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Wave Scattering by an Asymmetric Nonmonotonic Double Layer

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In this report, we give a detailed derivation of the eigenvalues and of the corresponding eigenfunctions of the collisionless Boltzamann equation governing the vibrations of a multispecies ionized gas. These eigenfunctions are worked out as a superposition of the singular, triply discretely degenerate and doubly continuously degenerate eigenfunctions of the free-streaming Vlasov operator (the Liouville operator) [1]. The superposition is carried out in the Fourier transformed velocity space, where the Liouville eigenfunctions are smooth. We prove that, by a judicious superposition of these Liouville eigenfunctions, a peculiar, non degenerate eigenfunction of the Vlasov operator can be worked out, such that its limit value, as the conjugate velocity coordinate tends to infinity, equals the permittivity of the ionized gas. Requiring that this limit vanish, as demanded by Lebesgue's lemma, yields the dispersion relation of electrostatic oscillations.

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I. INTRODUCTION

!!!!!!Write about water bags (the gamma dependent x intervals are nested)

The linear stability analysis of the zero-th order equilibrium distribution functions pertaining to nonlinear, collisionless Vlasov equilibria needs solve a linear eigenvalue problem for the operator governing their small-amplitude perturbations.

Besinceived as an initial value differential problem and solved by the Green function technique.

If the Green function has "virtual" poles in its non physical Riemann sheet, the corresponding oscillations (the virtual modes, not proper eigenfunctions) can be represented as a superposition of the eigenfunctions belonging the continuous spectrum [2, 3]. These are the Landau damped oscillations and appear in kinetic as well inhomogeneous fluid systems [2, 3]. They can be used to enhance resonance absorption of electromagnetic waves [4, 5] and to explain wave attenuation in non dissipative homogeneous stochasitic media as yet another form of Landau damping [6].

Another method [7] to treat linear oscillations about an equilibrium state consists in transforming the differential operator acting on the perturbations of the equilibrium into an integral operator. A regularization technique of the singular Cauchy kernel of this operator separates the subset of the eigenfunctions belonging to the continuous spectrum and the subset of virtual modes, these latter as again being due to the zeroes of a certain dispersion function in its non physical Riemann sheet. This method was applied to both electrostatic oscillations of a cold inhomogeneous plasma [8] and to fully elecromagnetic oscillations in a kinetic plasma [9].

In this report we use this approach, and in order to cater also for the recently discovered singularities [10] of the equilibrium distribution functions, we develop our treatment in the space of the Fourier transformed velocity[11–13], where they are well behaved.

This allows us to find a useful technique to reconstruct the permittivity of the medium directly from the functional shape of the eigenfunction in the transformed space.

II. NOTATIONS, ASSUMPTIONS AND BOUNDARY CONDITIONS

Let $\hat{\Phi}$ and $\hat{\phi}$, denote the steady state and perturbation electric potential in the plasma,

$$\Phi = (\hat{\Phi} - \min\hat{\Phi})/\Phi_0,\ \tilde{\phi} = \hat{\phi}/\Phi_0 \tag{1}$$

the corresponding quantities, normalized to $\Phi_0 = \max\hat{\Phi} - \min\hat{\Phi}$, e the elementary charge, $\alpha = \mathrm{e}, \mathrm{i}$ a label for the electron and ion quantities, $Z_\alpha e$ the particle charges,

$$-V_\mathrm{e} = Z_\mathrm{e}\Phi,\ -V_\mathrm{i} = Z_\mathrm{i}(\Phi - 1) = Z_\mathrm{i}(V_\mathrm{e}/|Z_\mathrm{e}| - 1), \tag{2}$$

the normalized electron and ion potential energies in the steady state potential Φ, m_e the electron's mass, n_0 a density scale,

$$n_0\sqrt{[m_\mathrm{e}/(e\Phi_0)]}(\tilde{F}_\alpha + \tilde{f}_\alpha)/|Z_\alpha| \tag{3}$$

the one-particle velocity distributions and $\tilde{F}_\alpha$ and $\tilde{f}_\alpha$ their normalized steady state and perturbation parts.

We assume that Φ depends on space coordinate x, $\tilde{\phi}$ on x and time t, $\tilde{F}$ on x and velocity coordinate v and $\tilde{f}$ on x, v, t, respectively normalized to

$$\lambda = \sqrt{[e\Phi_0/(4\pi n_0 e^2)]},\ v_0 = \sqrt{(e\Phi_0/m_\mathrm{e})},\ \omega_\mathrm{p}^{-1} = L/v_0, \tag{4}$$

and we denote by

$$\phi_\omega(x) = \int_{-\infty}^{\infty} \mathrm{d}t e^{-\mathrm{i}\omega t}\tilde{\phi}(x,t), \tag{5}$$

$$f_{\alpha\omega}(x,q) = \int_{-\infty}^{\infty} \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}v e^{\mathrm{i}(qv-\omega t)}\tilde{f}_\alpha(x,v,t), \tag{6}$$

$$F_\alpha(x,q) = \int_{-\infty}^{\infty} \mathrm{d}v e^{\mathrm{i}qv}\tilde{F}_\alpha(x,v) \tag{7}$$

the Fourier transforms of the electric potential perturbation and of the particle distributions.

We introduce the vector

$$|f_\omega\rangle = [f_{\mathrm{e}\omega}(x,q), f_{\mathrm{i}\omega}(x,q)]^\mathrm{T}, \tag{8}$$

the particle mass raito $\mu_\alpha = m_\alpha/m_\mathrm{e}$, the free streaming and interaction operators

$$S = \frac{\partial^2}{\partial x \partial q} - q\Phi' \begin{bmatrix} Z_\mathrm{e}/\mu_\mathrm{e} & 0 \\ 0 & Z_\mathrm{i}/\mu_\mathrm{i} \end{bmatrix}, \tag{9}$$

$$K = HD_x^{-1}P_0, \tag{10}$$

where a "$'$" denotes differentiation with respect to x,

$$H = -q \begin{bmatrix} Z_\mathrm{e}F_\mathrm{e}/\mu_\mathrm{e} & -Z_\mathrm{e}F_\mathrm{e}/\mu_\mathrm{e} \\ Z_\mathrm{i}F_\mathrm{i}/\mu_\mathrm{i} & -Z_\mathrm{i}F_\mathrm{i}/\mu_\mathrm{i} \end{bmatrix}, \tag{11}$$

and, given a generic function $g(x,q)$,

$$D_x^{-1}g = \int \mathrm{d}x' g(x',q), \tag{12}$$

$$P_0 g = g(x,0). \tag{13}$$

Using the above definitions, the linerized electron and ion Vlasov-Poisson equations are

$$\omega|f_\omega\rangle - S|f_\omega\rangle = K|f_\omega\rangle, \tag{14}$$

$$\phi''_\omega = P_0(f_{\mathrm{e}\omega} - f_{\mathrm{i}\omega}). \tag{15}$$

In the following, the solution of Eqs. (14) and (15) will be given in terms of the eigenfunctions (labelled by a superscript 0)

$$|\chi^{(0)s_\mathrm{e}}_{\mathrm{e}\sigma}\rangle = [\chi^{(0)s_\mathrm{e}}_{\mathrm{e}\sigma}(x,q), 0]^\mathrm{T}, \tag{16}$$

$$|\chi^{(0)s_\mathrm{i}}_{\mathrm{i}\sigma}\rangle = [0, \chi^{(0)s_\mathrm{i}}_{\mathrm{i}\sigma}(x,q)]^\mathrm{T}, \tag{17}$$

$$|\psi^{(0)s_\alpha}_{\alpha\sigma}\rangle = D_x^{-1}P_0|\chi^{(0)s_\alpha}_{\alpha\sigma}\rangle \tag{18}$$

of the free streaming equation (or also ballistic or Liouville equation (Eq. (14)), unperturbed by the interaction operator K),

$$\frac{\partial^2\chi^{(0)s_\alpha}_{\alpha\sigma}}{\partial x \partial q} - \frac{Z_\alpha}{\mu_\alpha}q\Phi'\chi^{(0)s_\alpha}_{\alpha\sigma} = \sigma\chi^{(0)s_\alpha}_{\alpha\sigma} \tag{19}$$

and of Poisson equation (Eq. (15)), which have σ as eigenvalue and satisfy the boundary conditions

if $|x| \to \infty$, $\Phi \to$ const. then

$$\chi^{(0)s_\alpha}_{\alpha\sigma} \to \rho^{s_\alpha}_{\alpha\sigma}e^{\mathrm{i}s_\alpha k_{\alpha\sigma}x},\ \psi^{(0)s_\alpha}_{\alpha\sigma} \to \epsilon^{s_\alpha}_{\alpha\sigma}e^{\mathrm{i}s_\alpha k_{\alpha\sigma}x}, \tag{20}$$

$$s_\alpha = \pm, \tag{21}$$

where $\rho^{s_\alpha}_{\alpha\sigma}$ and $\epsilon^{s_\alpha}_{\alpha\sigma}$ are complex quantities, and $k_{\alpha\sigma}$ is a real constant. In Eq. (18), $\psi^{(0)s_\alpha}_{\alpha\sigma}$ is proportional to the perturbation electric field generated by the particles of species α distributed according to the eigenfunction $\chi^{(0)s_\alpha}_{\alpha\sigma}$.

The conditions in Eqs. (20)-(21) are justified by observing that, when x takes large values, the steady state potential Φ of a double layer approaches a constant value and the plasma becomes homogeneous: Eqs. (20)-(21) prescribe that, in these conditions, the solution of Eqs. (15) and (19) approach a sinusoidal, right-moving ($s_\alpha = +$) or left-moving ($s_\alpha = -$) wave and that its spatial mean approaches zero.

A vanishing value of the boundary amplitudes $\rho^{s_\alpha}_{\alpha\sigma}$ and $\epsilon^{s_\alpha}_{\alpha\sigma}$ applies if, e.g. only outgoing or ingoing waves exist at the boundary or if $\chi^{(0)s_\alpha}_{\alpha\sigma}$ describes reflected or trapped particles which are unable to reach one or both boundaries. These conditions are appropriate for the scattering or also emission of radiation by the non monotonic double layer.

III. THE UNPERTURBED EIGENVALUE PROBLEM FOR THE NONMONOTONIC DOUBLE LAYER

The solution of Eq. (19) is

$$\chi^{(0)s_\alpha}_{\alpha\sigma\gamma_\alpha} = C_{\alpha\gamma_\alpha} e^{-\mathrm{i}s_\alpha\sigma\xi_{\alpha\gamma_\alpha}+\mathrm{i}s_\alpha q|B_{\alpha\gamma_\alpha}|}/|B_{\alpha\gamma_\alpha}|, \tag{22}$$

where $C_{\alpha\gamma_\alpha}$ is a normalization constant, $s_\alpha = \pm$ was defined in Eq. (21),

$$B_{\alpha\gamma_\alpha}(x) = s_\alpha\sqrt{\{2[\gamma_\alpha + V_\alpha(x)]/\mu_\alpha\}}, \tag{23}$$

$$\xi_{\alpha\gamma_\alpha}(x) = \int_{x_{\alpha\gamma_\alpha}}^{x} \frac{\mathrm{d}x'}{|B_{\alpha\gamma_\alpha}(x')|}, \tag{24}$$

γ_α and $x_{\alpha\gamma_\alpha}$ are real quantities, and a new label γ_α was introduced accordingly.

For $\chi^{(0)s_\alpha}_{\alpha\sigma\gamma_\alpha}$ in Eq. (22) not to be exponentially unbounded, $\gamma_\alpha + V_\alpha$ must be non negative. This implies thatt, if $\gamma_\alpha < 0$, then, in (Eq. (24)), x and $x_{\alpha\gamma_\alpha}$ take values only in γ_α-dependent intervals

$$a_{\alpha\gamma_\alpha} < x < b_{\alpha\gamma_\alpha}\ a_{\alpha\gamma_\alpha} < x_{\alpha\gamma_\alpha} < b_{\alpha\gamma_\alpha} \tag{25}$$

bounded, at least on one side, by points at which particles of species α are reflected. If $\gamma_\alpha > 0$, then $\gamma_\alpha + V_\alpha$ is always positive and particles of species α move over the whole x-interval: in this case, we set $a_{\alpha\gamma_\alpha} = -\infty$, $b_{\alpha\gamma_\alpha} = \infty$.

Specifically, for the steady state potential profile of the nonmonotonic double layer ([14], Fig. 1)

$$\Phi(x) = \{2\sqrt{U}/[(1+\sqrt{U})\coth(\kappa x/2) - (1-\sqrt{U})]\}^2 \tag{26}$$

$$0 < U < 1 \tag{27}$$

and for $\gamma_\mathrm{e} < 0$, electrons move over two disjoint, semi-infinite intervals labelled in the following by the subscript 1 or 2: thus the endpoint $b_{\mathrm{e}\gamma_\mathrm{e}(1)}$ (Eqs. (29) and (B7)) corresponds to the right electron reflection point and the left endpoint $a_{\mathrm{e}\gamma_\mathrm{e}(1)}$ (Eq. (29)) extends to the left boundary of the double layer; the left endpoint $a_{\mathrm{e}\gamma_\mathrm{e}(2)}$ (Eqs. (30) and (B8)) corresponds to the left electron reflection point and the endpoint $b_{\mathrm{e}\gamma_\mathrm{e}(2)}$ (Eq. (30)) extends to the right boundary of the double layer. For $\gamma_\mathrm{e} > 0$, electrons move over an infinite interval: the endpoints $a_{\mathrm{e}\gamma_\mathrm{e}(1)}$ and $b_{\mathrm{e}\gamma_\mathrm{e}(1)}$ (Eqs. (28)) extend respectively to the left and right boundary of the double layer. The electron eigenfunctions are thus defined in the following intervals

$$\begin{aligned}&\text{if } \gamma_\mathrm{e} > 0 \text{ then}\\ &-\infty = a_{\mathrm{e}\gamma_\mathrm{e}(1)} < x < b_{\mathrm{e}\gamma_\mathrm{e}(1)} = \infty,\ x_{\mathrm{e}\gamma_\mathrm{e}(1)} = 0,\end{aligned} \tag{28}$$

$$\begin{aligned}&\text{if } x < 0 \text{ and } -|Z_\mathrm{e}|U \le \gamma_\mathrm{e} < 0 \text{ then}\\ &-\infty = a_{\mathrm{e}\gamma_\mathrm{e}(1)} < x < b_{\mathrm{e}\gamma_\mathrm{e}(1)} = x_{\mathrm{e}\gamma_\mathrm{e}(1)} < 0,\end{aligned} \tag{29}$$

$$\begin{aligned}&\text{if } x > 0 \text{ and } -|Z_\mathrm{e}| \le \gamma_\mathrm{e} < 0 \text{ then}\\ &0 < x_{\mathrm{e}\gamma_\mathrm{e}(2)} = a_{\mathrm{e}\gamma_\mathrm{e}(2)} < x < b_{\mathrm{e}\gamma_\mathrm{e}(2)} = \infty.\end{aligned} \tag{30}$$

The lower γ_e bounds in Eqs. (29) and (30) are given by the requirement that, in each x-interval, $\gamma_\mathrm{e} + V_\mathrm{e} > 0$, i.e. $\gamma_\mathrm{e} \ge \min(-V_\mathrm{e})$ (Eqs. (1) and (2) and Fig. 1). The choice of the integration bound $x_{\mathrm{e}\gamma_\mathrm{e}}$, positioned at one of the electron reflection points, is so made that, as $\gamma_\mathrm{e} \to 0^-$, $x_{\mathrm{e}\gamma_\mathrm{e}(1)} = b_{\mathrm{e}\gamma_\mathrm{e}(1)}$ (Eq. (29)) and $x_{\mathrm{e}\gamma_\mathrm{e}(2)} = a_{\mathrm{e}\gamma_\mathrm{e}(2)}$ (Eq. (30)) coalesce at the position of the maximum steady state electron potential energy ($x = 0$, Fig. 1 and Appendix B for details) at which $x_{\mathrm{e}\gamma_\mathrm{e}}$ is based for all positive γ_e's (Eq. (28)).

For $\gamma_\mathrm{i} < -Z_\mathrm{i}(1-U)$, ions move over a finite interval: thus the endpoints $a_{\mathrm{i}\gamma_\mathrm{i}(1)}$ (Eqs. (33) and (B13)) and $b_{\mathrm{i}\gamma_\mathrm{i}(1)}$ (Eqs. (33) and (B12)) correspond respectively to the left and right ion reflection points. For $-Z_\mathrm{i}(1-U) < \gamma_\mathrm{i} < 0$, ions overcome the steady state potential barrier $-V_\mathrm{i} = -Z_\mathrm{i}(1-U)$ (Eqs. (26) and (2) and Fig. 1): they thus move over a semi-infinite interval and the endpoint $a_{\mathrm{i}\gamma_\mathrm{i}(1)}$ extends to the left boundary of the double layer (Eq. (32)). For $\gamma_\mathrm{i} > 0$, ions move over an infinite interval and $b_{\mathrm{i}\gamma_\mathrm{i}(1)}$ extends to the right boundary of the double layer (Eq. (31)). In all acases, the ion eigenfunctions are defined in one single interval:

$$\begin{aligned}&\text{if } \gamma_\mathrm{i} > 0 \text{ then}\\ &-\infty = a_{\mathrm{i}\gamma_\mathrm{i}(1)} < x < b_{\mathrm{i}\gamma_\mathrm{i}(1)} = \infty,\ x_{\mathrm{i}\gamma_\mathrm{i}(1)} = \infty,\end{aligned} \tag{31}$$

$$\begin{aligned}&\text{if } -Z_\mathrm{i}(1-U) \le \gamma_\mathrm{i} < 0 \text{ then}\\ &-\infty = a_{\mathrm{i}\gamma_\mathrm{i}(1)} < x < b_{\mathrm{i}\gamma_\mathrm{i}(1)} = x_{\mathrm{i}\gamma_\mathrm{i}(1)} < \infty,\end{aligned} \tag{32}$$

$$\begin{aligned}&\text{if } -Z_\mathrm{i} \le \gamma_\mathrm{i} < -Z_\mathrm{i}(1-U) \text{ then}\\ &-\infty < a_{\mathrm{i}\gamma_\mathrm{i}(1)} < x < b_{\mathrm{i}\gamma_\mathrm{i}(1)} = x_{\mathrm{i}\gamma_\mathrm{i}(1)} < \infty.\end{aligned} \tag{33}$$

The lower γ_i bound in Eq. (33) is given by the requirement that $\gamma_\mathrm{i} + V_\mathrm{i} > 0$, i.e. (Eqs. (1), (2) and Fig. 1) $\gamma_\mathrm{i} \ge \min(-V_\mathrm{i}) = -Z_\mathrm{i}$. The choice of the integration bound $x_{\mathrm{i}\gamma_\mathrm{i}}$, positioned at the ion reflection point, is so made that, as $\gamma_\mathrm{i} \to 0^-$, $x_{\mathrm{i}\gamma_\mathrm{i}(1)} = b_{\mathrm{i}\gamma_\mathrm{i}(1)}$ (Eq. (32)) approaches ∞, the position of the maximum steady state ion potential energy at which $x_{\mathrm{i}\gamma_\mathrm{i}(1)}$ is based for all positive γ_i's (Eq. (31)).

The above analysis of the reflection points induces a distinction of the electron eigenfunctions, which we make by a further label $\nu_{\mathrm{e}\gamma_\mathrm{e}}$, indicating the x-interval in which the eigenfunctions is defined:

$$\text{if } x < 0 \text{ then } \nu_{\mathrm{e}\gamma_\mathrm{e}} = 1, \tag{34}$$

$$\text{if } x > 0 \text{ then } \nu_{\mathrm{e}\gamma_\mathrm{e}} = 2. \tag{35}$$

To maintain uniqueness of notation, a label $\nu_{\mathrm{i}\gamma_\mathrm{i}}$ will be also introduced for the ion eigenfunctions, although, for the steady state potential profile of Eq. (26), it will take one only value. We set

$$\text{given } \alpha,\ \gamma_\alpha,\ \nu_{\alpha\gamma_\alpha} = 1\ldots N_{\alpha\gamma_\alpha},\ \nu'_{\alpha\gamma_\alpha} = 1\ldots N_{\alpha\gamma_\alpha}, \tag{36}$$

$$\text{if } \nu_{\alpha\gamma_\alpha} \ne \nu'_{\alpha\gamma_\alpha} \text{ then } \chi^{(0)s_\alpha}_{\alpha\sigma\gamma_\alpha(\nu_{\alpha\gamma_\alpha})} \ne \chi^{(0)s_\alpha}_{\alpha\sigma\gamma_\alpha(\nu'_{\alpha\gamma_\alpha})}, \tag{37}$$

where

$$\text{if} x < 0 \text{ and } -|Z_\mathrm{e}|U < \gamma_\mathrm{e} < 0 \text{ then } N_{\mathrm{e}\gamma_\mathrm{e}} = 2, \tag{38}$$
$$\text{if} x > 0 \text{ and } -|Z_\mathrm{e}| < \gamma_\mathrm{e} < 0 \text{ then } N_{\mathrm{e}\gamma_\mathrm{e}} = 2, \tag{39}$$
$$\text{if } \gamma_\mathrm{e} > 0 \text{ then } N_{\mathrm{e}\gamma_\mathrm{e}} = 1, \tag{40}$$
$$\text{if } \gamma_\mathrm{i} > -Z_\mathrm{i} \text{ then } N_{\mathrm{i}\gamma_\mathrm{i}} = 1. \tag{41}$$

In the following, the unperturbed eigenfunctions will be set to zero outside the intervals where they are defined. This leads to the two equivalent conditions:

$$\text{given } \gamma_\alpha > -|Z_\alpha| \text{ and } x \notin (a_{\alpha\gamma_\alpha(\nu_{\alpha\gamma_\alpha})}, b_{\alpha\gamma_\alpha(\nu_{\alpha\gamma_\alpha})}) \text{ then}$$
$$\chi^{(0)s_\alpha}_{\alpha\sigma\gamma_\alpha(\nu_{\alpha\gamma_\alpha})} = 0, \tag{42}$$
$$\text{given } x \in (a_{\alpha\gamma_\alpha(\nu_{\alpha\gamma_\alpha})}, b_{\alpha\gamma_\alpha(\nu_{\alpha\gamma_\alpha})}) \; x \text{ and } \gamma_\alpha < -V_\alpha(x) \text{ then}$$
$$\chi^{(0)s_\alpha}_{\alpha\sigma\gamma_\alpha(\nu_{\alpha\gamma_\alpha})} = 0. \tag{43}$$

For steady states endowed with asymmetric potential profiles, such as in Eq. (26), the above introduced distinction of the electron eigenfunctions is morphologically motivated: there isn't any way to relate the two eigenfunctions by symmetry considerations as, e.g., for bell-shaped solitary wave seady state potential profiles (in which the eigenfunctions are related by reflection-symmetry), or periodic steady state potential profiles (in which the eigenfunctions are related by translation-symmetry and collated in Bloch form).

That difference is also physically well grounded, and in fact useful to analyze situations in which perturbations of the electron distribution function are confined to one particular interval. These arise, e.g., when a low energy electron perturbing population is injected at only one plasma end it cannot overcome the potential barrier set by the potential of Eq. (26) at $x = 0$.

With the limitations analyzed in Eqs. (28)-(33), the phase of the eigenfunctions $\chi^{(0)s_\alpha}_{\alpha\sigma\gamma_\alpha}$ (Eq. (22)) is real and, being the normalization constants also real (Eqs. (68)-(70) and (73) below), the following properties are verified by inspection:

$$\bar{\chi}^{(0)s_\alpha}_{\alpha\sigma\gamma_\alpha}(x,q) = \chi^{(0)-s_\alpha}_{\alpha\sigma\gamma_\alpha}(x,q) = \chi^{(0)s_\alpha}_{\alpha(-\sigma)\gamma_\alpha}(x,-q), \tag{44}$$

One last property of the eigenfunctions arises, for $\gamma_\alpha < 0$, near a reflection point $a_{\alpha\gamma_\alpha}$ (Eqs. (30) and (33)) because, for $x \simeq a_{\alpha\gamma_\alpha}$, $B_{\alpha\gamma_\alpha}(x) \simeq 2s_\alpha\sqrt{(x - a_{\alpha\gamma_\alpha})}/\beta$ (β being a suitable constant, Eq. (23)) and thus, developing $\xi_{\alpha\gamma_\alpha}$ in Eq. (24), Eq. (22) gives

$$\text{if } x \simeq a_{\alpha\gamma_\alpha} \text{ then } \chi^{(0)s_\alpha}_{\alpha\sigma\gamma_\alpha} \simeq C_{\alpha\gamma_\alpha}\frac{s_\alpha - \mathrm{i}\sin(\sigma\beta\sqrt{[x - a_{\alpha\gamma_\alpha}]})}{2\sqrt{(x - a_{\alpha\gamma_\alpha})/\beta}}. \tag{45}$$

A similar expression holds when x approaches a reflection point $b_{\alpha\gamma_\alpha}$ (Eqs. (29), (32) and (33)).

Beside the particle distribution eigenfunctions $\chi^{(0)s_\alpha}_{\alpha\sigma\gamma_\alpha}$, in the following treatment we shall also need the electric field eigenfunctions (Eq. (18))

$$\psi^{(0)s_\alpha}_{\alpha\sigma\gamma_\alpha}(x) = D_x^{-1}P_0\chi^{(0)s_\alpha}_{\alpha\sigma\gamma_\alpha} =$$
$$C_{\alpha\gamma_\alpha}\int \mathrm{d}x'\frac{e^{-\mathrm{i}s_\alpha\sigma\xi_{\alpha\gamma_\alpha}(x')}}{|B_{\alpha\gamma_\alpha}(x')|} = \frac{C_{\alpha\gamma_\alpha}e^{-\mathrm{i}s_\alpha\sigma\xi_{\alpha\gamma_\alpha}(x)}}{-\mathrm{i}s_\alpha\sigma}, \tag{46}$$

where we used the change of integration variable

$$t = \xi_{\alpha\gamma_\alpha}(x) = \int_{x_{\alpha\gamma_\alpha}}^{x}\frac{\mathrm{d}x'}{|B_{\alpha\gamma_\alpha}(x')|}. \tag{47}$$

In the following treatment, $\psi^{(0)s_\alpha}_{\alpha\sigma\gamma_\alpha}$ will not be needed outside the x-intervals specified in Eqs. (28)-(33) and it can be set to zero there:

$$\text{given } \gamma_\alpha > -|Z_\alpha| \text{ and } x \notin (a_{\alpha\gamma_\alpha(\nu_{\alpha\gamma_\alpha})}, b_{\alpha\gamma_\alpha(\nu_{\alpha\gamma_\alpha})}) \text{ then}$$
$$\psi^{(0)s_\alpha}_{\alpha\sigma\gamma_\alpha(\nu_{\alpha\gamma_\alpha})} = 0, \tag{48}$$
$$\text{given } x \in (a_{\alpha\gamma_\alpha(\nu_{\alpha\gamma_\alpha})}, b_{\alpha\gamma_\alpha(\nu_{\alpha\gamma_\alpha})}) \; x \text{ and } \gamma_\alpha < -V_\alpha(x) \text{ then}$$
$$\psi^{(0)s_\alpha}_{\alpha\sigma\gamma_\alpha(\nu_{\alpha\gamma_\alpha})} = 0. \tag{49}$$

Properties analogous to Eqs. (37), (44) and (45) also hold for the electric field eigenfunctions $\psi^{(0)s_\alpha}_{\alpha\sigma\gamma_\alpha}$ (Eq. (46)):

$$\text{if } \nu_{\alpha\gamma_\alpha} \neq \nu'_{\alpha\gamma_\alpha} \text{ then } \psi^{(0)s_\alpha}_{\alpha\sigma\gamma_\alpha(\nu_{\alpha\gamma_\alpha})} \neq \chi^{(0)s_\alpha}_{\alpha\sigma\gamma_\alpha(\nu'_{\alpha\gamma_\alpha})}. \tag{50}$$
$$\bar{\psi}^{(0)s_\alpha}_{\alpha\sigma\gamma_\alpha}(x,q) = \psi^{(0)-s_\alpha}_{\alpha\sigma\gamma_\alpha}(x,q) = \psi^{(0)s_\alpha}_{\alpha(-\sigma)\gamma_\alpha}(x,-q), \tag{51}$$

and

$$\text{if } x \simeq a_{\alpha\gamma_\alpha} \text{ then } \psi^{(0)s_\alpha}_{\alpha\sigma\gamma_\alpha} \simeq C_{\alpha\gamma_\alpha}\frac{\mathrm{i}s_\alpha + \sin(\sigma\beta\sqrt{[x - a_{\alpha\gamma_\alpha}]})}{\sigma}. \tag{52}$$

In conclusion, the pair $\chi^{(0)s_\alpha}_{\alpha\sigma\gamma_\alpha}, \psi^{(0)s_\alpha}_{\alpha\sigma\gamma_\alpha}$ (Eqs. (22)-(24) and (46)), defined in the x-intervals specified in Eqs. (28)-(33), and vanishing otherwise (Eqs. (42), (43) and (48)), is the sought solution of the unperturbed Vlasov-Poisson problem (Eqs. (15) and (19)). Also, when x takes large values, the steady state potential Φ (Eq. (26)) approaches a constant value, the plasma becomes homogeneous, $\xi_{\alpha\gamma_\alpha}$ (Eq. (24)) is approximately proportional to (const. + x) and thus $\chi^{(0)s_\alpha}_{\alpha\sigma\gamma_\alpha}$ (Eq. (22)) and $\psi^{(0)s_\alpha}_{\alpha\sigma\gamma_\alpha}$ (Eq. (46)) meet the boundary conditions of Eq. (20).

