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Polyhedral models, Spatial bisimilarity, Spatial logics, Logical equivalence, Spatial model checking, Strong Bisimulation, Branching Bisimulation.

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Weak ±-Minimisation for Model Checking Polyhedra*

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Abstract. The work in this paper builds on the *polyhedral semantics* of the Spatial Logic for Closure Spaces (`SLCS`), and the geometric spatial model checker `PolyLogicA`. Polyhedral models are central in domains that exploit mesh processing, such as 3D computer graphics. A discrete representation of polyhedra is given by face-poset models, which are amenable to spatial model checking using the logical language $\texttt{SLCS}_\eta$ and `PolyLogicA`. In this work, we propose a procedure that computes the minimal model with respect to *weak ±-bisimilarity* – that is $\texttt{SLCS}_\eta$-logical equivalence – via a translation to LTSs and *branching bisimilarity*. Because of its reduced size, the minimal model makes geometric model checking more efficient. We provide a preliminary experimental validation of the approach exploiting the minimization capabilities of `mCRL2`.

Keywords: Polyhedral models · Spatial bisimilarity · Spatial logics · Logical equivalence · Spatial model checking · Strong Bisimulation · Branching Bisimulation.

1 Introduction and Related Work

Spatial and spatio-temporal model checking have recently been successfully employed in a variety of application areas, including Collective Adaptive Systems [19, 15], signals [26], images [18, 22, 1] and polyhedra [7, 14, 8, 9]. Interest in these methods for spatial analysis is increasing in computer science and in other domains, including initially unanticipated ones, such as medical imaging [4, 2].

* The authors are listed in alphabetical order, as they equally contributed to the work presented in this paper.
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Spatial model checking is a global technique: it comprises the automatic verification of properties, expressed in a suitable spatial logic, such as `SLCS` [17, 18], on each point of a spatial model. The logic `SLCS` has been defined originally for Čech closure spaces [27], a generalisation of topological spaces, and model checking algorithms have been developed for finite closure spaces also in combination with discrete time, leading to spatio-temporal model checking [15]. The spatial model checker `VoxLogicA`, proposed in [15], is very efficient in checking properties of large images –represented as symmetric finite closure models– expressed in `SLCS` [4, 3, 2]. For example, the automatic segmentation via a suitable `SLCS` formula characterising the white matter of the brain in a 3D MRI image consisting of circa 12M voxels (i.e. $256 \times 256 \times 181$), requires approximately 10 seconds, using `VoxLogicA` on a desktop computer [3].[5]

In [13, 20] several bisimulations for finite closure spaces have been studied, with the aim to improve the efficiency of model checking via model minimisation. These notions cover a spectrum from CM-bisimilarity, an equivalence based on *proximity* — similar to and inspired by topo-bisimilarity for topological models [5] — to its specialisation for quasi-discrete closure models, CMC-bisimilarity, to CoPa-bisimilarity, an equivalence based on *conditional reachability*. Each of these bisimilarities has been equipped with its logical characterisation.

The spatial model checking techniques mentioned above targeting grid-based structures have been extended to *polyhedral models* [6, 24]. Polyhedra are subsets in $\mathbb{R}^n$ generated by simplicial complexes, i.e. finite collections of simplexes satisfying certain conditions. A simplex is the convex hull of a set of affinely independent points. Given a set `PL` of proposition letters, a polyhedral model is obtained from a polyhedron by assigning a polyhedral subset to each proposition letter $p \in \texttt{PL}$, namely those points that "satisfy" p. Polyhedral models in $\mathbb{R}^3$ can be used for (approximately) representing objects in continuous 3D space. This is typical of many 3D visual computing techniques, where an object is split into suitable parts of different size. Such ways of splitting of an object are known as mesh techniques and include triangular surface meshes or tetrahedral volume meshes (see [23]).

In [6] a version of `SLCS` has been proposed for expressing spatial properties of points lying in polyhedral models, and in particular *conditional reachability* properties. Besides negation and conjunction, the particular logic, called $\texttt{SLCS}_\gamma$ in the sequel, provides the γ reachability operator. Informally, a point x in a polyhedral model satisfies the conditional reachability formula $\gamma(\Phi_1, \Phi_2)$ if there is a topological path starting from x, ending in a point y satisfying Φ_2, and such that all the intermediate points of the path between x and y satisfy Φ_1. Note that neither x nor y is required to satisfy Φ_1. Many other interesting properties, such as proximity (in the topological sense, i.e. "being in the topological closure of") or "being surrounded by" can be expressed using reachability (see [6]). A weaker version of conditional reachability, denoted by η, has been introduced

[5] Intel Core i9-9900K processor (with 8 cores and 16 threads) and 32GB of RAM. Note that `VoxLogicA` checks such logical specifications for *every* point in the model exploiting parallel execution, memoization, and state-of-the-art imaging libraries [4].

in [8, 9]. A point x in a polyhedral model satisfies the conditional reachability formula $\eta(\Phi_1, \Phi_2)$ if there is a topological path starting from x, ending in a point y satisfying Φ_2, and x and all the intermediate points of the path between x and y satisfy Φ_1. Thus now x is required to satisfy Φ_1. It should be clear to the reader that η can easily be expressed using γ and, in fact, in [8, 9] it has been shown that the logic where γ has been replaced by η ($\mathtt{SLCS}_\eta$, in the sequel), is strictly weaker than $\mathtt{SLCS}_\gamma$.

Interestingly, polyhedral models can conveniently be represented by discrete structures, the so-called *cell poset models*: each point of the polyhedron is mapped to a "cell", i.e. an element of the associated cell poset model. Furthermore, as it has already been shown for $\mathtt{SLCS}_\gamma$ [6], also $\mathtt{SLCS}_\eta$ can be interpreted on cell poset models so that the mapping from a polyhedral model into its cell poset model preserves and reflects the logic [8]: a point satisfies a formula of $\mathtt{SLCS}_\eta$ iff the cell which it is mapped to satisfies the formula. This result has paved the way to the definition and implementation of model checking techniques for $\mathtt{SLCS}_\eta$ on polyhedral models, by working on their discrete representations. The interested reader is referred to [6] for the description of the model checker `PolyLogicA`.

As in the case of traditional (temporal) model checking, efficiency of spatial model checking can be improved by suitable model minimisation techniques. In particular, we are interested in techniques based on spatial bisimilarity. In [8], *weak simplicial bisimilarity* on polyhedral models ($\approx_\triangle$) has been introduced and it has been shown that it enjoys the Hennessy-Milner property (HMP) with respect to $\mathtt{SLCS}_\eta$, i.e. $\approx_\triangle$ coincides with logical equivalence as induced by $\mathtt{SLCS}_\eta$, namely $\equiv_\eta$. In [8], a notion of bisimulation equivalence has been proposed for cell poset models as well, namely *weak ±-bisimilarity* ($\approx_\pm$, to be read as 'weak plus-minus' bisimilarity) such that two points in the polyhedral model are weakly simplicial bisimilar iff their cells are weakly ±-bisimilar. Also, it has been shown that on cell poset models $\approx_\pm$ coincides with $\equiv_\eta$.

In the present paper, we build upon the theoretical results of [8, 9] by showing a spatial model minimisation procedure based on weak ±-bisimilarity, namely *weak ±-minimisation*. The procedure uses an encoding of cell poset models into labelled transition systems (LTSs) following an approach that is similar to that presented in [16] for finite closure models. More precisely, in the case of cell poset models, there is a one-to-one correspondence between the states of the LTS and the cells of the poset model. It is shown that two cells are weakly ±-bisimilar in the poset model iff they — as states of the LTS — are branching bisimulation equivalent. This provides an effective way for computing the equivalence classes for the set of cells, from which the minimal model is built, on which $\mathtt{SLCS}_\eta$ model checking can be safely performed. Such a computation can be very efficient since efficient LTS minimisation tools are available for branching bisimulation, such as `mCRL2` [21]. As we will see in Section 5, this can bring to a drastic reduction of the size of the spatial model, thus increasing the practical efficiency of spatial model checking. For instance, a larger variant of the model in Figure 6a, composed of 6,154 cells, is reduced to a model consisting of 38 nodes, which is a reduction of two orders of magnitude. Summarising, the main original contributions are:

– Introduction of a novel polyhedral model minimisation procedure based on weak $\pm$-bisimilarity, including formal proofs of correctness of the procedure;
– Proof-of-concept of its practical potential and effectiveness through a prototype toolchain and spatial model checking examples. It is shown that the cell poset models can be drastically reduced by several orders of magnitude.

The paper is structured as follows. Section 2 presents relevant background and notation. Section 3 presents the minimisation approach. Section 4 gives an overview of the related toolchain and Section 5 presents applications of the minimisation procedure. Finally, Section 6 presents conclusions and plans for future work. Detailed proofs, further background and notational details are presented in the Appendix.

2 Preliminaries

In this section we collect general definitions and notation, and introduce some basic notions regarding the language $\texttt{SLCS}_\gamma$, its polyhedral and poset models, and the truth-preserving map $\mathbb{F}$ between these models. For further details we refer the reader to [6, 14, 9].

General notions and notation. For sets X and Y, a function $f : X \to Y$, and subsets $A \subseteq X$ and $B \subseteq Y$ we define the direct image $f(A)$ of A and the inverse image $f^{-1}(B)$ of B by $\{f(a) \,|\, a \in A\}$ and $\{a \,|\, f(a) \in B\}$, respectively. The restriction of f to A is denoted by $f|A$. The powerset of the set X is denoted by $\mathbf{2}^X$. For a relation $R \subseteq X \times X$ we let $R^- = \{(y, x) \,|\, (x, y) \in R\}$ denote its converse and we let $R^\pm$ denote $R \cup R^-$. In the remainder of the paper we assume a set $\texttt{PL}$ of proposition letters to be given. The sets of natural numbers and of real numbers are denoted by $\mathbb{N}$ and $\mathbb{R}$, respectively. We use standard interval notation: for $x, y \in \mathbb{R}$ we let $[x, y]$ be the set $\{r \in \mathbb{R} \,|\, x \leqslant r \leqslant y\}$, $[x, y) = \{r \in \mathbb{R} \,|\, x \leqslant r < y\}$, and so on. Intervals of $\mathbb{R}$ are equipped with the Euclidean topology. We use a similar notation for intervals over $\mathbb{N}$: for $n, m \in \mathbb{N}$ we use $[m; n]$ to denote the set $\{i \in \mathbb{N} \,|\, m \leqslant i \leqslant n\}$, $[m; n) = \{i \in \mathbb{N} \,|\, m \leqslant i < n\}$, and so on.

Topological notions. A simplex σ is the convex hull of $d + 1$ affinely independent points $\mathbf{v_0}, \ldots, \mathbf{v_d}$ in $\mathbb{R}^m$, also called vertices, where $d \leqslant m$, thus $\sigma = \{\, \lambda_0 \mathbf{v_0} + \ldots + \lambda_d \mathbf{v_d} \mid \lambda_0, \ldots, \lambda_d \in [0, 1] \text{ and } \sum_{i=0}^{d} \lambda_i = 1 \,\}$. For instance, a segment AB together with its end-points A and B is a simplex in $\mathbb{R}^m$, for $m \geqslant 1$. Any subset of the set of points characterising a simplex σ induces a simplex σ', and we write $\sigma' \sqsubseteq \sigma$, noting that $\sqsubseteq$ is a partial order, e.g. $A \sqsubseteq AB \sqsubseteq AB$.

The relative interior $\widetilde{\sigma}$ of a simplex σ is the same as σ "without its borders", i.e. $\widetilde{\sigma} = \{\, \lambda_0 \mathbf{v_0} + \ldots + \lambda_d \mathbf{v_d} \,|\, \lambda_0, \ldots, \lambda_d \in (0, 1] \text{ and } \sum_{i=0}^{d} \lambda_i = 1 \,\}$. For instance, the relative interior $\widetilde{AB}$ of the closed segment AB is the open segment AB, without the end-points A and B. The relative interior of a simplex is often called a *cell* and is, for $d > 0$, equal to the topological interior taken inside the affine hull of the simplex.[6] The partial order $\sqsubseteq$ is reflected on cells: $\widetilde{\sigma_1} \preccurlyeq \widetilde{\sigma_2}$ iff $\sigma_1 \sqsubseteq \sigma_2$. Note that $\widetilde{\sigma_1} \preccurlyeq \widetilde{\sigma_2}$ iff $\widetilde{\sigma_1} \in \mathcal{C}_T(\widetilde{\sigma_2})$, where $\mathcal{C}_T$ is the topological closure.

A simplicial complex K is a finite collection of simplexes in $\mathbb{R}^m$ such that (i) if $\sigma \in K$ and $\sigma' \sqsubseteq \sigma$ then also $\sigma' \in K$, and (ii) if $\sigma, \sigma' \in K$ and $\sigma \cap \sigma' \neq \emptyset$, then $\sigma \cap \sigma' \sqsubseteq \sigma$ and $\sigma \cap \sigma' \sqsubseteq \sigma'$. The cell poset of simplicial complex K is $(\widetilde{K}, \preccurlyeq)$ where $\widetilde{K}$ is the set $\{\, \widetilde{\sigma} \,|\, \sigma \in K\}$. The polyhedron $|K|$ of K is the set-theoretic union of the simplexes in K. Note that $|K|$ inherits the topology of $\mathbb{R}^m$.

A polyhedral model is a pair $(|K|, V)$ where $V : \mathtt{PL} \to \mathbf{2}^{|K|}$ maps every proposition letter $p \in \mathtt{PL}$ to the set of points of $|K|$ that satisfy p. It is required, for all $p \in \mathtt{PL}$, that $V(p)$ is always a union of cells in $\widetilde{K}$. Similarly, a poset model $(W, \preccurlyeq, \mathcal{V})$ is a poset equipped with a valuation function $\mathcal{V} : \mathtt{PL} \to \mathbf{2}^W$. Given a polyhedral model $\mathcal{P} = (|K|, V)$, we say that $(\widetilde{K}, \preccurlyeq, \mathcal{V})$ is the *cell poset model* of $\mathcal{P}$ iff $(\widetilde{K}, \preccurlyeq)$ is the cell poset of K and, for all $\widetilde{\sigma} \in \widetilde{K}$, we have: $\widetilde{\sigma} \in \mathcal{V}(p)$ iff $\widetilde{\sigma} \subseteq V(p)$. We let $\mathbb{F}(\mathcal{P})$ denote the cell poset model of $\mathcal{P}$ and, with a bit of overloading, for all $x \in |K|$, let $\mathbb{F}(x)$ denote the unique cell $\widetilde{\sigma}$ such that $x \in \widetilde{\sigma}$, then the map $\mathbb{F} : |K| \to \widetilde{K}$ is a continuous function [10, Corollary 3.4]. Note that poset models are a subclass of Kripke models. We say that $\mathcal{F}$ is a cell poset model to mean that there exists a polyhedral model $\mathcal{P}$ such that $\mathcal{F} = \mathbb{F}(\mathcal{P})$.

