leading force is actually established by the sign of α. Note that the three physical time-scales studied in Sec. II correspond to $\alpha = 0 (T_1)$, $\alpha = -1/2$ (T_2), and $\alpha = -1$ (T_3).

To derive the full dynamics of system (2) with the conservative force (4), we consider the nondimensional setting (8) where we take as timescale the general one $\mathbb{T}$ stated in (21) and, in analogy with the illustrative examples of Sec. II, as length-scale

$$\mathbb{L} = \sqrt{\frac{\sigma}{\gamma^2}}\mathbb{T}^{1/2}. \tag{22}$$

Moreover, we make the further substitution $V_S = \mathbb{M}^{\alpha/2}U_S$ such that the nondimensional Langevin equation with a harmonic oscillator (4) reads

$$dX_S = \mathbb{M}^{\alpha/2}U_S\,dS, \tag{23a}$$

$$dU_S = -\mathbb{M}^{1+3\alpha/2}X_S\,dS - \mathbb{M}^\alpha U_S\,dS + \mathbb{M}^{\alpha/2}\sqrt{2}\,dW_S. \tag{23b}$$

Given the dynamical meaning of the nondimensional parameter $\mathbb{M}$, system (23) results in an interpreting framework for the phenomenology of the Brownian motion and, more in general, of the Langevin-like dynamics.

B. Results

Now, we describe in a heuristic way the dynamics behind (23) in the limit $\mathbb{M} \to 0$. We distinguish five cases: $\alpha > 0$, $\alpha = 0$, $-1 < \alpha < 0$, $\alpha = -1$, and $\alpha < -1$.

Case: $\alpha > 0$ (strong viscous motion)

In system (23) above, all the terms involving $\mathbb{M}$ have positive exponents. Thus, when $\mathbb{M} \to 0$, the dynamics are

$$dX_S = 0, \tag{24a}$$

$$dU_S = 0. \tag{24b}$$

This means that the timescale is so short that the particle is stuck at the initial condition. Nothing has happened yet.

Case: $\alpha = 0$ (motion independent of the conservative force)

The only term lost in the Langevin Eq. (23) when $\mathbb{M} \to 0$ is the one including the conservative force. Thus, the resulting SDE is

$$dX_S = U_S\,dS, \tag{25a}$$

$$dU_S = -U_S\,dS + \sqrt{2}\,dW_S. \tag{25b}$$

For this short timescale, the particle moves as if there is no conservative force. Thus, we have the same Langevin dynamics for a free particle.

In the case $\alpha < 0$, we expect $\mathbb{M}^{-\alpha/2}U_S \to 0$ as $\mathbb{M} \to 0$. This can be proved using methods established in the literature.[11] For negative α, we rewrite (23). In particular, such two equations can be combined by multiplying the dU_S Eq. (23b) by $\mathbb{M}^{-\alpha/2}$, solving for the $U_S\,dS$ term and, finally, substituting it into the dX_S Eq. (23a) yields the SDE

$$d(X_S + \mathbb{M}^{-\alpha/2}U_S) = -\mathbb{M}^{1+\alpha}X_S\,dS + \sqrt{2}\,dW_S. \tag{26}$$

Looking at this last form of the nondimensional Langevin Eq. (26), we can explore the dynamics in the three cases: $-1 < \alpha < 0$, $\alpha = -1$, and $\alpha < -1$.

Case: $-1 < \alpha < 0$ (inviscid motion)

For $-1 < \alpha < 0$, and intermediate time-scales, the limit SDE is

$$dX = \sqrt{2}\,dW_S. \tag{27}$$

Thus, the particle diffuses according to a Wiener process, analogously to a free Brownian particle in the diffusive regime at large elapsed times.