According to the above considerations, the real number γ_α is a continuous degeneracy parameter; the species label α (Eqs. (16) and (17)), the phase sign s_α (Eq. (21)) and, for $\alpha = \mathrm{e}$, $\gamma_\mathrm{e} < 0$, the electron domain label $\nu_{\mathrm{e}\gamma_\mathrm{e}}$ (Eq. (36)) are discrete degeneracy parameters; two eigenfunctions having a different value of any of these parameters are solutions of Eqs. (15) and (19) corresponding to the same eigenvalue σ.

IV. ORTHOGONALITY OF THE UNPERTURBED EIGENFUBCTIONS

To analyze the orthogonality of two vectors $|f_{\alpha\gamma_\alpha}\rangle = [f_{\alpha\gamma_\alpha 1}(x,q), f_{\alpha\gamma_\alpha 2}(x,q)]^\mathrm{T}$ and $|f_{\beta\gamma'_\beta}\rangle = [f_{\beta\gamma'_\beta 1}(x,q), f_{\beta\gamma'_\beta 2}(x,q)]^\mathrm{T}$ such that

$$\text{if } x \notin (a_{\alpha\gamma_\alpha}, b_{\alpha\gamma_\alpha}) \text{ then } f_{\alpha\gamma_\alpha} = 0, \tag{53}$$
$$\text{if } x \notin (a_{\beta\gamma'_\beta}, b_{\beta\gamma'_\beta}) \text{ then } f_{\beta\gamma'_\beta} = 0, \tag{54}$$

we introduce their scalar product

$$\langle f_{\alpha\gamma_\alpha}|f_{\beta\gamma'_\beta}\rangle =$$
$$\int_{a_{\alpha\beta\gamma_\alpha\gamma'_\beta}}^{b_{\alpha\beta\gamma_\alpha\gamma'_\beta}}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}q(f_{\alpha\gamma_\alpha}, f_{\beta\gamma'_\beta}), \tag{55}$$
$$(f_{\alpha\gamma_\alpha}, f_{\beta\gamma'_\beta}) = \Re(f_{\alpha\gamma_\alpha 1}\bar{f}_{\beta\gamma'_\beta 1} + f_{\alpha\gamma_\alpha 2}\bar{f}_{\beta\gamma'_\beta 2}), \tag{56}$$

where $\Re$ and the overbar give the real part and complex conjugation and the x-integration bounds

$$a_{\alpha\beta\gamma_\alpha\gamma'_\beta} = \max(a_{\alpha\gamma_\alpha}, a_{\beta\gamma'_\beta}), \tag{57}$$
$$b_{\alpha\beta\gamma_\alpha\gamma'_\beta} = \min(b_{\alpha\gamma_\alpha}, b_{\beta\gamma'_\beta}). \tag{58}$$

delimit x-integration to the interval where the integrand does not identically vanish. The actual values of these bounds will be given in Appendix C

Because of the definition of the vectors $|\chi^{(0)s_\alpha}_{\alpha\sigma\gamma_\alpha}\rangle$ (Eqs. (16) and (17)), the quantity $\langle\chi^{(0)s_\alpha}_{\alpha\sigma\gamma_\alpha}|\chi^{s'_\beta}_{\beta\sigma'\gamma'_\beta}\rangle$ vanishes if $\beta \neq \alpha$. Also, substituting Eq. (22) into Eq. (55), the part involving q-integration

$$\int_{-\infty}^{\infty} \mathrm{d}q e^{\mathrm{i}q[s_\alpha|B_{\alpha\gamma_\alpha}(x)|-s'_\alpha|B_{\alpha\gamma'_\alpha}(x)|]} = 2\pi\delta(s_\alpha|B_{\alpha\gamma_\alpha}(x)| - s'_\alpha|B_{\alpha\gamma'_\alpha}(x)|) \tag{59}$$

vanishes if $s_\alpha|B_{\alpha\gamma_\alpha}(x)| \neq s'_\alpha|B_{\alpha\gamma'_\alpha}(x)|$, which certainly occurs when the phase signs s_α and s'_α (Eq. (21)) are different.

Furthermore, since any of the electron eigenfunctions $\chi^{(0)s_\mathrm{e}}_{\mathrm{e}\sigma\gamma_\mathrm{e}(1)}$, defined in domain 1, vanishes in domain 2 (Eqs. (29), (42) and (37)), it is orthogonal to any of the eigenfunctions $\chi^{(0)s_\mathrm{e}}_{\mathrm{e}\sigma'\gamma'_\mathrm{e}(2)}$ defined in domain 2, which in turn vanishes in domain 1 (Eqs. (30), (42) and (37)).

On the other hand, when $\beta = \alpha$ and $s'_\alpha = s_\alpha$, Eqs. (55) and (59) give

$$\langle\chi^{(0)s_\alpha}_{\alpha\sigma\gamma_\alpha}|\chi^{(0)s_\alpha}_{\alpha\sigma'\gamma'_\alpha}\rangle = 2\pi\mu_\alpha|C_{\alpha\gamma_\alpha}|^2\Re\int_{\eta_{\alpha\gamma_\alpha}}^{\zeta_{\alpha\gamma_\alpha}} \mathrm{d}te^{-\mathrm{i}s_\alpha(\sigma-\sigma')t}\delta(\gamma_\alpha - \gamma'_\alpha), \tag{60}$$

where we used the change of integration variable given in Eq. (47) and, according to Eqs. (28)-(42), (57) and (58), the t-integration limits $\eta_{\alpha\gamma_\alpha} = \xi_{\alpha\gamma_\alpha}(a_{\alpha\gamma_\alpha})$ and $\zeta_{\alpha\gamma_\alpha} = \xi_{\alpha\gamma_\alpha}(b_{\alpha\gamma_\alpha})$ are, for the electron eigenfunctions,

$$\text{if } \gamma_\mathrm{e} > 0 \text{ then } \eta_{\mathrm{e}\gamma_\mathrm{e}} = -\infty,\ \zeta_{\mathrm{e}\gamma_\mathrm{e}} = \infty, \tag{61}$$

$$\text{if } -|Z_\mathrm{e}| \le \gamma_\mathrm{e} < 0 \text{ and } x < b_{\mathrm{e}\gamma_\mathrm{e}(1)} \text{ then } \eta_{\mathrm{e}\gamma_\mathrm{e}} = -\infty,\ \zeta_{\mathrm{e}\gamma_\mathrm{e}} = 0, \tag{62}$$

$$\text{if } -|Z_\mathrm{e}| \le \gamma_\mathrm{e} < 0 \text{ and } x > a_{\mathrm{e}\gamma_\mathrm{e}(2)} \text{ then } \eta_{\mathrm{e}\gamma_\mathrm{e}} = 0,\ \zeta_{\mathrm{e}\gamma_\mathrm{e}} = \infty \tag{63}$$

and, for the ion eigenfunctions,

$$\text{if } \gamma_\mathrm{i} > 0 \text{ then } \eta_{\mathrm{i}\gamma_\mathrm{i}(1)} = -\infty,\ \zeta_{\mathrm{i}\gamma_\mathrm{i}(1)} = 0, \tag{64}$$

$$\text{if } -Z_\mathrm{i}(1-U) \le \gamma_\mathrm{i} < 0 \text{ then } \eta_{\mathrm{i}\gamma_\mathrm{i}(1)} = -\infty,\ \zeta_{\mathrm{i}\gamma_\mathrm{i}(1)} = 0, \tag{65}$$

$$\text{if } -Z_\mathrm{i} \le \gamma_\mathrm{i} < -Z_\mathrm{i}(1-U) \text{ then } \eta_{\mathrm{i}\gamma_\mathrm{i}(1)} = -T_{\mathrm{i}\gamma_\mathrm{i}},\ \zeta_{\mathrm{i}\gamma_\mathrm{i}(1)} = 0, \tag{66}$$

where

$$T_{\mathrm{i}\gamma_\mathrm{i}} = \int_{a_{\mathrm{i}\gamma_\mathrm{i}(1)}}^{b_{\mathrm{i}\gamma_\mathrm{i}(1)}} \mathrm{d}x/|b_{\mathrm{i}\gamma_\mathrm{i}(1)}(x)| < \infty. \tag{67}$$

We thus see that, when the t-integration interval in Eq. (60) is infinite (Eqs. (61)), the eigenfunctions may be made orthonormal for any real value of σ and σ' (continuous spectrum) by setting

$$\text{if } \gamma_\alpha > 0 \text{ then } C_{\alpha\gamma_\alpha} = 1/(2\pi\sqrt{\mu_\alpha}). \tag{68}$$

A continuous spectrum also arises when the t-integration interval is semi-infinite (Eqs. (62)-(65)). In this case orthonormality is ensured, provided

$$\text{if } -|Z_\mathrm{e}| \le \gamma_\mathrm{e} < 0 \text{ then } C_{\mathrm{e}\gamma_\mathrm{e}} = 1/(\sqrt{2}\pi\sqrt{\mu_\mathrm{e}}), \tag{69}$$

$$\text{if } -Z_\mathrm{i}(1-U) \le \gamma_\mathrm{i} < 0 \text{ then } C_{\mathrm{i}\gamma_\mathrm{i}} = 1/(\sqrt{2}\pi\sqrt{\mu_\mathrm{i}}). \tag{70}$$

Last, when the t-integration interval is finite (Eq. (66)) a discrete ion spectrum

$$\text{if } -Z_\mathrm{i} \le \gamma_\mathrm{i} < -Z_\mathrm{i}(1-U) \text{ then} \tag{71}$$

$$\sigma = \sigma_{\mathrm{i}\gamma_\mathrm{i}m} = 2\pi m/T_{\mathrm{i}\gamma_\mathrm{i}},\ \sigma' = \sigma_{\mathrm{i}\gamma_\mathrm{i}m'} = 2\pi m'/T_{\mathrm{i}\gamma_\mathrm{i}} \tag{72}$$

appears, where the mode numbers m, m' are integers. Orthonormality in Eq. (60) is thus attained by setting

$$\text{if } -Z_\mathrm{i} \le \gamma_\mathrm{i} < -Z_\mathrm{i}(1-U) \text{ then } C_{\mathrm{i}\gamma_\mathrm{i}} = 1/\sqrt{(2\pi\mu_\mathrm{i}T_{\mathrm{i}\gamma_\mathrm{i}})}. \tag{73}$$

In conclusion, the constants given in Eqs. (68)-(73) ensure that the orthonormality relation

$$\langle\chi^{(0)s_\alpha}_{\alpha\sigma\gamma_\alpha}|\chi^{(0)s'_\beta}_{\beta\sigma'\gamma'_\beta}\rangle = \delta_{\alpha\beta}\delta_{\sigma\sigma'}\delta_{\gamma_\alpha\gamma'_\beta}\delta_{s_\alpha s'_\beta} \tag{74}$$

holds, where $\delta_{aa'}$ is Kronecker's symbol if a, a' belong to a discrete set, and Dirac's $\delta(a - a')$ function if they belong to a continuous set.

Figs. 2 and 3 depict the eigenfunctions respectively corresponding to free and reflected electrons (Eqs. (28) and (29),(30)) and to free, reflected and trapped ions (Eqs. (31), (32) and (33)). They are plotted by inserting the model potential (Eq. (26)) into Eq. (2), and eventually into Eq. (22) and (60), through quadrature of Eq. (24), and by adopting the values of the normalization constants given in Eqs. (68)-(73).

V. THE PERTURBED EIGENVALUE PROBLEM FOR THE MULTIDOMAIN DOUBLE LAYER

We seek the eigenfunctions of the perturbed eigenvalue problem of Eq. (14) by an expansion in terms of the unperturbed eigenfunctions $\chi^{(0)}_{\alpha\sigma\gamma_\alpha}$ (Eqs. (22)-(24) and (68)-(73)) extended over their spectrum and over all of their possible degeneracy parameters. Taking into account Eq. (43), we write

$$|\chi_{\alpha\omega}\rangle = \sum_{\beta=\mathrm{e,i}}\int_{-V_\beta}^{\infty} \mathrm{d}\gamma'_\beta \sum_{\nu'_\beta=1}^{N_{\beta\gamma'_\beta}}\sum_{s_\beta=\pm}\sum_{\sigma'} X^{s_\beta}_{\beta(\nu'_\beta)}(\omega, \sigma', \gamma'_\beta)|\chi^{(0)s_\beta}_{\beta\sigma'\gamma'_\beta(\nu'_\beta)}\rangle, \tag{75}$$

where $X^{s_\beta}_{\beta(\nu'_\beta)}(\omega, \sigma', \gamma'_\beta)$ is a suitable coefficient, ν'_β is the coordinate domain label, and $N_{\beta\gamma'_\beta}$ was defined in Eqs. (38)-(41).

In Eq. (75), the integration over γ'_β includes the values of γ'_β specified in Eqs. (61)-(65) pertaining to the continuous spectrum: in this case, the sum over σ' continuously extends over the whole real axis. Furthermore, if x is such that $-V_\mathrm{i}(x) < -Z_\mathrm{i}(1-U)$ (i.e. $\Phi(x) < U$, Eq. (2)), then the integration over γ'_i extending from $-V_\mathrm{i}$ to $-Z_\mathrm{i}(1-U)$ pertains to the discrete ion spectrum (Eq. (66)): in this case the sum over σ' extends over that spectrum (Eq. (72)).

Next, we substitute Eq. (75) into Eq. (14) and we take the scalar product by $\langle\chi^{(0)s_\alpha}_{\alpha\sigma\gamma_\alpha(\nu_{\alpha\gamma_\alpha})}|$ according to Eqs. (55) and (56). Since the $|\chi^{(0)s_\alpha}_{\alpha\sigma\gamma_\alpha(\nu_{\alpha\gamma_\alpha})}\rangle$'s are orthonormal, we have

$$(\omega-\sigma)X^{s_\alpha}_{\alpha(\nu_{\alpha\gamma_\alpha})}(\omega,\sigma,\gamma_\alpha)= \langle\chi^{(0)s_\alpha}_{\alpha\sigma\gamma_\alpha(\nu_{\alpha\gamma_\alpha})}|\sum_{\beta=\mathrm{e,i}}\int_{-V_\beta}^{\infty}\mathrm{d}\gamma'_\beta\sum_{\nu'_\beta=1}^{N_{\beta\gamma'_\beta}}|\Psi^{(0)}_{\beta\gamma'_\beta(\nu'_\beta)}\rangle, \tag{76}$$

where (Eqs. (10), (11) and (46))

$$|\Psi^{(0)}_{\beta\gamma'_\beta(\nu'_\beta)}\rangle=\sum_{s_\beta=\pm}\sum_{\sigma'}X^{s_\beta}_{\beta(\nu'_\beta)}(\omega,\sigma',\gamma'_\beta)H|\psi^{(0)s_\beta}_{\beta\sigma'\gamma'_\beta(\nu'_\beta)}\rangle. \tag{77}$$

Now, $\chi^{(0)s_\alpha}_{\alpha\sigma\gamma_\alpha(\nu_{\alpha\gamma_\alpha})}$ vanishes outside the interval $(a_{\alpha\gamma_\alpha(\nu_{\alpha\gamma_\alpha})},b_{\alpha\gamma_\alpha(\nu_{\alpha\gamma_\alpha})})$ (Eq. (42)) and thus Eq. (76) reads

$$(\omega-\sigma)X^{s_\alpha}_{\alpha(\nu_{\alpha\gamma_\alpha})}(\omega,\sigma,\gamma_\alpha)= \sum_{\beta=\mathrm{e,i}}\int_{a_{\alpha\gamma_\alpha(\nu_{\alpha\gamma_\alpha})}}^{a_{\alpha\gamma_\alpha(\nu_{\alpha\gamma_\alpha})}}\mathrm{d}x\int_{-V_\beta(x)}^{\infty}\mathrm{d}\gamma'_\beta \sum_{\nu'_\beta=1}^{N_{\beta\gamma'_\beta}}h_{\alpha\beta(\nu_{\alpha\gamma_\alpha};\nu'_\beta)}(x,\sigma,\gamma_\alpha,\gamma'_\beta), \tag{78}$$

where, according to Eqs. (55) and (56), we set

$$h_{\alpha\beta(\nu_{\alpha\gamma_\alpha};\nu'_\beta)}(x,\omega,\sigma,\gamma_\alpha,\gamma'_\beta)= \int_{-\infty}^{\infty}\mathrm{d}q(\chi^{(0)s_\alpha}_{\alpha\sigma\gamma_\alpha(\nu_{\alpha\gamma_\alpha})}(x,q),\Psi^{(0)}_{\beta\gamma'_\beta(\nu'_\beta)}(\omega,x,q)). \tag{79}$$

Eq. (78) will now be developed in detail for electrons ($\alpha=\mathrm{e}$) in domain 1 ($\nu_{\mathrm{e}\gamma_\mathrm{e}}=1$) and for $\gamma_\mathrm{e}<0$:

$$(\omega-\sigma)X^{s_\mathrm{e}}_{\mathrm{e}(1)}(\omega,\sigma,\gamma_\mathrm{e})=\sum_{\beta=\mathrm{e,i}}H_{\mathrm{e}\beta(1)}(\omega,\sigma,\gamma_\mathrm{e}), \tag{80}$$

where

$$H_{\mathrm{ee}(1)}(\omega,\sigma,\gamma_\mathrm{e})=\int_{a_{\mathrm{e}\gamma_\mathrm{e}(1)}}^{b_{\mathrm{e}\gamma_\mathrm{e}(1)}}\mathrm{d}x\int_{-V_\mathrm{e}(x)}^{\infty}\mathrm{d}\gamma'_\mathrm{e}h_{\mathrm{ee}(1;1)}(x,\omega,\sigma,\gamma_\mathrm{e},\gamma'_\mathrm{e}), \tag{81}$$

$$H_{\mathrm{ei}(1)}(\omega,\sigma,\gamma_\mathrm{e})=\int_{a_{\mathrm{e}\gamma_\mathrm{e}(1)}}^{b_{\mathrm{e}\gamma_\mathrm{e}(1)}}\mathrm{d}x\int_{-V_\mathrm{i}(x)}^{\infty}\mathrm{d}\gamma'_\mathrm{i}h_{\mathrm{ei}(1;1)}(x,\omega,\sigma,\gamma_\mathrm{e},\gamma'_\mathrm{i}). \tag{82}$$

In the integral extending over negative γ'_e, we omitted the contribution of the vanishing quantity $h_{\mathrm{ee}(1;2)}$ (Eqs. (79) and (D2)). in that extending over positive γ'_e, we omitted the sum over the electron domain label ν'_e because, for $\gamma'_\mathrm{e}>0$, $N_{\mathrm{e}\gamma'_\mathrm{e}}=1$ (Eq. (40)). We also omitted the sum over the ion domain label $\nu_{\mathrm{i}\gamma'_\mathrm{i}}$ because $N_{\mathrm{i}\gamma'_\mathrm{i}}=1$ (Eq. (41)).

Taking into account that, in domain 1 (Eq. (29) and Fig. 1),

$$-V'_\mathrm{e}>0, \tag{83}$$
$$a_{\mathrm{e}\gamma_\mathrm{e}(1)}=-\infty,\ -V_\mathrm{e}(a_{\mathrm{e}\gamma_\mathrm{e}(1)})=-|Z_\mathrm{e}|U, \tag{84}$$
$$\text{if }-|Z_\mathrm{e}|U<\gamma_\mathrm{e}<0\text{ then }-V_\mathrm{e}(b_{\mathrm{e}\gamma_\mathrm{e}(1)})=\gamma_\mathrm{e}, \tag{85}$$
$$\text{if }-|Z_\mathrm{e}|U<\gamma'_\mathrm{e}<0\text{ then }[-V_\mathrm{e}]^{-1}(\gamma'_\mathrm{e})=b_{\mathrm{e}\gamma'_\mathrm{e}(1)}, \tag{86}$$

and inverting the integration order in Eq. (81) according to Eq. (E4), which applies when $-V_\mathrm{e}$ monotonically increases (Eq. (83)), and to Eqs. (84)-(86), we have

$$\text{if }\gamma_\mathrm{e}<0\text{ then }H_{\mathrm{ee}(1)}(\omega,\sigma,\gamma_\mathrm{e})= \int_{-|Z_\mathrm{e}|U}^{\gamma_\mathrm{e}}\mathrm{d}\gamma'_\mathrm{e}\int_{a_{\mathrm{e}\gamma_\mathrm{e}(1)}}^{b_{\mathrm{e}\gamma'_\mathrm{e}(1)}}\mathrm{d}xh_{\mathrm{ee}(1;1)}(x,\omega,\sigma,\gamma_\mathrm{e},\gamma'_\mathrm{e})+ \int_{\gamma_\mathrm{e}}^{0}\mathrm{d}\gamma'_\mathrm{e}\int_{a_{\mathrm{e}\gamma_\mathrm{e}(1)}}^{b_{\mathrm{e}\gamma_\mathrm{e}(1)}}\mathrm{d}xh_{\mathrm{ee}(1;1)}(x,\omega,\sigma,\gamma_\mathrm{e},\gamma'_\mathrm{e})+ \int_{0}^{\infty}\mathrm{d}\gamma'_\mathrm{e}\int_{a_{\mathrm{e}\gamma_\mathrm{e}(1)}}^{b_{\mathrm{e}\gamma_\mathrm{e}(1)}}\mathrm{d}xh_{\mathrm{ee}(1;1)}(x,\omega,\sigma,\gamma_\mathrm{e},\gamma'_\mathrm{e}). \tag{87}$$

In the integral extending over positive γ'_e, we omitted the sum over the electron domain label 1; because, for $\gamma'_\mathrm{e}>0$, $N_{\mathrm{e}\gamma'_\mathrm{e}}=1$ (Eq. (40)). Furthermore, in the first γ'_e-integral, we also extend integration from $\gamma'_\mathrm{e}=-|Z_\mathrm{e}|U$ down to $\gamma'_\mathrm{e}=-|Z_\mathrm{e}|<-|Z_\mathrm{e}|U$ (note that $U<1$, Eq. (27)): this does not change the value of that integral because $h_{\mathrm{ee}(1;1)}$ identically vanishes for $x<0$ and $\gamma'_\mathrm{e}<|Z_\mathrm{e}|U$ (Eqs. (79) and (D22)). Last, using the definitions of the x-integration bounds (Eqs. (57), (58) and (C4)), we rewrite Eq. (87) as

$$\text{if }\gamma_\mathrm{e}<0\text{ then }H_{\mathrm{ee}(1)}(\omega,\sigma,\gamma_\mathrm{e})= \int_{-|Z_\mathrm{e}|}^{0}\mathrm{d}\gamma'_\mathrm{e}\int_{a_{\mathrm{ee}\gamma_\mathrm{e}\gamma'_\mathrm{e}(1)}}^{b_{\mathrm{ee}\gamma_\mathrm{e}\gamma'_\mathrm{e}(1)}}\mathrm{d}xh_{\mathrm{ee}(1;1)}(x,\omega,\sigma,\gamma_\mathrm{e},\gamma'_\mathrm{e}). \tag{88}$$

In a similar way, taking into account that, in domain 1,

$$-V'_\mathrm{i}<0, \tag{89}$$
$$a_{\mathrm{e}\gamma_\mathrm{e}(1)}=-\infty,-V_\mathrm{i}(a_{\mathrm{e}\gamma_\mathrm{e}(1)})=-Z_\mathrm{i}(1-U), \tag{90}$$
$$\text{if }-|Z_\mathrm{e}|<\gamma_\mathrm{e}<0\text{ then }-V_\mathrm{i}(b_{\mathrm{e}\gamma_\mathrm{e}(1)})=\gamma^*_\mathrm{i}, \tag{91}$$
$$\text{if }-Z_\mathrm{i}<\gamma'_\mathrm{i}<Z_\mathrm{i}(1-U)\text{ then } [-V_\mathrm{i}]^{-1}(\gamma'_\mathrm{i})=a_{\mathrm{i}\gamma'_\mathrm{i}(1)}, \tag{92}$$

where

$$\gamma^*_\mathrm{i}=-Z_\mathrm{i}(1+\gamma_\mathrm{e}/|Z_\mathrm{e}|), \tag{93}$$

we invert the integration order in Eq. (82) according to Eq. (E7), which applies when $-V_\mathrm{i}$ monotonically decreases (Eq. (89)), and to Eqs. (90)-(92):

$$\text{if }\gamma_\mathrm{e}<0\text{ then }H_{\mathrm{ei}(1)}(\omega,\sigma,\gamma_\mathrm{e})= \int_{\gamma^*_\mathrm{i}}^{-Z_\mathrm{i}(1-U)}\mathrm{d}\gamma'_\mathrm{i}\int_{a_{\mathrm{i}\gamma'_\mathrm{i}(1)}}^{b_{\mathrm{e}\gamma_\mathrm{e}(1)}}\mathrm{d}xh_{\mathrm{ei}(1;1)}(x,\omega,\sigma,\gamma_\mathrm{e},\gamma'_\mathrm{i})+ \int_{-Z_\mathrm{i}(1-U)}^{\infty}\mathrm{d}\gamma'_\mathrm{i}\int_{a_{\mathrm{e}\gamma_\mathrm{e}(1)}}^{b_{\mathrm{e}\gamma_\mathrm{e}(1)}}\mathrm{d}xh_{\mathrm{ei}(1;1)}(x,\omega,\sigma,\gamma_\mathrm{e},\gamma'_\mathrm{i}). \tag{94}$$

Due to Eqs. (79) and (D9), the first γ'_i-integral of Eq. (94) remains unchanged if we replace its lower integration bound γ^*_i by $-Z_\mathrm{i}$ which, due to Eq. (D7), is certainly not larger than γ^*_i. In the second γ'_i-integral of Eq. (94), $a_{\mathrm{e}\gamma_\mathrm{e}(1)}=\infty$ (Eq. (29)) and so does $a_{\mathrm{i}\gamma'_\mathrm{i}(1)}$ because $\gamma'_\mathrm{i}>-Z_\mathrm{i}(1-U)$ (Eq. (31)). Thus $a_{\mathrm{e}\gamma_\mathrm{e}(1)}$ may well be replaced by $a_{\mathrm{i}\gamma'_\mathrm{i}(1)}$ in that integral. In turn $b_{\mathrm{e}\gamma_\mathrm{e}(1)}$ and $a_{\mathrm{i}\gamma'_\mathrm{i}(1)}$ may be renamed according to the definitions of the x-integration endpoints (Eq. (C5)), and Eq. (94) may be rewritten as

$$\text{if }\gamma_\mathrm{e}<0\text{ then }H_{\mathrm{ei}(1)}(\omega,\sigma,\gamma_\mathrm{e})= \int_{-|Z_\mathrm{i}|}^{\infty}\mathrm{d}\gamma'_\mathrm{i}\int_{a_{\mathrm{ei}\gamma_\mathrm{e}\gamma'_\mathrm{i}(1)}}^{b_{\mathrm{ei}\gamma_\mathrm{e}\gamma'_\mathrm{i}(1)}}\mathrm{d}xh_{\mathrm{ei}(1;1)}(x,\omega,\sigma,\gamma_\mathrm{e},\gamma'_\mathrm{i}). \tag{95}$$

Inserting Eqs. (88) and (95) into Eq. (80) we obtain

$$\text{if } \gamma_\mathrm{e} < 0 \text{ then } (\omega - \sigma) X^{s_\mathrm{e}}_{\mathrm{e}(1)}(\omega, \sigma, \gamma_\mathrm{e}) = \int_{-|Z_\mathrm{e}|}^{\infty} \mathrm{d}\gamma'_\mathrm{e} \int_{a_{\mathrm{ee}\gamma_\mathrm{e}\gamma'_\mathrm{e}(1)}}^{b_{\mathrm{ee}\gamma_\mathrm{e}\gamma'_\mathrm{e}(1)}} \mathrm{d}x h_{\mathrm{ee}(1;1)}(x, \omega, \sigma, \gamma_\mathrm{e}, \gamma'_\mathrm{e}) + \int_{-Z_\mathrm{i}}^{\infty} \mathrm{d}\gamma'_\mathrm{i} \int_{a_{\mathrm{ei}\gamma_\mathrm{e}\gamma'_\mathrm{i}(1)}}^{b_{\mathrm{ei}\gamma_\mathrm{e}\gamma'_\mathrm{i}(1)}} \mathrm{d}x h_{\mathrm{ei}(1;1)}(x, \omega, \sigma, \gamma_\mathrm{e}, \gamma'_\mathrm{i}), \tag{96}$$

Finally, reverting to the definitions of $h_{\mathrm{ee}(1;1)}$ and $h_{\mathrm{ei}(1;1)}$ (Eqs. (77) and (79)), we realize that Eq. (96) is a particular case (in which $\alpha = \mathrm{e}, \nu_{\mathrm{e}\gamma_\mathrm{e}} = 1$) of the general form involving only the basic quantites, i.e. the eigenfunctions $\chi^{(0)s_\alpha}_{\alpha\sigma\gamma_\alpha}$ and $\psi^{(0)s_\alpha}_{\alpha\sigma\gamma_\alpha}$:

$$(\omega - \sigma) X^{s_\alpha}_{\alpha(\nu_{\alpha\gamma_\alpha})}(\omega, \sigma, \gamma_\alpha) = \sum_{\beta=\mathrm{e,i}} \int_{-|Z_\beta|}^{\infty} \mathrm{d}\gamma'_\beta \sum_{s_\beta=\pm} \sum_{\sigma'} \sum_{\nu'_\beta=1}^{N_{\beta\gamma'_\beta}} G^{s_\alpha s_\beta}_{\alpha\beta(\nu_{\alpha\gamma_\alpha};\nu'_\beta)}(\sigma, \sigma', \gamma_\alpha, \gamma'_\beta) X^{s_\beta}_{\beta(\nu'_\beta)}(\omega, \sigma', \gamma'_\beta) \tag{97}$$

where

$$G^{s_\alpha s_\beta}_{\alpha\beta(\nu_{\alpha\gamma_\alpha};\nu'_\beta)}(\sigma, \sigma', \gamma_\alpha, \gamma'_\beta) = \langle \chi^{(0)s_\alpha}_{\alpha\sigma\gamma_\alpha(\nu_{\alpha\gamma_\alpha})} | H | \psi^{(0)s_\beta}_{\beta\sigma'\gamma'_\beta(\nu'_\beta)} \rangle, \tag{98}$$

and $\langle \ldots | \ldots \rangle$ precisely denotes the scalar product defined in Eqs. (55) and (56).