Fig. 1a shows a polyhedral model with three proposition letters, viz. **red**, **green** and **gray**, indicated by corresponding colours. The model is 'unpacked' into its cells in the middle of the Fig. 1b. The cells are collected in the cell poset model, whose Hasse diagram is shown in Fig. 1c.

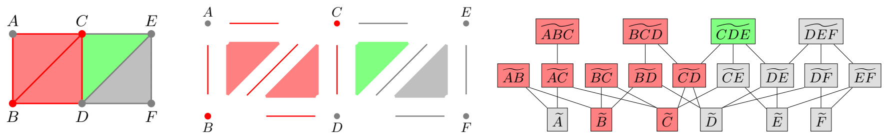


Fig. 1: A polyhedral model $\mathcal{P}$ (1a) with its cells (1b), and the Hasse diagram of the related cell poset model (1c).

In a topological space (X, τ), a topological path from $x \in X$ is a total, continuous function $\pi : [0,1] \to X$ such that $\pi(0) = x$. We call $\pi(0)$ and $\pi(1)$ the *starting* and *ending* point of π, respectively, while $\pi(r)$ is an *intermediate* point of π for $r \in (0,1)$. Fig. 2a shows (in blue) a path from a point x on the open segment $\widetilde{AB}$ to the vertex D in the polyhedral model of Fig 1a.

[6] But note that the relative interior of a simplex composed of just a single point is the point itself and not the empty set.

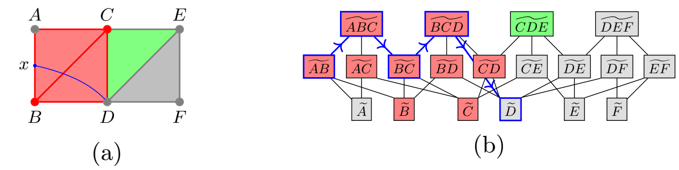


Fig. 2: (2a) A topological path from point x to vertex D in the polyhedral model $\mathcal{P}$ of Figure 1a. (2b) The corresponding $\pm$-path (in blue) in the Hasse diagram of the cell poset model $\mathbb{F}(\mathcal{P})$.

Topological paths are represented in cell posets by so-called $\pm$-paths, a subclass of undirected paths. For technical reasons,[7] in this paper we extend the definition given in [6] to general Kripke frames. Given a Kripke frame (W, R), an undirected path of length $\ell \in \mathbb{N}$ from w is a total function $\pi : [0;\ell] \to X$ such that $\pi(0) = x$ and $R^{\pm}(\pi(i), \pi(i+1))$ for all $i \in [0;\ell)$. The starting and ending points are $\pi(0)$ and $\pi(\ell)$, respectively, while $\pi(i)$ is an intermediate point for $i \in (0;\ell)$. The path is a *$\pm$-path* iff $\ell \geqslant 2$, $R(\pi(0), \pi(1))$, and $R^{-}(\pi(\ell-1), \pi(\ell))$.

The $\pm$-path[8] $(\widetilde{AB}, \widetilde{ABC}, \widetilde{BC}, \widetilde{BCD}, \widetilde{D})$, drawn in blue in Fig. 2b, faithfully represents the path from x shown in Fig. 2a. Note that a path π such that, say, $\pi(0) \in \widetilde{CD}$, $\pi(1) = E$, and $\pi((0,1)) \subseteq \widetilde{CDE}$, i.e. a path that "jumps immediately" to $\widetilde{CDE}$ after starting in $\widetilde{CD}$ cannot be represented in the poset by any undirected path π', of some length $\ell \geqslant 2$ such that $\pi'(0) \succ \pi'(1)$ (or $\pi'(\ell-1) \prec \pi'(\ell)$, for symmetry reasons), while it is correctly represented by the $\pm$-path $(\widetilde{CD}, \widetilde{CDE}, \widetilde{E})$, where $\widetilde{CD} \prec \widetilde{CDE} \succ \widetilde{E}$.

The logic $\mathtt{SLCS}_\eta$. The logic $\mathtt{SLCS}_\eta$, a version of $\mathtt{SLCS}$ for polyhedral models, has been introduced in [8]. Apart from proposition letters, negation, and conjunction, it has a single modal operator η, expressing conditional reachability. The satisfaction of $\eta(\Phi_1, \Phi_2)$, for a polyhedral model $\mathcal{P} = (|K|, V)$ and $x \in |K|$, is recalled below:

$$\begin{aligned}\mathcal{P}, x \models \eta(\Phi_1, \Phi_2) \Leftrightarrow & \text{ there is a topological path } \pi : [0,1] \to |K| \text{ with } \pi(0) = x, \\ & \mathcal{P}, \pi(r) \models \Phi_1 \text{ for all } r \in [0,1), \text{ and } \mathcal{P}, \pi(1) \models \Phi_2.\end{aligned}$$

We also recall the interpretation of $\mathtt{SLCS}_\eta$ on poset models. The satisfaction of $\eta(\Phi_1, \Phi_2)$, for a poset model $\mathcal{F} = (W, \preccurlyeq, \mathcal{V})$ and $w \in W$, is given by

$$\begin{aligned}\mathcal{F}, w \models \eta(\Phi_1, \Phi_2) \Leftrightarrow & \text{ there is a } \pm\text{-path } \pi : [0;\ell] \to W \text{ with } \pi(0) = w, \\ & \mathcal{F}, \pi(i) \models \Phi_1 \text{ for all } i \in [0;\ell), \text{ and } \mathcal{F}, \pi(\ell) \models \Phi_2.\end{aligned}$$

In [8] it has been proven that for each point $x \in |K|$ and formula Φ in $\mathtt{SLCS}_\eta$

[7] We are interested in model-checking structures resulting from the minimisation, via bisimilarity, of cell poset models, and such structures are often just (reflexive) Kripke models rather than poset models.

[8] For undirected path π of length ℓ we often use the sequence notation $(x_i)_{i=0}^{\ell}$ where $x_i = \pi(i)$ for $i \in [0;\ell]$.

we have $\mathcal{P}, x \models \Phi$ iff $\mathbb{F}(\mathcal{P}), \mathbb{F}(x) \models \Phi$. In addition, the notion of *weak simplicial bisimilarity*, as introduced in [8] for polyhedral models, enjoys the classical Hennessy-Milner property: two points $x_1, x_2 \in |K|$ are weakly simplicial bisimilar, written $x_1 \approx^{\mathcal{P}}_{\triangle} x_2$, iff they satisfy the same $\texttt{SLCS}_\eta$ formulas, i.e. they are equivalent with respect to the logic $\texttt{SLCS}_\eta$, written $x_1 \equiv^{\mathcal{P}}_{\eta} x_2$.

The result has been extended to the notion of *weak ±-bisimilarity* on finite poset models, a notion of bisimilarity based on ±-paths: $w_1, w_2 \in W$ are weakly ±-bisimilar, written $w_1 \approx^{\mathcal{F}}_{\pm} w_2$, iff they satisfy the same $\texttt{SLCS}_\eta$ formulas, i.e. $w_1 \equiv^{\mathcal{F}}_{\eta} w_2$ (see [8] for details). In summary, we have:

$$x_1 \approx^{\mathcal{P}}_{\triangle} x_2 \text{ iff } x_1 \equiv^{\mathcal{P}}_{\eta} x_2 \text{ iff } \mathbb{F}(x_1) \equiv^{\mathbb{F}(\mathcal{P})}_{\eta} \mathbb{F}(x_2) \text{ iff } \mathbb{F}(x_1) \approx^{\mathbb{F}(\mathcal{P})}_{\pm} \mathbb{F}(x_2).$$

As a closing remark, since ±-paths are defined on Kripke structures, the satisfaction relation of $\texttt{SLCS}_\eta$ on poset models extends naturally to Kripke structures.

3 Building the Minimal Model Modulo $\equiv_\eta$

In this section we present a minimisation procedure for finite poset models modulo weak ±-bisimilarity. Given a finite poset model $\mathcal{F} = (W, \preccurlyeq, \mathcal{V})$, the procedure consists of three steps:

Step 1: The poset model $\mathcal{F}$ is encoded as an LTS denoted $\mathbb{S}_C(\mathcal{F})$. The set of states of $\mathbb{S}_C(\mathcal{F})$ is W. The encoding ensures that logically equivalent points are mapped to branching bisimilar states. Thus, for points $w_1, w_2 \in W$ that are logically equivalent with respect to $\texttt{SLCS}_\eta$ in the poset model $\mathcal{F}$, i.e. $w_1 \equiv^{\mathcal{F}}_{\eta} w_2$, we must have that they are branching bisimilar as states in the LTS $\mathbb{S}_C(\mathcal{F})$, i.e. $w_1 \leftrightarrow^{\mathbb{S}_C(\mathcal{F})}_{b} w_2$.

Step 2: The LTS $\mathbb{S}_C(\mathcal{F})$ is reduced modulo branching bisimilarity using available software tools, such as mCRL2 [21]. This step yields the set of equivalence classes of W for $\leftrightarrow^{\mathbb{S}_C(\mathcal{F})}_{b}$. Because of the correspondence of logical equivalence and branching bisimilarity, we obtain $W/{\equiv^{\mathcal{F}}_{\eta}}$.

Step 3: The minimal model is built. It turns out that this model is not necessarily a poset model (see the example in Fig. 5 in Section 5). However, it is a reflexive Kripke model of the form $(W/{\equiv^{\mathcal{F}}_{\eta}}, R)$ where R is a relation induced by the ordering $\preccurlyeq$ of $\mathcal{F}$.

In the remainder of this section we focus on Step 1 and Step 3.

3.1 The Encoding of $\mathcal{F}$ as $\mathbb{S}_C(\mathcal{F})$

We obtain the LTS $\mathbb{S}_C(\mathcal{F})$ from the poset $\mathcal{F}$ as follows.

Definition 1. *For finite poset model $\mathcal{F} = (W, \preccurlyeq, \mathcal{V})$ and symbols $\tau, \mathbf{c}, \mathbf{d} \notin \texttt{PL}$, the LTS $\mathbb{S}_C(\mathcal{F})$ is defined by $\mathbb{S}_C(\mathcal{F}) = (S, L, \rightarrow)$ where*

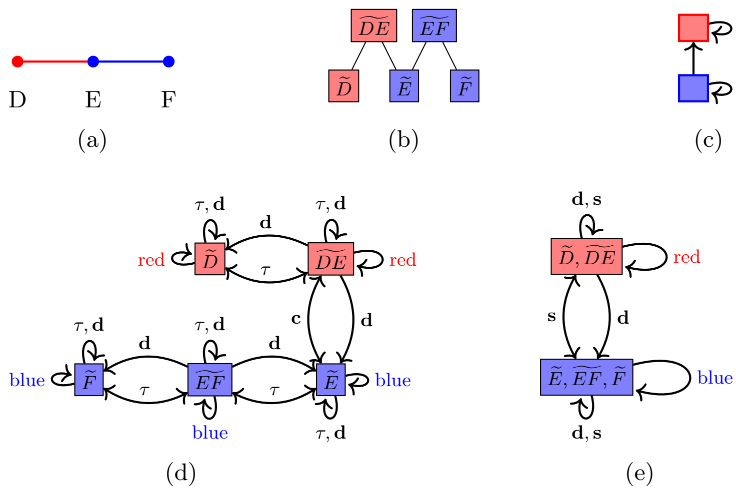


Fig. 3: (3a) A polyhedral model $\mathcal{P}'$; (3b) poset model $\mathcal{F}' = \mathbb{F}(\mathcal{P}')$; (3c) minimal Kripke model $\mathcal{F}'_{min}$; (3d) the LTS $\mathbb{S}_C(\mathcal{F}')$ obtained from $\mathcal{F}'$ by the encoding of Def. 1; (3e) The LTS $\mathbb{S}_A(\mathcal{F}')$ obtained from $\mathcal{F}'$ by the encoding of Def. 3. Note that whenever $s \xrightarrow{\ell} s'$ and $s' \xrightarrow{\ell} s$ a "double transition" $s \xleftrightarrow{\ell} s'$ is drawn in the figure between s and s'.

- *the set of states S is the set W;*
- *the set of labels L consists of* $\mathtt{PL} \cup \{\tau, \mathbf{c}, \mathbf{d}\}$*;*
- *the transition relation $\rightarrow$ is the smallest relation on $S \times L \times S$ induced by the following transition rules.*

$$\text{(PLC)}\ \frac{w \in \mathcal{V}(p)}{w \xrightarrow{p} w} \qquad \text{(TAU)}\ \frac{w \preccurlyeq^{\pm} w' \quad \mathcal{V}^{-1}(\{w\}) = \mathcal{V}^{-1}(\{w'\})}{w \xrightarrow{\tau} w'}$$

$$\text{(CNG)}\ \frac{w \preccurlyeq^{\pm} w' \quad \mathcal{V}^{-1}(\{w\}) \neq \mathcal{V}^{-1}(\{w'\})}{w \xrightarrow{\mathbf{c}} w'} \qquad \text{(DWN)}\ \frac{w \succcurlyeq w'}{w \xrightarrow{\mathbf{d}} w'} \quad \bullet$$

Fig. 3a shows an example of a simple polyhedral model $\mathcal{P}'$. The LTS $\mathbb{S}_C(\mathcal{F}')$ associated to the poset model $\mathcal{F}'$ of $\mathcal{P}'$ is shown in Fig. 3d.[9]

In order to show that the above definition establishes that $w_1 \equiv^{\mathcal{F}}_{\eta} w_2$ iff $w_1 \leftrightarroweq^{\mathbb{S}_C(\mathcal{F})}_{b} w_2$, it is convenient to consider an intermediate structure, that is an LTS too. We denote this second LTS by $\mathbb{S}_A(\mathcal{F})$. This structure helps in the proofs to separate concerns related to the various equivalences that are involved. Suppose that points w_1 and w_2 in $\mathcal{F}$ are encoded by the states s_1 and s_2 in $\mathbb{S}_A(\mathcal{F})$,

[9] Note that τ self-loops in LTSs are irrelevant since we are working modulo branching bisimilarity. In this paper we focus mainly on correctness, while in future work we will address optimisation of the encoding procedures.

respectively. We will have that points w_1 and w_2 are logically equivalent in $\mathcal{F}$ with respect to $\mathtt{SLCS}_\eta$ iff states s_1 and s_2 are strong bisimilar (in the sense of [25]) in $\mathbb{S}_A(\mathcal{F})$, written $s_1 \simeq^{\mathbb{S}_A(\mathcal{F})} s_2$. Furthermore, it will hold that s_1 and s_2 are strongly bisimilar in $\mathbb{S}_A(\mathcal{F})$ iff w_1 and w_2 are branching bisimilar in $\mathbb{S}_C(\mathcal{F})$, thus providing the correctness of the construction.