Case: $\alpha = -1$ (motion mildly governed by the conservative force)

For the critical case $\alpha = -1$, the particle obeys the classical over-damped dynamics of the Smoluchowski–Kramers approximation

$$dX_S = -X_S\,dS + \sqrt{2}\,dW_S. \tag{28}$$

Case: $\alpha < -1$ (motion strongly governed by the conservative force)

When $\alpha < -1$, and long time-scales, the leading order effects are due to the conservative force. Thus,

$$X_S = 0, \tag{29}$$

and the particle is stuck in the potential well.

C. SDE solutions ($-1 \le \alpha \le 0$) and infinitesimal generators ($\alpha > 0$)

It is worthy to note that when $-1 \le \alpha \le 0$ the SDE (26) can be solved in a manner adopted in similar literature studies[36,37] and the asymptotic behavior can be derived. When $\alpha > 0$, the infinitesimal generators can be analyzed to prove the small-parameter limits. Thus, here we sketch in a heuristic way how the results derived on the basis of a SDE hold also on the basis of the corresponding partial differential equation. Technical details are available in the literature.[23,38,39]

In particular, the solution of (26) is

$$X_S = X_0 + \mathbb{M}^{-\alpha/2}(1 - e^{-S\mathbb{M}^\alpha})U_0 - \mathbb{M}^{1+\alpha}\int_0^S [1 - e^{-(S-\xi)\mathbb{M}^\alpha}]X_\xi \, d\xi + \sqrt{2}\int_0^S [1 - e^{-(S-\xi)\mathbb{M}^\alpha}]\, dW_\xi, \quad -1 \le \alpha \le 0, \tag{30}$$

and the limit as $\mathbb{M} \to 0$ can be taken.

For what $\alpha > 0$ concerns, the infinitesimal generators can be analyzed to prove limits. The details on the technicalities for a rigorous proof are established in the following literature.[23,38,39] In particular, the backward Kolmogorov equations are used to analyze the limit. The starting SDE is Eq. (23). For the analysis of the backward Kolmogorov equation let $g(x', u', s'|x, u, s) = g$ be the probability density of the distribution of the position and rescaled velocity (x', u') of the particle at time s' given their values at (x, u) and time $s < s'$. The backward Kolmogorov equation for (23) is

$$\frac{\partial g}{\partial t} = \left(\mathbb{M}^{\alpha/2}\mathcal{L}_1 + \mathbb{M}^\alpha \mathcal{L}_2 + \mathbb{M}^{1+3\alpha/2}\mathcal{L}_3\right)g, \tag{31}$$

where

$$\mathcal{L}_1 = -u\frac{\partial}{\partial x} + \frac{\partial^2}{\partial u^2}, \tag{32a}$$

$$\mathcal{L}_2 = -u\frac{\partial}{\partial u}, \tag{32b}$$

$$\mathcal{L}_3 = -x\frac{\partial}{\partial u}. \tag{32c}$$

Since $\alpha > 0$, then, every term in (31) has a positive exponent of $\mathbb{M}$. Consider for g the ansatz: $g = g_0 + \mathbb{M}^{\alpha/2}g_1 + \mathbb{M}^\alpha g_2 + \cdots$.[38,39] Thus, to leading order

$$\frac{\partial g_0}{\partial t} = 0, \quad \Rightarrow \quad g_0(x, u, t) = g_0(x, u). \tag{33}$$

Thus, since $g_0(x, u)$, this is the initial condition for the Brownian particles' position and (rescaled) velocity. The next-order equation is

$$\frac{\partial g_1}{\partial t} = \mathcal{L}_1 g_0 = -u\frac{\partial g_0}{\partial x} + \frac{\partial^2 g_0}{\partial u^2}. \tag{34}$$

This is advection and diffusion of the initial condition. Thus, the particle is initially stuck at its initial condition and is being moved at order $\mathbb{M}$ via advection and diffusion due to the initial velocity and the random perturbation. This homogenization approach is not fully rigorous. To prove a limit theorem more elaborate techniques can be used.[40,41]