A similar procedure applies, in Eqs. (80)-(82), to the electron egenfunctions for negative γ_e in domain 2 (Appendix F), to the electron egenfunctions for positive γ_e (Appendix G) and to the ion eigenfunctions (Appendix H) an it also leads to Eqs. (97) and (98).

VI. THE INTEGRAL KERNELS

Taking into account the symmetry relations in Eq. (44) and that K in Eq. (10) is real, the following relations hold for the integral kernels (Eq. (98))

$$G^{--}_{\alpha\beta}(\sigma, \sigma', \gamma_\alpha, \gamma'_\beta) = G^{++}_{\alpha\beta}(\sigma, \sigma', \gamma_\alpha, \gamma'_\beta), \tag{99}$$

$$G^{+-}_{\alpha\beta}(\sigma, \sigma', \gamma_\alpha, \gamma'_\beta) = G^{-+}_{\alpha\beta}(\sigma, \sigma', \gamma_\alpha, \gamma'_\beta) = G^{++}_{\alpha\beta}(\sigma, -\sigma', \gamma_\alpha, \gamma'_\beta). \tag{100}$$

We now proceed to the calculation of the matrix elements defined in Eq. (98), starting with $G^{++}_{\mathrm{ee}}(\sigma, \sigma', \gamma_\mathrm{e}, \gamma'_\mathrm{e})$. Since we can write (Eqs. (22) and (46))

$$\chi^{(0)s_\alpha}_{\alpha\sigma\gamma_\alpha} = e^{\mathrm{i}s_\alpha q |B_{\alpha\gamma_\alpha}|}(-\mathrm{i}s_\alpha\sigma)\psi^{(0)s_\alpha}_{\alpha\sigma\gamma_\alpha}/|B_{\alpha\gamma_\alpha}|, \tag{101}$$

and (Eqs. (10), (22) and (18))

$$K|\chi^{(0)s_\mathrm{e}}_{\mathrm{e}\sigma'\gamma'_\mathrm{e}}\rangle = -q \begin{bmatrix} Z_\mathrm{e} F_\mathrm{e} \psi^{(0)s_\mathrm{e}}_{\mathrm{e}\sigma'\gamma'_\mathrm{e}}/\mu_\mathrm{e} \\ Z_\mathrm{i} F_\mathrm{i} \psi^{(0)s_\mathrm{i}}_{\mathrm{e}\sigma'\gamma'_\mathrm{e}}/\mu_\mathrm{i} \end{bmatrix}, \tag{102}$$

$$K|\chi^{(0)s_\mathrm{i}}_{\mathrm{i}\sigma'\gamma'_\mathrm{i}}\rangle = q \begin{bmatrix} Z_\mathrm{e} F_\mathrm{e} \psi^{(0)s_\mathrm{e}}_{\mathrm{i}\sigma'\gamma'_\mathrm{i}}/\mu_\mathrm{e} \\ Z_\mathrm{i} F_\mathrm{i} \psi^{(0)s_\mathrm{i}}_{\mathrm{i}\sigma'\gamma'_\mathrm{i}}/\mu_\mathrm{i} \end{bmatrix}, \tag{103}$$

Eq. (98) gives

$$G^{++}_{\mathrm{ee}}(\sigma, \sigma', \gamma_\mathrm{e}, \gamma'_\mathrm{e}) = -(Z_\mathrm{e}/\mu_\mathrm{e}) \times \Re \int_{a_{\mathrm{ee}\gamma_\mathrm{e}\gamma'_\mathrm{e}}}^{b_{\mathrm{ee}\gamma_\mathrm{e}\gamma'_\mathrm{e}}} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}q \bar{\chi}^{(0)+}_{\mathrm{e}\sigma\gamma_\mathrm{e}} q F_\mathrm{e}(q) \psi^{(0)+}_{\mathrm{e}\sigma'\gamma'_\mathrm{e}} = -(Z_\mathrm{e}/\mu_\mathrm{e}) \Re \left\{ \int_{a_{\mathrm{ee}\gamma_\mathrm{e}\gamma'_\mathrm{e}}}^{b_{\mathrm{ee}\gamma_\mathrm{e}\gamma'_\mathrm{e}}} \mathrm{d}x [(\mathrm{i}\sigma)\bar{\psi}^{(0)+}_{\mathrm{e}\sigma\gamma_\mathrm{e}} \psi^{(0)+}_{\mathrm{e}\sigma'\gamma'_\mathrm{e}}/|B_{\mathrm{e}\gamma_\mathrm{e}}|] \times \int_{-\infty}^{\infty} \mathrm{d}q e^{-\mathrm{i}q|B_{\mathrm{e}\gamma_\mathrm{e}}|} q F_\mathrm{e}(q) \right\}. \tag{104}$$

Taking into account that (Eq. (7)) F_α is the Fourier transform of the steady state distribution function $\tilde{F}_\alpha$ (in turn a function of the particle energy $w = \mu_\alpha v^2/2 - V_\alpha$), we have

$$\int_{-\infty}^{\infty} \mathrm{d}q e^{-\mathrm{i}s_\alpha q|B_{\alpha\gamma_\alpha}|} q F_\mathrm{e}(q) = 2\pi\mathrm{i}[\partial\tilde{F}_\alpha/\partial v]|_{v=s_\alpha|B_{\alpha\gamma_\alpha}|} = 2\pi\mathrm{i}[\mu_\alpha v \partial\tilde{F}_\alpha/\partial w]|_{v=s_\alpha|B_{\alpha\gamma_\alpha}|} = 2\pi\mathrm{i}s_\alpha|B_{\alpha\gamma_\alpha}|[\mu_\alpha \partial\tilde{F}_\alpha/\partial w]|_{v=s_\alpha|B_{\alpha\gamma_\alpha}|}. \tag{105}$$

Now, we set

$$\tilde{F}'_\alpha(w) = \partial\tilde{F}_\alpha(w)/\partial w = [\partial\tilde{F}_\alpha(\mu_\alpha v^2/2 + V_\alpha)/\partial v]/(\mu_\alpha v), \tag{106}$$

so that

$$[\partial\tilde{F}_\alpha/\partial w]|_{v=s_\alpha|B_{\alpha\gamma_\alpha}|} = [\tilde{F}'_\alpha(\mu_\alpha v^2/2 - V_\alpha)]|_{v=s_\alpha|B_{\alpha\gamma_\alpha}|} = \tilde{F}'_\alpha(\mu_\alpha[2(\gamma_\alpha + V_\alpha)/\mu_\alpha]/2 - V_\alpha) = \tilde{F}'_\alpha(\gamma_\alpha) \tag{107}$$

and Eq. (105) reduces to

$$\int_{-\infty}^{\infty} \mathrm{d}q e^{-\mathrm{i}s_\alpha q|B_{\alpha\gamma_\alpha}|} q F_\alpha = 2\pi\mathrm{i}\mu_\alpha s_\alpha |B_{\alpha\gamma_\alpha}| \tilde{F}'_\alpha(\gamma_\alpha). \tag{108}$$

Since $\tilde{F}'_\alpha(\gamma_\alpha)$ is a real function independent of x, Eq. (104) reads

$$G^{++}_{\mathrm{ee}}(\sigma, \sigma', \gamma_\mathrm{e}, \gamma'_\mathrm{e}) = 2\pi\sigma(Z_\mathrm{e}/\mu_\mathrm{e}) \times \mu_\mathrm{e}\tilde{F}'_\mathrm{e}(\gamma_\mathrm{e}) \Re \int_{a_{\mathrm{ee}\gamma_\mathrm{e}\gamma'_\mathrm{e}}}^{b_{\mathrm{ee}\gamma_\mathrm{e}\gamma'_\mathrm{e}}} \mathrm{d}x \bar{\psi}^{(0)+}_{\mathrm{e}\sigma\gamma_\mathrm{e}} \psi^{(0)+}_{\mathrm{e}\sigma'\gamma'_\mathrm{e}}, \tag{109}$$

or also, taking into account Eq. (46),

$$G^{++}_{\mathrm{ee}}(\sigma, \sigma', \gamma_\mathrm{e}, \gamma'_\mathrm{e}) = 2\pi\sigma C_{\mathrm{e}\gamma_\mathrm{e}} C_{\mathrm{e}\gamma'_\mathrm{e}} Z_\mathrm{e} \tilde{F}'_\mathrm{e}(\gamma_\mathrm{e}) \times \Re \int_{a_{\mathrm{ee}\gamma_\mathrm{e}\gamma'_\mathrm{e}}}^{b_{\mathrm{ee}\gamma_\mathrm{e}\gamma'_\mathrm{e}}} \mathrm{d}x \frac{1}{\mathrm{i}\sigma} e^{\mathrm{i}\sigma\xi_{\mathrm{e}\gamma_\mathrm{e}}(x)} \frac{1}{-\mathrm{i}\sigma'} e^{-\mathrm{i}\sigma'\xi_{\mathrm{e}\gamma'_\mathrm{e}}(x)}. \tag{110}$$

Here, due to Eqs. (C4) and (C4), the electron-electron kernel (Eq. (110)) reduces to

$$G^{++}_{\mathrm{ee}}(\sigma, \sigma', \gamma_\mathrm{e}, \gamma'_\mathrm{e}) = 2\pi C_{\mathrm{e}\gamma_\mathrm{e}} C_{\mathrm{e}\gamma'_\mathrm{e}} \frac{1}{\sigma'} Z_\mathrm{e} \tilde{F}'_\mathrm{e}(\gamma_\mathrm{e}) \times \Re \int_{a_{\mathrm{e}\,\min(\gamma_\mathrm{e},\gamma'_\mathrm{e})}}^{b_{\mathrm{e}\,\min(\gamma_\mathrm{e},\gamma'_\mathrm{e})}} \mathrm{d}x e^{\mathrm{i}\sigma\xi_{\mathrm{e}\gamma_\mathrm{e}}(x) - \mathrm{i}\sigma'\xi_{\mathrm{e}\gamma'_\mathrm{e}}(x)}. \tag{111}$$

In a similar way, from Eqs. (98) and (103), we find the ion-ion kernel:

$$G^{++}_{\mathrm{ii}}(\sigma, \sigma', \gamma_\mathrm{i}, \gamma'_\mathrm{i}) = -2\pi C_{\mathrm{i}\gamma_\mathrm{i}} C_{\mathrm{i}\gamma'_\mathrm{i}} \frac{1}{\sigma'} Z_\mathrm{i} \tilde{F}'_\mathrm{i}(\gamma_\mathrm{i}) \times \Re \int_{a_{\mathrm{i}\,\min(\gamma_\mathrm{i},\gamma'_\mathrm{i})}}^{b_{\mathrm{i}\,\min(\gamma_\mathrm{i},\gamma'_\mathrm{i})}} \mathrm{d}x e^{\mathrm{i}\sigma\xi_{\mathrm{i}\gamma_\mathrm{i}}(x) - \mathrm{i}\sigma'\xi_{\mathrm{i}\gamma'_\mathrm{i}}(x)}. \tag{112}$$

Yaking the integration bounds from Eqs. (C5)-(??), the cross-species kernels are

$$G_{\rm ei}^{++}(\sigma,\sigma',\gamma_{\rm e},\gamma_{\rm i}') = -2\pi C_{{\rm e}\gamma_{\rm e}} C_{{\rm i}\gamma_{\rm i}'} \frac{1}{\sigma'} Z_{\rm e}\tilde{F}_{\rm e}'(\gamma_{\rm e}) \times$$
$$\Re \int_{a_{{\rm ei}\gamma_{\rm e}\gamma_{\rm i}'}}^{b_{{\rm ei}\gamma_{\rm e}\gamma_{\rm i}'}} {\rm d}x e^{{\rm i}\sigma\xi_{{\rm e}\gamma_{\rm e}}(x)-{\rm i}\sigma'\xi_{{\rm i}\gamma_{\rm i}'}(x)}. \tag{113}$$

and

$$G_{\rm ie}^{++}(\sigma,\sigma',\gamma_{\rm i},\gamma_{\rm e}') = 2\pi C_{{\rm i}\gamma_{\rm i}} C_{{\rm e}\gamma_{\rm e}'} \frac{1}{\sigma'} Z_{\rm i}\tilde{F}_{\rm i}'(\gamma_{\rm i}) \times$$
$$\Re \int_{a_{{\rm ie}\gamma_{\rm i}\gamma_{\rm e}'}}^{b_{{\rm ie}\gamma_{\rm i}\gamma_{\rm e}'}} {\rm d}x e^{{\rm i}\sigma\xi_{{\rm i}\gamma_{\rm i}}(x)-{\rm i}\sigma'\xi_{{\rm e}\gamma_{\rm e}'}(x)} \tag{114}$$

For combinations of the signs s_α and s_β' different from $++$, the matrix elements $G_{\alpha\beta}^{s_\alpha s_\beta'}$ in Eq. (98) can be calculated from Eqs. (111)-(112), using the symmetry relations in Eqs. (99) and (100).

We end this section by pointing out that, because of the particle masses $\mu_{\rm e,i}$ appearing in the denominators of the matrix elements in Eqs. (111)-(112), the largest of them is $G_{\rm ee}$, followed by $G_{\rm e,i}$, $G_{\rm ie}$ and, last, $G_{\rm ii}$.

VII. THE HOMOGENEOUS ELECTRON INTEGRAL KERNELS

To better understand the nature of the kernels $G_{\alpha\beta}$ (Eqs. (111)-(112)), and the way they contribute to the coefficients X_α (Eq. (??)) and to the eigenfunctions $\chi_{\alpha\omega}$ (Eq. (75)), we consider first a simplified situation in which the ion mass is infinite and the potential ϕ is a constant which, without loss of generality, we set to zero.

Eq. (97) and (98) also arise, by a trivial exchange of the x and γ_β' order of integration in Eq. (76), when the steady state potential Φ (and thus V_α) is a constant. In this case, there are no reflection points and, in Eqs. (55), (56) and (97), we must set

$$\text{if } \Phi = \text{const. then}$$
$$a_{\alpha\beta\gamma_\alpha\gamma_\beta'} = -\infty,\ b_{\alpha\beta\gamma_\alpha\gamma_\beta'} = \infty,\ \gamma_{\alpha\beta 0(\nu_{\alpha\gamma_\alpha})} = 0. \tag{115}$$

In this case, from Eqs. (23) and (2), we have

$$|B_{{\rm e}\gamma_{\rm e}}| = \surd(2\gamma_{\rm e}/\mu_{\rm e}) \tag{116}$$

and the integral in Eq. (111) reduces to

$$\int_{-\infty}^{\infty} {\rm d}x e^{{\rm i}\sigma\int {\rm d}x/|B_{{\rm e}\gamma_{\rm e}}| - {\rm i}\sigma'\int {\rm d}x/B_{{\rm e}\gamma_{\rm e}'}} =$$
$$\int_{-\infty}^{\infty} {\rm d}x e^{\pm{\rm i}\{\sigma/\surd(2\gamma_{\rm e}/\mu_{\rm e})-\sigma'/\surd(2\gamma_{\rm e}'/\mu_{\rm e})\}\times} =$$
$$2\pi\delta(\sigma/\surd(2\gamma_{\rm e}/\mu_{\rm e}) - \sigma'/\surd(2\gamma_{\rm e}'/\mu_{\rm e})). \tag{117}$$

Using the identity

$$\delta(f(x)) = [|{\rm d}f/{\rm d}x|_{x=-\infty}]^{-1}\delta(x - -\infty), \tag{118}$$

where $-\infty$ is the root of $f(x) = 0$, we transform Eq. (117) into

$$\int_{-\infty}^{\infty} {\rm d}x e^{{\rm i}\sigma\int {\rm d}x/|B_{{\rm e}\gamma_{\rm e}}| - {\rm i}\sigma'\int {\rm d}x/B_{{\rm e}\gamma_{\rm e}'}} =$$
$$2\pi \frac{1}{|\partial[-\sigma'/\surd(2\gamma_{\rm e}'/\mu_{\rm e})]\partial\gamma_{\rm e}'|}\delta(\gamma_{\rm e}' - [\sigma'/\sigma]^2\gamma_{\rm e}) =$$
$$2\pi \frac{1}{|-\sigma'\surd(\mu_{\rm e}/2)\partial[1/\surd(\gamma_{\rm e}')]\partial\gamma_{\rm e}'|}\delta(\gamma_{\rm e}' - [\sigma'/\sigma]^2\gamma_{\rm e}) =$$
$$2\pi \frac{1}{|-\sigma'\surd(\mu_{\rm e}/2)[-(\gamma_{\rm e}')^{-3/2}/2]|}\delta(\gamma_{\rm e}' - [\sigma'/\sigma]^2\gamma_{\rm e}) =$$
$$2\pi \frac{\mu_{\rm e}|\gamma_{\rm e}'|^{3/2}}{|-\sigma'(\mu_{\rm e}/2)^{3/2}|}\delta(\gamma_{\rm e}' - [\sigma'/\sigma]^2\gamma_{\rm e}) =$$
$$2\pi \frac{\mu_{\rm e}|2\gamma_{\rm e}'/\mu_{\rm e}|^{3/2}}{|-\sigma'|}\delta(\gamma_{\rm e}' - [\sigma'/\sigma]^2\gamma_{\rm e}) =$$
$$2\pi \frac{\mu_{\rm e}[\sigma']^2|2\gamma_{\rm e}/\mu_{\rm e}|^{3/2}}{|\sigma|^3}\delta(\gamma_{\rm e}' - [\sigma'/\sigma]^2\gamma_{\rm e}). \tag{119}$$

Therefore, Eq. (111) gives

$$G_{\rm ee}^{++}(\sigma,\sigma',\gamma_{\rm e},\gamma_{\rm e}') = \frac{1}{\sigma'}\frac{Z_{\rm e}\tilde{F}_{\rm e}'(\gamma_{\rm e})}{\surd\mu_{\rm e}\surd\mu_{\rm e}} \times$$
$$\Re\frac{\mu_{\rm e}[\sigma']^2|2\gamma_{\rm e}/\mu_{\rm e}|^{3/2}}{|\sigma|^3}\delta(\gamma_{\rm e}' - [\sigma'/\sigma]^2\gamma_{\rm e}) \tag{120}$$

or

$$G_{\rm ee}^{++}(\sigma,\sigma',\gamma_{\rm e},\gamma_{\rm e}') =$$
$$\sigma' g_{\rm ee}(\sigma,\gamma_{\rm e})\delta(\gamma_{\rm e}' - [\sigma'/\sigma]^2\gamma_{\rm e}), \tag{121}$$

where

$$g_{\rm ee}(\sigma,\gamma_{\rm e}) = Z_{\rm e}|2\gamma_{\rm e}/\mu_{\rm e}|^{3/2}\tilde{F}_{\rm e}'(\gamma_{\rm e})/|\sigma|^3. \tag{122}$$

Likewise, using Eqs. (99) and (100), we have

$$G_{\rm ee}^{--}(\sigma,\sigma',\gamma_{\rm e},\gamma_{\rm e}') = G_{\rm ee}^{++}(\sigma,\sigma',\gamma_{\rm e},\gamma_{\rm e}') \tag{123}$$
$$G_{\rm ee}^{+-}(\sigma,\sigma',\gamma_{\rm e},\gamma_{\rm e}') = G_{\rm ee}^{-+}(\sigma,\sigma',\gamma_{\rm e},\gamma_{\rm e}') =$$
$$-\sigma' g_{\rm ee}(\sigma,\gamma_{\rm e})\delta(\gamma_{\rm e}' - [\sigma'/\sigma]^2\gamma_{\rm e}). \tag{124}$$

VIII. THE ELECTRON HOMOGENEOUS EIGENVALUE PROBLEM

If, as assumed in Section VII, the ion mass is infinite, in Eq. (??) taken for electrons ($\alpha = {\rm e}$) only the term involving $G_{\rm ee}^{s_{\rm e}s_{\rm e}'}$ survives and we have

$$(\sigma - \omega)X_{\rm e}^{s_{\rm e}}(\omega,\sigma,\gamma_{\rm e}) = \sum_{s_{\rm e}'=\pm}\int_{-\infty}^{\infty}{\rm d}\sigma'\int_0^{\infty}{\rm d}\gamma_{\rm e}'$$
$$G_{\rm ee}^{s_{\rm e}s_{\rm e}'}(\sigma,\sigma',\gamma_{\rm e},\gamma_{\rm e}')X_{\rm e}^{s_{\rm e}'}(\omega,\sigma',\gamma_{\rm e}'). \tag{125}$$

If, in Eq. (125), we change the signs of ω and σ and then the sign of the integration variable σ', and we take into account that the matrix elements $G_{\rm ee}$, as defined in Eqs. (121)-(124), are even functions of σ and odd functions of σ', we easily prove that $X_{\rm e}^{\pm}(-\omega,-\sigma,\gamma_{\rm e})$ and $X_{\rm e}^{\pm}(\omega,\sigma,\gamma_{\rm e})$ satisfy the same linear equation, and thus

$$X_{\rm e}^{\pm}(-\omega,\sigma,\gamma_{\rm e}) = X_{\rm e}^{\pm}(\omega,-\sigma,\gamma_{\rm e}). \tag{126}$$

Substituting the matrix elements $G_{\rm ee}^{s_{\rm e}s'_{\rm e}}$ from Eqs. (121)-(124) into Eq. (125) and, performing the $\gamma'_{\rm e}$ integration, we find

$$(\omega-\sigma)X_{\rm e}^{\pm}(\omega,\sigma,\gamma_{\rm e}) = g_{\rm ee}(\sigma,\gamma_{\rm e})\int_{-\infty}^{\infty}{\rm d}\sigma'\sigma' \times$$
$$[X_{\rm e}^{\pm}(\omega,\sigma',[\sigma'/\sigma]^2\gamma_{\rm e}) - X_{\rm e}^{\mp}(\omega,\sigma',[\sigma'/\sigma]^2\gamma_{\rm e})]. \quad (127)$$

Eqs. (127) is the eigenvalue problem for the electron eigenfunctions in a homogeneous medium.

IX. THE SUPERPOSITION COEFFICIENTS $X_{\rm e}^{\pm}$

Because of the symmetry relation given in Eq. (126), the eigenvalue problem in Eq. (127) needs be solved only for $\omega > 0$. We seek solutions to this equation in the form

$$X_{\rm e}^{\pm}(\omega,\sigma,\gamma_{\rm e}) = \Lambda_{\rm e}^{\pm}(\sigma,\gamma_{\rm e})\delta(\sigma-\omega) - {\rm P}\frac{Y_{\rm e}^{\pm}(\sigma,\gamma_{\rm e})}{\sigma-\omega}, \quad (128)$$
for $\omega > 0$.

Substituting Eq. (128) into Eqs. (127), we have

$$Y_{\rm e}^{\pm}(\sigma,\gamma_{\rm e}) = g_{\rm ee}(\sigma,\gamma_{\rm e})\int_{-\infty}^{\infty}{\rm d}\sigma'\sigma' \times$$
$$\left[\Lambda_{\rm e}^{\pm}(\sigma,[\sigma'/\sigma]^2\gamma_{\rm e})\delta(\sigma'-\omega) - {\rm P}\frac{Y_{\rm e}^{\pm}(\sigma',[\sigma'/\sigma]^2\gamma_{\rm e})}{\sigma'-\omega} - \Lambda_{\rm e}^{\mp}(\sigma,[\sigma'/\sigma]^2\gamma_{\rm e})\delta(\sigma'-\omega) + {\rm P}\frac{Y_{\rm e}^{\mp}(\sigma',[\sigma'/\sigma]^2\gamma_{\rm e})}{\sigma'-\omega}\right], \quad (129)$$
for $\omega > 0$

and, carrying out the σ' integration,

$$Y_{\rm e}^{\pm}(\sigma,\gamma_{\rm e}) = g_{\rm ee}(\sigma,\gamma_{\rm e}) \times$$
$$\left\{\omega\Lambda_{\rm e}^{\pm}(\sigma,[\omega/\sigma]^2\gamma_{\rm e}) - {\rm P}\int_{-\infty}^{\infty}{\rm d}\sigma'\sigma'\frac{Y_{\rm e}^{\pm}(\sigma',[\sigma'/\sigma]^2\gamma_{\rm e})}{\sigma'-\omega} - \omega\Lambda_{\rm e}^{\mp}(\sigma,[\omega/\sigma]^2\gamma_{\rm e}) + {\rm P}\int_{-\infty}^{\infty}{\rm d}\sigma'\sigma'\frac{Y_{\rm e}^{\mp}(\sigma',[\sigma'/\sigma]^2\gamma_{\rm e})}{\sigma'-\omega}\right\}, \quad (130)$$
for $\omega > 0$.

This integral equation is solved by setting

$$Y_{\rm e}^{\pm}(\sigma,\gamma_{\rm e}) = \sigma g_{\rm ee}(\sigma,\gamma_{\rm e}) \quad (131)$$

and

$$\frac{\sigma}{\omega} = \Lambda_{\rm e}^{\pm}(\sigma,[\omega/\sigma]^2\gamma_{\rm e}) - {\rm P}\int_{-\infty}^{\infty}{\rm d}\sigma'\sigma'\frac{\sigma'}{\omega}\frac{g_{\rm ee}(\sigma',[\sigma'/\sigma]^2\gamma_{\rm e})}{\sigma'-\omega}, \quad (132)$$
for $\omega > 0$.

We now use the identity

$$\frac{\sigma'}{\omega}\frac{1}{\sigma'-\omega} = \frac{1}{\sigma'-\omega} + \frac{1}{\omega} \quad (133)$$

and, in substituting

$$g_{\rm ee}(\sigma',[\sigma'/\sigma]^2\gamma_{\rm e}) = \frac{Z_{\rm e}|2\gamma_{\rm e}/\mu_{\rm e}|^{3/2}\tilde{F}'_{\rm e}([\sigma'/\omega]^2\gamma_{\rm e})}{|\sigma'|^3} \quad (134)$$

from Eq. (122) into Eq. (132), we take into account that $\sigma' g_{\rm ee}(\sigma',[\sigma'/\sigma]^2\gamma_{\rm e})$ is an odd function of σ', and integrates to 0. Thus,

$$\Lambda_{\rm e}^{\pm}(\sigma,\gamma_{\rm e}) = 1 + \frac{Z_{\rm e}|2\gamma_{\rm e}/\mu_{\rm e}|^{3/2}}{\sigma^3}{\rm P}\int_{-\infty}^{\infty}{\rm d}\sigma'\sigma'\frac{\tilde{F}'_{\rm e}([\sigma'/\omega]^2\gamma_{\rm e})}{\sigma'-\sigma}, \quad (135)$$
for $\sigma > 0$,

where, since $\Lambda_{\rm e}^{\pm}$ multiplies $\delta(\sigma-\omega)$, we replaced ω by σ and $\omega > 0$ by $\sigma > 0 \Rightarrow |\sigma| = \sigma$.

From Eqs. (131) and (135) it is thus seen that

$$X_{\rm e}^{+}(\omega,\sigma,\gamma_{\rm e}) = X_{\rm e}^{-}(\omega,\sigma,\gamma_{\rm e}), \quad (136)$$
for $\omega > 0$.

X. THE SPACE FOURIER TRANSFORM OF THE PERTURBED ELECTRON EIGENFUNCTION

Neglecting the $O(\mu_{\rm i}^{-1/2})$ ionic contributions (see discussion at the end of Section VI), Eq. (75) reads

$$\chi_{{\rm e}\omega}(x,q) = \int_{-\infty}^{\infty}{\rm d}\sigma'\int_0^{\infty}{\rm d}\gamma'_{\rm e} \times$$
$$[X_{\rm e}^{+}(\omega,\sigma',\gamma'_{\rm e})\chi^{(0)+}_{{\rm e}\sigma'\gamma'_{\rm e}}(x,q) + X_{\rm e}^{-}(\omega,\sigma',\gamma'_{\rm e})\chi^{(0)-}_{{\rm e}\sigma'\gamma'_{\rm e}}(x,q)]. \quad (137)$$

Because of the symmetry relations given in Eq. (44), we have

$$\chi_{{\rm e}(-\omega)}(x,q) = \int_{-\infty}^{\infty}{\rm d}\sigma'\int_0^{\infty}{\rm d}\gamma'_{\rm e} \times$$
$$[X_{\rm e}^{+}(\omega,-\sigma',\gamma'_{\rm e})\chi^{(0)+}_{{\rm e}\sigma'\gamma'_{\rm e}}(x,q) + X_{\rm e}^{-}(\omega,-\sigma',\gamma'_{\rm e})\chi^{(0)-}_{{\rm e}\sigma'\gamma'_{\rm e}}(x,q)], \quad (138)$$

or, changing the sign of the integration variable σ'

$$\chi_{{\rm e}(-\omega)}(x,q) = \int_{-\infty}^{\infty}{\rm d}\sigma'\int_0^{\infty}{\rm d}\gamma'_{\rm e} \times$$
$$[X_{\rm e}^{+}(\omega,\sigma',\gamma'_{\rm e})\chi^{(0)+}_{{\rm e}(-\sigma')\gamma'_{\rm e}}(x,q) + X_{\rm e}^{-}(\omega,\sigma',\gamma'_{\rm e})\chi^{(0)-}_{{\rm e}(-\sigma')\gamma'_{\rm e}}(x,q)], \quad (139)$$

and using the symmetry relations given in Eq. (44),

$$\chi_{{\rm e}(-\omega)}(x,q) = \int_{-\infty}^{\infty}{\rm d}\sigma'\int_0^{\infty}{\rm d}\gamma'_{\rm e} \times$$
$$[X_{\rm e}^{+}(\omega,\sigma',\gamma'_{\rm e})\bar{\chi}^{(0)+}_{{\rm e}\sigma'\gamma'_{\rm e}}(x,-q) + X_{\rm e}^{-}(\omega,\sigma',\gamma'_{\rm e})\bar{\chi}^{(0)-}_{{\rm e}\sigma'\gamma'_{\rm e}}(x,-q)]. \quad (140)$$

Finally, taking into account that the coefficients $X_{\rm e}^{\pm}$ are real, we have that

$$\chi_{{\rm e}(-\omega)}(x,q) = \bar{\chi}_{{\rm e}\omega}(x,-q). \quad (141)$$

Because of this symmetry relation, we may restrict the analysis of Eq. (137) for $\omega > 0$ only.