LTS $\mathbb{S}_A(\mathcal{F})$ is more abstract than $\mathbb{S}_C(\mathcal{F})$. Define $\Theta = \{\, \mathcal{V}^{-1}(\{w\}) \,|\, w \in W \,\}$ and consider, for $\alpha \in \Theta$, the α-connected components of $\mathcal{F}$. Then, each state s of $\mathbb{S}_A(\mathcal{F})$ is an α-connected component of $\mathcal{F}$, for some α as above. So, we group together all the points in W that can reach one another only via a path in $\mathcal{F}$ composed of elements all satisfying exactly the same proposition letters.

The above intuition is formalised by the following definition.

Definition 2. *Given finite poset model $(W, \preccurlyeq, \mathcal{V})$, we define relation $\rightleftharpoons \subseteq W \times W$ as the set of pairs (w_1, w_2) such that an undirected path π of some length ℓ exists with $\pi(0) = w_1, \pi(\ell) = w_2$, and $\mathcal{V}^{-1}(\{\pi(i)\}) = \mathcal{V}^{-1}(\{\pi(j)\})$, for all $i, j \in [0; \ell]$.*

The relevant definitions lead straightforwardly to the following observation.

Proposition 1. *Let $(W, \preccurlyeq, \mathcal{V})$ be a finite poset model. Then $\rightleftharpoons$ is an equivalence relation on W.* □

We are ready to actually define the encoding to the more "abstract" LTS.

Definition 3. *Given finite poset model $\mathcal{F} = (W, \preccurlyeq, \mathcal{V})$, and $\mathbf{s}, \mathbf{d} \notin \mathtt{PL}$, we define the LTS $\mathbb{S}_A(\mathcal{F}) = (S, L, \rightarrow)$ where*

- *the set S of states is the quotient $W/\rightleftharpoons$ of W modulo $\rightleftharpoons$;*
- *the set L of labels is $\mathbf{2}^{\mathtt{PL}} \cup \{\mathbf{s}, \mathbf{d}\}$;*
- *the transition relation is the smallest relation on $W \times L \times W$ induced by the following transition rules:*

$$\text{(PL)}\ [w]_{\rightleftharpoons} \xrightarrow{\mathcal{V}^{-1}(\{w\})} [w]_{\rightleftharpoons}$$

$$\text{(Step)}\ \frac{w \preccurlyeq^{\pm} w'}{[w]_{\rightleftharpoons} \xrightarrow{\mathbf{s}} [w']_{\rightleftharpoons}} \qquad \text{(Down)}\ \frac{w \succcurlyeq w'}{[w]_{\rightleftharpoons} \xrightarrow{\mathbf{d}} [w']_{\rightleftharpoons}}$$

•

An example of $\mathbb{S}_A(\mathcal{F})$ is shown in Fig. 3e. The following theorem ensures that the points w_1 and w_2 are logically equivalent in $\mathcal{F}$ with respect to $\mathtt{SLCS}_\eta$ if and only if their equivalence classes $[w_1]_{\rightleftharpoons}$ and $[w_2]_{\rightleftharpoons}$ are strongly bisimilar in $\mathbb{S}_A(\mathcal{F})$.

Theorem 1. *Let $\mathcal{F} = (W, \preccurlyeq, \mathcal{V})$ be a finite poset model. For all $w_1, w_2 \in W$ it holds that $[w_1]_{\rightleftharpoons} \simeq^{\mathbb{S}_A(\mathcal{F})} [w_2]_{\rightleftharpoons}$ iff $w_1 \equiv_\eta^{\mathcal{F}} w_2$.*

The implication from left to right is proven by induction on $\mathtt{SLCS}_\eta$ formulas. For the reverse direction one shows that the relation B on the set of states of $\mathbb{S}_A(\mathcal{F})$ such that $B(s_1, s_2)$ iff $w_1 \equiv_\eta w_2$ for some $w_1 \in s_1$, $w_2 \in s_2$, is a strong bisimulation.

The following theorem ensures that $[w_1]_{\rightleftharpoons}$ and $[w_2]_{\rightleftharpoons}$ are strongly bisimilar in $\mathbb{S}_A(\mathcal{F})$ if and only if w_1 and w_2 are branching bisimilar in $\mathbb{S}_C(\mathcal{F})$.

Theorem 2. *Let $\mathcal{F} = (W, \preccurlyeq, \mathcal{V})$ be a finite poset model. For all $w_1, w_2 \in W$ it holds that $[w_1]_{\rightleftharpoons} \simeq^{\mathbb{S}_A(\mathcal{F})} [w_2]_{\rightleftharpoons}$ iff $w_1 \leftrightarrow_b^{\mathbb{S}_C(\mathcal{F})} w_2$.*

To prove the implication from left to right we show that the relation B_C on W such that $B_C(w_1, w_2)$ iff $[w_1]_{\rightleftharpoons} \simeq^{\mathbb{S}_A(\mathcal{F})} [w_2]_{\rightleftharpoons}$ is a branching bisimulation. For the reverse implication one shows that the relation B_A on $W/{\rightleftharpoons}$ with $B_A(s_1, s_2)$ iff $w_1 \leftrightarrow_b^{\mathbb{S}_C(\mathcal{F})} w_2$ for some $w_1 \in s_1$, $w_2 \in s_2$, is a strong bisimulation.

From Theorems 1 and 2 we finally obtain our claim:

Corollary 1. *Let $\mathcal{F} = (W, \preccurlyeq, \mathcal{V})$ be a finite poset model. For all $w_1, w_2 \in W$ the following holds: $w_1 \equiv_\eta^{\mathcal{F}} w_2$ iff $w_1 \leftrightarrow_b^{\mathbb{S}_C(\mathcal{F})} w_2$.*

Now that we have characterised logical equivalence $\equiv_\eta$ for $\texttt{SLCS}_\eta$ for the points of a poset model $\mathcal{F}$ in terms of branching bisimilarity $\leftrightarrow_b$ for the LTS $\mathbb{S}_C(\mathcal{F})$, we can compute the minimal LTS modulo branching bisimilarity with standard techniques available, such as branching equivalence minimisation provided by the `mCRL2` tool set.

3.2 Building the Minimal Model

Via the correspondence of $\texttt{SLCS}_\eta$-equivalence for a poset model and branching bisimilarity of its encoding, one can obtain the equivalence classes of $\texttt{SLCS}_\eta$ by identifying the branching bisimilar states in the LTS. With the equivalence classes modulo $\equiv_\eta$ for the poset model available, we can consider the ensued quotient model. We obtain a Kripke model that is minimal with respect to $\equiv_\eta$, but which is not necessarily a poset model.

Definition 4 **($\mathcal{F}_{\min}$).** *For finite poset model $\mathcal{F} = (W, \preccurlyeq, \mathcal{V})$ let the Kripke model $\mathcal{F}_{\min} = (W_{\min}, R_{\min}, \mathcal{V}_{\min})$ have*

- *set of points $W_{\min} = W/{\equiv_\eta}$, the equivalence classes of W with respect to $\equiv_\eta$,*
- *accessibility relation $R_{\min} \subseteq W_{\min} \times W_{\min}$ satisfying*
$$R([w_1], [w_2]) \text{ iff } w_1' \preccurlyeq w_2' \text{ for some } w_1' \equiv_\eta w_1 \text{ and } w_2' \equiv_\eta w_2$$
for $w_1, w_2 \in W$, and
- *valuation $\mathcal{V}_{\min} : \texttt{PL} \to \mathbf{2}^{W_{\min}}$ such that*
$$\mathcal{V}_{\min}(p) = \{\, [w] \in W_{\min} \mid w' \in \mathcal{V}(p) \text{ for some } w' \equiv_\eta w \,\}$$
for $p \in \texttt{PL}$. •

Clearly, $\mathcal{F}_{\min}$ is a reflexive Kripke model. Reflexivity of the accessibility relation $R_{\min}$ is immediate from reflexivity of the ordering $\preccurlyeq$. Furthermore, it is minimal with respect to $\texttt{SLCS}_\eta$ by definition of $\equiv_\eta$ and $W_{/\equiv_\eta}$. An example of the minimal Kripke model of the polyhedral model in Fig. 3a is shown in Fig. 3c. The following theorem ensures that the model defined above is sound and complete with respect to the logic, so that the minimisation procedure is correct. It is proven by induction on the structure of Φ.

Theorem 3. *Given finite poset model $\mathcal{F} = (W, \preccurlyeq, \mathcal{V})$ let $\mathcal{F}_{\min}$ be defined as in Definition 4. Then, for each $w \in W$ and* $\mathtt{SLCS}_\eta$ *formula Φ the following holds: $\mathcal{F}, w \models \Phi$ iff $\mathcal{F}_{\min}, [w]_{\equiv_\eta} \models \Phi$.*

Finally, the following theorem turns out to be useful for simplifying the procedure for the effective construction of $\mathcal{F}_{min}$:

Theorem 4. *For any poset model $\mathcal{F} = (W, \preccurlyeq, \mathcal{V})$ and $\mathcal{F}_{\min}$ as of Def. 4 and for all $\alpha_1, \alpha_2 \in W_{\min}$, it holds that $R_{min}(\alpha_1, \alpha_2)$ if and only if $\alpha_2 \xrightarrow{\mathbf{d}} \alpha_1$ is a transition of the minimal LTS obtained from $\mathbb{S}_C(\mathcal{F})$ via branching equivalence.*

4 An Experimental Minimisation Toolchain

In this section we provide a brief overview of an experimental toolchain to study the minimisation procedure for polyhedral models and to illustrate the practical potential of the theory presented in the previous section. The further development and a thorough analysis of the toolchain will be the subject of future work. Fig. 4 illustrates the elements of the toolchain that, starting from a polyhedral model in `json` format, produces the set of equivalence classes and the minimal Kripke model. The former may serve as input for the `PolyVisualizer` tool[10] [6], a polyhedra visualizer, to inspect the results, whereas the latter can be used for spatial model checking through an adapted version of `PolyLogicA` that can check spatial properties on Kripke models using ±-*paths* instead of regular paths. The toolchain is also able to map the results obtained on the minimal Kripke model back to the original polyhedral model, because of the direct correspondence between the states of the Kripke model and the equivalence classes.

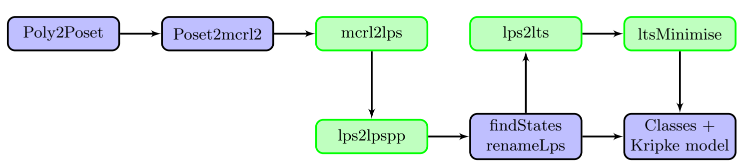


Fig. 4: Tool chain for polyhedral model minimisation. Parts in green are command line operations of the `mCRL2` tool suite. Parts in blue are developed in Python in the context of the current paper.

The toolchain uses several command line operations provided by the `mCRL2` tool suite [11] (shown in green in Fig. 4) and a number of operations developed in the context of this paper (shown in blue in Fig. 4). The prototype aims to demonstrate the feasibility of our approach from a qualitative perspective, providing support for examples that illustrate the practical usefulness of the theory.

[10] http://ggrilletti2.scienceontheweb.net/polyVisualizer/polyVisualizer_static_maze.html

Further performance issues, computational complexity and a full implementation of the approach will be addressed in future work. The operation `Poly2Poset` transforms the polyhedral model into a poset model. The operation `Poset2mcrl2` encodes the poset model into a `mCRL2` specification of an LTS following the procedure defined in Definition 1. The operations `mcrl2lps` and `lps2lts` transform the encoding into a linearised LTS-representation which is then minimised (`ltsMinimise`) via branching bisimulation. The operation `lps2lpspp` provides a textual version of the linear process which is used to obtain the correspondence between internal state labels of the minimised LTS and the cells of the original polyhedral model present in the equivalence classes. The latter, in turn, are essential for the generation of the result files of model checking the minimised model and form the input to the `PolyVisualizer` (together with the original polyhedral model and a colour definition file). Figs. 6 and 7 in the next section show an example. Maintaining the relation between internal state labels of the minimised LTS and the original states of the poset and polyhedral model is the most tricky part of the toolchain as such internal state labels are assigned dynamically in the `lps2lts` procedure. This aspect is dealt with by the `findStates` and `renameLps` procedures.

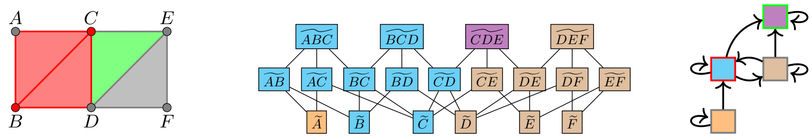
5 Minimisation at Work

Fig. 5a shows the model which was presented in Fig. 1a. Its poset representation is shown in Fig. 5b with equivalence classes indicated in different colours.[11] The minimal Kripke model is shown in Fig. 5c. In the latter, the colours of the borders of the elements (red, green, and grey) recall the original atomic propositions used in Fig. 5a, whereas the colour of the interior reflects the colour of the equivalence class as used in Fig. 5b. Note that vertex A is not part of the equivalence class of the other grey points. It can be distinguished from, for example, grey point D because D satisfies $\texttt{SLCS}_\eta$-formula $\phi_1 = \eta(\mathbf{grey} \vee \mathbf{green}, \mathbf{green})$ whereas point A does *not* satisfy ϕ_1. Note also that formula $\phi_2 = \eta(\mathbf{grey} \vee \mathbf{red}, \mathbf{red})$ is satisfied by D, but also by point E (actually by any grey point, including A).