IV. CONCLUSION

In this paper, in analogy with the study of transport and diffusion in continuum mechanics, we propose to study the motion of a Brownian particle by means of a nondimensional parameter. In particular, we define it through the ratio between the conservative and non-conservative forces. More specifically, we analyzed the Langevin equation for the motion of a Brownian particle under the action of the conservative external force of a harmonic oscillator, constant friction and constant noise amplitude. We determined all the possible time-scales $\mathbb{T}$ (21), and the associated length-scales $\mathbb{L}$ (22), on the basis of the nondimensional parameter $\mathbb{M}^\alpha$ (7) with exponent $\alpha \in \mathbb{R}$. We derived the nondimensionalised Langevin equation encoding all the possible regimes (23). Through this nondimensionalisation we identified five different regimes provided by α. They correspond to very short time-scales $\alpha > 0$ when the particle slowly advecs according to its initial velocity, short timescale $\alpha = 0$ when the particle moves according to the Langevin equation in the absence of an external conservative force as a free Brownian particle, intermediate time-scales $-1 < \alpha < 0$ when the particle diffuses as a Wiener process, that is as a free Brownian particle at large elapsed times, long timescale $\alpha = -1$ when the over-damped limit holds, and very long time-scales $\alpha < -1$ when the particle is trapped by the conservative external force.

The effectiveness of this novel formulation allows for studying the whole rich phenomenology in a unitary framework that is controlled by the single parameter α rather than by the introduction of three different parameters embodying three different time-scales, namely T_1, T_2, and T_3 as they are defined in (5). A novel result emerging from our dimensional analysis is that the inviscid regime during which the particle moves as a (Wiener) diffusive process is not associated with a timescale (see, T_2) but to a continuous spectrum of time-scales constrained by $-1 < \alpha < 0$. Moreover, another quantitative result concerns the fact that from the standard analysis leading to the three times-scales T_1, T_2, and T_3 three regimes emerge: ballistic $t < T_1$; diffusive $T_1 < t < T_3$; confined $t > T_3$. The proposed approach allows for establishing five regimes. In fact, the very initial (strong viscous motion) and the very final (strongly affected by the conservative force) regimes are temporally and spatially quantified by the two constraints $\alpha > 0$ and $\alpha < -1$, respectively. To conclude, notwithstanding its easy-to-follow derivation, this new scenario paves the path for the development of a comprehensive analysis of arbitrary Langevin-type processes through a proper nondimensional number.

In Secs. III B and III C, we give arguments justifying the limiting equations of each case. However, the arguments presented are not fully rigorous. To prove a limit theorem using the solution of the SDE directly, one must use proper techniques.[11,24,36] To prove a weaker form of convergence using the infinitesimal operator, like shown in Sec. III C, one must use homogenization techniques.[38,40,41]

Here, the analysis is done in one dimension and we choose the harmonic oscillator to allow for simple nondimensional analysis. However, a similar scaling can be chosen for arbitrary conservative force[37] but, if viscosity or noise amplitude are state dependent, i.e., $\gamma = \gamma(x)$ or $\sigma = \sigma(x)$, then, the limiting equation will vary. For example, if the over-damped timescale $T_1 = m/\gamma$ is cosidered together with varying friction, the limiting equation will include a noise-induced drift term.[23,24] Thus, the heuristics presented here are not expected to hold.

Hence, since a noise-induced drift term arises for the overdamped timescale with $\alpha = 0$, in future perspective one interesting follow on question is: for general state-dependent viscosity and noise amplitude, what are the limiting SDEs for other regimes provided by α?

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Scott Hottovy: Formal analysis (equal); Investigation (equal); Methodology (equal); Writing – original draft (equal); Writing – review & editing (equal). **Paolo Paradisi:** Conceptualization (equal); Writing – original draft (equal); Writing – review & editing (equal). **Gianni Pagnini:** Conceptualization (equal); Formal analysis (equal); Investigation (equal); Methodology (equal); Writing – original draft (equal); Writing – review & editing (equal).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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