When $\omega > 0$, further using Eq. (136), we reduce Eq. (137) to

$$\chi_{\mathrm{e}\omega}(x,q) = \int_{-\infty}^{\infty} \mathrm{d}\sigma' \int_{0}^{\infty} \mathrm{d}\gamma_{\mathrm{e}}' X_{\mathrm{e}}^{+}(\omega,\sigma',\gamma_{\mathrm{e}}') \times [\chi^{(0)+}_{\mathrm{e}\sigma'\gamma_{\mathrm{e}}'}(x,q) + \chi^{(0)-}_{\mathrm{e}\sigma'\gamma_{\mathrm{e}}'}(x,q)]. \quad (142)$$

Substituting the zero order eigenfunctions from Eqs. (**??**)-(**??**) taken for electrons ($\alpha = \mathrm{e}$), Eq. (142) becomes

$$\chi_{\mathrm{e}\omega}(x,q) = \frac{1}{(2\pi\sqrt{\mu_{\mathrm{e}}})} \int_{-\infty}^{\infty} \mathrm{d}\sigma' \int_{0}^{\infty} \mathrm{d}\gamma_{\mathrm{e}}' X_{\mathrm{e}}^{+}(\omega,\sigma',\gamma_{\mathrm{e}}') \times \frac{1}{B_{\mathrm{e}\gamma_{\mathrm{e}}'}} \left[e^{\mathrm{i}(-\sigma' x/B_{\mathrm{e}\gamma_{\mathrm{e}}'} + qB_{\mathrm{e}\gamma_{\mathrm{e}}'})} + e^{-\mathrm{i}(-\sigma' x/B_{\mathrm{e}\gamma_{\mathrm{e}}'} + qB_{\mathrm{e}\gamma_{\mathrm{e}}'})} \right]. \quad (143)$$

In the integral involving $e^{\mathrm{i}(-\sigma' x/B_{\mathrm{e}\gamma_{\mathrm{e}}'} + qB_{\mathrm{e}\gamma_{\mathrm{e}}'})}$, we make the substitution (we remind that in the homogeneous case being considered, $|B_{\mathrm{e}\gamma_{\mathrm{e}}}| = \sqrt{(2\gamma_{\mathrm{e}}/\mu_{\mathrm{e}})}$)

$$|B_{\mathrm{e}\gamma_{\mathrm{e}}}| = \sqrt{(2\gamma_{\mathrm{e}}/\mu_{\mathrm{e}})} = v > 0, \gamma_{\mathrm{e}} = \mu_{\mathrm{e}} v^2/2, \quad (144)$$

$$\sigma = kv, \quad (145)$$

Since $v > 0$, the limits for the new variables are

$$\gamma_{\mathrm{e}} = 0 \Rightarrow v = 0, \ \gamma_{\mathrm{e}} = \infty \Rightarrow v = \infty, \quad (146)$$

$$\sigma = -\infty \Rightarrow k = -\infty, \ \sigma = \infty \Rightarrow k = \infty \quad (147)$$

Likewise, in the integral involving $e^{-\mathrm{i}(-\sigma x/B_{\mathrm{e}\gamma_{\mathrm{e}}'} + qB_{\mathrm{e}\gamma_{\mathrm{e}}'})}$ in Eq. (143), we set

$$|B_{\mathrm{e}\gamma_{\mathrm{e}}}| = \sqrt{(2\gamma_{\mathrm{e}}/\mu_{\mathrm{e}})} = -v > 0, \gamma_{\mathrm{e}} = \mu_{\mathrm{e}} v^2/2 \quad (148)$$

$$\sigma = -kv. \quad (149)$$

Since $v < 0$, the limits for the new variables now are

$$\gamma_{\mathrm{e}} = 0 \Rightarrow v = 0, \ \gamma_{\mathrm{e}} = \infty \Rightarrow v = -\infty, \quad (150)$$

$$\sigma = -\infty \Rightarrow k = \infty, \ \sigma = \infty \Rightarrow k = -\infty. \quad (151)$$

The Jacobian of these transformations is

$$\begin{vmatrix} \partial\sigma/\partial k & \partial\sigma/\partial v \\ \partial\gamma_{\mathrm{e}}/\partial k & \partial\gamma_{\mathrm{e}}/\partial v \end{vmatrix} = \begin{vmatrix} \pm v & \pm k \\ 0 & \mu_{\mathrm{e}} v \end{vmatrix} = \pm\mu_{\mathrm{e}} v^2, \quad (152)$$

where the upper sign is for the transformation given in Eqs. (144) and (145) and the lower sign is for that given in Eqs. (148) and (149). In this way, Eq. (143) becomes

$$\chi_{\mathrm{e}\omega}(x,q) = \frac{\mu_{\mathrm{e}}}{(2\pi\sqrt{\mu_{\mathrm{e}}})} \times \left\{ \int_{-\infty}^{\infty} \mathrm{d}k' \int_{0}^{\infty} \mathrm{d}v' v' X_{\mathrm{e}}^{+}(\omega, k'v', \mu_{\mathrm{e}}[v']^2/2) \times e^{\mathrm{i}(-k'x+qv')} - \int_{\infty}^{-\infty} \mathrm{d}k' \int_{0}^{-\infty} \mathrm{d}v' v' X_{\mathrm{e}}^{+}(\omega, k'v', \mu_{\mathrm{e}}[v']^2/2) e^{\mathrm{i}(-k'x+qv')} \right\} = \frac{\sqrt{\mu_{\mathrm{e}}}}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}k' e^{-\mathrm{i}k'x} \times \int_{-\infty}^{\infty} \mathrm{d}v' |v'| X_{\mathrm{e}}^{+}(\omega, k'v', \mu_{\mathrm{e}}[v']^2/2) e^{\mathrm{i}qv'}. \quad (153)$$

This shows that

$$\chi_{\mathrm{e}k\omega}(q) = \sqrt{\mu_{\mathrm{e}}} \int_{-\infty}^{\infty} \mathrm{d}v' |v'| X_{\mathrm{e}}^{+}(\omega, kv', \mu_{\mathrm{e}}[v']^2/2) e^{\mathrm{i}qv'} \quad (154)$$

is the space Fourier transform of $\chi_{\mathrm{e}\omega}(x,q)$ according to the definitions

$$\chi_{\mathrm{e}k\omega}(q) = \int_{-\infty}^{\infty} \mathrm{d}x e^{\mathrm{i}kx} \chi_{\mathrm{e}\omega}(x,q), \quad (155)$$

$$\chi_{\mathrm{e}\omega}(x,q) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}k e^{-\mathrm{i}kx} \chi_{\mathrm{e}k\omega}(q). \quad (156)$$

Changing the sign of k and of the integration variable v' in Eq. (154) and taking into account that X^+ is real, we prove that

$$\chi_{\mathrm{e}(-k)\omega}(q) = \bar{\chi}_{\mathrm{e}k\omega}(q) = \chi_{\mathrm{e}k\omega}(-q). \quad (157)$$

Using this result and the symmetry relations established in Eq. (141), we also prove that

$$\chi_{\mathrm{e}k(-\omega)}(q) = \bar{\chi}_{\mathrm{e}(-k)\omega}(-q) = \chi_{\mathrm{e}k\omega}(-q) = \bar{\chi}_{\mathrm{e}k\omega}(q), \quad (158)$$

so that we may restrict the analysis of Eq. (154) for $k > 0$ and $\omega > 0$ only.

Substituting X^+ from Eq. (I24)

$$X_{\mathrm{e}}^{+}(\omega, kv, \mu_{\mathrm{e}} v^2/2) = \frac{1}{k}\delta(v - \omega/k) - \frac{Z_{\mathrm{e}}}{\mu_{\mathrm{e}} k^2} \frac{1}{2\mathrm{i}\pi} \times \left[\frac{\int_0^\infty \mathrm{d}q' q' F_{\mathrm{e}}(q') e^{-\mathrm{i}q'v}}{k[v - (\omega + \mathrm{i}0)/k]} + \frac{\int_{-\infty}^0 \mathrm{d}q' q' F_{\mathrm{e}}(q') e^{-\mathrm{i}q'v}}{k[v - (\omega - \mathrm{i}0)/k]} \right], \quad (159)$$

for $\omega > 0$ and $k > 0$

into Eq. (154), we distinguish two cases. If $q - q' > 0$, we close the v'-integration path in Eq. (154) in the upper half complex v' plane by an anti-clock-wise half-circle whose radius we let become infinite. The integral certainly converges because $e^{\mathrm{i}(q-q')v'}$ is bounded over the whole upper half v' plane: in particular the contribution from the integration along the half circle vanishes when its radius tends to infinity because, on that half circle, the integrand

$$v' X^+ \approx \frac{1}{k} \frac{Z_{\mathrm{e}}}{\mu_{\mathrm{e}} k^2} \frac{1}{2\mathrm{i}\pi} e^{\mathrm{i}qv'} \int_{-\infty}^{\infty} \mathrm{d}q' q' F_{\mathrm{e}}(q') e^{-\mathrm{i}q'v'} \quad (160)$$

either phase-mixes to 0 for $\Re v' \neq 0, |v'| \to \infty$ or vanishes exponentially for $\Re v' = 0, |v'| \to \infty$. Furthermore, since only the pole at $v' = \omega/k + \mathrm{i}0$ lies within the integration path, only the first term in the square brackets in Eq. (159) gives a non zero contribution to the integral in Eq. (154). In that term, $q' > 0$ and thus the assumption $q - q' > 0$ we made for the present sub-case, implies that $q > 0$. By the residue theorem we thus find

$$\chi_{\mathrm{e}k\omega}(q) = \frac{1}{k} \frac{\omega}{k} e^{\mathrm{i}q[(\omega+\mathrm{i}0)/k]} \times \left[1 - \frac{Z_{\mathrm{e}}}{\mu_{\mathrm{e}} k^2} \int_0^q \mathrm{d}q' q' F_{\mathrm{e}}(q') e^{-\mathrm{i}q'[\omega+\mathrm{i}0)/k]} \right], \quad (161)$$

for $\omega > 0, \ k > 0$ and $q > 0$.

Likewise, if $q - q' < 0$, we close the v'-integration path in Eq. (154) by a clock-wise half-circle of infinite radius in the lower half complex v' plane. In this case, only the pole at $v' = \omega/k - \mathrm{i}0$ lies within the integration path and only the second term in the second line of Eq. (159) gives a non zero contribution to the integral in Eq. (154). In that term, $q' < 0$ and thus the assumption $q - q' < 0$ we made for the present sub-case, implies that if $q<0$. By the residue theorem we thus find

$$\begin{aligned}&\chi_{\mathrm{e}k\omega}(q)=\frac{1}{k}\frac{\omega}{k}e^{\mathrm{i}q[(\omega-\mathrm{i}0)/k]}\times\\&\left[1+\frac{Z_{\mathrm{e}}}{\mu_{\mathrm{e}}k^2}\int_q^0\mathrm{d}q'q'F_{\mathrm{e}}(q')e^{-\mathrm{i}q'[(\omega-\mathrm{i}0)/k]}\right],\qquad(162)\\&\text{for }\omega>0,\ k>0\text{ and }q<0.\end{aligned}$$

Clearly, once the limit to the real axis in the exponential factors is taken, Eqs. (161) and (162) give the same result.

In conclusion we may state that

$$\begin{aligned}&\chi_{\mathrm{e}k\omega}(q)=\frac{\sqrt{\mu_{\mathrm{e}}}\omega}{k^2}\times\\&e^{\mathrm{i}q\omega/k}\left[1-\frac{Z_{\mathrm{e}}}{\mu_{\mathrm{e}}k^2}\int_0^q\mathrm{d}q'q'F_{\mathrm{e}}(q')e^{-\mathrm{i}q'\omega/k}\right]\qquad(163)\\&\text{for }\omega>0,\ k>0\end{aligned}$$

is the eigenfunction of the perturbed Vlasov operator for $k>0$, $\omega>0$ and for all values of q. For the other signs of k and ω the eigenfunction is to be calculated according to the symmetry relations given in Eqs. (157) and (158).

A further symmetry relation may be established as follows:

$$\begin{aligned}&\chi_{\mathrm{e}k\omega}(-q)=\frac{\sqrt{\mu_{\mathrm{e}}}\omega}{k^2}\times\\&e^{-\mathrm{i}q\omega/k}\left[1-\frac{Z_{\mathrm{e}}}{\mu_{\mathrm{e}}k^2}\int_0^{-q}\mathrm{d}q'q'F_{\mathrm{e}}(q')e^{-\mathrm{i}q'\omega/k}\right]\qquad(164)\\&\text{for }\omega>0,\ k>0.\end{aligned}$$

As a way of check, we change the sign of the integration variable q' and taking into account that $F_{\mathrm{e}}(-q)=\bar{F}_{\mathrm{e}}(q)$, Eq. (164) reads

$$\begin{aligned}&\chi_{\mathrm{e}k\omega}(-q)=\frac{\sqrt{\mu_{\mathrm{e}}}\omega}{k^2}\times\\&e^{-\mathrm{i}q\omega/k}\left[1-\frac{Z_{\mathrm{e}}}{\mu_{\mathrm{e}}k^2}\int_0^{q}\mathrm{d}q'q'\bar{F}_{\mathrm{e}}(q')e^{\mathrm{i}q'\omega/k}\right]=\\&\bar{\chi}_{\mathrm{e}k\omega}(q),\qquad(165)\\&\text{for }\omega>0,\ k>0.\end{aligned}$$

which complies with the symmetry relation established in Eq. (157).

We may now state the main result of our work. Since $\chi_{\mathrm{e}k\omega}(q)$, the Fourier transform of a smooth function of velocity, must vanish as $q\to\infty$ by Lebesgue's lemma, from Eq. (163) we have that

$$1-\frac{Z_{\mathrm{e}}}{\mu_{\mathrm{e}}k^2}\int_0^\infty\mathrm{d}q'q'F_{\mathrm{e}}(q')e^{-\mathrm{i}q'\omega/k}=0.\qquad(166)$$

XI. THE PERTURBED ELECTRON EIGENFUNCTION

From the analysis in Section X, we see that the perturbed electron eigenfunction corresponding to the eigenvalue ω, which may obviously range ove the entire real axis, has none of the degeneracies affecting the unperturbed eigenfunctions given in Eqs. (??)-(??). We may thus state that the spectrum of the Vlasov operator is continuous and simple.

Using the symmetry relations established in Eqs. (157), we rewrite Eq. (156) as

$$\chi_{\mathrm{e}\omega}(x,q)=\frac{1}{\pi}\Re\int_0^\infty\mathrm{d}ke^{-\mathrm{i}kx}\chi_{\mathrm{e}k\omega}(q).\qquad(167)$$

Further using the other symmetry established in Eq. (157) we see that

$$\chi_{\mathrm{e}\omega}(-x,q)=\chi_{\mathrm{e}\omega}(x,-q),\qquad(168)$$

so that we may restrict our analysis to the case $x>0$.

In substituting for $\chi_{\mathrm{e}k\omega}(q)$, we first consider the case $q>0$. Using Eq. (161)

$$\begin{aligned}&\chi_{\mathrm{e}\omega}(x,q)=\sqrt{\mu_{\mathrm{e}}}\frac{1}{\pi}\Re\int_0^\infty\mathrm{d}k\frac{\omega}{k^2}e^{-\mathrm{i}kx}\times\\&\left[e^{\mathrm{i}q(\omega+\mathrm{i}0)/k}-\frac{Z_{\mathrm{e}}}{\mu_{\mathrm{e}}k^2}\times\right.\\&\left.\int_0^q\mathrm{d}q'q'F_{\mathrm{e}}(q')e^{\mathrm{i}(q-q')(\omega+\mathrm{i}0)/k}\right]\qquad(169)\\&\text{for }\omega>0,\ q>0\text{ and }x>0.\end{aligned}$$

Exchanging the order of the k and q' integration Eq. (169) reads

$$\begin{aligned}&\chi_{\mathrm{e}\omega}(x,q)=\sqrt{\mu_{\mathrm{e}}}\frac{1}{\pi}\Re\left[\int_0^\infty\mathrm{d}k\frac{\omega}{k^2}e^{-\mathrm{i}(kx-q(\omega+\mathrm{i}0)/k)}-\right.\\&\frac{Z_{\mathrm{e}}}{\mu_{\mathrm{e}}}\int_0^q\mathrm{d}q'q'F_{\mathrm{e}}(q')\times\\&\left.\int_0^\infty\mathrm{d}k\frac{\omega}{k^4}e^{-\mathrm{i}[kx-(q-q')(\omega+\mathrm{i}0)/k]}\right]\qquad(170)\\&\text{for }\omega>0,\ q>0\text{ and }x>0.\end{aligned}$$

We now use the identity

$$\begin{aligned}&\int_0^\infty\mathrm{d}te^{-pt}t^{\nu-1}e^{-a/(4t)}=\\&2[a/(4p)]^{\nu/2}K_\nu([ap]^{1/2})\qquad(171)\\&\text{for }\Re a>0.\end{aligned}$$

The first k-integral in Eq. (170) reduces to the general form given in Eq. (171) for

$$a=4\mathrm{i}[(\omega+\mathrm{i}0)q],\ p=\mathrm{i}x,\ \nu=-1\qquad(172)$$

and, taking the limit to the real ω-axis, we have

$$\begin{aligned}&\Re\int_0^\infty\mathrm{d}k\frac{\omega}{k^2}e^{-\mathrm{i}(kx-q(\omega+\mathrm{i}0)/k)}=\\&2\omega\Re([(\omega+\mathrm{i}0)q/x]^{-1/2}K_{-1}(2[-(\omega+\mathrm{i}0)xq]^{1/2}))=\\&-\pi\omega[\omega q/x]^{-1/2}\Re H^{(2)}_{-1}(2[\omega xq]^{1/2})=\\&\pi\omega[\omega q/x]^{-1/2}J_{-1}(2[\omega xq]^{1/2})=\\&\pi\omega[\omega q/x]^{-1/2}J_1(2[\omega xq]^{1/2})\qquad(173)\\&\text{for }\omega>0,\ q>0\text{ and }x>0,\end{aligned}$$

where J, K and $H^{(2)}$ denote the Bessel, the modified Bessel and the Hankel functions respectively. Notice that the integral defined in Eq. (173) is well behaved as $q\to0$.

Likewise, the second k-integral in Eq. (170) reduces to the general form given in Eq. (171) for $\nu=-3$ and we can write

$$\begin{aligned}&\Re\int_0^\infty\mathrm{d}k\frac{\omega}{k^4}e^{-\mathrm{i}[kx-(q-q')(\omega+\mathrm{i}0)/k]}=\\&-\pi\omega[\omega(q-q')/x]^{-3/2}J_3(2[\omega x(q-q')]^{1/2})\qquad(174)\\&\text{for }\omega>0,\ q>q'>0\text{ and }x>0.\end{aligned}$$

Notice that the integral given in Eq. (174) is well behaved as $q \to q'$.

When $q < 0$, we use Eq. (162) to write

$$\chi_{e\omega}(x,q) = \sqrt{\mu_e}\frac{1}{\pi}\Re\left[\int_0^\infty dk\frac{\omega}{k^2}e^{-i(kx-q(\omega-i0)/k)}+ \frac{Z_e}{\mu_e}\int_q^0 dq' q' F_e(q')\times \int_0^\infty dk\frac{\omega}{k^4}e^{-i[kx-(q-q')(\omega-i0)/k]}\right] \tag{175}$$

for $\omega > 0,\ q < 0$ and $x > 0$.

The first k-integral in Eq. (175) reduces to the general form given in Eq. (171) for

$$a = 4i[(\omega - i0)q],\ p = ix,\ \nu = -1 \tag{176}$$

and, taking the limit to the real ω-axis, we have

$$\Re\int_0^\infty dk\frac{\omega}{k^2}e^{-i(kx-q(\omega-i0)/k)} = 2\omega\Re([(\omega-i0)q/x]^{-1/2}K_{-1}(2[-(\omega-i0)xq]^{1/2})) = 2\omega\Re(i[\omega|q|/x]^{-1/2}K_{-1}(2[\omega x|q|]^{1/2})) = 0 \tag{177}$$

for $\omega > 0,\ q < 0$ and $x > 0$.

Likewise, the second k-integral in Eq. (175) reduces to the general form given in Eq. (171) for $\nu = -3$ and we can write

$$\Re\int_0^\infty dk\frac{\omega}{k^4}e^{-i[kx-(q-q')(\omega+i0)/k]} = 0. \tag{178}$$

for $\omega > 0,\ q < q' < 0$ and $x > 0$.

XII. CONCLUSIONS

In this way, the electron eigenfunction pertaining to the particles reflected at $x = a_{e\gamma_e(1)}$ (Eq. (29)) is different from (and indeed orthogonal to, as we shall see) the one pertaining to the particles reflected at $x = a_{e\gamma_e(2)}$ (Eq. (30)), even for the same values of σ, γ_e and s_e. No such difference needs be introduced for the ion eigenfunctions and if $\gamma_e > 0$, because no reflection points exist and the electron eigenfunction is defined for $-\infty < x < \infty$.

In this work we described a technique to find the permittivity of an electron gas to electrostatic perturbations. This technique is worked out in the space of the Fourier transformed velocity coordinate, the q-space and it is based upon a judicious construction of the eigenfunctions of the Vlasov operator as a superposition of those of its free-streaming part (the Liouville operator)[1].

The choice of the superposition coefficients provides the eigenfunctions in a form akin to that found in Ref. [11] by an entirely different method.

The peculiarity of these eigenfunctions is that their limit value, as the coordinate q tends to infinity, is proportional to the permittivity of the electron gas to electrostatic perturbations, as found, e.g., in Ref. [15]. Since this limit must vanish, by Lebesgue's lemma, our technique directly provides the dispersion equation for the electrostatic perturbations.

As a way of example, we applied this technique to an homogeneous ionized electron gas over a neutralizing, infinite mass, ion background, but its general formulation may well be used in the case of an inhomogeneous, electron and finite mass ion gas.

Appendix A: Reduction of Ampére's Equation

In physicsl units, Ampére's law for the electric field $\hat{E}$ and currnet $\hat{j}$ is

$$\frac{\partial\hat{E}}{\partial\hat{t}} = -4\pi\hat{J} = -4\pi\hat{j} = -4\pi\sum_\alpha\hat{j}_\alpha$$
$$\hat{j}_\alpha = Q_\alpha\int_{-\infty}^\infty d\hat{v}\hat{v}\hat{f}_\alpha, \tag{A1}$$

where the sum is extended over all the particle species α of charge Q_α. Using the notation of Section II, and setting

$$\hat{E} = -\Phi_0\phi', \tag{A2}$$

we have

$$\frac{\Phi_0}{\lambda}\omega_p\frac{\partial\tilde{\phi}'}{\partial t} = 4\pi n_0 e v_0\sum_\alpha\frac{Z_\alpha}{|Z_\alpha|}\tilde{j}_\alpha,$$

where

$$\tilde{j}_\alpha = \int_{-\infty}^\infty dv v\tilde{f}_\alpha, \tag{A3}$$

are the normalised particle fluxes. Eq. (A3) reduces to

$$\frac{\Phi_0}{v_0}\omega_p^2\frac{\partial\tilde{\phi}'}{\partial t} = 4\pi n_0 e v_0\sum_\alpha\frac{Z_\alpha}{|Z_\alpha|}\tilde{j}_\alpha, \tag{A4}$$

or

$$\frac{\partial\tilde{\phi}'}{\partial t} = \frac{4\pi n_0 e^2}{m_e\omega_p^2}\frac{m_e v_0^2}{e\Phi_0}\sum_\alpha\frac{Z_\alpha}{|Z_\alpha|}\tilde{j}_\alpha = \sum_\alpha\frac{Z_\alpha}{|Z_\alpha|}\tilde{j}_\alpha. \tag{A5}$$

Now, from Eq. (5), we see that $\partial\tilde{\phi}/\partial t$ is the inverse ω-Fourier transform of $i\omega\phi_\omega$. Also, taking the q-derivative of both sides of Eq. (6)

$$\frac{\partial f_{\alpha\omega}}{\partial q} = \int_{-\infty}^\infty d\omega\int_{-\infty}^\infty dv i v e^{i(qv-\omega t)}\tilde{f}_\alpha, \tag{A6}$$

we see that $v\tilde{f}_\alpha$ is the inverse q- and ω-Fourier transform of $-i\partial f_{\alpha\omega}/\partial q$. Thus, taking the direct t-Fourier transform of Eq. (A5), we have

$$i\omega\phi'_\omega = \sum_\alpha\frac{Z_\alpha}{|Z_\alpha|}j_{\alpha\omega}, \tag{A7}$$

where

$$j_{\alpha\omega} = \frac{-i}{2\pi}\int_{-\infty}^\infty dv\int_{-\infty}^\infty dq e^{-iqv}\frac{\partial f_{\alpha\omega}}{\partial q}. \tag{A8}$$

If the q-integral converges, we may invert the order of v and q integration and, using the identity $\int_{-\infty}^\infty dv e^{-iqv} = 2\pi\delta(q)$, we find

$$\phi'_\omega = -i(j_{e\omega} - j_{i\omega})/\omega,\ j_\alpha = -i\partial f_{\alpha\omega}/\partial q|_{q=0}. \tag{A9}$$

Appendix B: Reflection Points and Eigenfunction Phases in the Nonmonotonic Double Layer

To reduce the integral in Eq. (24), we first make the substitution

$$2\sqrt{U}/[(1+\sqrt{U})\coth(\kappa x/2)-(1-\sqrt{U})]=u, \tag{B1}$$

whence

$$\begin{aligned}&(1+\sqrt{U})\coth(\kappa x/2)u=2\sqrt{U}+(1-\sqrt{U})u,\\&\coth(\kappa x/2)=(1-\sqrt{U}+2\sqrt{U}/u)/(1+\sqrt{U}),\end{aligned} \tag{B2}$$

and

$$\begin{aligned}\kappa\frac{\mathrm{d}x}{\mathrm{d}u}&=\frac{2}{1-[(1-\sqrt{U}+2\sqrt{U}/u)/(1+\sqrt{U})]^2}\frac{-2\sqrt{U}/u^2}{1+\sqrt{U}}\\&=\frac{2(1+\sqrt{U})^2}{(2\sqrt{U}-2\sqrt{U}/u)(2+2\sqrt{U}/u)}\frac{-2\sqrt{U}/u^2}{1+\sqrt{U}}=\\&\frac{1+\sqrt{U}}{(1-u)(u+\sqrt{U})}=\frac{1}{1-u}+\frac{1}{u+\sqrt{U}}=G(u).\end{aligned} \tag{B3}$$

Then we insert the potential Φ (Eq. (26)) into the electron potential energy $-V_e=-|Z_e|\Phi$ (Eq. (2)) and then in the electron eigenfunction phase (Eqs. (23) and (24)) to get

$$\xi_{e\gamma_e}=\frac{\sqrt{\mu_e}}{\sqrt{2}\kappa}\int_{u_{e\gamma_e}}^{u}\mathrm{d}t\frac{G(t)}{\sqrt{(\gamma_e+|Z_e|t^2)}}, \tag{B4}$$

where $G(t)$ was defined in Eq. (B3) and $u_{e\gamma_e}$ is a constant to be determined so that $\xi_{e\gamma_e}$ in Eq. (B4) be real, as follows:

$$\text{if } -|Z_e|\le\gamma_e<0 \text{ then}$$

$$\text{either } u<u_{e\gamma_e(1)}=-\sqrt{|\gamma_e/Z_e|}, \tag{B5}$$

$$\text{or}\quad u>u_{e\gamma_e(2)}=\sqrt{|\gamma_e/Z_e|}. \tag{B6}$$

To the quantities $u_{e\gamma_e(1)}$ and $u_{e\gamma_e(2)}$ respectively correspond, through Eq. (B2), the values of the reflection points $b_{e\gamma_e(1)}$ and $a_{e\gamma_e(2)}$ reported in Eqs. (29) and (30) of the main text:

$$\text{if } -|Z_e|\le\gamma_e<0 \text{ then}$$

$$b_{e\gamma_e(1)}=\frac{2}{\kappa}\coth^{-1}\left(\frac{1-\sqrt{U}-2\sqrt{U}/\sqrt{|\gamma_e/Z_e|}}{1+\sqrt{U}}\right), \tag{B7}$$

$$a_{e\gamma_e(2)}=\frac{2}{\kappa}\coth^{-1}\left(\frac{1-\sqrt{U}+2\sqrt{U}/\sqrt{|\gamma_e/Z_e|}}{1+\sqrt{U}}\right). \tag{B8}$$

As $\gamma_e\to0^-$ in Eq. (B5) and (B6), $u_{e\gamma_e(1)}$ and $u_{e\gamma_e(2)}$ coalesce to zero. This is why, for $\gamma_e>0$, we set

$$\text{if } \gamma_e>0 \text{ then } u_{e\gamma_e}=0. \tag{B9}$$

To this value of $u_{e\gamma_e}$ corresponds, through Eq. (B2), the value of $x_{e\gamma_e}=0$ reported in Eq. (28) of the main text.