Recall that in the minimal model the Kripke states represent equivalence classes modulo $\equiv_\eta$. There is a transition in the Kripke model (Fig. 5c) between two states, say x and y, respectively, if and only if there is a cell in the class related to x and one in class related to y that are connected in the poset (see Fig. 5b). This is the standard way to build such minimal models (see Def. 4). In Fig. 5b it is easy to see that cell $\widetilde{D}$ (brown) is a face of $\widetilde{CD}$ (cyan) and $\widetilde{C}$ (cyan) is a face of $\widetilde{CE}$ (brown), so they are mutually 'below' each other. This explains the presence of a loop in the minimal Kripke structure between the brown and the cyan class (see Fig. 5c).

The example in Fig. 6a shows a simple symmetric 3D cube composed of one white 'room' in the middle surrounded by 26 green 'rooms' in a snapshot of

[11] Note that such colours have only an illustrative purpose. In particular, they are not related to the colours expressing the evaluation of proposition letters.



(a) (b) (c)

Fig. 5: Polyhedral model (5a), its classes in the poset (5b) and its minimised Kripke model (5c).

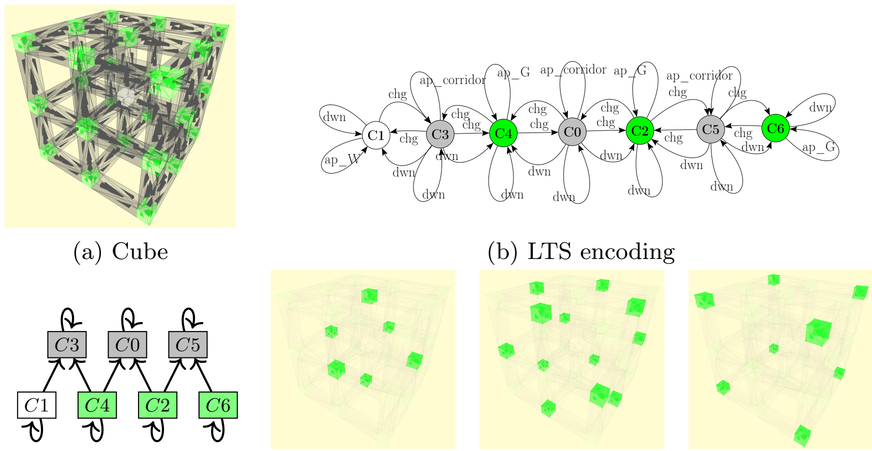
the `PolyVisualizer` tool. Rooms are connected by grey 'corridors' as shown in the figure. In total, the structure consists of 2,620 cells. Fig. 6b shows the minimal LTS with respect to branching bisimilarity as produced by `mCRL2`. (The numbering of the states is as generated by `mCRL2`). It has 7 states: one white state $C1$, three grey ones ($C3$, $C0$, and $C5$) and three green states ($C4$, $C2$, and $C6$). The white state represents the class of all the cells of the white room. Transition labels *chg* and *dwn* denote **c** and **d**, whereas ap_X denotes atomic proposition X. Green state $C4$ (visualised on the original polyhedron in Fig. 6d) represents the the class of green rooms that are directly connected to the white room by a corridor. Green state $C2$ (visualised in Fig. 6e) represents the class of green rooms situated on the edges of the cube. Green state $C6$ (visualised in Fig. 6f) represents the class of green rooms situated at the corners of the cube. Fig. 6c shows the minimal Kripke model modulo $\equiv_\eta$.

It is not difficult to find $\texttt{SLCS}_\eta$ formulas that distinguish the various green classes. For example, the cells in $C4$ satisfy $\Phi_1 = \eta(\texttt{green} \vee \eta(\texttt{grey}, \texttt{white}), \texttt{white})$, whereas no cell in $C2$ or $C6$ satisfies Φ_1. To distinguish class $C2$ from $C6$ and $C4$, one can observe that cells in $C2$ satisfy $\Phi_2 = \eta(\texttt{green} \vee \eta(\texttt{grey}, \Phi_1), \Phi_1)$ whereas those in $C3$ do not satisfy Φ_2. Figure 7 shows the result of PolyLogicA model checking for the formulas Φ_1 (see Fig. 7b) and Φ_2 (see Fig. 7c).[12]

Table 1 provides a more detailed insight in the time performance of the various components of the toolchain on models of the cube of different sizes, all with green rooms forming the outer frame of the cube and white rooms positioned inside the cube. Note the substantial reduction in size (several orders of magnitude) of the minimised model, where the number of states corresponds to the number of equivalence classes, compared to the full model (nr. of cells). This leads to a similar reduction in model checking time (see last two lines of Table 1). However, regarding the minimisation procedure itself, there seems to be a bottleneck of performance in `lps2lts`, whereas the time to encode and minimise the model (see `ltsMinimise`) is actually very small. Note that the minimised model, once obtained, can be used for multiple model checking sessions. Future work will address further improvements of the efficiency of the constituents of the minimisation procedure, even if the current results are already very encour-

[12] All tests were performed on a workstation equipped with an Intel(R) Core(TM) i9-9900K CPU @ 3.60 GHz (8 cores, 16 threads).

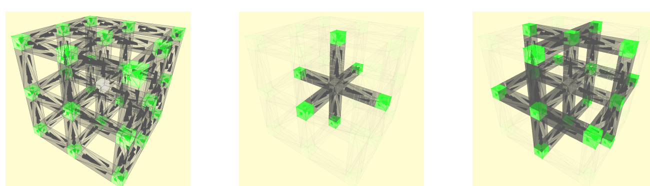
aging. More specifically, the `lps2lts` step might be avoided by implementing our encoding directly into the binary `mCRL2` LTS format. This requires usage of the `mCRL2` `C++` *application programming interface*, and is left to future work.



(a) Cube (b) LTS encoding

(c) Minimal model (d) C4 (e) C2 (f) C6

Fig. 6: Cube with 27 rooms: 26 green and one white in the middle.



(a) (b) Result ϕ_1 (c) Result ϕ_2

Fig. 7: (7a) The 3D cube. Results of `PolyLogicA` model checking of the formulas ϕ_1 (7b) and ϕ_2 (7c) on the minimised model mapped back onto the full 3D cube with `PolyVisualizer`.

6 Conclusions

Polyhedral models are widely used in domains that exploit mesh processing such as 3D computer graphics. These models are typically huge, consisting of very many cells. Spatial model checking of such models is an interesting, novel

Table 1: Performance for 3D cube example. All times are in seconds.

	Cube 3x3x3	Cube 3x5x3	Cube 3x5x4	Cube 5x5x5
Nr. of classes	7	21	38	21
Nr. of cells	2,619	3,568	6,145	13,375
Nr. of vertices	216	288	480	1,000
`poly2poset`	0.35	0.34	0.43	1.10
`loadData`	0.00	0.00	0.01	0.02
`poset2mcrl2`	0.16	0.30	0.42	0.95
`mcrl2lps`	1.71	3.51	5.42	23.72
`lps2lpspp`	0.24	0.41	0.57	1.95
`findStates`	0.17	0.31	0.41	4.18
`renamelps`	0.54	0.95	1.34	4.47
`lps2lts`	21.41	78.26	135.22	794.33
`ltsMinimise`	0.06	0.23	0.24	0.35
`createJsonFiles`	6.35	51.37	160.53	587.99
`createModelFile`	0.01	0.01	0.01	0.03
Model checking original model	8.76	24.90	64.50	671.30
Model checking minimised model	0.02	0.03	0.03	0.03

approach to verify properties of such models and visualise the results in a graphically appealing way. In previous work the polyhedral model checker `PolyLogicA` was developed for this purpose.

In order to reduce model checking time and computing resources, we have proposed an effective procedure that computes the minimal model, modulo logical equivalence with respect to the logic $\texttt{SLCS}_\eta$, of a polyhedral model. Such minimised models are also amenable to model checking with `PolyLogicA`. The procedure has been formalised and proven correct. A prototype implementation of the procedure has been developed in the form of a toolchain, that also involves operations provided by the `mCRL2` toolset, to study the practical feasibility of the approach and to identify possible bottlenecks. We have also shown how the model checking results of the minimal model can be projected back onto the original polyhedral model. This provides a direct 3D visual inspection of the results through a polyhedra visualizer.

In future work we aim at a more sophisticated implementation of the procedure, possibly using in a more direct way the minimisation operations provided by `mCRL2` and integrating the various steps in the procedure. Furthermore, we would be interested in extending $\texttt{SLCS}_\eta$ with further operators, for example those concerning notions of distance.

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Appendix

This appendix contains the proofs for all the results presented in Sections 2 and 3 (Section B) and presents detailed background information and results, as well as notational details (Section A).

A Background and Notation in Detail

For sets X and Y, a function $f : X \to Y$, and subsets $A \subseteq X$ and $B \subseteq Y$, we define $f(A)$ and $f^{-1}(B)$ as $\{f(a) \,|\, a \in A\}$ and $\{a \,|\, f(a) \in B\}$, respectively. The *restriction* of f on A is denoted by $f|A$. The powerset of X is denoted by $\mathbf{2}^X$. For relation $R \subseteq X \times X$ we let R^- denote its converse and $R^\pm$ denote $R \cup R^-$. In the sequel, we assume that a set $\mathtt{PL}$ of *proposition letters* is fixed. The set of natural numbers and that of real numbers are denoted by $\mathbb{N}$ and $\mathbb{R}$ respectively. We use the standard interval notation: for $x, y \in \mathbb{R}$ we let $[x, y]$ be the set $\{r \in \mathbb{R} \,|\, x \leq r \leq y\}$, $[x, y) = \{r \in \mathbb{R} \,|\, x \leq r < y\}$ and so on, where $[x, y]$ is equipped with the Euclidean topology inherited from $\mathbb{R}$. We use a similar notation for intervals over $\mathbb{N}$: for $n, m \in \mathbb{N}$ $[m; n]$ denotes the set $\{i \in \mathbb{N} \,|\, m \leq i \leq n\}$, $[m; n)$ denotes the set $\{i \in \mathbb{N} \,|\, m \leq i < n\}$, and similarly for $(m; n]$ and $(m; n)$.

A *topological space* is a pair (X, τ) where X is a set (of *points*) and τ is a collection of subsets of X satisfying the following axioms: (i) $\emptyset, X \in \tau$, (ii) for any index set I, $\bigcup_{i\in I} A_i \in \tau$ if each $A_i \in \tau$, and (iii) for any finite index set I, $\bigcap_{i\in I} A_i \in \tau$ if each $A_i \in \tau$. We let $\mathcal{C}_T$ denote the topological closure operator.

A *Kripke frame* is a pair (W, R) where W is a set and $R \subseteq W \times W$, the *accessibility* relation on W.

A *Kripke model* is a tuple $(W, R, \mathcal{V})$ where (W, R) is a Kripke frame and $\mathcal{V} : \mathtt{PL} \to \mathbf{2}^W$ is the valuation function, assigning to each $p \in \mathtt{PL}$ the set $\mathcal{V}(p)$ of elements of W where p holds.

In the context of the present paper, it is convenient to view a *partially ordered set* — poset, in the sequel — $(W, \preceq)$ as a Kripke frame where the relation $\preceq \subseteq W \times W$ is a partial order, i.e. it is reflexive, transitive and anti-symmetric. Similarly we define a *poset model* as a Kripke model where the accessibility relation is a partial order. For partial orders $(W, \preceq)$, we use the standard notation, i.e.: $\preceq^-$ will be denoted by $\succeq$, $w_1 \prec w_2$ denotes $w_1 \preceq w_2$ and $w_1 \neq w_2$, and similarly for $\succ$.

A.1 Simplexes, Simplicial Complexes, Polyhedra, and Polyhedral Models

The notions of simplex, simplicial complex and polyhedron form the basis for geometrical reasoning in a finite setting, amenable to polyhedral model-checking and related techniques [6]. A *simplex* is the convex hull of a set of affinely independent points[13], namely the vertices of the simplex.

[13] $\mathbf{v_0}, \ldots, \mathbf{v_d}$ are affinely independent if $\mathbf{v_1} - \mathbf{v_0}, \ldots, \mathbf{v_d} - \mathbf{v_0}$ are linearly independent. In particular, this condition implies that $d \leq m$.

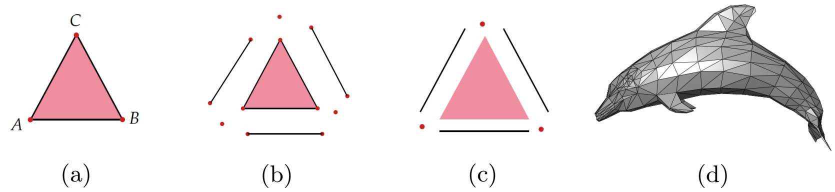


Fig. 8: (8a) A simplicial complex (actually a simplex itself). (8b) Decomposed into its simplexes as faces. (8c) Partitioned into its cells. (8d) A triangular surface mesh of a dolphin [12].

Definition 5 (Simplex). *A* simplex σ *of* dimension d *is the convex hull of a finite set* $\{\mathbf{v_0}, \ldots, \mathbf{v_d}\} \subseteq \mathbb{R}^m$ *of* $d+1$ *affinely independent points, i.e.:*

$$\sigma = \{\lambda_0 \mathbf{v_0} + \ldots + \lambda_d \mathbf{v_d} \mid \lambda_0, \ldots, \lambda_d \in [0,1] \text{ and } \sum_{i=0}^{d} \lambda_i = 1\}.$$

The barycentre b_σ *of* σ *is defined as follows:* $b_\sigma = \sum_{i=0}^{d} \frac{1}{d+1}\mathbf{v_i}$. *Given a simplex* σ *with vertices* $\mathbf{v_0}, \ldots, \mathbf{v_d}$, *any subset of* $\{\mathbf{v_0}, \ldots, \mathbf{v_d}\}$ *spans a simplex* σ' *in turn: we say that* σ' *is a* face *of* σ, *written* $\sigma' \sqsubseteq \sigma$. •

Clearly, $\sqsubseteq$ is a partial order. Note that a simplex is a subset of the ambient space $\mathbb{R}^m$ and so it inherits its topological structure.