Similar considerations apply to the ion eigenfunction phase (Eq. (24)). Inserting the potential Φ (Eq. (26)) into the ion potential energy $-V_i=-Z_i(1-\Phi)$ (Eq. (2)) and then in Eqs. (23) and (24), we have

$$\xi_{i\gamma_i}=\frac{\sqrt{\mu_i}}{\sqrt{2}\kappa}\int_{u_{i\gamma_i}}^{u}\mathrm{d}t\frac{G(t)}{\sqrt{(\gamma_i+Z_i-Z_it^2)}}, \tag{B10}$$

where $G(t)$ was defined in Eq. (B3). If $\gamma_i<0$, then, for $\xi_{i\gamma_i}$ in Eq. (B10) to be real, the following conditions must be met

$$\begin{aligned}&\text{if } -Z_i\le\gamma_i<0 \text{ then}\\&-u_{i\gamma_i}<u<u_{i\gamma_i}=\sqrt{(1+\gamma_i/Z_i)}.\end{aligned} \tag{B11}$$

To the quantities $u_{i\gamma_i}$ and $-u_{i\gamma_i}$ respectively correspond, through Eq. (B2), the values of the reflection points $x_{i\gamma_i(1)}=b_{i\gamma_i(1)}$ and $x_{i\gamma_i(1)}=a_{i\gamma_i(1)}$ reported in Eqs. (32) and (33) of the main text:

$$\begin{aligned}&\text{if } -Z_i\le\gamma_i<0 \text{ then } b_{i\gamma_i(1)}=\\&\frac{2}{\kappa}\coth^{-1}\left(\frac{1-\sqrt{U}+2\sqrt{[U/(1+\gamma_i/Z_i)]}}{1+\sqrt{U}}\right),\end{aligned} \tag{B12}$$

$$\begin{aligned}&\text{if } -Z_i\le\gamma_i<-Z_i(1-U) \text{ then } a_{i\gamma_i(1)}=\\&\frac{2}{\kappa}\coth^{-1}\left(\frac{1-\sqrt{U}-2\sqrt{[U/(1+\gamma_i/Z_i)]}}{1+\sqrt{U}}\right).\end{aligned} \tag{B13}$$

Here, the reflection point $a_{i\gamma_i(1)}$ exists as a real quantity only if $-Z_i\le\gamma_i<-Z_i(1-U)$, because the argument of $\coth^{-1}$ must be smaller than -1.

As $\gamma_i\to0^-$ in Eq. (B12), $u_{i\gamma_i}\to1$. This is why, for $\gamma_i>0$, we set

$$\text{if } \gamma_i>0 \text{ then } u_{i\gamma_i}=1. \tag{B14}$$

To this value of $u_{i\gamma_i}$ corresponds, through Eq. (B2), the value of $x_{i\gamma_i(1)}=\infty$ reported in Eq. (31) of the main text.

We now arrange Eqs. (B4) and (B10) in a single formula

$$\begin{aligned}\xi_\alpha&=\frac{\sqrt{\mu_\alpha}}{\sqrt{2}\kappa}\int_{u_{\alpha\gamma_\alpha}}^{u}\mathrm{d}t\frac{G(t)}{\sqrt{(Y_\alpha-Z_\alpha t^2)}}=\\&\frac{\sqrt{\mu_\alpha}}{\sqrt{2}\kappa}[I_1(t)+I_2(t)]_{t=u_{\alpha\gamma_\alpha}}^{t=u},\end{aligned} \tag{B15}$$

where

$$Y_e=\gamma_e,\ Y_i=Z_i+\gamma_i, \tag{B16}$$

$$\begin{aligned}I_1(u)&=\int\frac{\mathrm{d}u}{\sqrt{(Y_\alpha-Z_\alpha u^2)}}\frac{1}{1-u}=\\&-\int\frac{\mathrm{d}s}{s\sqrt{(Y_\alpha-Z_\alpha+2Z_\alpha s-Z_\alpha s^2)}},\end{aligned} \tag{B17}$$

$$\text{with } s=1-u, \tag{B18}$$

$$\begin{aligned}I_2(u)&=\int\frac{\mathrm{d}u}{\sqrt{(Y_\alpha-Z_\alpha u^2)}}\frac{1}{u+\sqrt{U}}=\\&\int\frac{\mathrm{d}r}{r\sqrt{(Y_\alpha-Z_\alpha a-2Z_\alpha r\sqrt{U}-Z_\alpha r^2)}},\end{aligned} \tag{B19}$$

$$\text{with } r=u+\sqrt{U}. \tag{B20}$$

These two integrals may in turn be written in a single formula

$$I_1(u)=-I(-1,1-u),\ I_2(u)=I(\sqrt{U},u+\sqrt{U}), \tag{B21}$$

where

$$I(b,w)=\int\frac{\mathrm{d}w}{wR},\ R=\sqrt{(W_\alpha-2Z_\alpha bw-Z_\alpha w^2)}, \tag{B22}$$

$$W_\alpha=Y_\alpha-Z_\alpha b^2. \tag{B23}$$

Following Ref. [16, formula 2.266 p. 84], we have

$$\text{if } W_\alpha > 0 \text{ then } I(b,w) = \frac{1}{\sqrt{W_\alpha}} \ln \frac{w}{W_\alpha - Z_\alpha bw + \sqrt{W_\alpha}R}, \tag{B24}$$

$$\text{if } W_\alpha < 0 \text{ then } I(b,w) = \frac{1}{\sqrt{|W_\alpha|}} \tan^{-1} \frac{W_\alpha + Z_\alpha bw}{\sqrt{|W_\alpha|}R}. \tag{B25}$$

Appendix C: Relative Positions of the Reflection Points and Nonvanishing Domains for $h_{\alpha\beta}$

To calculate the x-integration bounds in Eqs. (55)-(58)

$$a_{\alpha\beta\gamma_\alpha\gamma'_\beta} = \max(a_{\alpha\gamma_\alpha}, a_{\beta\gamma'_\beta}), \tag{C1}$$

$$b_{\alpha\beta\gamma_\alpha\gamma'_\beta} = \min(b_{\alpha\gamma_\alpha}, b_{\beta\gamma'_\beta}) \tag{C2}$$

we need determine the endpoints $a_{\alpha\gamma_\alpha}$, $b_{\alpha\gamma_\alpha}$, $a_{\beta\gamma'_\beta}$, $b_{\beta\gamma'_\beta}$ of the intervals in which the eigenfunctions do not identically vanish (Eqs. (42) and (48)) and their relative position. These endpoints are either reflection points or boundaries of the double layer and they were defined in Eqs. (28)-(30) for electrons and in Eqs (31)-(33) for ions.

Specifically, when $\gamma_\mathrm{e} < 0$ and $\gamma'_\mathrm{e} < 0$ and when $\gamma_\mathrm{i} < -Z_\mathrm{i}(1-U)$ and $\gamma_\mathrm{i} < -Z'_\mathrm{i}(1-U)$ in domain 1 or $\gamma_\mathrm{i} < 0$ and $\gamma'_\mathrm{i} < 0$ in domain 2, inspection of Fig. 1 shows that

$$\text{if } \gamma_\alpha \le \gamma'_\alpha \text{ then } a_{\alpha\gamma'_\alpha} \le a_{\alpha\gamma_\alpha},\ b_{\alpha\gamma'_\alpha} \ge b_{\alpha\gamma_\alpha}, \tag{C3}$$

When γ_e and/or γ'_e exceed 0, then $a_{\mathrm{e}\gamma_\mathrm{e}}$ and/or $a_{\mathrm{e}\gamma'_\mathrm{e}}$ approach $-\infty$ and $b_{\mathrm{e}\gamma_\mathrm{e}}$ or $b_{\mathrm{e}\gamma'_\mathrm{e}}$ approach ∞ (Eq. (28)). Also, when γ_i and/or γ'_i exceed $-Z_\mathrm{i}(1-U)$, but remain negative, then $a_{\mathrm{i}\gamma_\mathrm{i}(1)}$ and/or $a_{\mathrm{i}\gamma'_\mathrm{i}(1)}$ = approach $-\infty$ (Eq. (31)). When γ_i and/or γ'_i exceed 0, then $b_{\mathrm{i}\gamma_\mathrm{i}(1)}$ and/or $b_{\mathrm{i}\gamma'_\mathrm{i}(1)}$ approach ∞ (Eqs. (31)-(32)). In all cases, the relations in Eq. (C3) holds, possibly with the equal sign holding. This proves Eq. (C4) below.

When $\alpha \neq \beta$, the labels (1) or (2) need be applied to $a_{\alpha\beta\gamma_\alpha\gamma'_\beta}$ and $b_{\alpha\beta\gamma_\alpha\gamma'_\beta}$ when the endpoints $a_{\mathrm{e}\gamma_\mathrm{e}(1)}$, $b_{\mathrm{e}\gamma_\mathrm{e}(1)}$, $a_{\mathrm{e}\gamma'_\mathrm{e}(1)}$, $b_{\mathrm{e}\gamma'_\mathrm{e}(1)}$ (Eq. (29)) or $a_{\mathrm{e}\gamma_\mathrm{e}(2)}$, $b_{\mathrm{e}\gamma_\mathrm{e}(2)}$, $a_{\mathrm{e}\gamma'_\mathrm{e}(2)}$, $b_{\mathrm{e}\gamma'_\mathrm{e}(2)}$ (Eq. (30)) are used in Eqs. (C1) and (C2).

Specifically, in domain 1, for all values of γ_e, $a_{\mathrm{e}\gamma_\mathrm{e}(1)} = -\infty$ (Eqs. (28) and (29)). If $\gamma'_\mathrm{i} < -Z_\mathrm{i}(1-U)$, then $a_{\mathrm{i}\gamma'_\mathrm{i}(1)} > -\infty$ and, if $\gamma'_\mathrm{i} > -Z_\mathrm{i}(1-U)$, then $a_{\mathrm{i}\gamma'_\mathrm{i}(1)} = -\infty$ (Eq. (32)): in both cases, $a_{\mathrm{e}\gamma_\mathrm{e}(1)}$ never exceeds $a_{\mathrm{i}\gamma'_\mathrm{i}(1)}$. This proves the first of Eq. (C5) below.

In domain 2, for all values of γ_e, $b_{\mathrm{e}\gamma_\mathrm{e}(2)} = \infty$ (Eqs. (28) and (30)). If $\gamma'_\mathrm{i} < 0$, then $b_{\mathrm{i}\gamma'_\mathrm{i}(1)} < \infty$ and, if $\gamma'_\mathrm{i} > 0$, then $b_{\mathrm{i}\gamma'_\mathrm{i}(1)} = \infty$ (Eq. (28)): in both cases, $b_{\mathrm{i}\gamma'_\mathrm{i}(1)}$ never exceeds $b_{\mathrm{e}\gamma_\mathrm{e}(2)}$. This proves the first of Eq. (C6) below.

In domian 1, for $\gamma_\mathrm{e} < 0$ and for all values of γ'_i, $b_{\mathrm{i}\gamma'_\mathrm{i}(1)} > 0 > b_{\mathrm{e}\gamma_\mathrm{e}(1)}$ (Eqs. (29) and (32)). This proves the first of Eq. (C7) below.

In domain 2, for $\gamma_\mathrm{e} < 0$ and for all values of γ'_i, $a_{\mathrm{i}\gamma'_\mathrm{i}(1)} < 0 < a_{\mathrm{e}\gamma_\mathrm{e}(2)}$ (Eqs. (30) and (33)). This proves the first of Eq. (C8) below.

If $\gamma_\mathrm{e} > 0$, then $b_{\mathrm{e}\gamma_\mathrm{e}(1)} = \infty$ (Eq. (28)); If $\gamma'_\mathrm{i} < 0$, then $b_{\mathrm{i}\gamma'_\mathrm{i}} < \infty$ and, if $\gamma'_\mathrm{i} > 0$, then $b_{\mathrm{i}\gamma'_\mathrm{i}} = \infty$. In both cases $b_{\mathrm{i}\gamma'_\mathrm{i}}$ never exceeds $b_{\mathrm{e}\gamma_\mathrm{e}(1)}$: this proves the first of Eq. (C9);

Last, if $\gamma_\mathrm{e} > 0$, then $a_{\mathrm{e}\gamma_\mathrm{e}(2)} = -\infty$ (Eq. (28)); If $\gamma'_\mathrm{i} < -Z_\mathrm{i}(1-U)$, then $a_{\mathrm{i}\gamma'_\mathrm{i}} > -\infty$ and, if $\gamma'_\mathrm{i} > -Z_\mathrm{i}(1-U)$, then $a_{\mathrm{i}\gamma'_\mathrm{i}} = -\infty$. In both cases $a_{\mathrm{e}\gamma_\mathrm{e}(2)}$ never exceeds $a_{\mathrm{i}\gamma'_\mathrm{i}}$: this proves the first of Eq. (C10).

The ordering relations between the endpoints thus give the following x-integration bounds.

$$a_{\alpha\alpha\gamma_\alpha\gamma'_\alpha} = a_{\alpha\,\min(\gamma_\alpha,\gamma'_\alpha)},\ b_{\alpha\alpha\gamma_\alpha\gamma'_\alpha} = b_{\alpha\,\min(\gamma_\alpha,\gamma'_\alpha)}. \tag{C4}$$

$$a_{\mathrm{ei}\gamma_\mathrm{e}\gamma'_\mathrm{i}(1)} = a_{\mathrm{i}\gamma'_\mathrm{i}(1)},\ a_{\mathrm{ie}\gamma_\mathrm{i}\gamma'_\mathrm{e}(1)} = a_{\mathrm{i}\gamma_\mathrm{i}(1)}, \tag{C5}$$

$$b_{\mathrm{ei}\gamma_\mathrm{e}\gamma'_\mathrm{i}(2)} = b_{\mathrm{i}\gamma'_\mathrm{i}(1)},\ b_{\mathrm{ie}\gamma_\mathrm{i}\gamma'_\mathrm{e}(2)} = b_{\mathrm{i}\gamma_\mathrm{i}(1)}. \tag{C6}$$

if $\gamma_\mathrm{e} < 0$ or $\gamma'_\mathrm{e} < 0$ then

$$b_{\mathrm{ei}\gamma_\mathrm{e}\gamma'_\mathrm{i}(1)} = b_{\mathrm{e}\gamma_\mathrm{e}(1)},\ b_{\mathrm{ie}\gamma_\mathrm{i}\gamma'_\mathrm{e}(1)} = b_{\mathrm{e}\gamma'_\mathrm{e}(1)}, \tag{C7}$$

$$a_{\mathrm{ei}\gamma_\mathrm{e}\gamma'_\mathrm{i}(2)} = a_{\mathrm{e}\gamma_\mathrm{e}(2)},\ a_{\mathrm{ie}\gamma_\mathrm{i}\gamma'_\mathrm{e}(2)} = a_{\mathrm{e}\gamma'_\mathrm{e}(2)}, \tag{C8}$$

if $\gamma_\mathrm{e} > 0$ or $\gamma'_\mathrm{e} > 0$ then

$$b_{\mathrm{ei}\gamma_\mathrm{e}\gamma'_\mathrm{i}(1)} = b_{\mathrm{i}\gamma'_\mathrm{i}(1)},\ b_{\mathrm{ie}\gamma_\mathrm{i}\gamma'_\mathrm{e}(1)} = b_{\mathrm{i}\gamma_\mathrm{i}(1)}. \tag{C9}$$

$$b_{\mathrm{ei}\gamma_\mathrm{e}\gamma'_\mathrm{i}(2)} = b_{\mathrm{i}\gamma'_\mathrm{i}(1)},\ a_{\mathrm{ie}\gamma_\mathrm{i}\gamma'_\mathrm{e}(2)} = a_{\mathrm{i}\gamma_\mathrm{i}(1)}. \tag{C10}$$

The other relations appearing in the second columns of Eqs. (C5)-(C10) are reported for ease of reference and they follow from the obvious symmetry of Eqs. (C1) and (C2):

$$a_{\alpha\beta\gamma_\alpha\gamma'_\beta} = a_{\beta\alpha\gamma_\beta\gamma'_\alpha},\ b_{\alpha\beta\gamma_\alpha\gamma'_\beta} = b_{\beta\alpha\gamma_\beta\gamma'_\alpha}. \tag{C11}$$

Appendix D: Nonvanishing x-intervals for $h_{\alpha\beta}$

The contribution of the unperturbed oscillations of particles of species β to the perturbed oscillations of particles of species α depends on the x-intervals where the functions $h_{\alpha\beta(\nu_{\alpha\gamma_\alpha};\nu'_\beta)}(x,\sigma,\gamma_\alpha,\gamma'_\beta)$ (Eqs. (77))-(79) do not identically vanish. In turn, the extent of these intervals depends on the degeneracy parameters γ_α and γ'_β, as we shall presently determine.

Specifically, in Eq. (77), $\Psi^{(0)}_{\beta\gamma'_\beta(\nu'_\beta)}$ is based on the unperturbed electric potential eigenfunctions $\psi^{(0)s_\beta}_{\beta\sigma'\gamma'_\beta(\nu'_\beta)}$, all of which involve the same value of γ'_β so that, due to Eq. (48),

$$\text{given } \gamma'_\beta \text{ and } x \notin (a_{\beta\gamma'_\beta}, b_{\beta\gamma'_\beta}) \text{ then } \Psi^{(0)}_{\beta\gamma'_\beta} = 0. \tag{D1}$$

It immediately follows from Eqs. (29), (30), (42) and (D1) that

$$\text{if } \gamma_\mathrm{e} < 0, \gamma_\mathrm{e} < 0 \text{ and } \nu_{\mathrm{e}\gamma_\mathrm{e}} \neq \nu_{\mathrm{e}\gamma_\mathrm{e}} \text{ then } h_{\mathrm{ee}(\nu_{\mathrm{e}\gamma_\mathrm{e}};\nu_{\mathrm{e}\gamma_\mathrm{e}})}(x,\sigma,\gamma_\mathrm{e},\gamma_\mathrm{e}) = 0, \tag{D2}$$

and

$$\text{if } x > 0 \text{ and } \gamma_\mathrm{e} < 0 \text{ then } h_{\alpha\mathrm{e}(\nu_{\alpha\gamma_\alpha};1)}(x,\sigma,\gamma_\alpha,\gamma_\mathrm{e}) = 0, \tag{D3}$$

$$\text{if } x < 0 \text{ and } \gamma_\mathrm{e} < 0 \text{ then } h_{\alpha\mathrm{e}(\nu_{\alpha\gamma_\alpha};2)}(x,\sigma,\gamma_\alpha,\gamma_\mathrm{e}) = 0. \tag{D4}$$

We further observe that, since, in domain 1, $-V_\mathrm{i}$ is a decreasing function of x (Eq. (2) and Fig. 1), given two points a and b, $-V_\mathrm{i}(a) < -V_\mathrm{i}(b) \Rightarrow a > b$. Then, from the relations

$$\text{if } x < 0 \text{ and } -|Z_\mathrm{e}| < \gamma_\mathrm{e} < 0 \text{ then } -V_\mathrm{i}(b_{\mathrm{e}\gamma_\mathrm{e}(1)}) = \gamma^*_\mathrm{i}, \tag{D5}$$

$$\text{if } x < 0 \text{ and } -Z_\mathrm{i} < \gamma_\mathrm{i} < Z_\mathrm{i}(1-U) \text{ then } -V_\mathrm{i}(a_{\mathrm{i}\gamma_\mathrm{i}(1)}) = \gamma_\mathrm{i}, \tag{D6}$$

where

$$\gamma_{\rm i}^* = -Z_{\rm i}(1+\gamma_{\rm e}/|Z_{\rm e}|), \tag{D7}$$

we have

$$\text{if } x<0,\ \gamma_{\rm e}<0 \text{ and } \gamma_{\rm i}<\gamma_{\rm i}^* \text{ then } a_{{\rm i}\gamma_{\rm i}(1)} > b_{{\rm e}\gamma_{\rm e}(1)}. \tag{D8}$$

In this circumstance, the interval $(a_{{\rm e}\gamma_{\rm e}(1)}, b_{{\rm e}\gamma_{\rm e}(1)})$ where $\chi^{(0)s_\alpha}_{\alpha\sigma\gamma_\alpha} \neq 0$ (Eq. (42)) and $(a_{{\rm i}\gamma_{\rm i}(1)}, b_{{\rm i}\gamma_{\rm i}(1)})$ where $\Psi^{(0)}_{{\rm i}\gamma_{\rm i}} \neq 0$ (Eq. (D1)) are disjoint, so that, according to Eqs. (D1) and (79),

$$\text{if } x<0,\ \gamma_{\rm e}<0 \text{ and } \gamma_{\rm i}<\gamma_{\rm i}^* \text{ then } h_{{\rm ei}(1;1)}(x,\sigma,\gamma_{\rm e},\gamma_{\rm i}) = 0. \tag{D9}$$

A similar argument applies in domain 2 where $-V_{\rm i}$ monotonically increases, so that $-V_{\rm i}(b) < -V_{\rm i}(a) \Rightarrow a>b$. Then, from the relations

$$\text{if } x>0 \text{ and } -|Z_{\rm e}|<\gamma_{\rm e}<0 \text{ then } -V_{\rm i}(a_{{\rm e}\gamma_{\rm e}(2)}) = \gamma_{\rm i}^*, \tag{D10}$$
$$\text{if } x>0 \text{ and } -Z_{\rm i}<\gamma_{\rm i}<0 \text{ then } -V_{\rm i}(b_{{\rm i}\gamma_{\rm i}(1)}) = \gamma_{\rm i}, \tag{D11}$$

we have

$$\text{if } x>0,\ \gamma_{\rm e}<0 \text{ and } \gamma_{\rm i}<\gamma_{\rm i}^* \text{ then } a_{{\rm e}\gamma_{\rm e}(2)} > b_{{\rm i}\gamma_{\rm i}(1)}. \tag{D12}$$

In this circumstance, the interval $(a_{{\rm e}\gamma_{\rm e}(2)}, b_{{\rm e}\gamma_{\rm e}(2)})$ where $\chi^{(0)s_{\rm e}}_{{\rm e}\sigma\gamma_{\rm e}} \neq 0$ (Eq. (42)) and $(a_{{\rm i}\gamma_{\rm i}(1)}, b_{{\rm i}\gamma_{\rm i}(1)})$ where $\Psi^{(0)}_{{\rm i}\gamma_{\rm i}} \neq 0$ (Eq. (D1)) are disjoint, so that, according to Eqs. (D1) and (79),

$$\text{if } x>0,\ \gamma_{\rm e}<0 \text{ and } \gamma_{\rm i}<\gamma_{\rm i}^* \text{ then } h_{{\rm ei}(2;1)}(x,\sigma,\gamma_{\rm e},\gamma_{\rm i}) = 0. \tag{D13}$$

We further observe that, since, in domain 1, $-V_{\rm e}$ is an increasing function of x (Eq. (2) and Fig. 1), given two points a and b, $-V_{\rm e}(a) > -V_{\rm e}(b) \Rightarrow a>b$. Then, from the relations

$$\text{if } -Z_{\rm i}<\gamma_{\rm i}<-Z_{\rm i}(1-U) \text{ then } -V_{\rm e}(a_{{\rm i}\gamma_{\rm i}(1)}) = \gamma_{\rm e}^*, \tag{D14}$$
$$\text{if } \gamma_{\rm i}>-Z_{\rm i}(1-U) \text{ then } -V_{\rm e}(a_{{\rm i}\gamma_{\rm i}(1)}) = -|Z_{\rm e}|U, \tag{D15}$$
$$\text{if } \gamma_{\rm e}<0 \text{ then } -V_{\rm e}(b_{{\rm e}\gamma_{\rm e}(1)}) = \gamma_{\rm e}, \tag{D16}$$

we have

$$\text{if } x<0 \text{ and } \gamma_{\rm e}<-V_{\rm ea} \text{ then } a_{{\rm i}\gamma_{\rm i}(1)} > b_{{\rm e}\gamma_{\rm e}(1)}, \tag{D17}$$

where

$$-V_{\rm ea} = \max(\gamma_{\rm e}^*, -|Z_{\rm e}|U) \text{ i.e.} \tag{D18}$$
$$\text{if } -Z_{\rm i}<\gamma_{\rm i}<-Z_{\rm i}(1-U) \text{ then } -V_{\rm ea} = \gamma_{\rm e}^*, \tag{D19}$$
$$\text{if } \gamma_{\rm i}>-Z_{\rm i}(1-U) \text{ then } -V_{\rm ea} = -|Z_{\rm e}|U \tag{D20}$$

and

$$\gamma_{\rm e}^* = -|Z_{\rm e}|(1+\gamma_{\rm i}/Z_{\rm i}), \tag{D21}$$

In this circumstance, the interval $(a_{{\rm i}\gamma_{\rm i}(1)}, b_{{\rm i}\gamma_{\rm i}(1)})$ where $\chi^{(0)s_{\rm i}}_{{\rm i}\sigma\gamma_{\rm i}} \neq 0$ (Eq. (42)) and $(a_{{\rm e}\gamma_{\rm e}(1)}, b_{{\rm e}\gamma_{\rm e}(1)})$ where $\Psi^{(0)}_{{\rm e}\gamma_{\rm e}(1)} \neq 0$ (Eq. (D1)) are disjoint, so that, according to Eqs. (D1) and (79),

$$\text{if } x<0 \text{ and } \gamma_{\rm e}<-V_{\rm ea} \text{ then } h_{{\rm ie}(1;1)}(x,\sigma,\gamma_{\rm i},\gamma_{\rm e}) = 0, \tag{D22}$$

Last, since, in domain 2, $-V_{\rm e}$ is an decreasing function of x (Eq. (2) and Fig. 1), given two points a and b, $-V_{\rm e}(b) > -V_{\rm e}(a) \Rightarrow a>b$. Then, from the relations

$$\text{if } -Z_{\rm i}<\gamma_{\rm i}<0 \text{ then } -V_{\rm e}(b_{{\rm i}\gamma_{\rm i}(1)}) = \gamma_{\rm e}^*, \tag{D23}$$
$$\text{if } \gamma_{\rm i}>0 \text{ then } b_{{\rm i}\gamma_{\rm i}(1)} = \infty \text{ and } -V_{\rm e}(b_{{\rm i}\gamma_{\rm i}(1)}) = -|Z_{\rm e}|, \tag{D24}$$
$$\text{if } -|Z_{\rm e}|<\gamma_{\rm e}<0 \text{ then } V_{\rm e}(a_{{\rm e}\gamma_{\rm e}(2)}) = \gamma_{\rm e}, \tag{D25}$$

we have

$$\text{if } x>0 \text{ and } \gamma_{\rm e}<-V_{\rm eb} \text{ then } a_{{\rm e}\gamma_{\rm e}(2)} > b_{{\rm i}\gamma_{\rm i}(1)}. \tag{D26}$$

where

$$-V_{\rm eb} = \max(\gamma_{\rm e}^*, -|Z_{\rm e}|) \text{ i.e.} \tag{D27}$$
$$\text{if } -Z_{\rm i}<\gamma_{\rm i}<0 \text{ then } -V_{\rm eb} = \gamma_{\rm e}^*, \tag{D28}$$
$$\text{if } \gamma_{\rm i}>0 \text{ then } -V_{\rm eb} = -|Z_{\rm e}|. \tag{D29}$$

In this circumstance, the interval $(a_{{\rm i}\gamma_{\rm i}(1)}, b_{{\rm i}\gamma_{\rm i}(1)})$ where $\chi^{(0)s_{\rm i}}_{{\rm i}\sigma\gamma_{\rm i}} \neq 0$ (Eq. (42)) and $(a_{{\rm e}\gamma_{\rm e}(2)}, b_{{\rm e}\gamma_{\rm e}(2)})$ where $\Psi^{(0)}_{{\rm e}\gamma_{\rm e}(2)} \neq 0$ (Eq. (D1)) are disjoint, so that, according to Eqs. (D1) and (79),

$$\text{if } x>0 \text{ and } \gamma_{\rm e}<-V_{\rm eb} \text{ then } h_{{\rm ie}(1;2)}(x,\sigma,\gamma_{\rm i},\gamma_{\rm e}) = 0, \tag{D30}$$

Appendix E: Inversion of the Integration Order

Given real numbers $a<b$ and c and the generic functions $f(x,y)$ and $g(x)<c$, when g is not a constant, we write

$$I = \int_a^b {\rm d}x \int_{g(x)}^c {\rm d}y f(x,y) = \int_{g(a)}^{g(b)} {\rm d}t h'(t) \int_t^c {\rm d}y f(h(t),y), \tag{E1}$$

where $t=g(x)$, $h(t)=g^{-1}(t)$ and $h'(t) = [g'(x)|_{x=h(t)}]^{-1}$. If g is an increasing function of x, then $g(b)>t>g(a)$ and we further write

$$I = \int_{g(a)}^{g(b)} {\rm d}t h'(t) \int_t^{g(b)} {\rm d}y f(h(t),y) + \int_{g(a)}^{g(b)} {\rm d}t h'(t) \int_{g(b)}^c {\rm d}y f(h(t),y). \tag{E2}$$

Then we interchange the order of integration according to Fubini's rule:

$$I = \int_{g(a)}^{g(b)} {\rm d}y \int_{g(a)}^{y} {\rm d}t h'(t) f(h(t),y) + \int_{g(b)}^{c} {\rm d}y \int_{g(a)}^{g(b)} {\rm d}t h'(t) f(h(t),y) \tag{E3}$$

and, reverting to the x variable, we finally get

$$\text{if } b>a,\ g'(x)\neq 0 \text{ in } (a,b) \text{ and } g(b)>g(a) \text{ then}$$
$$\int_a^b {\rm d}x \int_{g(x)}^c {\rm d}y f(x,y) = \int_{g(a)}^{g(b)} {\rm d}y \int_a^{g^{-1}(y)} {\rm d}x f(x,y) + \int_{g(b)}^c {\rm d}y \int_a^b {\rm d}x f(x,y). \tag{E4}$$

On the other hand, if g is a decreasing function of x, then $g(a) > t > g(b)$ and, in place of Eq. (E2), we write

$$I = -\int_{g(b)}^{g(a)} \mathrm{d}t h'(t) \int_{t}^{g(a)} \mathrm{d}y f(h(t), y) - \int_{g(b)}^{g(a)} \mathrm{d}t h'(t) \int_{g(a)}^{c} \mathrm{d}y f(h(t), y). \tag{E5}$$

We again interchange the order of integration, so that

$$I = -\int_{g(b)}^{g(a)} \mathrm{d}y \int_{g(b)}^{y} \mathrm{d}t h'(t) f(h(t), y) - \int_{g(a)}^{c} \mathrm{d}y \int_{g(b)}^{g(a)} \mathrm{d}t h'(t) f(h(t), y) \tag{E6}$$

and we finally revert to the x variable to get

$$\text{if } b > a,\ g'(x) \neq 0 \text{ in } (a, b) \text{ and } g(a) > g(b) \text{ then} \\ \int_{a}^{b} \mathrm{d}x \int_{g(x)}^{c} \mathrm{d}y f(x, y) = \int_{g(b)}^{g(a)} \mathrm{d}y \int_{g^{-1}(y)}^{b} \mathrm{d}x f(x, y) + \int_{g(a)}^{c} \mathrm{d}y \int_{a}^{b} \mathrm{d}x f(x, y). \tag{E7}$$

Appendix F: The integral Equation for the Perturbed Electron Eigenfunction Coefficients in Domain 2 and for $\gamma_{\mathrm{e}} < 0$

Eq. (78) for electrons ($\alpha = \mathrm{e}$) for $\gamma_{\mathrm{e}} < 0$ in domain 2 ($\nu_{\mathrm{e}\gamma_{\mathrm{e}}} = 2$) reads

$$(\omega - \sigma) X_{\mathrm{e}(2)}^{s_{\mathrm{e}}}(\omega, \sigma, \gamma_{\mathrm{e}}) = \sum_{\beta=\mathrm{e,i}} H_{\mathrm{e}\beta(2)}(\omega, \sigma, \gamma_{\mathrm{e}}) \tag{F1}$$

where

$$H_{\mathrm{ee}(2)}(\omega, \sigma, \gamma_{\mathrm{e}}) = \int_{a_{\mathrm{e}\gamma_{\mathrm{e}}(2)}}^{b_{\mathrm{e}\gamma_{\mathrm{e}}(2)}} \mathrm{d}x \int_{-V_{\mathrm{e}}(x)}^{\infty} \mathrm{d}\gamma'_{\mathrm{e}} h_{\mathrm{ee}(2;2)}(x, \omega, \sigma, \gamma_{\mathrm{e}}, \gamma'_{\mathrm{e}}), \tag{F2}$$

$$H_{\mathrm{ei}(2)}(\omega, \sigma, \gamma_{\mathrm{e}}) = \int_{a_{\mathrm{e}\gamma_{\mathrm{e}}(2)}}^{b_{\mathrm{e}\gamma_{\mathrm{e}}(2)}} \mathrm{d}x \int_{-V_{\mathrm{i}}(x)}^{\infty} \mathrm{d}\gamma'_{\mathrm{i}} h_{\mathrm{ei}(2;1)}(x, \omega, \sigma, \gamma_{\mathrm{e}}, \gamma'_{\mathrm{i}}), \tag{F3}$$

In the integral extending over negative γ'_{e}, we omitted the contribution of the vanishing quantity $h_{\mathrm{ee}(2;1)}$ (Eqs. (79) and (D2)). in that extending over positive γ'_{e}, we omitted the sum over the electron domain label ν'_{e} because, for $\gamma'_{\mathrm{e}} > 0$, $N_{\mathrm{e}\gamma'_{\mathrm{e}}} = 1$ (Eq. (40)). We also omitted the sum over the ion domain label $\nu_{\mathrm{i}\gamma'_{\mathrm{i}}}$ because $N_{\mathrm{i}\gamma'_{\mathrm{i}}} = 1$ (Eq. (41)).

In domain 2 we have (Fig. 1 and Eqs. (2), (26) and (30))

$$-V'_{\mathrm{e}} < 0, \tag{F4}$$
$$b_{\mathrm{e}\gamma_{\mathrm{e}}(2)} = \infty,\ -V_{\mathrm{e}}(b_{\mathrm{e}\gamma_{\mathrm{e}}(2)}) = -|Z_{\mathrm{e}}|, \tag{F5}$$
$$\text{if } -|Z_{\mathrm{e}}| < \gamma_{\mathrm{e}} < 0 \text{ then } -V_{\mathrm{e}}(a_{\mathrm{e}\gamma_{\mathrm{e}}(2)}) = \gamma_{\mathrm{e}}, \tag{F6}$$
$$\text{if } -|Z_{\mathrm{e}}| < \gamma'_{\mathrm{e}} < 0 \text{ then } [-V_{\mathrm{e}}]^{-1}(\gamma'_{\mathrm{e}}) = a_{\mathrm{e}\gamma'_{\mathrm{e}}(2)}, \tag{F7}$$

Inverting the integration order in Eq. (F2) according to Eq. (E7), which applies when $-V_{\mathrm{e}}$ monotoniclly decreases (Eq. (F4)), and to Eqs. (F5)-(F7), we have

$$\text{if } \gamma_{\mathrm{e}} < 0 \text{ then } H_{\mathrm{ee}(2)}(\omega, \sigma, \gamma_{\mathrm{e}}) = \int_{-|Z_{\mathrm{e}}|}^{\gamma_{\mathrm{e}}} \mathrm{d}\gamma'_{\mathrm{e}} \int_{a_{\mathrm{e}\gamma'_{\mathrm{e}}(2)}}^{b_{\mathrm{e}\gamma_{\mathrm{e}}(2)}} \mathrm{d}x h_{\mathrm{ee}(2;2)}(x, \omega, \sigma, \gamma_{\mathrm{e}}, \gamma'_{\mathrm{e}}) + \int_{\gamma_{\mathrm{e}}}^{0} \mathrm{d}\gamma'_{\mathrm{e}} \int_{a_{\mathrm{e}\gamma_{\mathrm{e}}(2)}}^{b_{\mathrm{e}\gamma_{\mathrm{e}}(2)}} \mathrm{d}x h_{\mathrm{ee}(2;2)}(x, \omega, \sigma, \gamma_{\mathrm{e}}, \gamma'_{\mathrm{e}}) + \int_{0}^{\infty} \mathrm{d}\gamma'_{\mathrm{e}} \int_{a_{\mathrm{e}\gamma_{\mathrm{e}}(2)}}^{b_{\mathrm{e}\gamma_{\mathrm{e}}(2)}} \mathrm{d}x h_{\mathrm{ee}(2;1)}(x, \omega, \sigma, \gamma_{\mathrm{e}}, \gamma'_{\mathrm{e}}). \tag{F8}$$

Using the definitions of the integration endpoints in Eq. (C4), we rewrite Eq. (F8) as

$$\text{if } \gamma_{\mathrm{e}} < 0 \text{ then } H_{\mathrm{ee}(2)}(\omega, \sigma, \gamma_{\mathrm{e}}) = \int_{-|Z_{\mathrm{e}}|}^{\infty} \mathrm{d}\gamma'_{\mathrm{e}} \int_{a_{\mathrm{ee}\gamma_{\mathrm{e}}\gamma'_{\mathrm{e}}(2)}}^{b_{\mathrm{ee}\gamma_{\mathrm{e}}\gamma'_{\mathrm{e}}(2)}} \mathrm{d}x h_{\mathrm{ee}(2;2)}(x, \omega, \sigma, \gamma_{\mathrm{e}}, \gamma'_{\mathrm{e}}). \tag{F9}$$

In a similar way, taking into account that, in domain 2,

$$-V'_{\mathrm{i}} > 0, \tag{F10}$$
$$b_{\mathrm{e}\gamma_{\mathrm{e}}(2)} = \infty,\ -V_{\mathrm{i}}(b_{\mathrm{e}\gamma_{\mathrm{e}}(2)}) = 0, \tag{F11}$$
$$\text{if } -|Z_{\mathrm{e}}| < \gamma_{\mathrm{e}} < 0 \text{ then } -V_{\mathrm{i}}(a_{\mathrm{e}\gamma_{\mathrm{e}}(2)}) = \gamma_{\mathrm{i}}^{*}, \tag{F12}$$
$$\text{if } -Z_{\mathrm{i}} < \gamma'_{\mathrm{i}} < 0 \text{ then } [-V_{\mathrm{i}}]^{-1}(\gamma'_{\mathrm{i}}) = b_{\mathrm{i}\gamma'_{\mathrm{i}}(1)}, \tag{F13}$$

where γ_{i}^{*} was defined in Eq. (D7), we invert the integration order in Eq. (F3) according to Eq. (E4), which applies when when $-V_{\mathrm{i}}$ monotonically increases (Eq. (F10)), and to Eqs. (F11)-(F13):

$$\text{if } \gamma_{\mathrm{e}} < 0 \text{ then } H_{\mathrm{ei}(2)}(\omega, \sigma, \gamma_{\mathrm{e}}) = \int_{\gamma_{\mathrm{i}}^{*}}^{0} \mathrm{d}\gamma'_{\mathrm{i}} \int_{a_{\mathrm{e}\gamma_{\mathrm{e}}(2)}}^{b_{\mathrm{i}\gamma'_{\mathrm{i}}(1)}} \mathrm{d}x h_{\mathrm{ei}(2;1)}(x, \omega, \sigma, \gamma_{\mathrm{e}}, \gamma'_{\mathrm{i}}) + \int_{0}^{\infty} \mathrm{d}\gamma'_{\mathrm{i}} \int_{a_{\mathrm{e}\gamma_{\mathrm{e}}(2)}}^{b_{\mathrm{e}\gamma_{\mathrm{e}}(2)}} \mathrm{d}x h_{\mathrm{ei}(2;1)}(x, \omega, \sigma, \gamma_{\mathrm{e}}, \gamma'_{\mathrm{i}}). \tag{F14}$$

Due to Eq. (D13), the first x-integral of Eq. (F14) remains unchanged if we replace its lower integration bound γ_{i}^{*} by $-Z_{\mathrm{i}}$ which, due to Eq. (D7), is certainly not larger than γ_{i}^{*}. Also, in the second integral of Eq. (F14), $\gamma'_{\mathrm{i}} > 0$ and thus $b_{\mathrm{i}\gamma'_{\mathrm{i}}(1)} = \infty$ (Eq. (31)): since also $b_{\mathrm{e}\gamma_{\mathrm{e}}(2)} = \infty$ (Eq. (30)) it may well be replaced by $b_{\mathrm{i}\gamma'_{\mathrm{i}}(1)}$. In turn, $a_{\mathrm{e}\gamma_{\mathrm{e}}(2)}$ may be renamed to $a_{\mathrm{ei}\gamma_{\mathrm{e}}\gamma'_{\mathrm{i}}(2)}$ (Eq. (C8)) and $b_{\mathrm{i}\gamma'_{\mathrm{i}}(1)}$ to $b_{\mathrm{ei}\gamma_{\mathrm{e}}\gamma'_{\mathrm{i}}(2)}$ (Eq. (C6)) so that Eq. (F14) may be rewritten as

$$\text{if } \gamma_{\mathrm{e}} < 0 \text{ then } H_{\mathrm{ei}(2)}(\omega, \sigma, \gamma_{\mathrm{e}}) = \int_{-Z_{\mathrm{i}}}^{\infty} \mathrm{d}\gamma'_{\mathrm{i}} \int_{a_{\mathrm{ei}\gamma_{\mathrm{e}}\gamma'_{\mathrm{i}}(2)}}^{b_{\mathrm{ei}\gamma_{\mathrm{e}}\gamma'_{\mathrm{i}}(2)}} \mathrm{d}x h_{\mathrm{ei}(2;1)}(x, \omega, \sigma, \gamma_{\mathrm{e}}, \gamma'_{\mathrm{i}}). \tag{F15}$$

Inserting Eqs. (F9) and (F15) into Eq. (F1), we have

$$\text{if } \gamma_{\mathrm{e}} < 0 \text{ then } (\omega - \sigma) X_{\mathrm{e}(2)}^{s_{\mathrm{e}}}(\omega, \sigma, \gamma_{\mathrm{e}}) = \int_{-|Z_{\mathrm{e}}|}^{\infty} \mathrm{d}\gamma'_{\mathrm{e}} \int_{a_{\mathrm{ee}\gamma_{\mathrm{e}}\gamma'_{\mathrm{e}}(2)}}^{b_{\mathrm{ee}\gamma_{\mathrm{e}}\gamma'_{\mathrm{e}}(2)}} \mathrm{d}x h_{\mathrm{ee}(2;2)}(x, \omega, \sigma, \gamma_{\mathrm{e}}, \gamma'_{\mathrm{e}}) + \int_{-Z_{\mathrm{i}}}^{\infty} \mathrm{d}\gamma'_{\mathrm{i}} \int_{a_{\mathrm{ei}\gamma_{\mathrm{e}}\gamma'_{\mathrm{i}}(2)}}^{b_{\mathrm{ei}\gamma_{\mathrm{e}}\gamma'_{\mathrm{i}}(2)}} \mathrm{d}x h_{\mathrm{ei}(2;1)}(x, \omega, \sigma, \gamma_{\mathrm{e}}, \gamma'_{\mathrm{i}}). \tag{F16}$$

Finally, reverting to the definitions of $h_{\mathrm{ee}(2;2)}$ and $h_{\mathrm{ei}(2;1)}$ (Eqs. (77) and (79)), Eq. (F16) appears as a particular case of Eq. (97).

Appendix G: The integral Equation for the Perturbed Electron Eigenfunction Coefficients for $\gamma_{\rm e} > 0$

For electrons ($\alpha = {\rm e}$), $\nu_{{\rm e}\gamma_{\rm e}} = 1$ is the only value available when $\gamma_{\rm e} > 0$ (Eqs. (36) and (40)) in Eq. (80), which thus reads

$$(\omega - \sigma) X^{s_{\rm e}}_{{\rm e}(1)}(\omega, \sigma, \gamma_{\rm e}) = \sum_{\beta={\rm e,i}} H_{{\rm e}\beta(1)}(\omega, \sigma, \gamma_{\rm e}), \tag{G1}$$

where

$$H_{\rm ee(1)}(\omega, \sigma, \gamma_{\rm e}) = \int_{a_{{\rm e}\gamma_{\rm e}(1)}}^{b_{{\rm e}\gamma_{\rm e}(1)}} {\rm d}x \int_{-V_{\rm e}(x)}^{\infty} {\rm d}\gamma'_{\rm e} \sum_{\nu'_{\rm e}=1}^{N_{{\rm e}\gamma'_{\rm e}}} h_{{\rm ee}(1;\nu'_{\rm e})}(x, \omega, \sigma, \gamma_{\rm e}, \gamma'_{\rm e}) \tag{G2}$$

$$H_{\rm ei(1)}(\omega, \sigma, \gamma_{\rm e}) = \int_{a_{{\rm e}\gamma_{\rm e}(1)}}^{b_{{\rm e}\gamma_{\rm e}(1)}} {\rm d}x \int_{-V_{\rm i}(x)}^{\infty} {\rm d}\gamma'_{\rm i} h_{{\rm ei}(1;1)}(x, \omega, \sigma, \gamma_{\rm e}, \gamma'_{\rm i}), \tag{G3}$$

and we omitted the sum over the ion domain label $\nu'_{\rm i}$ because $N_{{\rm i}\gamma'_{\rm i}} = 1$ (Eq. (41)).

The x-integration interval in Eq. (G2) will now be split into two parts:

$$\text{if } \gamma_{\rm e} > 0 \text{ then } H_{\rm ee(1)} = H^{\rm left}_{\rm ee(1)} + H^{\rm right}_{\rm ee(1)}. \tag{G4}$$

The left part ends at $x = 0$ and the right part starts at $x = 0$. In each part $-V_\beta(x)$ is a monotonic function and we may invert the x and γ'_β order of integration. Specifically, in the left part of the x-integration, we have (Fig. 1 and Eqs. (2), (26) and (29)),

$$-V'_{\rm e} > 0, \tag{G5}$$
$$a_{{\rm e}\gamma_{\rm e}(1)} = -\infty,\ -V_{\rm e}(a_{{\rm e}\gamma_{\rm e}(1)}) = -|Z_{\rm e}|U, \tag{G6}$$
$$-V_{\rm e}(0) = 0, \tag{G7}$$
$$\text{if } \gamma'_{\rm e} < 0 \text{ then } [-V_{\rm e}]^{-1}(\gamma'_{\rm e}) = b_{{\rm e}\gamma'_{\rm e}(1)}, \tag{G8}$$

and we invert the integration order in Eq. (G2) according to Eq. (E4), which applies when Eq. (G5) holds, and to Eqs. (G6)-(G8):

$$\text{if } \gamma_{\rm e} > 0 \text{ then } H^{\rm left}_{\rm ee(1)}(\omega, \sigma, \gamma_{\rm e}) = \int_{-|Z_{\rm e}|U}^{0} {\rm d}\gamma'_{\rm e} \int_{a_{{\rm e}\gamma_{\rm e}(1)}}^{b_{{\rm e}\gamma'_{\rm e}(1)}} {\rm d}x h_{{\rm ee}(1;1)}(x, \omega, \sigma, \gamma_{\rm e}, \gamma'_{\rm e}) + \int_{0}^{\infty} {\rm d}\gamma'_{\rm e} \int_{a_{{\rm e}\gamma_{\rm e}(1)}}^{0} {\rm d}x h_{{\rm ee}(1;1)}(x, \omega, \sigma, \gamma_{\rm e}, \gamma'_{\rm e}). \tag{G9}$$

In the second integral of Eq. (G10), extending over positive $\gamma'_{\rm e}$, we omitted the sum over the electron domain label $\nu'_{\rm e}$ because, for $\gamma'_{\rm e} > 0$, $N_{{\rm e}\gamma'_{\rm e}} = 1$ (Eqs. (36) and (40)). In the first integral of Eq. (G10), extending over negative $\gamma'_{\rm e}$, we omitted the contribution of $h_{{\rm ee}(1;2)}$ which identically vanishes for $x < 0$ and $\gamma'_{\rm e} < 0$ (Eq. (D2)); Due to Eq. (D22), in the first $\gamma'_{\rm e}$-integral the lower integration bound $-|Z_{\rm e}|U$ may be replaced by $-|Z_{\rm e}|U$ which, due to Eq. (27)), is smaller than $|Z_{\rm e}|U$. In the first x-integral, we may also rename the lower x-integration bound $a_{{\rm e}\gamma_{\rm e}(1)}$ (the same appearing in Eq. (G2)) to $a_{{\rm e}\gamma'_{\rm e}(1)}$: both equal $-\infty$ for $x < 0$ (Eq. (28) and (30)). Eq. (G10) may thus be rearranged as

$$\text{if } \gamma_{\rm e} > 0 \text{ then } H^{\rm left}_{\rm ee(1)}(\omega, \sigma, \gamma_{\rm e}) = \int_{-|Z_{\rm e}|}^{0} {\rm d}\gamma'_{\rm e} \int_{a_{{\rm e}\gamma'_{\rm e}(1)}}^{b_{{\rm e}\gamma'_{\rm e}(1)}} {\rm d}x h_{{\rm ee}(1;1)}(x, \omega, \sigma, \gamma_{\rm e}, \gamma'_{\rm e}) + \int_{0}^{\infty} {\rm d}\gamma'_{\rm e} \int_{a_{{\rm e}\gamma_{\rm e}(1)}}^{0} {\rm d}x h_{{\rm ee}(1;1)}(x, \omega, \sigma, \gamma_{\rm e}, \gamma'_{\rm e}). \tag{G10}$$

In the right part of the x-integration, we have (Fig. 1 and Eqs. (2), (26) and (29)),

$$-V'_{\rm e} < 0, \tag{G11}$$
$$b_{{\rm e}\gamma_{\rm e}(1)} = \infty,\ -V_{\rm e}(b_{{\rm e}\gamma_{\rm e}(1)}) = -|Z_{\rm e}|, \tag{G12}$$
$$-V_{\rm e}(0) = 0, \tag{G13}$$
$$\text{if } -|Z_{\rm e}| < \gamma'_{\rm e} < 0 \text{ then } [-V_{\rm e}]^{-1}(\gamma'_{\rm e}) = a_{{\rm e}\gamma'_{\rm e}(2)}, \tag{G14}$$

and we invert the integration order in Eq. (G2) according to Eq. (E7), which applies when Eq. (G11) holds, and to Eqs. (G12)-(G14):

$$\text{if } \gamma_{\rm e} > 0 \text{ then } H^{\rm right}_{\rm ee(1)}(\omega, \sigma, \gamma_{\rm e}) = \int_{-|Z_{\rm e}|}^{0} {\rm d}\gamma'_{\rm e} \int_{a_{{\rm e}\gamma'_{\rm e}(2)}}^{b_{{\rm e}\gamma_{\rm e}(2)}} {\rm d}x h_{{\rm ee}(1;2)}(x, \omega, \sigma, \gamma_{\rm e}, \gamma'_{\rm e}) + \int_{0}^{\infty} {\rm d}\gamma'_{\rm e} \int_{0}^{b_{{\rm e}\gamma_{\rm e}(1)}} {\rm d}x h_{{\rm ee}(1;1)}(x, \omega, \sigma, \gamma_{\rm e}, \gamma'_{\rm e}). \tag{G15}$$

In the second integral of Eq. (G16), extending over positive $\gamma'_{\rm e}$, we omitted the sum over the electron domain label $\nu'_{\rm e}$ (Eq. (G2)) because, for $\gamma'_{\rm e} > 0$, $N_{{\rm e}\gamma'_{\rm e}} = 1$ (Eqs. (36) and (40)). In the first integral of Eq. (G16), extending over negative $\gamma'_{\rm e}$, we omitted the contribution of the vanishing quantity $h_{{\rm ee}(1;1)}$ which identically vanishes for $x < 0$ and $\gamma'_{\rm e} < 0$ (Eq. (D2)). In the first x-integral of Eq. (G16) we may rename the upper x-integration bound $b_{{\rm e}\gamma_{\rm e}(1)}$ (the same appearing in Eq. (G3)) to $b_{{\rm e}\gamma'_{\rm e}(2)}$: both equal ∞ for $x > 0$ (Eq. (28) and (30)). Eq. (G15) may thus be rewritten as

$$\text{if } \gamma_{\rm e} > 0 \text{ then } H^{\rm right}_{\rm ee(1)}(\omega, \sigma, \gamma_{\rm e}) = \int_{-|Z_{\rm e}|}^{0} {\rm d}\gamma'_{\rm e} \int_{a_{{\rm e}\gamma'_{\rm e}(2)}}^{b_{{\rm e}\gamma'_{\rm e}(2)}} {\rm d}x h_{{\rm ee}(1;2)}(x, \omega, \sigma, \gamma_{\rm e}, \gamma'_{\rm e}) + \int_{0}^{\infty} {\rm d}\gamma'_{\rm e} \int_{0}^{b_{{\rm e}\gamma_{\rm e}(1)}} {\rm d}x h_{{\rm ee}(1;1)}(x, \omega, \sigma, \gamma_{\rm e}, \gamma'_{\rm e}). \tag{G16}$$

Inserting Eqs. (G10) and (G16) into Eq. (G4), we get

$$\text{if } \gamma_{\rm e} > 0 \text{ then } H_{\rm ee(1)}(\omega, \sigma, \gamma_{\rm e}) = \int_{-|Z_{\rm e}|}^{0} {\rm d}\gamma'_{\rm e} \int_{a_{{\rm e}\gamma'_{\rm e}(1)}}^{b_{{\rm e}\gamma'_{\rm e}(1)}} {\rm d}x h_{{\rm ee}(1;1)}(x, \omega, \sigma, \gamma_{\rm e}, \gamma'_{\rm e}) + \int_{-|Z_{\rm e}|}^{0} {\rm d}\gamma'_{\rm e} \int_{a_{{\rm e}\gamma'_{\rm e}(2)}}^{b_{{\rm e}\gamma'_{\rm e}(2)}} {\rm d}x h_{{\rm ee}(1;2)}(x, \omega, \sigma, \gamma_{\rm e}, \gamma'_{\rm e}) + \int_{0}^{\infty} {\rm d}\gamma'_{\rm e} \int_{a_{{\rm e}\gamma_{\rm e}(1)}}^{b_{{\rm e}\gamma_{\rm e}(1)}} {\rm d}x h_{{\rm ee}(1;1)}(x, \omega, \sigma, \gamma_{\rm e}, \gamma'_{\rm e}). \tag{G17}$$

In the first two $\gamma'_{\rm e}$-integrals of Eq. (G17), $\gamma'_{\rm e} < \gamma_{\rm e}$. According to Eq. (C4), the x-integration bounds $a_{{\rm e}\gamma'_{\rm e}(\nu'_{\rm e})}$, $b_{{\rm e}\gamma'_{\rm e}(\nu'_{\rm e})}$ (for $\nu'_{\rm e} = 1, 2$) may be renamed to $a_{{\rm ee}\gamma_{\rm e}\gamma'_{\rm e}(\nu'_{\rm e})}$, $b_{{\rm ee}\gamma_{\rm e}\gamma'_{\rm e}(\nu'_{\rm e})}$ respectively. In the third $\gamma'_{\rm e}$-integral of Eq. (G17), $\gamma'_{\rm e} > 0$

and, by assumption, $\gamma_\mathrm{e} > 0$; thus $a_{\mathrm{e}\gamma_\mathrm{e}(1)} = -\infty = a_{\mathrm{e}\gamma'_\mathrm{e}(1)}$, $b_{\mathrm{e}\gamma_\mathrm{e}(1)} = \infty = b_{\mathrm{e}\gamma'_\mathrm{e}(1)}$ (Eq. (28)) so that, again according to Eq. (C4), $a_{\mathrm{e}\gamma_\mathrm{e}(1)}$, $b_{\mathrm{e}\gamma_\mathrm{e}(1)}$ may be renamed to $a_{\mathrm{ee}\gamma_\mathrm{e}\gamma'_\mathrm{e}(1)}$, $b_{\mathrm{ee}\gamma_\mathrm{e}\gamma'_\mathrm{e}(1)}$ respectively. As a result, Eq. (G17) may be rewritten as

$$\text{if } \gamma_\mathrm{e} > 0 \text{ then } H_{\mathrm{ee}(1)}(\omega,\sigma,\gamma_\mathrm{e}) = \int_{-|Z_\mathrm{e}|}^{\infty} \mathrm{d}\gamma'_\mathrm{e} \sum_{\nu'_\mathrm{e}=1}^{N_{\mathrm{e}\gamma'_\mathrm{e}}} \int_{a_{\mathrm{ee}\gamma_\mathrm{e}\gamma'_\mathrm{e}(\nu'_\mathrm{e})}}^{b_{\mathrm{ee}\gamma_\mathrm{e}\gamma'_\mathrm{e}(\nu'_\mathrm{e})}} \mathrm{d}x h_{\mathrm{ee}(1;\nu'_\mathrm{e})}(x,\omega,\sigma,\gamma_\mathrm{e},\gamma'_\mathrm{e}). \tag{G18}$$

We now turn to Eq. (G3) which and again we split the x-integration:

$$H_{\mathrm{ei}(1)} = H^{\mathrm{left}}_{\mathrm{ei}(1)} + H^{\mathrm{right}}_{\mathrm{ei}(1)}. \tag{G19}$$

Taking into account that, in the left part of the x-integration,

$$-V'_\mathrm{i} < 0, \tag{G20}$$
$$a_{\mathrm{e}\gamma_\mathrm{e}(1)} = -\infty, -V_\mathrm{i}(a_{\mathrm{e}\gamma_\mathrm{e}(1)}) = -Z_\mathrm{i}(1-U), \tag{G21}$$
$$-V_\mathrm{i}(0) = -Z_\mathrm{i}, \tag{G22}$$
$$\text{if } -Z_\mathrm{i} < \gamma'_\mathrm{i} < Z_\mathrm{i}(1-U) \text{ then } [-V_\mathrm{i}]^{-1}(\gamma'_\mathrm{i}) = a_{\mathrm{i}\gamma'_\mathrm{i}(1)}, \tag{G23}$$

we invert the integration order in Eq. (G3) according to Eq. (E7), which applies when Eq. (G20) holds, and to Eqs. (G21)-(G23):

$$\text{if } \gamma_\mathrm{e} > 0 \text{ then } H^{\mathrm{left}}_{\mathrm{ei}(1)}(\omega,\sigma,\gamma_\mathrm{e}) = \int_{-Z_\mathrm{i}}^{-Z_\mathrm{i}(1-U)} \mathrm{d}\gamma'_\mathrm{i} \int_{a_{\mathrm{i}\gamma'_\mathrm{i}(1)}}^{0} \mathrm{d}x h_{\mathrm{ei}(1;1)}(x,\omega,\sigma,\gamma_\mathrm{e},\gamma'_\mathrm{i}) + \int_{-Z_\mathrm{i}(1-U)}^{\infty} \mathrm{d}\gamma'_\mathrm{i} \int_{a_{\mathrm{e}\gamma_\mathrm{e}(1)}}^{0} \mathrm{d}x h_{\mathrm{ei}(1;1)}(x,\omega,\sigma,\gamma_\mathrm{e},\gamma'_\mathrm{i}). \tag{G24}$$