The *relative interior* $\widetilde{\sigma}$ of a simplex σ plays a similar role as the notion of "interior" in topology and is defined as follows:

Definition 6 (Relative Interior of a Simplex). *Given a simplex* σ *with vertices* $\{\mathbf{v_0}, \ldots, \mathbf{v_d}\}$ *the* relative interior $\widetilde{\sigma}$ *of* σ *is the following set:*

$$\{\lambda_0 \mathbf{v_0} + \ldots + \lambda_d \mathbf{v_d} \mid \lambda_0, \ldots, \lambda_d \in (0,1] \text{ and } \sum_{i=0}^{d} \lambda_i = 1\}.$$

•

We write $\widetilde{\sigma}' \preceq \widetilde{\sigma}$ whenever $\sigma' \sqsubseteq \sigma$. Note that the topological closure $\mathcal{C}_T(\widetilde{\sigma})$ of the relative interior $\widetilde{\sigma}$ of a simplex σ is σ itself, that $\widetilde{\sigma}' \preceq \widetilde{\sigma}$ if and only if $\widetilde{\sigma}'$ is included in $\mathcal{C}_T(\widetilde{\sigma})$ and that $\preceq$ is a partial order as well.

The notion of *simplicial complex* builds upon that of simplex and is the fundamental tool for constructing complex geometrical objects as sets of points in $\mathbb{R}^m$, namely polyhedra, out of simplexes.

Definition 7 (Simplicial Complex and Polyhedron). *A* simplicial complex K *is a finite collection of simplexes of* $\mathbb{R}^m$ *such that: (i) if* $\sigma \in K$ *and* $\sigma' \sqsubseteq \sigma$ *then also* $\sigma' \in K$; *(ii) if* $\sigma, \sigma' \in K$ *and* $\sigma \cap \sigma' \neq \emptyset$, *then* $\sigma \cap \sigma' \sqsubseteq \sigma$ *and* $\sigma \cap \sigma' \sqsubseteq \sigma'$. *The* polyhedron $|K|$ *of* K *is the set-theoretic union of the simplexes in* K. •

We recall that the polyhedron $|K|$ is a subset of the ambient space $\mathbb{R}^m$ and so it inherits the topological structure of $\mathbb{R}^m$. We furthermore recall that different simplicial complexes can give rise to the same polyhedron.

In the polyhedral semantics of SLCS proposed in [6], all the points of a polyhedral model that belong to the same cell are required to satisfy the same set of atomic proposition letters. This is reflected in the definition below:

Definition 8 (Polyhedral Model). *For a simplicial complex K and a set of proposition letters* PL*, a* polyhedral model *is a pair $(|K|, V)$ where $V : \texttt{PL} \to \mathbf{2}^{|K|}$ is a valuation function assigning to each proposition letter p the set $V(p)$ of the points that satisfy p. It is required that, for all $p \in \texttt{PL}$, $V(p)$ is a union of cells in $\widetilde{K}$.* •

So, polyhedra are topological spaces and polyhedral models are a subclass of topological models.

A.2 Cell Posets and Cell Poset Models

Relations $\sqsubseteq$ and $\preceq$ on simplexes are inherited by simplicial complexes: relation $\sqsubseteq$ on simplicial complex K is the union of the face relations on the simplexes composing K, and similarly for $\preceq$. We let $\widetilde{K}$ be the set $\{\widetilde{\sigma} \mid \sigma \in K\}$ of all the relative interiors of the simplexes of K. The elements of $\widetilde{K}$ are called *cells*. It is easy to see that $(\widetilde{K}, \preceq)$ is a poset.

Definition 9 (Cell Poset). *Given a simplicial complex K, the* cell poset *of K is the poset $(\widetilde{K}, \preceq)$.* •

Note and that $\widetilde{K}$ forms a partition of polyhedron $|K|$. By definition of partition, each $x \in |K|$ belongs to a unique cell. We define the mapping $\mathbb{F} : |K| \to \widetilde{K}$ by letting $\mathbb{F}(x)$ be such a unique cell. Note that $\mathbb{F}$ is a continuous function.

Example 1. Fig. 8a shows a triangle as an example of a simplicial complex. Its simplexes in the face relation are shown in Fig. 8b. The triangle can be partitioned into 7 cells (see Fig. 8c): its interior ($\widetilde{ABC}$, an open triangle), three open segments ($\widetilde{AB}, \widetilde{BC}, \widetilde{AC}$, the sides without endpoints) and the (singletons of the) three vertices ($\widetilde{A}, \widetilde{B}, \widetilde{C}$). Each vertex is in the $\preceq$ relation with two open segments (and the open triangle itself), and each open segment is in the $\preceq$ relation with the open triangle. The figure shows also a small example of a simplicial complex (actually a triangular surface mesh) of a dolphin (Fig. 8d). ⋄

The following definition characterises the discrete representation of polyhedral models we will use in the rest of the paper.

Definition 10 (Cell Poset Model). *Given a polyhedral model $\mathcal{P} = (|K|, V)$, the* cell poset model *of $\mathcal{P}$ is the poset model $(W, \preceq, \mathcal{V})$ where $(W, \preceq) = (\widetilde{K}, \preceq)$ is the cell poset of K and $\widetilde{\sigma} \in \mathcal{V}(p)$ if and only if $\widetilde{\sigma} \subseteq V(p)$.* •

With a little bit of overloading, given polyhedral model $\mathcal{P} = (|K|, V)$ we extend the notation $\mathbb{F}$ in the obvious way, i.e. we let $\mathbb{F}(\mathcal{P}) = (W, \preceq, \mathcal{V})$ as defined in Definition 10 on page 21.

Example 2. Figure 1a shows an example of a polyhedral model $\mathcal{P}$ with atomic propositions **red**, **green** and **gray**. The Hasse diagram of the cell poset model $\mathbb{F}(\mathcal{P})$ associated to $\mathcal{P}$ is shown in Figure 1c. ⋄

A.3 Labelled Transition Systems

Definition 11. *A* labelled transition system, *LTS for short, is a tuple* $(S, L, \longrightarrow)$ *where* S *is a non-empty set of* states, L *is a non-empty set of* transition labels *and* $\longrightarrow \subseteq S \times L \times S$ *is the transition relation.* •

Definition 12 (Strong Bisimulation and Strong Equivalence). *Given LTS* $\mathbb{S} = (S, L, \longrightarrow)$ *a binary relation* $B \subseteq S \times S$ *is a* strong bisimulation *if, for all* $s_1, s_2 \in S$*, if* $B(s_1, s_2)$ *then the following holds:*

1. *if* $s_1 \stackrel{\lambda}{\longrightarrow} s_1'$ *for some* λ *and* s_1*, then* s_2' *exists such that* $s_2 \stackrel{\lambda}{\longrightarrow} s_2'$ *and* $B(s_1', s_2')$*, and*
2. *if* $s_2 \stackrel{\lambda}{\longrightarrow} s_2'$ *for some* λ *and* s_2*, then* s_1' *exists such that* $s_1 \stackrel{\lambda}{\longrightarrow} s_1'$ *and* $B(s_1', s_2')$.

We say that s_1 *and* s_2 *are* strongly equivalent *in* $\mathbb{S}$*, written* $s_1 \simeq^{\mathbb{S}} s_2$ *if a strong bisimulation* B *exists such that* $B(s_1, s_2)$. •

It has been shown that $\sim^{\mathbb{S}}$ is the union of all strong bisimulations in $\mathbb{S}$, it is the largest strong bisimulation and it is an equivalence relation [25].

Definition 13 (Branching Bisimulation and Equivalence). *Given LTS* $\mathbb{S} = (S, L, \longrightarrow)$ *such that* $\tau \in L$ *a binary relation* $B \subseteq S \times S$ *is a* branching bisimulation *iff, for all* $s, t, s' \in S$*, and* $\lambda \in L$*, whenever* $B(s, t)$ *and* $s \stackrel{\lambda}{\longrightarrow} s'$*, it holds that: (i)* $B(s', t)$ *and* $\lambda = \tau$*, or (ii)*$B(s, \bar{t}), B(s', t')$ *and* $t \stackrel{\tau^*}{\longrightarrow} \bar{t}$*,* $\bar{t} \stackrel{\lambda}{\longrightarrow} t'$*, for some* $\bar{t}, t' \in S$.

Two states $s, t \in S$ *are called* branching equivalent *in* $\mathbb{S}$*, written* $s \leftrightarroweq_b^{\mathbb{S}} t$ *if* $B(s, t)$ *for some branching bisimulation* B *for* S.

We will omit the superscript $\mathbb{S}$ in $\simeq^{\mathbb{S}}$ and $\leftrightarroweq_b^{\mathbb{S}}$ when this will not cause confusion.

A.4 Paths

Paths play a crucial role in the present paper. In the sequel, we provide definitions for the different kinds of paths we will use later on in the paper and we prove some useful properties of theirs.

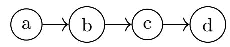


Fig. 9: A simple finite Kripke frame. Arrows in the figure represent the accessibility relation R.

Definition 14 (Topological Path). *Given a topological space* (X, τ) *and* $x \in X$*, a* topological path *from* x *is a total, continuous function* $\pi : [0,1] \to X$ *such that* $\pi(0) = x$*. We call* x *the* starting point *of* π*. The* ending point *of* π *is* $\pi(1)$*, while for any* $r \in (0,1)$*,* $\pi(r)$ *is an* intermediate point *of* π*.* •

Definition 15 (Paths Over Kripke Frames). *Given a Kripke frame* (W, R) *and* $w \in W$*:*

- *An* undirected path *from* w*, of length* $\ell \in \mathbb{N}$*, is a total function* $\pi : [0;\ell] \to W$ *such that* $\pi(0) = w$ *and, for all* $i \in [0;\ell)$*,* $R^{\pm}(\pi(i), \pi(i+1))$*;*
- *A* $\downarrow$-path *(to be read as "down path") from* w*, of length* $\ell \geq 1$*, is an undirected path* π *from* w *of length* ℓ *such that* $R^{-}(\pi(\ell-1), \pi(\ell))$*;*
- *A* $\pm$-path *(to be read as "plus-minus path") from* w*, of length* $\ell \geq 2$*, is a* $\downarrow$*-path* π *from* w *of length* ℓ *such that* $R(\pi(0), \pi(1))$*;*
- *An* $\updownarrow$-path *(to be read as "up-down path") from* w*, of length* 2ℓ*, for* $\ell \geq 1$*, is a* $\pm$*-path* π *of length* 2ℓ *such that* $R(\pi(2i), \pi(2i+1))$ *and* $R^{-}(\pi(2i+1), \pi(2i+2))$*, for all* $i \in [0;\ell)$*.* •

We call w *the* starting point *of* π*. The* ending point *of* π *is* $\pi(\ell)$*, while for any* $i \in (0;\ell)$*,* $\pi(i)$ *is an* intermediate point *of* π*.* •

Below, we will show some facts regarding the relationship among $\updownarrow$-paths, $\pm$-paths and $\downarrow$-paths, but first we need to introduce some notation and operations on paths over Kripke frames. For undirected path π of length ℓ we often use the sequence notation $(w_i)_{i=0}^{\ell}$ where $w_i = \pi(i)$ for all $i \in [0;\ell]$.

Definition 16 (Operations on Paths). *Given a Kripke frame* (W, R) *and paths* $\pi' = (w'_i)_{i=0}^{\ell'}$ *and* $\pi'' = (w''_i)_{i=0}^{\ell''}$*, with* $w'_{\ell'} = w''_0$*, the* sequentialisation $\pi' \cdot \pi'' : [0;\ell' + \ell''] \to W$ *of* π' *with* π'' *is the path from* w'_0 *defined as follows:*

$$(\pi' \cdot \pi'')(i) = \begin{cases} \pi'(i), & \text{if } i \in [0;\ell'], \\ \pi''(i - \ell'), & \text{if } i \in [\ell';\ell' + \ell'']. \end{cases}$$

For path $\pi = (w_i)_{i=0}^{\ell}$ *and* $k \in [0;\ell]$ *we define the* k*-shift of* π*, denoted by* $\pi{\uparrow}k$*, as follows:* $\pi{\uparrow}k = (w_{j+k})_{j=0}^{\ell-k}$ *and, for* $0 < m \leq \ell$*, we let* $\pi{\leftarrow}m$ *denote the path obtained from* π *by inserting a copy of* $\pi(m)$ *immediately before* $\pi(m)$ *itself. In other words, we have:* $\pi{\leftarrow}m = (\pi|[0;m]) \cdot ((\pi(m), \pi(m)) \cdot (\pi{\uparrow}m))$*. Finally, any path* $\pi|[0;k]$*, for some* $k \in [0;\ell]$*, is a* (non-empty) prefix *of* π*.* •

Example 3. For Kripke frame $(\{a, b, c, d\}, R)$ with $R = \{(a, b), (b, c), (c, d)\}$ (see Figure 9), path (a, b, c) of length 2 and path (c, d) of length 1, we have that $(a, b, c) \cdot (c, d) = (a, b, c, d)$, of length 3, $(a) \cdot (a, b) = (a, b)$, $(a) \cdot (a) = (a)$. Note the difference between sequentialisation and concatenation '++': for instance, (a, b)++$(c) = (a, b, c)$ whereas $(a, b) \cdot (c)$ is undefined since $b \neq c$, (a)++(a) is (a, a) whereas $(a) \cdot (a) = (a)$. We have $(a, b, c)\uparrow 1 = (b, c)$ and $(a, b, c)\uparrow 2 = (c)$ while $(a, b, c)\leftarrow 1 = (a, b, b, c)$. Paths $(a), (a, b), (a, b, c)$ are all the (non-empty) prefixes of (a, b, c). ⋄

As it is clear from Def. 15, every $\uparrow\!\downarrow$-path is also a $\pm$-path, that is also a $\downarrow$-path. Furthermore, the lemmas below ensure that, for reflexive Kripke frames:

- for every $\pm$-path there is a $\uparrow\!\downarrow$-path with the same starting and ending points and with the same set of intermediate points, occurring in the same order (Lemma 2 below);
- for every $\downarrow$-path there is a $\uparrow\!\downarrow$-path with the same starting and ending points and with the same set of intermediate points, occurring in the same order (Lemma 3 below);
- for every $\downarrow$-path there is a $\pm$-path with the same starting and ending points and with the same set of intermediate points, occurring in the same order (Lemma 1 below).

Lemma 1. *Given a reflexive Kripke frame (W, R) and a $\downarrow$-path $\pi : [0; \ell] \to W$, there is a $\pm$-path $\pi' : [0; \ell''] \to W$, for some ℓ', and a total, surjective, monotonic, non-decreasing function $f : [0; \ell'] \to [0; \ell]$ with $\pi'(j) = \pi(f(j))$ for all $j \in [0; \ell']$.*

Proof. See [9]

Lemma 2. *Given a reflexive Kripke frame (W, R) and a $\pm$-path $\pi : [0; \ell] \to W$, there is a $\uparrow\!\downarrow$-path $\pi' : [0; \ell'] \to W$, for some ℓ', and a total, surjective, monotonic non-decreasing function $f : [0; \ell'] \to [0; \ell]$ such that $\pi'(j) = \pi(f(j))$ for all $j \in [0; \ell']$.*

Proof. We proceed by induction on the length ℓ of $\pm$-path π.
Base case: $\ell = 2$.
In this case, by definition of $\pm$-path, we have $R(\pi(0), \pi(1))$ and $R^{-}(\pi(1), \pi(2))$, which, by definition of $\uparrow\!\downarrow$-path, implies that π itself is an $\uparrow\!\downarrow$-path and $f : [0; \ell] \to [0; \ell]$ is just the identity function.

Induction step. We assume the assertion holds for all $\pm$-paths of length ℓ and we prove it for $\ell+1$. Let $\pi : [0; \ell+1] \to W$ be a $\pm$-path. Then $R^{-}(\pi(\ell), \pi(\ell+1))$, since π is a $\pm$-path. We consider the following cases:
Case A: $R^{-}(\pi(\ell - 1), \pi(\ell))$ and $R^{-}(\pi(\ell), \pi(\ell + 1))$.
In this case, consider the prefix $\pi_1 = \pi|[0; \ell]$ of π, noting that π_1 is a $\pm$-path of length ℓ. By the Induction Hypothesis there is an $\uparrow\!\downarrow$-path π'_1 of some length ℓ'_1 and a total, surjective, monotonic non-decreasing function $g : [0; \ell'_1] \to [0; \ell]$ such that $\pi'_1(j) = \pi_1(g(j)) = \pi(g(j))$ for all $j \in [0; \ell'_1]$. Note that $\pi'_1(\ell'_1) = \pi(\ell)$ so that the sequentialisation of π'_1 with the two-element path $(\pi(\ell), \pi(\ell + 1))$ is

well-defined. Consider path $\pi' = (\pi'_1 \cdot (\pi(\ell), \pi(\ell+1))) \leftarrow \ell'_1$, of length $\ell'_1 + 2$ consisting of π'_1 followed by $\pi(\ell)$ followed in turn by $\pi(\ell+1)$. In other words, $\pi' = (\pi'_1(0) \dots \pi'_1(\ell'_1), \pi(\ell), \pi(\ell+1))$, with $\pi'_1(\ell'_1) = \pi(\ell)$ — recall that R is reflexive. It is easy to see that π' is an $\updownarrow$-path and that function $f : [0; \ell'_1 + 2] \to [0; \ell+1]$, with $f(j) = g(j)$ for $j \in [0; \ell'_1]$, $f(\ell'_1 + 1) = \ell$ and $f(\ell'_1 + 2) = \ell + 1$, is total, surjective, and monotonic non-decreasing.
Case B: $R(\pi(\ell-1), \pi(\ell))$ and $R^-(\pi(\ell), \pi(\ell+1))$.
In this case the prefix $\pi|[0;\ell]$ of π is *not* a ±-path. We then consider the path consisting of prefix $\pi|[0;\ell-1]$ where we add a copy of $\pi(\ell-1)$, i.e. the path $\pi_1 = (\pi|[0;\ell-1]) \leftarrow (\ell-1)$ — we can do that because R is reflexive. Note that π_1 is a ±-path and has length ℓ. By the Induction Hypothesis there is an $\updownarrow$-path π'_1 of some length ℓ'_1 and a total, surjective, monotonic non-decreasing function $g : [0; \ell'_1] \to [0; \ell]$ such that $\pi'_1(j) = \pi_1(g(j)) = \pi(g(j))$ for all $j \in [0; \ell'_1]$. Consider path $\pi' = \pi'_1 \cdot (\pi(\ell-1), \pi(\ell), \pi(\ell+1))$, of length $\ell'_1 + 2$, that is well defined since $\pi'_1(\ell'_1) = \pi(\ell-1)$ by definition of π_1. In other words, $\pi' = (\pi'_1(0), \dots, \pi'_1(\ell'_1), \pi(\ell), \pi(\ell+1))$, with $\pi'_1(\ell'_1) = \pi(\ell-1)$. Path π' is an $\updownarrow$-path. In fact $\pi'|[0;\ell'_1] = \pi'_1$ is an $\updownarrow$-path. Furthermore, $\pi'(\ell'_1) = \pi(\ell-1)$, $R(\pi(\ell-1), \pi(\ell))$, $R^-(\pi(\ell)), \pi(\ell+1)$ and $\pi(\ell+1) = \pi'(\ell'_1 + 2)$. Finally, function $f : [0; \ell'_1 + 2] \to [0; \ell+1]$, with $f(j) = g(j)$ for $j \in [0; \ell'_1]$, $f(\ell'_1 + 1) = \ell$ and $f(\ell'_1 + 2) = \ell + 1$, is total, surjective, and monotonic non-decreasing.

Lemma 3. *Given a reflexive Kripke frame* (W, R) *and a* $\downarrow$*-path* $\pi : [0; \ell] \to W$*, there is a* $\updownarrow$*-path* $\pi' : [0; \ell''] \to W$*, for some* ℓ'*, and a total, surjective, monotonic non-decreasing function* $f : [0; \ell'] \to [0; \ell]$ *such that* $\pi'(j) = \pi(f(j))$ *for all* $j \in [0; \ell']$.

Proof. The proof is carried out by induction on the length ℓ of π.
Base case. $\ell = 1$. Suppose $\ell = 1$, i.e. $\pi : [0;1] \to W$ with $R^-(\pi(0), \pi(1))$. Then let $\pi' : [0;2] \to W$ be such that $\pi'(0) = \pi'(1) = \pi(0)$ and $\pi'(2) = \pi(1)$ — we can do that since R is reflexive — and $f : [0;2] \to [0;1]$ be such that $f(0) = f(1) = 0$ and $f(2) = 1$. Clearly π' is an $\updownarrow$-path and $\pi'(j) = \pi(f(j))$ for all $j \in [0;2]$.
Induction step. We assume the assertion holds for all $\downarrow$-paths of length ℓ and we prove it for $\ell + 1$. Let $\pi : [0; \ell+1] \to W$ a $\downarrow$-path and suppose the assertion holds for all $\downarrow$-paths of length ℓ. In particular, it holds for $\pi \uparrow 1$, i.e., there is an $\updownarrow$-path π'' of some length ℓ'' with $\pi''(0) = \pi(1)$, and total, monotonic non-decreasing surjection $g : [0; \ell''] \to W$ such that $\pi''(j) = \pi(g(j))$ for all $j \in [0; \ell'']$. Suppose $R(\pi(0), \pi(1))$ does not hold. Then, since R is reflexive, we let $\pi' = (\pi(0), \pi(0), \pi(1)) \cdot \pi''$ and $f : [0; \ell''+2] \to [0; \ell+1]$ with $f(0) = f(1) = 0$ and $f(j) = g(j-2)$ for all $j \in [2; \ell''+2]$. If instead $R(\pi(0), \pi(1))$, then we let $\pi' = (\pi(0), \pi(1), \pi(1)) \cdot \pi''$ and $f : [0; \ell''+2] \to [0; \ell+1]$ with $f(0) = 0, f(1) = 1$ and $f(j) = g(j-2)$ for all $j \in [2; \ell''+2]$.

A.5 The χ Formula

It is useful to define a "characteristic" $\mathtt{SLCS}_\eta$ formula $\chi(w)$ that is satisfied by all and only those w' such that $w' \equiv_\eta w$.

Definition 17. *Given a finite poset model* $(W, \preceq, \mathcal{V})$, $w_1, w_2 \in W$, *define* $\texttt{SLCS}_\eta$ *formula* δ_{w_1,w_2} *as follows: if* $w_1 \equiv_\eta w_2$, *then set* $\delta_{w_1,w_2} = \texttt{true}$, *otherwise pick some* $\texttt{SLCS}_\eta$ *formula* ψ *such that* $\mathcal{F}, w_1 \models \psi$ *and* $\mathcal{F}, w_2 \models \neg\psi$, *and set* $\delta_{w_1,w_2} = \psi$. *For* $w \in W$ *define* $\chi(w) = \bigwedge_{w' \in W} \delta_{w,w'}$. •

Proposition 2. *Given a finite poset model* $(W, \preceq, \mathcal{V})$, *for* $w_1, w_2 \in W$, *it holds that*

$$\mathcal{F}, w_2 \models \chi(w_1) \text{ if and only if } w_1 \equiv_\eta w_2. \tag{1}$$

The following result states that to evaluate an $\texttt{SLCS}_\eta$ formula $\eta(\Phi_1, \Phi_2)$ in a poset model, it does not matter whether one considers $\pm$-paths or $\downarrow$-paths.

Proposition 3. *Given a finite poset* $\mathcal{F} = (W, \preceq, \mathcal{V})$, $w \in W$ *and an* $\texttt{SLCS}_\eta$ *formula* $\eta(\Phi_1, \Phi_2)$ *the following statements are equivalent:*

1. *There exists a* $\pm$*-path* $\pi : [0;\ell] \to W$ *for some* ℓ *with* $\pi(0) = w$, $\mathcal{F}, \pi(\ell) \models \Phi_2$ *and* $\mathcal{F}, \pi(i) \models \Phi_1$ *for all* $i \in [0;\ell)$.
2. *There exists a* $\downarrow$*-path* $\pi : [0;\ell] \to W$ *for some* ℓ *with* $\pi(0) = w$, $\mathcal{F}, \pi(2\ell) \models \Phi_2$ *and* $\mathcal{F}, \pi(i) \models \Phi_1$ *for all* $i \in [0;\ell)$.

Proof. The equivalence of statements (1) and (2) follows directly from Lemma 1 and the fact that $\pm$-paths are also $\downarrow$-paths.

A.6 Further Minimisation Example

Fig. 10a shows an example of a blue triangle with one red edge and one red vertex. Its cell poset model is shown in Fig. 10b. In Fig. 10c the nodes of the poset model that are in the same equivalence class modulo $\equiv_\eta$ are given the same colour.[14] The minimal Kripke model is shown in Fig. 10d. The colours of the borders of the nodes in Fig. 10d correspond to the proposition letters of the model in Fig. 10a whereas the interior colour of the nodes correspond to the colour of the corresponding equivalence classes in Fig. 10c. Note that the minimal model itself is not a poset.

[14] Note that such colours have only an illustrative purpose. In particular, they have nothing to do with the colours expressing the evaluation of proposition letters.

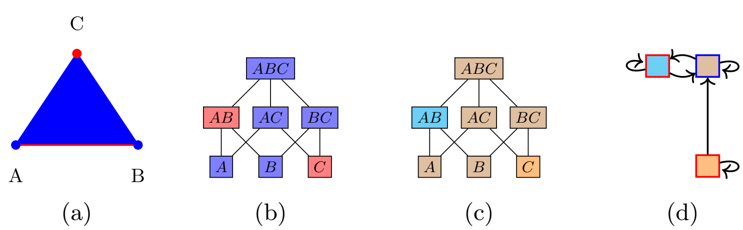


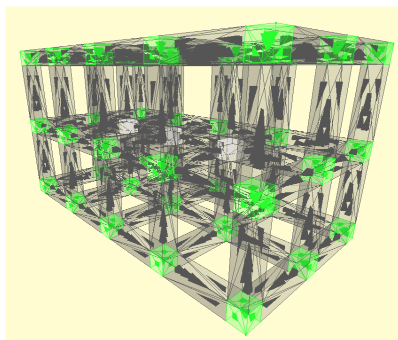
Fig. 10: A blue triangle with red vertex and a red side (10a), its poset model (10b), poset model with equivalence classes (10c) and minimal Kripke model (10d).

A.7 Additional Material Cube Examples of Sect. 5

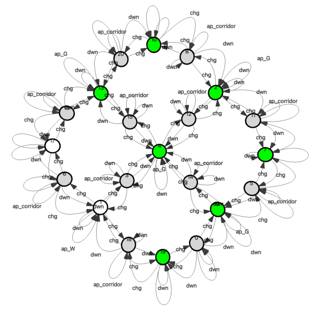
Below the spatial logic specification in `ImgQl` is shown, that was used for model checking the various cube-variants in Table 1 in Sect. 5 with `PolyLogicA`. `ImgQl` is the input language of `PolyLogicA` in which spatial logic properties of $\mathtt{SLCS}_\eta$ can be expressed. In the specification below, first the polyhedral model is loaded in json format. Then the definition of the operator η follows, which can be expressed in terms of the built-in reachability operator `through`, which, in turn, represents operator γ. After that, the atomic propositions `green`, `white` and `corridor` are defined. This is followed by a number of properties for the cube that should be self-explanatory. They include the formulas for ϕ_1 and ϕ_2 that were introduced in Sect. 5. Finally, the lines starting by `save` are defining which results to save in a file. Such files contain the name of a property and for each property a list of true/false items, one for each cell in the polyhedral model and in the order in which these cells are defined in that polyhedral model.

```
load model = "mazeG1W3x3Model.json"

// Define eta in terms of gamma (through):
let eta(x,y) = x & through(x,y)

let green       = ap("G")
let white       = ap("W")
let corridor    = ap("corridor")


let greenOrWhite  = (green | white)

let oneStepToWhite   = eta((green | eta(corridor,white)),white)
let twoStepsToWhite  = eta((green | eta(corridor,oneStepToWhite)), oneStepToWhite) & (!oneStepToWhite)
let threeStepsToWhite = eta((green | eta(corridor,twoStepsToWhite)), twoStepsToWhite) & (!twoStepsToWhite) & (!oneStepToWhite)

let phi1 = eta((green | eta(corridor,white)),white)
let phi2 = eta((green | eta(corridor,oneStepToWhite)), oneStepToWhite)

let greenThree = green & (!oneStepToWhite) & (!twoStepsToWhite)

//save "greenOrWhite"   greenOrWhite

save "phi1" phi1
save "phi2" phi2

save "oneStepToWhite" oneStepToWhite
save "twoStepsToWhite" twoStepsToWhite
save "threeStepsToWhite" threeStepsToWhite

//save "greenOneStepToWhite" green & oneStepToWhite
//save "greenTwoStepsToWhite" green & twoStepsToWhite
//save "greenThreeStepsToWhite" green & threeStepsToWhite
//save "greenThree" greenThree
```

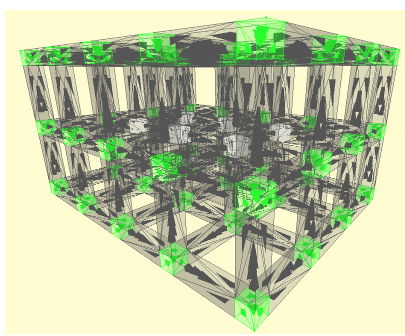
Figure 11 shows the 3x5x3 and the 3x5x4 cubes and their minimised LTSs. Note that in the LTSs not all transition labels are shown to avoid cluttering. However, states corresponding to corridors, green rooms and white rooms, are shown in grey, green and white, respectively.