In a similar way, taking into account that, on the right part of the integration,

$$-V'_\mathrm{i} > 0, \tag{G25}$$
$$b_{\mathrm{e}\gamma_\mathrm{e}(2)} = \infty, \; -V_\mathrm{i}(b_{\mathrm{e}\gamma_\mathrm{e}(2)}) = 0, \tag{G26}$$
$$-V_\mathrm{i}(0) = -Z_\mathrm{i}, \tag{G27}$$
$$\text{if } -Z_\mathrm{i} < \gamma'_\mathrm{i} < 0 \text{ then } [-V_\mathrm{i}]^{-1}(\gamma'_\mathrm{i}) = b_{\mathrm{i}\gamma'_\mathrm{i}(1)}, \tag{G28}$$

we invert the integration order in Eq. (G3) according to Eq. (E4), which applies when Eq. (G25) holds, and to Eqs. (G26)- (G28):

$$\text{if } \gamma_\mathrm{e} > 0 \text{ then } H^{\mathrm{right}}_{\mathrm{ei}(1)}(\omega,\sigma,\gamma_\mathrm{e}) = \int_{-Z_\mathrm{i}}^{0} \mathrm{d}\gamma'_\mathrm{i} \int_{0}^{b_{\mathrm{i}\gamma'_\mathrm{i}(1)}} \mathrm{d}x h_{\mathrm{ei}(1;1)}(x,\omega,\sigma,\gamma_\mathrm{e},\gamma'_\mathrm{i}) + \int_{0}^{\infty} \mathrm{d}\gamma'_\mathrm{i} \int_{0}^{b_{\mathrm{e}\gamma_\mathrm{e}(1)}} \mathrm{d}x h_{\mathrm{ei}(1;1)}(x,\omega,\sigma,\gamma_\mathrm{e},\gamma'_\mathrm{i}). \tag{G29}$$

Inserting Eqs. (G24) and (G29) into Eq. (G3), we write

$$\begin{aligned}\text{if } \gamma_\mathrm{e} > 0 \text{ then } H^{\mathrm{left}}_{\mathrm{ei}(1)}(\omega,\sigma,\gamma_\mathrm{e}) = {}& \int_{-Z_\mathrm{i}}^{-Z_\mathrm{i}(1-U)} \mathrm{d}\gamma'_\mathrm{i} \int_{a_{\mathrm{i}\gamma'_\mathrm{i}(1)}}^{0} \mathrm{d}x h_{\mathrm{ei}(1;1)}(x,\omega,\sigma,\gamma_\mathrm{e},\gamma'_\mathrm{i}) + \\ & \int_{-Z_\mathrm{i}(1-U)}^{0} \mathrm{d}\gamma'_\mathrm{i} \int_{a_{\mathrm{e}\gamma_\mathrm{e}(1)}}^{0} \mathrm{d}x h_{\mathrm{ei}(1;1)}(x,\omega,\sigma,\gamma_\mathrm{e},\gamma'_\mathrm{i}) + \\ & \int_{0}^{\infty} \mathrm{d}\gamma'_\mathrm{i} \int_{a_{\mathrm{e}\gamma_\mathrm{e}(1)}}^{0} \mathrm{d}x h_{\mathrm{ei}(1;1)}(x,\omega,\sigma,\gamma_\mathrm{e},\gamma'_\mathrm{i}) + \\ & \int_{-Z_\mathrm{i}}^{-Z_\mathrm{i}(1-U)} \mathrm{d}\gamma'_\mathrm{i} \int_{0}^{b_{\mathrm{i}\gamma'_\mathrm{i}(1)}} \mathrm{d}x h_{\mathrm{ei}(1;1)}(x,\omega,\sigma,\gamma_\mathrm{e},\gamma'_\mathrm{i}) + \\ & \int_{-Z_\mathrm{i}(1-U)}^{0} \mathrm{d}\gamma'_\mathrm{i} \int_{0}^{b_{\mathrm{i}\gamma'_\mathrm{i}(1)}} \mathrm{d}x h_{\mathrm{ei}(1;1)}(x,\omega,\sigma,\gamma_\mathrm{e},\gamma'_\mathrm{i}) + \\ & \int_{0}^{\infty} \mathrm{d}\gamma'_\mathrm{i} \int_{0}^{b_{\mathrm{e}\gamma_\mathrm{e}(1)}} \mathrm{d}x h_{\mathrm{ei}(1;1)}(x,\omega,\sigma,\gamma_\mathrm{e},\gamma'_\mathrm{i}).\end{aligned} \tag{G30}$$

i.e.

$$\begin{aligned}\text{if } \gamma_\mathrm{e} > 0 \text{ then } H^{\mathrm{left}}_{\mathrm{ei}(1)}(\omega,\sigma,\gamma_\mathrm{e}) = {}& \int_{-Z_\mathrm{i}}^{-Z_\mathrm{i}(1-U)} \mathrm{d}\gamma'_\mathrm{i} \int_{a_{\mathrm{i}\gamma'_\mathrm{i}(1)}}^{b_{\mathrm{i}\gamma'_\mathrm{i}(1)}} \mathrm{d}x h_{\mathrm{ei}(1;1)}(x,\omega,\sigma,\gamma_\mathrm{e},\gamma'_\mathrm{i}) + \\ & \int_{-Z_\mathrm{i}(1-U)}^{0} \mathrm{d}\gamma'_\mathrm{i} \int_{a_{\mathrm{e}\gamma_\mathrm{e}(1)}}^{b_{\mathrm{i}\gamma'_\mathrm{i}(1)}} \mathrm{d}x h_{\mathrm{ei}(1;1)}(x,\omega,\sigma,\gamma_\mathrm{e},\gamma'_\mathrm{i}) + \\ & \int_{0}^{\infty} \mathrm{d}\gamma'_\mathrm{i} \int_{a_{\mathrm{e}\gamma_\mathrm{e}(1)}}^{b_{\mathrm{e}\gamma_\mathrm{e}(1)}} \mathrm{d}x h_{\mathrm{ei}(1;1)}(x,\omega,\sigma,\gamma_\mathrm{e},\gamma'_\mathrm{i})\end{aligned} \tag{G31}$$

In the second and third γ'_i-integrals of Eq. (G31), $\gamma'_\mathrm{i} > -Z_\mathrm{i}(1-U)$ and, by assumption, $\gamma_\mathrm{e} > 0$. Thus $a_{\mathrm{e}\gamma_\mathrm{e}(1)}$ takes the same value ($-\infty$) of $a_{\mathrm{i}\gamma'_\mathrm{i}(1)}$ (Eqs. (28) and (32)) and may be replaced by it. Furthermore, in the third γ'_i-integral of Eq. (G31), $\gamma'_\mathrm{i} > 0$ and thus $b_{\mathrm{e}\gamma_\mathrm{e}(1)}$ takes the same value (∞) of $b_{\mathrm{i}\gamma'_\mathrm{i}(1)}$ (Eq. (31) and may be replaced by it.

In this way, all three γ'_i-integrals in Eq. (G31) now have the same x-integration bounds $a_{\mathrm{i}\gamma'_\mathrm{i}(1)}$ and $b_{\mathrm{i}\gamma'_\mathrm{i}(1)}$. According to Eqs. (C5) and (??), we rename these bounds to $a_{\mathrm{ei}\gamma_\mathrm{e}\gamma'_\mathrm{i}(1)}$ and $b_{\mathrm{ei}\gamma_\mathrm{e}\gamma'_\mathrm{i}(1)}$ respectively, so that Eq. (G32) reduces to

$$\text{if } \gamma_\mathrm{e} > 0 \text{ then } H_{\mathrm{ei}(1)}(\omega,\sigma,\gamma_\mathrm{e}) = \int_{-Z_\mathrm{i}}^{\infty} \mathrm{d}\gamma'_\mathrm{i} \int_{a_{\mathrm{ei}\gamma_\mathrm{e}\gamma'_\mathrm{i}(1)}}^{b_{\mathrm{ei}\gamma_\mathrm{e}\gamma'_\mathrm{i}(1)}} \mathrm{d}x h_{\mathrm{ei}(1;1)}(x,\omega,\sigma,\gamma_\mathrm{e},\gamma'_\mathrm{i}). \tag{G32}$$

Finally, inserting Eqs. (G18) and (G32) into Eq. (G1), and using the definitions of $h_{\mathrm{ee}(1;\nu'_\mathrm{e})}$ and $h_{\mathrm{ei}(1;1)}$ (Eqs. (77) and (79)), we see that the result fits in the general formula given in Eq. (97).

Appendix H: The Integral Equation for the Perturbed Ion Eigenfunction Coefficients

For ions ($\alpha = \mathrm{i}$), $\nu_{\mathrm{i}\gamma_\mathrm{i}} = 1$ is the only value available (Eqs. (36) and (41)) in Eq. (78), which thus reads

$$(\omega - \sigma) X^{s_\mathrm{i}}_{\mathrm{i}(1)}(\omega,\sigma,\gamma_\mathrm{i}) = \sum_{\beta=\mathrm{i,e}} H_{\mathrm{i}\beta(1)}(\omega,\sigma,\gamma_\mathrm{i}) \tag{H1}$$

where

$$H_{\mathrm{ii}(1)}(\omega,\sigma,\gamma_\mathrm{i}) = \int_{a_{\mathrm{i}\gamma_\mathrm{i}(1)}}^{b_{\mathrm{i}\gamma_\mathrm{i}(1)}} \mathrm{d}x \int_{-V_\mathrm{i}(x)}^{\infty} \mathrm{d}\gamma'_\mathrm{i} h_{\mathrm{ii}(1;1)}(x,\omega,\sigma,\gamma_\mathrm{i},\gamma'_\mathrm{i}), \tag{H2}$$

$$H_{\mathrm{ie}(1)}(\omega,\sigma,\gamma_\mathrm{i}) = \int_{a_{\mathrm{i}\gamma_\mathrm{i}(1)}}^{b_{\mathrm{i}\gamma_\mathrm{i}(1)}} \mathrm{d}x \int_{-V_\mathrm{e}(x)}^{\infty} \mathrm{d}\gamma'_\mathrm{e} \sum_{\nu'_\mathrm{e}=1}^{N_{\mathrm{e}\gamma'_\mathrm{e}}} h_{\mathrm{ie}(1;\nu'_\mathrm{e})}(x,\omega,\sigma,\gamma_\mathrm{i},\gamma'_\mathrm{e}), \tag{H3}$$

and we omitted the sum over the ion domain label $\nu_{\rm i}'$ because $N_{{\rm i}\gamma_{\rm i}'}=1$ (Eq. (41)).

In Eq. (H2), x-integration will now be split into two parts:

$$H_{{\rm ii}(1)}=H^{\rm left}_{{\rm ii}(1)}+H^{\rm right}_{{\rm ii}(1)}, \tag{H4}$$

The left part runs in domain 1, up to $x=0$ and the right part runs in domain 2, starting at $x=0$. In each part $-V_\beta(x)$ is a monotonic function and we may invert the x and γ_β' order of integration.

Specifically, in the left part, we have (Fig. 1 and Eqs. (2), (26) and (29)),

$$-V_{\rm i}'<0, \tag{H5}$$
$$\text{if } -Z_{\rm i}<\gamma_{\rm i}<-Z_{\rm i}(1-U) \text{ then}$$
$$-V_{\rm i}(a_{{\rm i}\gamma_{\rm i}(1)})=\gamma_{\rm i}, \tag{H6}$$
$$\text{if } \gamma_{\rm i}>-Z_{\rm i}(1-U) \text{ then}$$
$$a_{{\rm i}\gamma_{\rm i}(1)}=-\infty,\ -V_{\rm i}(a_{{\rm i}\gamma_{\rm i}(1)})=-Z_{\rm i}(1-U), \tag{H7}$$
$$-V_{\rm i}(0)=-Z_{\rm i}, \tag{H8}$$
$$\text{if } -Z_{\rm i}<\gamma_{\rm i}'<-Z_{\rm i}(1-U) \text{ then}$$
$$[-V_{\rm i}]^{-1}(\gamma_{\rm i}')=a_{{\rm i}\gamma_{\rm i}'(1)}. \tag{H9}$$

Inverting the integration order in Eq. (H2) according to Eq. (E7), which applies when $-V_{\rm i}$ monotonically decreases (Eq. (H5)), and to Eqs. (H6)-(H9), we have, for both $\gamma_{\rm i}<-Z_{\rm i}(1-U)$ and $\gamma_{\rm i}>-Z_{\rm i}(1-U)$,

$$H^{\rm left}_{{\rm ii}(1)}(\omega,\sigma,\gamma_{\rm i})=$$
$$\int_{-Z_{\rm i}}^{-V_{{\rm i}a}}{\rm d}\gamma_{\rm i}'\int_{a_{{\rm i}\gamma_{\rm i}'(1)}}^{0}{\rm d}x h_{{\rm ii}(1;1)}(x,\omega,\sigma,\gamma_{\rm i},\gamma_{\rm i}')+$$
$$\int_{-V_{{\rm i}a}}^{\infty}{\rm d}\gamma_{\rm i}'\int_{a_{{\rm i}\gamma_{\rm i}(1)}}^{0}{\rm d}x h_{{\rm ii}(1;1)}(x,\omega,\sigma,\gamma_{\rm i},\gamma_{\rm i}'), \tag{H10}$$

where

$$-V_{{\rm i}a}=\min(\gamma_{\rm i},-Z_{\rm i}[1-U]) \text{ i.e.} \tag{H11}$$
$$\text{if } -Z_{\rm i}<\gamma_{\rm i}<-Z_{\rm i}[1-U] \text{ then } -V_{{\rm i}a}=\gamma_{\rm i}, \tag{H12}$$
$$\text{if } \gamma_{\rm i}>-Z_{\rm i}[1-U] \text{ then } -V_{{\rm i}a}=-Z_{\rm i}[1-U]. \tag{H13}$$

In the right part of the integration, the ion parameters are (Fig. 1 and Eqs. (2), (26) and (29)),

$$-V_{\rm i}'>0,\ -V_{\rm i}(0)=-Z_{\rm i}, \tag{H14}$$
$$\text{if } -Z_{\rm i}<\gamma_{\rm i}<0 \text{ then } -V_{\rm i}(b_{{\rm i}\gamma_{\rm i}(1)})=\gamma_{\rm i}, \tag{H15}$$
$$\text{if } \gamma_{\rm i}>0 \text{ then } b_{{\rm i}\gamma_{\rm i}(1)}=\infty \text{ and } -V_{\rm i}(b_{{\rm i}\gamma_{\rm i}(1)})=0, \tag{H16}$$
$$-V_{\rm i}(0)=-Z_{\rm i}, \tag{H17}$$
$$\text{if } -Z_{\rm i}<\gamma_{\rm i}'<0 \text{ then } [-V_{\rm i}]^{-1}(\gamma_{\rm i}')=b_{{\rm i}\gamma_{\rm i}'(1)}, \tag{H18}$$

Inverting the integration order in Eq. (H2) according to Eq. (E4), which applies when $-V_{\rm i}$ monotonically increases (Eq. (H14)), and to Eqs. (H15)-(H18), we have, for both $\gamma_{\rm i}<0$ and $\gamma_{\rm i}>0$,

$$H^{\rm right}_{{\rm ii}(1)}(\omega,\sigma,\gamma_{\rm i})=$$
$$\int_{-Z_{\rm i}}^{-V_{{\rm i}b}}{\rm d}\gamma_{\rm i}'\int_0^{b_{{\rm i}\gamma_{\rm i}'(1)}}{\rm d}x h_{{\rm ii}(1;1)}(x,\omega,\sigma,\gamma_{\rm i},\gamma_{\rm i}')+$$
$$\int_{-V_{{\rm i}b}}^{\infty}{\rm d}\gamma_{\rm i}'\int_0^{b_{{\rm i}\gamma_{\rm i}(1)}}{\rm d}x h_{{\rm ii}(1;1)}(x,\omega,\sigma,\gamma_{\rm i},\gamma_{\rm i}'), \tag{H19}$$

where

$$-V_{{\rm i}b}=\min(\gamma_{\rm i},0) \text{ i.e.} \tag{H20}$$
$$\text{if } -Z_{\rm i}<\gamma_{\rm i}<0 \text{ then } -V_{{\rm i}b}=\gamma_{\rm i}, \tag{H21}$$
$$\text{if } \gamma_{\rm i}>0 \text{ then } -V_{{\rm i}b}=0. \tag{H22}$$

We now add both sides of Eqs. (H10) and (H19), as in Eq. (H4), and we first consider the case $-Z_{\rm i}<\gamma_{\rm i}<-Z_{\rm i}(1-U)$. In this case, in Eq. (H10), $-V_{{\rm i}a}=\gamma_{\rm i}=-V_{{\rm i}b}$ (Eqs. (H11) and (H20)) and we have

$$\text{if } -Z_{\rm i}<\gamma_{\rm i}<-Z_{\rm i}(1-U) \text{ then } H_{{\rm ii}(1)}(\omega,\sigma,\gamma_{\rm i})=$$
$$\int_{-Z_{\rm i}}^{-V_{{\rm i}b}}{\rm d}\gamma_{\rm i}'\int_{a_{{\rm i}\gamma_{\rm i}'(1)}}^{0}{\rm d}x h_{{\rm ii}(1;1)}(x,\omega,\sigma,\gamma_{\rm i},\gamma_{\rm i}')+$$
$$\int_{-V_{{\rm i}b}}^{\infty}{\rm d}\gamma_{\rm i}'\int_{a_{{\rm i}\gamma_{\rm i}(1)}}^{0}{\rm d}x h_{{\rm ii}(1;1)}(x,\omega,\sigma,\gamma_{\rm i},\gamma_{\rm i}')+$$
$$\int_{-Z_{\rm i}}^{-V_{{\rm i}b}}{\rm d}\gamma_{\rm i}'\int_0^{b_{{\rm i}\gamma_{\rm i}'(1)}}{\rm d}x h_{{\rm ii}(1;1)}(x,\omega,\sigma,\gamma_{\rm i},\gamma_{\rm i}')+$$
$$\int_{-V_{{\rm i}b}}^{\infty}{\rm d}\gamma_{\rm i}'\int_0^{b_{{\rm i}\gamma_{\rm i}(1)}}{\rm d}x h_{{\rm ii}(1;1)}(x,\omega,\sigma,\gamma_{\rm i},\gamma_{\rm i}'), \tag{H23}$$

i.e.

$$\text{if } -Z_{\rm i}<\gamma_{\rm i}<-Z_{\rm i}(1-U) \text{ then } H_{{\rm ii}(1)}(\omega,\sigma,\gamma_{\rm i})=$$
$$\int_{-Z_{\rm i}}^{-V_{{\rm i}b}}{\rm d}\gamma_{\rm i}'\int_{a_{{\rm i}\gamma_{\rm i}'(1)}}^{b_{{\rm i}\gamma_{\rm i}'(1)}}{\rm d}x h_{{\rm ii}(1;1)}(x,\omega,\sigma,\gamma_{\rm i},\gamma_{\rm i}')+$$
$$\int_{-V_{{\rm i}b}}^{\infty}{\rm d}\gamma_{\rm i}'\int_{a_{{\rm i}\gamma_{\rm i}(1)}}^{b_{{\rm i}\gamma_{\rm i}(1)}}{\rm d}x h_{{\rm ii}(1;1)}(x,\omega,\sigma,\gamma_{\rm i},\gamma_{\rm i}'). \tag{H24}$$

On the other hand, if $\gamma_{\rm i}>-Z_{\rm i}(1-U)$, then, in Eq. (H10), $-V_{{\rm i}a}=-Z_{\rm i}(1-U)$ (Eq. (H11)) and we rewrite Eq. (H10) as

$$\text{if } \gamma_{\rm i}>-Z_{\rm i}(1-U) \text{ then } H^{\rm left}_{{\rm ii}(1)}(\omega,\sigma,\gamma_{\rm i})=$$
$$\int_{-Z_{\rm i}}^{-Z_{\rm i}(1-U)}{\rm d}\gamma_{\rm i}'\int_{a_{{\rm i}\gamma_{\rm i}'(1)}}^{0}{\rm d}x h_{{\rm ii}(1;1)}(x,\omega,\sigma,\gamma_{\rm i},\gamma_{\rm i}')+$$
$$\int_{-Z_{\rm i}(1-U)}^{-V_{{\rm i}b}}{\rm d}\gamma_{\rm i}'\int_{a_{{\rm i}\gamma_{\rm i}(1)}}^{0}{\rm d}x h_{{\rm ii}(1;1)}(x,\omega,\sigma,\gamma_{\rm i},\gamma_{\rm i}')+$$
$$\int_{-V_{{\rm i}b}}^{\infty}{\rm d}\gamma_{\rm i}'\int_{a_{{\rm i}\gamma_{\rm i}(1)}}^{0}{\rm d}x h_{{\rm ii}(1;1)}(x,\omega,\sigma,\gamma_{\rm i},\gamma_{\rm i}') \tag{H25}$$

and Eq. (H19) as (note that, if $\gamma_{\rm i}>-Z_{\rm i}(1-U)$, $-V_{{\rm i}b}>-Z_{\rm i}(1-U)$)

$$\text{if } \gamma_{\rm i}>-Z_{\rm i}(1-U) \text{ then } H^{\rm right}_{{\rm ii}(1)}(\omega,\sigma,\gamma_{\rm i})=$$
$$\int_{-Z_{\rm i}}^{-Z_{\rm i}(1-U)}{\rm d}\gamma_{\rm i}'\int_0^{b_{{\rm i}\gamma_{\rm i}'(1)}}{\rm d}x h_{{\rm ii}(1;1)}(x,\omega,\sigma,\gamma_{\rm i},\gamma_{\rm i}')+$$
$$\int_{-Z_{\rm i}(1-U)}^{-V_{{\rm i}b}}{\rm d}\gamma_{\rm i}'\int_0^{b_{{\rm i}\gamma_{\rm i}'(1)}}{\rm d}x h_{{\rm ii}(1;1)}(x,\omega,\sigma,\gamma_{\rm i},\gamma_{\rm i}')+$$
$$\int_{-V_{{\rm i}b}}^{\infty}{\rm d}\gamma_{\rm i}'\int_0^{b_{{\rm i}\gamma_{\rm i}(1)}}{\rm d}x h_{{\rm ii}(1;1)}(x,\omega,\sigma,\gamma_{\rm i},\gamma_{\rm i}'). \tag{H26}$$

Adding both sides of Eqs. (H25) and (H26), as in Eq. (H4), we have

$$\text{if } \gamma_{\rm i}>-Z_{\rm i}(1-U) \text{ then } H_{{\rm ii}(1)}(\omega,\sigma,\gamma_{\rm i})=$$

$$\int_{-Z_\mathrm{i}}^{-Z_\mathrm{i}(1-U)} \mathrm{d}\gamma'_\mathrm{i} \int_{a_{\mathrm{i}\gamma'_\mathrm{i}(1)}}^{b_{\mathrm{i}\gamma'_\mathrm{i}(1)}} \mathrm{d}x h_{\mathrm{ii}(1;1)}(x,\omega,\sigma,\gamma_\mathrm{i},\gamma'_\mathrm{i}) +$$
$$\int_{-Z_\mathrm{i}(1-U)}^{-V_{\mathrm{i}b}} \mathrm{d}\gamma'_\mathrm{i} \int_{a_{\mathrm{i}\gamma_\mathrm{i}(1)}}^{b_{\mathrm{i}\gamma'_\mathrm{i}(1)}} \mathrm{d}x h_{\mathrm{ii}(1;1)}(x,\omega,\sigma,\gamma_\mathrm{i},\gamma'_\mathrm{i}) +$$
$$\int_{-V_{\mathrm{i}b}}^{\infty} \mathrm{d}\gamma'_\mathrm{i} \int_{a_{\mathrm{i}\gamma_\mathrm{i}(1)}}^{b_{\mathrm{i}\gamma_\mathrm{i}(1)}} \mathrm{d}x h_{\mathrm{ii}(1;1)}(x,\omega,\sigma,\gamma_\mathrm{i},\gamma'_\mathrm{i}). \tag{H27}$$

In the second integral of Eq. (H27), γ'_i exceeds $-Z_\mathrm{i}(1-U)$ and so does, by assumption, γ_i. Then (Eq. (32)), $a_{\mathrm{i}\gamma'_\mathrm{i}(1)} = a_{\mathrm{i}\gamma_\mathrm{i}(1)} = -\infty$: $a_{\mathrm{i}\gamma_\mathrm{i}(1)}$ may well be replaced by $a_{\mathrm{i}\gamma'_\mathrm{i}(1)}$ and Eq. (H27) by

$$\text{if } \gamma_\mathrm{i} > -Z_\mathrm{i}(1-U) \text{ then } H_{\mathrm{ii}(1)}(\omega,\sigma,\gamma_\mathrm{i}) =$$
$$\int_{-Z_\mathrm{i}}^{-V_{\mathrm{i}b}} \mathrm{d}\gamma'_\mathrm{i} \int_{a_{\mathrm{i}\gamma'_\mathrm{i}(1)}}^{b_{\mathrm{i}\gamma'_\mathrm{i}(1)}} \mathrm{d}x h_{\mathrm{ii}(1;1)}(x,\omega,\sigma,\gamma_\mathrm{i},\gamma'_\mathrm{i}) +$$
$$\int_{-V_{\mathrm{i}b}}^{\infty} \mathrm{d}\gamma'_\mathrm{i} \int_{a_{\mathrm{i}\gamma_\mathrm{i}(1)}}^{b_{\mathrm{i}\gamma_\mathrm{i}(1)}} \mathrm{d}x h_{\mathrm{ii}(1;1)}(x,\omega,\sigma,\gamma_\mathrm{i},\gamma'_\mathrm{i}), \tag{H28}$$

which coincides with Eq. (H24). Further using the definitions of the integration endpoints $a_{\alpha\alpha\gamma_\alpha\gamma'_\alpha}, b_{\alpha\alpha\gamma_\alpha\gamma'_\alpha}$ (Eq. (C4)), both Eq. (H24) and Eq. (H28) may be reduced to a single formula:

$$H_{\mathrm{ii}(1)}(\omega,\sigma,\gamma_\mathrm{i}) = \int_{-Z_\mathrm{i}}^{\infty} \mathrm{d}\gamma'_\mathrm{i} \int_{a_{\mathrm{ii}\gamma_\mathrm{i}\gamma'_\mathrm{i}(1)}}^{a_{\mathrm{ii}\gamma_\mathrm{i}\gamma'_\mathrm{i}(1)}} \mathrm{d}x h_{\mathrm{ii}(1;1)}(x,\omega,\sigma,\gamma_\mathrm{i},\gamma'_\mathrm{i}), \tag{H29}$$

We now revert to Eq. (H3) and again we split the x-integration:

$$H_{\mathrm{ie}(1)} = H^{\mathrm{left}}_{\mathrm{ie}(1)} + H^{\mathrm{right}}_{\mathrm{ie}(1)}. \tag{H30}$$

Taking into account that, in the left part of the integration, the electron eigenfunctions and reflection points are those of domain 1 and that

$$-V'_\mathrm{e} > 0, \tag{H31}$$
$$\text{if } -Z_\mathrm{i} < \gamma_\mathrm{i} < -Z_\mathrm{i}(1-U) \text{ then}$$
$$-V_\mathrm{e}(a_{\mathrm{i}\gamma_\mathrm{i}(1)}) = \gamma^*_\mathrm{e}, \tag{H32}$$
$$\text{if } \gamma_\mathrm{i} > -Z_\mathrm{i}(1-U) \text{ then}$$
$$-V_\mathrm{e}(a_{\mathrm{i}\gamma_\mathrm{i}(1)}) = -|Z_\mathrm{e}|U, \tag{H33}$$
$$-V_\mathrm{e}(0) = 0, \tag{H34}$$
$$\text{if } \gamma'_\mathrm{e} < 0 \text{ then } [-V_\mathrm{e}]^{-1}(\gamma'_\mathrm{e}) = b_{\mathrm{e}\gamma'_\mathrm{e}(1)}, \tag{H35}$$

where γ^*_e was defined in Eq. (D21), we invert the integration order in Eq. (H3) according to Eq. (E4), which applies when $-V_\mathrm{e}$ monotonicly increases (Eq. (H31)), and to Eqs. (H32)-(H9), we have, for both $\gamma_\mathrm{i} < -Z_\mathrm{i}(1-U)$ and $\gamma_\mathrm{i} > -Z_\mathrm{i}(1-U)$:

$$H^{\mathrm{left}}_{\mathrm{ie}(1)}(\omega,\sigma,\gamma_\mathrm{i}) =$$
$$\int_{-V_{\mathrm{e}a}}^{0} \mathrm{d}\gamma'_\mathrm{e} \int_{a_{\mathrm{i}\gamma_\mathrm{i}(1)}}^{b_{\mathrm{e}\gamma'_\mathrm{e}(1)}} \mathrm{d}x h_{\mathrm{ie}(1;1)}(x,\omega,\sigma,\gamma_\mathrm{i},\gamma'_\mathrm{e}) +$$
$$\int_{0}^{\infty} \mathrm{d}\gamma'_\mathrm{e} \int_{a_{\mathrm{i}\gamma_\mathrm{i}(1)}}^{0} \mathrm{d}x h_{\mathrm{ie}(1;1)}(x,\omega,\sigma,\gamma_\mathrm{i},\gamma'_\mathrm{e}), \tag{H36}$$

where $-V_{\mathrm{e}a}$ was given in Eq. (D18). In the first integral of Eq. (H36), we omitted the contribution of $h_{\mathrm{ie}(1;2)}(x,\omega,\sigma,\gamma_\mathrm{i},\gamma'_\mathrm{e})$ because, for $\gamma'_\mathrm{e} < 0$, it is non zero only for $x > 0$, i.e. outside the range of x-integration (Eq. (D4)). In the second integral, we omitted the sum over the electron domain label ν'_e because, for $\gamma'_\mathrm{e} > 0$, $N_{\mathrm{e}\gamma'_\mathrm{e}} = 1$ (Eq. (40)).