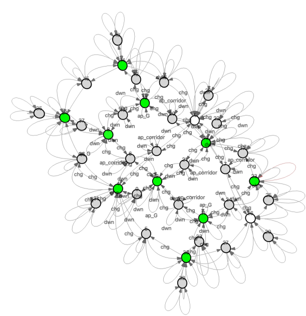
(a) Cube 3x5x3



(b) Minimised LTS



(c) Cube 3x5x4



(d) Minimised LTS

Fig. 11: Cubes of dimension 3x5x3 (Fig. 11a) and 3x5x4 (Fig. 11c) and their respective minimal LTSs (Figs. 11b and 11d).

B Detailed Proofs

B.1 Proof of Theorem 1

Theorem 1. *Let $\mathcal{F} = (W, \preceq, \mathcal{V})$ be a finite poset model. For all $w_1, w_2 \in W$ the following holds: $[w_1]_{\rightleftharpoons} \simeq^{\mathbb{S}_A(\mathcal{F})} [w_2]_{\rightleftharpoons}$ if and only if $w_1 \equiv^{\mathcal{F}}_{\eta} w_2$.*

Proof. We first prove that if $[w_1]_{\rightleftharpoons} \simeq^{\mathbb{S}_A(\mathcal{F})} [w_2]_{\rightleftharpoons}$ then $w_1 \equiv^{\mathcal{F}}_{\eta} w_2$. We proceed by induction on $\mathtt{SLCS}_\eta$ formulas and consider only the case $\eta(\Phi_1, \Phi_2)$, since the others are straightforward. Suppose $[w_1]_{\rightleftharpoons} \simeq^{\mathbb{S}_A(\mathcal{F})} [w_2]_{\rightleftharpoons}$ and $\mathcal{F}, w_1 \models \eta(\Phi_1, \Phi_2)$. Since $\mathcal{F}, w_1 \models \eta(\Phi_1, \Phi_2)$, there is (a ±-path, and so, by Proposition 3) a $\downarrow$-path π_1 from w_1 of some length $\ell_1 \geq 1$ such that $\mathcal{F}, \pi_1(\ell_1) \models \Phi_2$ and $\mathcal{F}, \pi_1(i) \models \Phi_1$ for all $i \in [0; \ell_1)$. At this point, we use induction on ℓ_1, together with induction on the formulas, for showing that also $\mathcal{F}, w_2 \models \eta(\Phi_1, \Phi_2)$ holds.

Base case: $\ell_1 = 1$.
In this case we have $\mathcal{F}, w_1 \models \Phi_1$ and $\mathcal{F}, \pi_1(1) \models \Phi_2$, with $w_1 \succeq \pi_1(1)$. Moreover, by the Induction Hypothesis on formulas, we also have $\mathcal{F}, w_2 \models \Phi_1$. In addition, by Rule (Down), we get $[w_1]_{\rightleftharpoons} \xrightarrow{\mathbf{d}} [\pi_1(1)]_{\rightleftharpoons}$. Since $[w_1]_{\rightleftharpoons} \simeq [w_2]_{\rightleftharpoons}$ by hypothesis, we also get $[w_2]_{\rightleftharpoons} \xrightarrow{\mathbf{d}} [w_2']_{\rightleftharpoons}$, for some $[w_2']_{\rightleftharpoons}$ with $[w_2']_{\rightleftharpoons} \simeq [\pi_1(1)]_{\rightleftharpoons}$. Note that, by definition of $\rightleftharpoons$ and since $[w_2]_{\rightleftharpoons} \xrightarrow{\mathbf{d}} [w_2']_{\rightleftharpoons}$, there is a path π_2' from w_2 of some length ℓ_2' such that $\pi_2'(j) \rightleftharpoons w_2$ for all $j \in [0; \ell_2']$ and $\pi_2'(\ell_2') \succeq w_2''$, with $w_2'' \in [w_2']_{\rightleftharpoons}$. Recalling that $\mathcal{F}, w_2 \models \Phi_1$, by Lemma 4 below, we also get that $\mathcal{F}, \pi_2'(j) \models \Phi_1$ for all $j \in [0; \ell_2']$. Recalling also that $\mathcal{F}, \pi_1(1) \models \Phi_2$, again by the Induction Hypothesis on formulas, from $[w_2']_{\rightleftharpoons} \simeq [\pi_1(1)]_{\rightleftharpoons}$, we get $\mathcal{F}, w_2' \models \Phi_2$ and, by Lemma 4 below, we also get $\mathcal{F}, w_2'' \models \Phi_2$. Consider now path $\pi_2 : [0; \ell_2' + 1] \to W$ defined as follows:

$$\pi_2(j) = \begin{cases} \pi_2'(j), & \text{if } j \in [0; \ell_2'], \\ \\ w_2'', & \text{if } j = \ell_2' + 1. \end{cases}$$

Clearly π_2 is a $\downarrow$-path from w_2 since π_2' is an undirected path and $\pi_2(\ell_2') \succeq \pi_2(\ell_2' + 1)$. Furthermore, we have shown above that $\mathcal{F}, \pi_2(\ell_2' + 1) \models \Phi_2$ and $\mathcal{F}, \pi_2(j) \models \Phi_1$ for all $j \in [0; \ell_2' + 1)$. Thus, we have that $\mathcal{F}, w_2 \models \eta(\Phi_1, \Phi_2)$, witnessed by π_2.

Induction step: We assume the assertion holds for $\ell_1 = n$, for $n \geq 1$ and we show it holds for $\ell_1 = n + 1$.
Since $w_1 \preccurlyeq^{\pm} \pi_1(1)$, by Rule (Step), we have that $[w_1]_{\rightleftharpoons} \xrightarrow{\mathbf{s}} [\pi_1(1)]_{\rightleftharpoons}$, and since, by hypothesis, $[w_1]_{\rightleftharpoons} \simeq [w_2]_{\rightleftharpoons}$, we also know that $[w_2]_{\rightleftharpoons} \xrightarrow{\mathbf{s}} [w_2']_{\rightleftharpoons}$ for some w_2' such that $[w_2']_{\rightleftharpoons} \simeq [\pi_1(1)]_{\rightleftharpoons}$. Note, furthermore, that $\mathcal{F}, \pi_1(1) \models \eta(\Phi_1, \Phi_2)$ since $\ell_1 \geq 2$ and that this is witnessed by $\pi_1 \uparrow 1$, which is a $\downarrow$-path of length n. Thus, by the Induction Hypothesis on ℓ_1, we get that $\mathcal{F}, w_2' \models \eta(\Phi_1, \Phi_2)$ since $[w_2']_{\rightleftharpoons} \simeq [\pi_1(1)]_{\rightleftharpoons}$ (see above). From $[w_2]_{\rightleftharpoons} \xrightarrow{\mathbf{s}} [w_2']_{\rightleftharpoons}$, by Rule (Step), we know that $w \in [w_2]_{\rightleftharpoons}$ and $w' \in [w_2']_{\rightleftharpoons}$ exist such that $w \preccurlyeq^{\pm} w'$. Since

$w \in [w_2]_{\rightleftharpoons}$ an undirected path π_2' exists from w_2 to w, of some length ℓ_2', such that $\pi_2'(j) \rightleftharpoons w_2$ for all $j \in [0;\ell_2']$. By the Induction Hypothesis on formulas, we know that $\mathcal{F}, w_2 \models \Phi_1$, and so, by Lemma 4 below, we get also $\mathcal{F}, \pi_2'(j) \models \Phi_1$ for all $j \in [0;\ell_2']$. Moreover, since $\mathcal{F}, w_2' \models \eta(\Phi_1, \Phi_2)$ (see above) and $w' \rightleftharpoons w_2'$, again by Lemma 4, we get $\mathcal{F}, w' \models \eta(\Phi_1, \Phi_2)$. This means that there is a $\pm$-path π_2'' from w' of some length ℓ_2'' witnessing $\mathcal{F}, w' \models \eta(\Phi_1, \Phi_2)$. Define π_2 as follows: $\pi_2' \cdot (w, w') \cdot \pi_2''$. It is easy to see that π_2 is a $\downarrow$-path witnessing $\mathcal{F}, w_2 \models \eta(\Phi_1, \Phi_2)$.

Now we prove that if $w_1 \equiv_\eta^{\mathcal{F}} w_2$ then $[w_1]_{\rightleftharpoons} \simeq^{\mathbb{S}_A(\mathcal{F})} [w_2]_{\rightleftharpoons}$. We do this by showing that the following binary relation B on W is a strong bisimulation:

$$B = \{(s_1, s_2) \in S \times S \,|\, \text{there are } w_1 \in s_1, w_2 \in s_2 \text{ such that } w_1 \equiv_\eta w_2\}.$$

Let, without loss of generality, $s_1 = [w_1]_{\rightleftharpoons}$ and $s_2 = [w_2]_{\rightleftharpoons}$, for some $w_1, w_2 \in W$ with $w_1 \equiv_\eta w_2$ and suppose $B([w_1]_{\rightleftharpoons}, [w_2]_{\rightleftharpoons})$, with $w_1 \equiv_\eta w_2$. We distinguish three cases:

Case A: $[w_1]_{\rightleftharpoons} \xrightarrow{\alpha} [w_1']_{\rightleftharpoons}$ with $\alpha \in \mathbf{2}^{\mathtt{PL}}$.
If $[w_1]_{\rightleftharpoons} \xrightarrow{\alpha} [w_1']_{\rightleftharpoons}$ for some $\alpha \in \mathbf{2}^{\mathtt{PL}}$ and $w_1' \in W$, then, by Rule (PL), we know that $[w_1']_{\rightleftharpoons} = [w_1]_{\rightleftharpoons}$. Furthermore, since $w_1 \equiv_\eta w_2$, we also know that $\mathcal{V}^{-1}(\{w_2\}) = \mathcal{V}^{-1}(\{w_1\}) = \alpha$. In addition, again by Rule (PL), we get that $[w_2]_{\rightleftharpoons} \xrightarrow{\alpha} [w_2]_{\rightleftharpoons}$ and, by hypothesis $B([w_1]_{\rightleftharpoons}, [w_2]_{\rightleftharpoons})$.

Case B: $[w_1]_{\rightleftharpoons} \xrightarrow{\mathbf{d}} [w_1']_{\rightleftharpoons}$
If $[w_1]_{\rightleftharpoons} \xrightarrow{\mathbf{d}} [w_1']_{\rightleftharpoons}$ for some $w_1' \in W$, then, by Rule (Down) there are $w \in [w_1]_{\rightleftharpoons}$ and $w' \in [w_1']_{\rightleftharpoons}$ such that $w \succeq w'$. Note that (w, w') is a $\downarrow$-path witnessing $\mathcal{F}, w \models \eta(\chi(w), \chi(w'))$, where χ is as in Definition 17 on page 26. Since $w \rightleftharpoons w_1$, we have that $\mathcal{F}, w_1 \models \eta(\chi(w), \chi(w'))$ holds, by Lemma 4. Moreover, since, by hypothesis, $w_1 \equiv_\eta w_2$, we also have $\mathcal{F}, w_2 \models \eta(\chi(w), \chi(w'))$. Then a $\pm$-path $\pi : [0;\ell] \to W$ from w_2 such that $\mathcal{F}, \pi(\ell) \models \chi(w')$ and $\mathcal{F}, \pi(j) \models \chi(w)$ for all $j \in [0;\ell)$. This in turn, by Proposition 2, means that $\pi(\ell) \equiv_\eta w'$ and $\pi(j) \equiv_\eta w$ for all $j \in [0;\ell)$, By Lemma 4, since $w' \rightleftharpoons w_1'$, we get $w' \equiv_\eta w_1'$, and by transitivity, since $\pi(\ell) \equiv_\eta w'$ (see above), we also have $\pi(\ell) \equiv_\eta w_1'$. Similarly, we get $\pi(j) \equiv_\eta w \equiv_\eta w_1$, which implies $\mathcal{V}^{-1}(\{\pi(j)\}) = \mathcal{V}^{-1}(\{w_1\})$, for all $j \in [0;\ell)$. Recall that $w_1 \equiv_\eta w_2$, which implies $\mathcal{V}^{-1}(w_2) = \mathcal{V}^{-1}(\{w_1\})$ and so we get also $\mathcal{V}^{-1}(\{\pi(j)\}) = \mathcal{V}^{-1}(\{w_2\})$, for all $j \in [0;\ell)$. In addition, for all $j \in [0;\ell)$ we have that $\pi|[0;j]$ connects $\pi(0) = w_2$ to $\pi(j)$. This means that, for all $j \in [0;\ell)$, $\pi(j) \in [w_2]_{\rightleftharpoons} = [\pi(\ell-1)]_{\rightleftharpoons}$ and since $\pi(\ell-1) \succeq \pi(\ell)$, by Rule (Down) we deduce $[\pi(\ell-1)]_{\rightleftharpoons} \xrightarrow{\mathbf{d}} [\pi(\ell)]_{\rightleftharpoons}$, that is $[w_2]_{\rightleftharpoons} \xrightarrow{\mathbf{d}} [\pi(\ell)]_{\rightleftharpoons}$. Recall that $\pi(\ell) \equiv_\eta w_1'$, so that, by definition of relation B, we finally get $B([w_1']_{\rightleftharpoons}, [\pi(\ell)]_{\rightleftharpoons})$.