In a similar way, taking into account that, in the left part of the integration, the electron eigenfunctions and reflection points are those of domain 2 and

$$-V'_\mathrm{e} < 0, -V_\mathrm{e}(0) = 0, \tag{H37}$$
$$\text{if } -Z_\mathrm{i} < \gamma_\mathrm{i} < 0 \text{ then } -V_\mathrm{e}(b_{\mathrm{i}\gamma_\mathrm{i}(1)}) = \gamma^*_\mathrm{e}, \tag{H38}$$
$$\text{if } \gamma_\mathrm{i} > 0 \text{ then } b_{\mathrm{i}\gamma_\mathrm{i}(1)} = \infty \text{ and } -V_\mathrm{e}(b_{\mathrm{i}\gamma_\mathrm{i}(1)}) = -|Z_\mathrm{e}|, \tag{H39}$$
$$\text{if } -|Z_\mathrm{e}| < \gamma'_\mathrm{e} < 0 \text{ then } [-V_\mathrm{e}]^{-1}(\gamma'_\mathrm{e}) = a_{\mathrm{e}\gamma'_\mathrm{e}(2)}, \tag{H40}$$

and inverting the integration order in Eq. (H3) according to Eq. (E7), which applies when $-V_\mathrm{e}$ monotonically decreases (Eq. (H37)), and to Eqs (H38)-(H40), we have, for both $\gamma_\mathrm{i} < 0$ and $\gamma_\mathrm{i} > 0$:

$$H^{\mathrm{right}}_{\mathrm{ie}(1)}(\omega,\sigma,\gamma_\mathrm{i}) =$$
$$\int_{-V_{\mathrm{e}b}}^{0} \mathrm{d}\gamma'_\mathrm{e} \int_{a_{\mathrm{e}\gamma'_\mathrm{e}(2)}}^{b_{\mathrm{i}\gamma_\mathrm{i}(1)}} \mathrm{d}x h_{\mathrm{ie}(1;2)}(x,\omega,\sigma,\gamma_\mathrm{i},\gamma'_\mathrm{e}) +$$
$$\int_{0}^{\infty} \mathrm{d}\gamma'_\mathrm{e} \int_{0}^{b_{\mathrm{i}\gamma_\mathrm{i}(1)}} \mathrm{d}x h_{\mathrm{ie}(1;1)}(x,\omega,\sigma,\gamma_\mathrm{i},\gamma'_\mathrm{e}), \tag{H41}$$

where $-V_{\mathrm{e}b}$ was given in Eq. (D27)

In the first integral of Eq. (H41), we omitted the contribution of $h_{\mathrm{ie}(1;1)}(x,\omega,\sigma,\gamma_\mathrm{i},\gamma'_\mathrm{e})$ because, for $\gamma'_\mathrm{e} < 0$, it is non zero only for $x < 0$, i.e. outside the range of x-integration (Eq. (D3)). In the second integral, we omitted the sum over the electron domain label ν'_e, because, for $\gamma'_\mathrm{e} > 0$, $N_{\mathrm{e}\gamma'_\mathrm{e}} = 1$ (Eq. (40)).

In a similar way, adding both sides of Eqs. (H36) and (H41), as in Eq. (H30), we have

$$H_{\mathrm{ie}(1)}(\omega,\sigma,\gamma_\mathrm{i}) =$$
$$\int_{-V_{\mathrm{e}a}}^{0} \mathrm{d}\gamma'_\mathrm{e} \int_{a_{\mathrm{i}\gamma_\mathrm{i}(1)}}^{b_{\mathrm{e}\gamma'_\mathrm{e}(1)}} \mathrm{d}x h_{\mathrm{ie}(1;1)}(x,\omega,\sigma,\gamma_\mathrm{i},\gamma'_\mathrm{e}) +$$
$$\int_{-V_{\mathrm{e}b}}^{0} \mathrm{d}\gamma'_\mathrm{e} \int_{a_{\mathrm{e}\gamma'_\mathrm{e}(2)}}^{b_{\mathrm{i}\gamma_\mathrm{i}(1)}} \mathrm{d}x h_{\mathrm{ie}(1;2)}(x,\omega,\sigma,\gamma_\mathrm{i},\gamma'_\mathrm{e}) +$$
$$\int_{0}^{\infty} \mathrm{d}\gamma'_\mathrm{e} \int_{a_{\mathrm{i}\gamma_\mathrm{i}(1)}}^{b_{\mathrm{i}\gamma_\mathrm{i}(1)}} \mathrm{d}x h_{\mathrm{ie}(1;1)}(x,\omega,\sigma,\gamma_\mathrm{i},\gamma'_\mathrm{e}). \tag{H42}$$

Due to Eq. (D22), the first x-integral of Eq. (H42) remains unchanged if we replace its lower integration bound $-V_{\mathrm{e}a}$ by $-|Z_\mathrm{e}|$ which, due to Eqs. (27), (D18) and (D27), is certainly not larger than $-V_{\mathrm{e}a}$. Eq. (D30)), justifies the replacement of $-V_{\mathrm{e}b}$ by $-|Z_\mathrm{e}|$ in the second x-integral of Eq. (D22). Last, we rename the x-integration bounds in Eq. (H42), according to Eqs. (C5)-(??) and we have

$$H_{\mathrm{ie}(1)}(\omega,\sigma,\gamma_\mathrm{i}) =$$
$$\int_{-|Z_\mathrm{e}|}^{0} \mathrm{d}\gamma'_\mathrm{e} \int_{a_{\mathrm{ie}\gamma_\mathrm{i}\gamma'_\mathrm{e}(1)}}^{b_{\mathrm{ie}\gamma_\mathrm{i}\gamma'_\mathrm{e}(1)}} \mathrm{d}x h_{\mathrm{ie}(1;1)}(x,\omega,\sigma,\gamma_\mathrm{i},\gamma'_\mathrm{e}) +$$
$$\int_{-|Z_\mathrm{e}|}^{0} \mathrm{d}\gamma'_\mathrm{e} \int_{a_{\mathrm{ie}\gamma_\mathrm{i}\gamma'_\mathrm{e}(2)}}^{b_{\mathrm{ie}\gamma_\mathrm{i}\gamma'_\mathrm{e}(2)}} \mathrm{d}x h_{\mathrm{ie}(1;2)}(x,\omega,\sigma,\gamma_\mathrm{i},\gamma'_\mathrm{e}) +$$
$$\int_{0}^{\infty} \mathrm{d}\gamma'_\mathrm{e} \int_{a_{\mathrm{ie}\gamma_\mathrm{i}\gamma'_\mathrm{e}(1)}}^{b_{\mathrm{ie}\gamma_\mathrm{i}\gamma'_\mathrm{e}(1)}} \mathrm{d}x h_{\mathrm{ie}(1;1)}(x,\omega,\sigma,\gamma_\mathrm{i},\gamma'_\mathrm{e}) \tag{H43}$$

i.e.

$$H_{\mathrm{ie}(1)}(\omega,\sigma,\gamma_\mathrm{i}) =$$

$$\int_{-|Z_e|}^{\infty} d\gamma_e' \sum_{\nu_e'=1}^{N_{e\gamma_e'}} \int_{a_{ie\gamma_i\gamma_e'(\nu_e')}}^{b_{ie\gamma_i\gamma_e'(\nu_e')}} dx h_{ie(1;\nu_e')}(x,\omega,\sigma,\gamma_i,\gamma_e'), \quad \text{(H44)}$$

where $N_{e\gamma_e'}$ was given in Eqs. (38)-(40).

Inserting Eqs. (H29) and (H44) into Eq. (H1), and using the definitions of $h_{ee(1;\nu_e')}$ and $h_{ei(1;1)}$ (Eqs. (77) and (79)), we see that the result fits in the general formula given in Eq. (97).

Appendix I: Reduction of the Superposition Coefficients X_e^+

In this appendix, we give some useful expressions for Y_e and for Λ_e found in Section IX which will be useful in the following. Because of Eq. (136), only the quantities with the + superscript will be given.

Making the substitution

$$\sqrt{(2\gamma_e/\mu_e)} = v > 0 \quad \text{(I1)}$$

into Eq. (131) (taking into account also Eq. (122)) and into Eq. (135) gives

$$Y_e^+(\sigma, \mu_e v^2/2) = \frac{Z_e v^3 \tilde{F}_e'(\mu_e v^2/2)}{\sigma^2}, \quad \text{(I2)}$$

and

$$\Lambda_e^+(\sigma,\mu_e v^2/2) = 1 + \frac{Z_e v^3}{\sigma^3} \mathrm{P} \int_{-\infty}^{\infty} d\sigma' \sigma' \frac{\tilde{F}_e'([\sigma'/\omega]^2 \mu_e v^2/2)}{\sigma' - \sigma}, \quad \text{(I3)}$$

for $\sigma > 0$. (I4)

Making the substitution

$$\sigma' = \sigma u/v, \text{ for } \sigma > 0, \quad \text{(I5)}$$

Eq. (I4) reads

$$\Lambda_e^+(\sigma,\mu_e v^2/2) = 1 + \frac{Z_e v^2}{\sigma^2} \mathrm{P} \int_{-\infty}^{\infty} du u \frac{\tilde{F}_e'(\mu_e u^2/2)}{u - v}, \quad \text{(I6)}$$

which extends also to negative values of σ.

Finally, since $F_e' = \mu_e v \partial \tilde{F}/\partial v$ (Eq. (106)), Eqs. (I2) and (I6) reduce to

$$Y_e^+(\sigma,\mu_e v^2/2) = \frac{Z_e v^2 \partial \tilde{F}_e(\mu_e v^2/2)/\partial v}{\mu_e \sigma^2}, \quad \text{(I7)}$$

and

$$\Lambda_e^+(\sigma,\mu_e v^2/2) = 1 + \frac{Z_e v^2}{\mu_e \sigma^2} \mathrm{P} \int_{-\infty}^{\infty} du \frac{\partial \tilde{F}_e(\mu_e u^2/2)/\partial u}{u - v}. \quad \text{(I8)}$$

Inserting Eqs. (I7) and (I8) into Eq. (128) we finally have

$$X_e^+(\omega,\sigma,\mu_e v^2/2) = \left[1 + \frac{Z_e v^2}{\mu_e \sigma^2} \mathrm{P} \int_{-\infty}^{\infty} du \frac{\partial \tilde{F}_e(\mu_e u^2/2)/\partial u}{u - v}\right] \delta(\sigma - \omega) - \frac{Z_e v^2}{\mu_e \sigma^2} \frac{\partial \tilde{F}_e(\mu_e v^2/2)}{\partial v} \mathrm{P} \frac{1}{\sigma - \omega}, \quad \text{(I9)}$$

for $\omega > 0$.

We also give the expressions of Y_e^+ and Λ_e^+ using quantities defined in the Fourier transformed velocity space. In the present homogeneous case (Eq. (116)), Eq. (108) taken of electrons ($\alpha = \mathrm{e}$) and for $s_e = +$, reduces to

$$\tilde{F}_e'(\mu_e v^2/2) = \frac{1}{2i\pi\mu_e} \frac{1}{v} \int_{-\infty}^{\infty} dq q F_e(q) e^{-iqv}, \quad \text{(I10)}$$

where Eq. (I1) was used, and Eq. (I2) reads

$$Y_e^+(\sigma,\mu_e v^2/2) = \frac{1}{2i\pi} \frac{Z_e v^2}{\mu_e \sigma^2} \int_{-\infty}^{\infty} dq q F_e(q) e^{-iqv}. \quad \text{(I11)}$$

Taking into account that $F(q)$ is the Fourier transform of a real function, so that $F(-q) = \bar{F}(q)$, Eq. (I11) further reduces to

$$Y_e^+(\sigma,\mu_e v^2/2) = \frac{1}{\pi} \frac{Z_e v^2}{\mu_e \sigma^2} \Im \int_0^{\infty} dq q F_e(q) e^{-iqv}. \quad \text{(I12)}$$

Changing the order of the u and q integration, Eq. (I6) reduces to

$$\Lambda_e^+(\sigma,\mu_e v^2/2) = 1 + \frac{1}{2i\pi\mu_e} \frac{Z_e v^2}{\sigma^2} \int_{-\infty}^{\infty} dq q F_e(q) \times \mathrm{P} \int_{-\infty}^{\infty} du \frac{e^{-iqu}}{u - v}. \quad \text{(I13)}$$

Using the identity

$$\mathrm{P} \int_{-\infty}^{\infty} du \frac{e^{-iqu}}{u - v} = -i\pi \mathrm{sign}(q) e^{-iqv}, \quad \text{(I14)}$$

and taking again into account that $F(-q) = \bar{F}(q)$, Eq. (I13) further reduces to

$$\Lambda_e^+(\sigma,\mu_e v^2/2) = 1 - \frac{Z_e v^2}{\mu_e \sigma^2} \Re \int_0^{\infty} dq q F_e(q) e^{-iqv}. \quad \text{(I15)}$$

Inserting Eqs. (I12) and (I15) into Eq. (128) we finally have

$$X_e^+(\omega,\sigma,\mu_e v^2/2) = \left[1 - \frac{Z_e v^2}{\mu_e \sigma^2} \Re \int_0^{\infty} dq q F_e(q) e^{-iqv}\right] \delta(\sigma - \omega) - \frac{1}{\pi} \frac{Z_e v^2}{\mu_e \sigma^2} \Im \int_0^{\infty} dq q F_e(q) e^{-iqv} \mathrm{P} \frac{1}{\sigma - \omega}, \quad \text{(I16)}$$

for $\omega > 0$.

Setting

$$\zeta = \sigma - \omega, \; Q(\sigma, v) = \frac{Z_e v^2}{\mu_e \sigma^2} \int_0^{\infty} dq q F_e(q) e^{-iqv}, \quad \text{(I17)}$$

Eq. (I16) reads

$$X_e^+ = \delta(\zeta) - \frac{1}{\pi} \left[\pi \delta(\zeta) \Re Q(\sigma, v) + \mathrm{P} \frac{1}{\zeta} \Im Q(\sigma, v)\right]. \quad \text{(I18)}$$

According to Plemelj's formulas

$$\frac{1}{\zeta \pm i0} = \lim_{\epsilon \to 0^+} \frac{1}{\zeta \pm i\epsilon} = \mp i\pi \delta(\zeta) + \mathrm{P} \frac{1}{\zeta}, \quad \text{(I19)}$$

we may write

$$2\mathrm{i}\pi\delta(\zeta) = \frac{1}{\zeta - \mathrm{i}0} - \frac{1}{\zeta + \mathrm{i}0}, \tag{I20}$$

$$2\mathrm{P}\frac{1}{\zeta} = \frac{1}{\zeta - \mathrm{i}0} + \frac{1}{\zeta + \mathrm{i}0}, \tag{I21}$$

and thus Eq. (I18) may be arranged as

$$\begin{aligned} X_\mathrm{e}^+ &= \delta(\zeta) - \frac{1}{2\mathrm{i}\pi} \times \\ &\left[\left(\frac{1}{\zeta - \mathrm{i}0} - \frac{1}{\zeta + \mathrm{i}0}\right)\Re Q(\sigma, v) + \right.\\ &\left.\left(\frac{\mathrm{i}}{\zeta - \mathrm{i}0} + \frac{\mathrm{i}}{\zeta + \mathrm{i}0}\right)\Im Q(\sigma, v)\right] = \\ &\delta(\zeta) - \frac{1}{2\mathrm{i}\pi} \times \\ &\left[\frac{1}{\zeta - \mathrm{i}0}(\Re Q(\sigma, v) + \mathrm{i}\Im Q(\sigma, v)) - \right.\\ &\left.\frac{1}{\zeta + \mathrm{i}0}(\Re Q(\sigma, v) - \mathrm{i}\Im Q(\sigma, v))\right] = \\ &\delta(\zeta) - \frac{1}{2\mathrm{i}\pi}\left[\frac{Q(\sigma, v)}{\zeta - \mathrm{i}0} - \frac{\bar{Q}(\sigma, v)}{\zeta + \mathrm{i}0}\right], \end{aligned} \tag{I22}$$

or, reintroducing the quantities defined in Eq. (I17), as

$$\begin{aligned} &X_\mathrm{e}^+(\omega, \sigma, \mu_\mathrm{e} v^2/2) = \delta(\sigma - \omega) - \frac{1}{2\mathrm{i}\pi}\frac{Z_\mathrm{e} v^2}{\mu_\mathrm{e}\sigma^2} \times \\ &\left[\frac{\int_0^\infty \mathrm{d}q q F_\mathrm{e}(q) e^{-\mathrm{i}qv}}{\sigma - \omega - \mathrm{i}0} - \frac{\int_0^\infty \mathrm{d}q q \bar{F}_\mathrm{e}(q) e^{\mathrm{i}qv}}{\sigma - \omega + \mathrm{i}0}\right], \\ &\text{for } \omega > 0. \end{aligned} \tag{I23}$$

Changing the sign of the integration variable in the numerator of the second term in the square brackets and taking into account that $\bar{F}(-q) = F(q)$, Eq. (I23) reads

$$\begin{aligned} &X_\mathrm{e}^+(\omega, \sigma, \mu_\mathrm{e} v^2/2) = \delta(\sigma - \omega) - \frac{1}{2\mathrm{i}\pi}\frac{Z_\mathrm{e} v^2}{\mu_\mathrm{e}\sigma^2} \times \\ &\left[\frac{\int_0^\infty \mathrm{d}q q F_\mathrm{e}(q) e^{-\mathrm{i}qv}}{\sigma - \omega - \mathrm{i}0} - \frac{\int_{-\infty}^0 \mathrm{d}q q F_\mathrm{e}(q) e^{-\mathrm{i}qv}}{\sigma - \omega + \mathrm{i}0}\right], \\ &\text{for } \omega > 0. \end{aligned} \tag{I24}$$

This form of the coefficient clearly points out the perturbation contribution, proportional to the electron charge Z_e, to the eigenfunction of the Vlasov operator.

Again using Eq. (I20), Eq. (I22) can be arranged as

$$\begin{aligned} &X_\mathrm{e}^+(\omega, \sigma, \mu_\mathrm{e} v^2/2) = \\ &\frac{1}{2\mathrm{i}\pi}\left[\frac{S(\sigma, v)}{\zeta - \mathrm{i}0} - \frac{\bar{S}(\sigma, v)}{\zeta + \mathrm{i}0}\right] = \frac{1}{\pi}\Im\frac{S(\sigma, v)}{\zeta - \mathrm{i}0}, \end{aligned} \tag{I25}$$

where

$$S(\sigma, v) = 1 - Q(\sigma, v) = 1 - \frac{Z_\mathrm{e} v^2}{\mu_\mathrm{e}\sigma^2}\int_0^\infty \mathrm{d}q q F_\mathrm{e}(q) e^{-\mathrm{i}qv}. \tag{I26}$$

Reintroducing the quantities defined in Eq. (I17), Eq. (I25) becomes

$$\begin{aligned} &X_\mathrm{e}^+(\omega, \sigma, \mu_\mathrm{e} v^2/2) = \\ &\frac{1}{\pi}\Im\frac{1 - \frac{Z_\mathrm{e} v^2}{\mu_\mathrm{e}\sigma^2}\int_0^\infty \mathrm{d}q q F_\mathrm{e}(q) e^{-\mathrm{i}qv}}{\sigma - \omega - \mathrm{i}0} \\ &\text{for } \omega > 0. \end{aligned} \tag{I27}$$

Finally, using the identities

$$\frac{1}{\zeta + \mathrm{i}0} = -\mathrm{i}\int_0^\infty \mathrm{d}t e^{\mathrm{i}(\zeta + \mathrm{i}0)t}, \tag{I28}$$

$$\frac{1}{\zeta - \mathrm{i}0} = \mathrm{i}\int_{-\infty}^0 \mathrm{d}t e^{\mathrm{i}(\zeta - \mathrm{i}0)t}, \tag{I29}$$

Eq. (I25) may be further rearranged as

$$\begin{aligned} &X_\mathrm{e}^+(\omega, \sigma, \mu_\mathrm{e} v^2/2) = \\ &\frac{1}{2\pi}\left[S(\sigma, v)\int_{-\infty}^0 \mathrm{d}t e^{\mathrm{i}(\sigma - \omega - \mathrm{i}0)t} + \right.\\ &\left.\bar{S}(\sigma, v)\int_0^\infty \mathrm{d}t e^{\mathrm{i}(\sigma - \omega + \mathrm{i}0)t}\right] = \\ &\frac{1}{\pi}\Re\left[S(\sigma, v)\int_{-\infty}^0 \mathrm{d}t e^{\mathrm{i}(\sigma - \omega - \mathrm{i}0)t}\right], \\ &\text{for } \omega > 0. \end{aligned} \tag{I30}$$

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[1] L. Palumbo and L. Nocera, *The Eigenfunctions of the Inhomogeneous Free-streaming Collisionless Boltzmann Operator in the Fourier Transformed Velocity Space*, Tech. Rep. (Institute for Chemical and Physical Processes, National Research Council, Via Moruzzi 1, I-56024 Pisa, Italy, 2014).
[2] E. Barston, Annals of Physics **29**, 283 (1964).
[3] Z. Sedláček, J. Plasma Phys. **5**, 239 (1971).
[4] Z. Sedláček and B. Roberts, J. Plasma Phys. **41**, 97 (1989).
[5] S. Poedts, W. Kerner, and M. Goossens, J. Plasma Phys. **42**, 27 (1989).
[6] L. Nocera, M. Medręk, and K. Murawski, Astron. Astrophys. **373**, 301 (2001).
[7] Z. Sedláček, Czech. J. Phys. B **23 B**, 892 (1973).
[8] Z. Sedláček, J. Plasma Phys. **6**, 187 (1971).
[9] Z. Sedláček, Czech. J. Phys. B **22 B**, 439 (1972).
[10] L. Nocera, Ann. Phys. **323**, 2482 (2008).
[11] Z. Sedláček and L. Nocera, J. Plasma Phys. **48**, 367 (1992).
[12] Z. Sedláček and L. Nocera, Phys. Lett. A **296**, 117 (2002).
[13] L. Nocera, Phys. Lett. A **352**, 244 (2006).
[14] L. Nocera and L. Palumbo, Phys. Plasmas **20**, 012107 (2013).

[15] K. Case, Ann. Phys. **7**, 349 (1959).
[16] I. Gradshteyn and I. Ryzhik, *Table of Integrals, Series and Products* (Academic Press, New York, 1981).

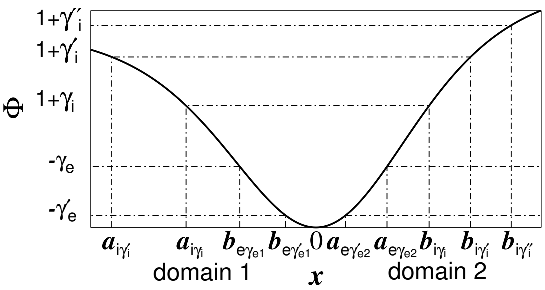

Figure 1.

Figure Captions

1. A typical waveform of a nonmonotonic double layer steady state potential. Shown are the reflection points $a_{\mathrm{e}\gamma_{\mathrm{e}}}, |B_{\mathrm{e}\gamma_{\mathrm{e}}}|$ of the electron and $a_{\mathrm{i}\gamma_{\mathrm{i}}(1)}, b_{\mathrm{e}\gamma_{\mathrm{i}}}$ of the ion eigenfunctions for several values of their respective degeneracy parameters γ_{e} and γ_{i}. The eigenfunctions are defined only in the x-domains where the horizontal dash-dotted lines originating from a reflection point are drawn.

2. Panel (a): the real (solid bold line) and imaginary (dashed bold line) parts of the eigenfunctions for free electrons subject to the nonmonotonic double layer steady state potential of Eq. (26) where $a = 0.25$, $\kappa = 2$. The other parameters in Eqs. (22) and (23) are $q = 4$, $\gamma_{\mathrm{e}} = 0.3, s_{\mathrm{e}} = +$, $\sigma = 4$, $Z_{\mathrm{e}} = -1$, $\mu_{\mathrm{e}} = 1$, $x_{\mathrm{e}\gamma_{\mathrm{e}}} = 0$. Panel ($b$): same as in ($a$), but for the reflected electrons and for $a = 0.64$, $\gamma_{\mathrm{e}} = -0.3$, $x_{\mathrm{e}\gamma_{\mathrm{e}}} = b_{\mathrm{e}0}$ for $x < 0$ and $x_{\mathrm{e}\gamma_{\mathrm{e}}} = a_{\mathrm{e}0}$ for $x > 0$. Near the reflection points $a_{\mathrm{e}\gamma_{\mathrm{e}}}$ and $|B_{\mathrm{e}\gamma_{\mathrm{e}}}|$, the real part of the eigenfunction diverges, whereas its imaginary part remains finite (Eq. (45)). Superimposed in panel (b) is the steady state equilibrium electron potential energy profile (thin solid line).

3. Panel (a): the real (solid bold line) and imaginary (dashed bold line) parts of the eigenfunctions for free ions subject to the nonmonotonic double layer steady state potential of Eq. (26) where $a = 0.25$, $\kappa = 2$. The other parameters in Eq. (22) and (23) are $q = 4$, $\gamma_{\mathrm{e}} = 0.3, s_{\mathrm{i}} = +$, $\sigma = 4/\sqrt{\mu_{\mathrm{i}}}$, $Z_{\mathrm{i}} = 1$, $\mu_{\mathrm{i}} = 1833$, $x_{\mathrm{i}\gamma_{\mathrm{i}}(1)} = \infty$. Panel ($b$): same as in ($a$), but for the reflected ions and for $a = 0.64$, $\gamma_{\mathrm{i}} = -0.3$, $x_{\mathrm{i}\gamma_{\mathrm{i}}(1)} = b_{\mathrm{i}0}$. Panel ($c$): same as in ($b$), but for the trapped ions and for $\gamma_{\mathrm{i}} = -0.7$, $x_{\mathrm{i}\gamma_{\mathrm{i}}(1)} = b_{\mathrm{i}0}$. Near the reflection points $a_{\mathrm{i}\gamma_{\mathrm{i}}(1)}$ and $b_{\mathrm{i}\gamma_{\mathrm{i}}(1)}$, the real part of the eigenfunction diverges, whereas its imaginary part remains finite (Eq. (45)). Superimposed in panels (b) and (c) is the steady state equilibrium ion potential energy profile (thin solid line).

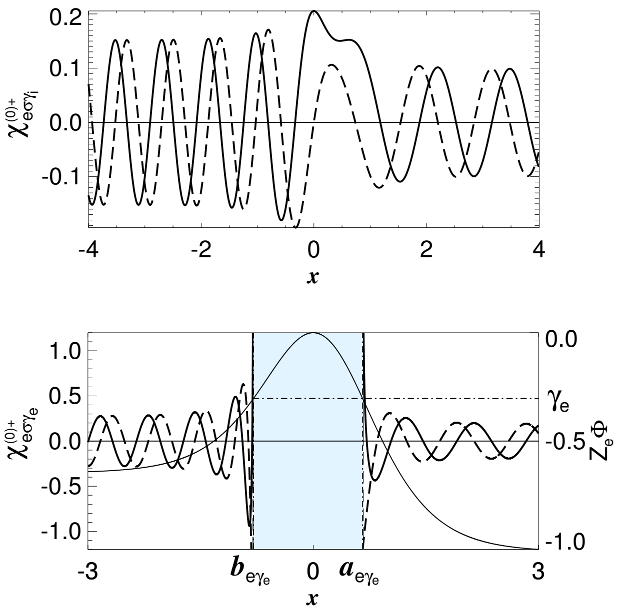

Figure 2.

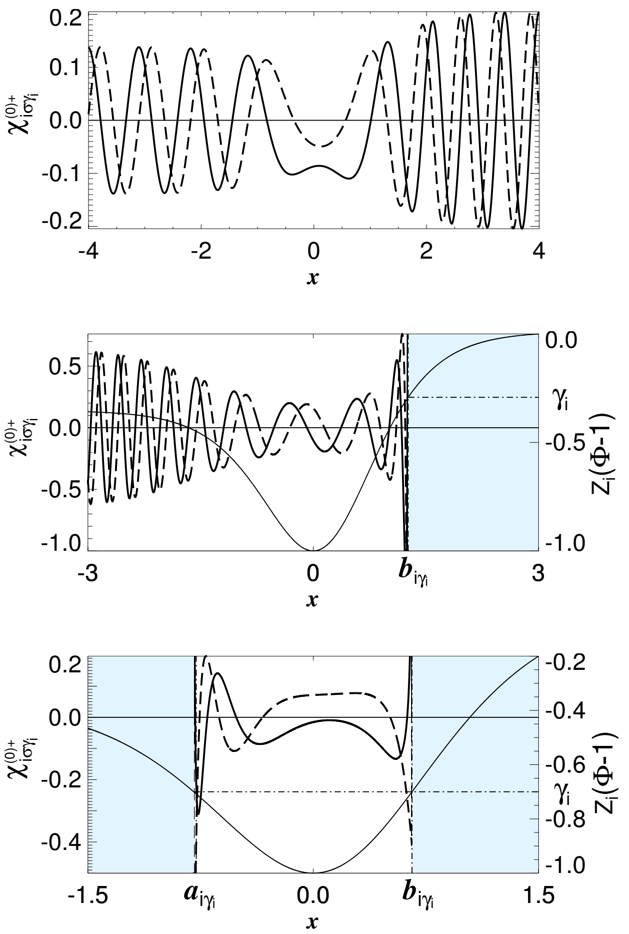

Figure 3.