Case C: $[w_1]_{\rightleftharpoons} \xrightarrow{\mathbf{s}} [w_1']_{\rightleftharpoons}$
Suppose, finally, that $[w_1]_{\rightleftharpoons} \xrightarrow{\mathbf{s}} [w_1']_{\rightleftharpoons}$ for some $w_1' \in W$. We distinguish two cases:
Case C1: $w_1' \in [w_1]_{\rightleftharpoons}$. In this case, by Lemma 4 below, we have also $w_1' \equiv_\eta w_1$. Furthermore, $w_1 \equiv_\eta w_2$ by hypothesis, thus we get $w_1' \equiv_\eta w_2$. But then, since $w_2 \preccurlyeq^{\pm} w_2$, by Rule (Step), we know that $[w_2]_{\rightleftharpoons} \xrightarrow{\mathbf{s}} [w_2]_{\rightleftharpoons}$ and since $w_1' \equiv_\eta w_2$,

by definition of relation B, we finally get $B([w_1']_{\rightleftharpoons}, [w_2]_{\rightleftharpoons})$.
Case C2: $w_1' \notin [w_1]_{\rightleftharpoons}$. We know there are $w \in [w_1]_{\rightleftharpoons}$ and $w' \in [w_1']_{\rightleftharpoons}$ such that $w \preccurlyeq^{\pm} w'$. Since $w \rightleftharpoons w_1$, then $\mathcal{V}^{-1}(\{w\}) = \mathcal{V}^{-1}(\{w_1\})$ and since $w' \rightleftharpoons w_1'$, then $\mathcal{V}^{-1}(\{w'\}) = \mathcal{V}^{-1}(\{w_1'\})$. Furthermore, since $w \preccurlyeq^{\pm} w'$, there is path (w, w') connecting w with w'. So there is a path connecting w_1 to w_1' and if $\mathcal{V}^{-1}(\{w_1\}) = \mathcal{V}^{-1}(\{w_1'\})$ would hold, it could not be that $w_1' \notin [w_1]_{\rightleftharpoons}$. Consequently, it must be $\mathcal{V}^{-1}(\{w_1\}) \neq \mathcal{V}^{-1}(\{w_1'\})$, which in turn implies $w_1 \not\equiv_{\eta} w_1'$. We note that the following holds:

$$\mathcal{F}, w_1 \models \eta(\chi(w_1), \eta(\chi(w_1) \vee \chi(w_1'), \chi(w_1')))$$

and, since $w_1 \equiv_{\eta} w_2$ we also have

$$\mathcal{F}, w_2 \models \eta(\chi(w_1), \eta(\chi(w_1) \vee \chi(w_1'), \chi(w_1'))).$$

Let π a $\pm$-path from w_2 witnessing the above formula and let k be the first index such that $\mathcal{F}, \pi(k) \models \chi(w_1')$. We have that, for all $j \in [0; k)$, $\mathcal{F}, \pi(j) \models \chi(w_1)$ and $\pi|[0; j]$ connects $\pi(0) = w_2$ to $\pi(j)$. Furthermore, for all such j, we have $\pi(j) \equiv_{\eta} w_1$, by Proposition 2, which entails $\mathcal{V}^{-1}(\{\pi(j)\}) = \mathcal{V}^{-1}(\{w_1\})$. Thus $\pi(j) \in [w_2]_{\rightleftharpoons}$ for all $j \in [0; k)$ and since $\pi(k-1) \preccurlyeq^{\pm} \pi(k)$ we have, by Rule (Step) $[w_2]_{\rightleftharpoons} \xrightarrow{\mathbf{s}} [\pi(k)]_{\rightleftharpoons}$. Finally, recalling that, again by Proposition 2, $w_1' \equiv_{\eta} \pi(k)$, we get $B([w_1']_{\rightleftharpoons}, [\pi(k)]_{\rightleftharpoons})$.

Lemma 4. *Given finite poset model $\mathcal{F} = (W, \preceq, \mathcal{V})$ and $w_1, w_2 \in W$ the following holds: if $w_1 \rightleftharpoons w_2$, then $w_1 \equiv_{\eta} w_2$.*

Proof. By induction on the structure of $\texttt{SLCS}_{\eta}$ formulas. We show only the case for $\eta(\Phi_1, \Phi_2)$ since the others are straightforward. Suppose $\mathcal{F}, w_1 \models \eta(\Phi_1, \Phi_2)$. Then there is a $\pm$-path π from w_1 of some length ℓ such that $\mathcal{F}, \pi(\ell) \models \Phi_2$ and $\mathcal{F}, \pi(i) \models \Phi_1$ for all $i \in [0; \ell)$. In particular, we have that $\mathcal{F}, w_1 \models \Phi_1$. So, by the Induction Hypothesis, since $w_1 \rightleftharpoons w_2$, we get that also $\mathcal{F}, w_2 \models \Phi_1$. In addition, by definition of $\rightleftharpoons$, and given that $w_2 \rightleftharpoons w_1$, there is an undirected path π' of some length ℓ' such that $\pi'(0) = w_2, \pi(\ell') = w_1$ and $\mathcal{V}^{-1}(\{\pi'(i)\}) = \mathcal{V}^{-1}(\{\pi'(j)\})$, for all $i, j \in [0; \ell']$. Note that, by definition of $\rightleftharpoons$, we have that $\pi'(k) \rightleftharpoons w_1$ for all $k \in [0; \ell']$. Thus, again by the Induction Hypothesis, we also get $\mathcal{F}, \pi'(k) \models \Phi_1$ for all $k \in [0; \ell']$. Clearly, the sequentialisation $\pi' \cdot \pi$ of π' with π is a $\downarrow$-path since π is a $\pm$-path. Furthermore, by Lemma 1, there is a $\pm$-path π'' with the same starting and ending points as $\pi' \cdot \pi$, and with the same set of intermediate points, occurring in the same order. Thus π'' witnesses $\mathcal{F}, w_2 \models \eta(\Phi_1, \Phi_2)$.

B.2 Proof of Theorem 2

Theorem 2. *Let $\mathcal{F} = (W, \preceq, \mathcal{V})$ be a finite poset model. For all $w_1, w_2 \in W$ the following holds: $[w_1]_{\rightleftharpoons} \simeq^{\mathbb{S}_A(\mathcal{F})} [w_2]_{\rightleftharpoons}$ if and only if $w_1 \leftrightarrows_b^{\mathbb{S}_C(\mathcal{F})} w_2$.*

Proof. We first prove that if $[w_1]_{\rightleftharpoons} \simeq^{\mathbb{S}_A(\mathcal{F})} [w_2]_{\rightleftharpoons}$ then $w_1 \leftrightarrow_b^{\mathbb{S}_C(\mathcal{F})} w_2$. We show that the following relation is a branching bisimulation:

$$B_C = \{(w_1, w_2) \in W \times W \mid [w_1]_{\rightleftharpoons} \simeq^{\mathbb{S}_A(\mathcal{F})} [w_2]_{\rightleftharpoons}\}.$$

Let us assume $B_C(w_1, w_2)$. We have to consider a few cases:

Case A: $w_1 \xrightarrow{p} w_1$.
If $w_1 \xrightarrow{p} w_1$, then, by Rule (PLC), we have $p \in \mathcal{V}^{-1}(\{w_1\})$. By definition of B_C and by hypothesis we know that $[w_1]_{\rightleftharpoons} \simeq [w_2]_{\rightleftharpoons}$ and so, by Lemma 5 below, we get $\mathcal{V}^{-1}(\{w_1\}) = \mathcal{V}^{-1}(\{w_2\})$. It follows then that $p \in \mathcal{V}^{-1}(\{w_2\})$ and, again by Rule (PLC), we finally get $w_2 \xrightarrow{p} w_2$, which is the required mimicking step since $B(w_1, w_2)$.

Case B: $w_1 \xrightarrow{\tau} w_1'$.
If $w_1 \xrightarrow{\tau} w_1'$ for some $w_1' \in W$, then, by Rule (TAU), we know that $w_1 \preccurlyeq^{\pm} w_1'$, with $\mathcal{V}^{-1}(\{w_1\}) = \mathcal{V}^{-1}(\{w_1'\})$, which, by definition of $\rightleftharpoons$, means $[w_1']_{\rightleftharpoons} = [w_1]_{\rightleftharpoons}$ and since $[w_1]_{\rightleftharpoons} \simeq^{\mathbb{S}_A(\mathcal{F})} [w_2]_{\rightleftharpoons}$ by definition of B_C, given that $B_C(w_1, w_2)$, we get $[w_1']_{\rightleftharpoons} \simeq^{\mathbb{S}_A(\mathcal{F})} [w_2]_{\rightleftharpoons}$. This, in turn, again by definition of B_C, means $B_C(w_1', w_2)$.

Case C: $w_1 \xrightarrow{\mathbf{c}} w_1'$.
If $w_1 \xrightarrow{\mathbf{c}} w_1'$ for some $w_1' \in W$, then, by Rule (CNG), we know that $w_1 \preccurlyeq^{\pm} w_1'$, with $\mathcal{V}^{-1}(\{w_1\}) \neq \mathcal{V}^{-1}(\{w_1'\})$, and, by Rule (Step), we have $[w_1]_{\rightleftharpoons} \xrightarrow{\mathbf{s}} [w_1']_{\rightleftharpoons}$. Since, by definition of B_C and by hypothesis, $[w_1]_{\rightleftharpoons} \simeq^{\mathbb{S}_A(\mathcal{F})} [w_2]_{\rightleftharpoons}$, we also have $[w_2]_{\rightleftharpoons} \xrightarrow{\mathbf{s}} [w_2']_{\rightleftharpoons}$ for some $[w_2']_{\rightleftharpoons} \simeq^{\mathbb{S}_A(\mathcal{F})} [w_1']_{\rightleftharpoons}$. From $[w_2]_{\rightleftharpoons} \xrightarrow{\mathbf{s}} [w_2']_{\rightleftharpoons}$, by Rule (Step), we know there are $w_3 \in [w_2]_{\rightleftharpoons}$ and $w_3' \in [w_2']_{\rightleftharpoons}$ such that $w_3 \preccurlyeq^{\pm} w_3'$. By Lemma 5 below, since $[w_1]_{\rightleftharpoons} \simeq^{\mathbb{S}_A(\mathcal{F})} [w_2]_{\rightleftharpoons}$ by hypothesis and $[w_1']_{\rightleftharpoons} \simeq^{\mathbb{S}_A(\mathcal{F})} [w_2']_{\rightleftharpoons}$ (see above), we have $\mathcal{V}^{-1}(\{w_1\}) = \mathcal{V}^{-1}(\{w_2\})$ and $\mathcal{V}^{-1}(\{w_1'\}) = \mathcal{V}^{-1}(\{w_2'\})$ and since $\mathcal{V}^{-1}(\{w_1\}) \neq \mathcal{V}^{-1}(\{w_1'\})$ (see above), we get $\mathcal{V}^{-1}(\{w_2\}) = \mathcal{V}^{-1}(\{w_1\}) \neq \mathcal{V}^{-1}(\{w_1'\}) = \mathcal{V}^{-1}(\{w_2'\})$. Consequently, since $w_3 \in [w_2]_{\rightleftharpoons}$ and $w_3' \in [w_2']_{\rightleftharpoons}$, we also finally get that $\mathcal{V}^{-1}(\{w_3\}) \neq \mathcal{V}^{-1}(\{w_3'\})$. Thus, by rule (CNG), we know that $w_3 \xrightarrow{\mathbf{c}} w_3'$. Now, since $w_3 \in [w_2]_{\rightleftharpoons}$, by definition of $\rightleftharpoons$ and by construction of $\mathbb{S}_C(\mathcal{F})$ we know there are $s_0, \dots s_n \in W$ with $s_0 = w_2$, $s_n = w_3$ such that $s_i \xrightarrow{\tau} s_{i+1}$ and $s_{i+1} \xrightarrow{\tau} s_i$, for all $i \in [0; n)$. We note that $B_C(w_1, s_i)$ for all $i \in [0; n]$. In fact for each $i \in [0; n]$ we have that $[s_i]_{\rightleftharpoons} = [w_2]_{\rightleftharpoons}$ by definition of $\rightleftharpoons$ and we also know that $[w_2]_{\rightleftharpoons} \simeq^{\mathbb{S}_A(\mathcal{F})} [w_1]_{\rightleftharpoons}$, since $B_C(w_1, w_2)$ by hypothesis. Thus we get $[s_i]_{\rightleftharpoons} \simeq^{\mathbb{S}_A(\mathcal{F})} [w_1]_{\rightleftharpoons}$, i.e. $B_C(w_1, s_i)$. Furthermore, we also note that $B_C(w_1', w_3')$. In fact $[w_3']_{\rightleftharpoons} = [w_2']_{\rightleftharpoons}$, since $w_3' \in [w_2']_{\rightleftharpoons}$. Furthermore, $[w_2']_{\rightleftharpoons} \simeq^{\mathbb{S}_A(\mathcal{F})} [w_1']_{\rightleftharpoons}$ (see above). So, we get $[w_3']_{\rightleftharpoons} \simeq^{\mathbb{S}_A(\mathcal{F})} [w_1']_{\rightleftharpoons}$, i.e. $B_C(w_1', w_3')$. In conclusion, we have that if $w_1 \xrightarrow{\mathbf{c}} w_1'$ for some $w_1' \in W$, then $w_2 = s_0 \xrightarrow{\tau} s_1 \xrightarrow{\tau} \dots \xrightarrow{\tau} s_n = w_3 \xrightarrow{\mathbf{c}} w_3'$ with $B_C(w_1', w_3')$ and $B_C(w_1, s_i)$ for all $i \in [0; n]$.

Case D: $w_1 \xrightarrow{\mathbf{d}} w_1'$.
If $w_1 \xrightarrow{\mathbf{d}} w_1'$ for some $w_1' \in W$, then, by Rule (DWN), we know that $w_1 \succeq w_1'$,

and, by Rule (Down), we have $[w_1]_{\rightleftharpoons} \xrightarrow{\mathbf{d}} [w_1']_{\rightleftharpoons}$. Since, by definition of B_C and by hypothesis, $[w_1]_{\rightleftharpoons} \simeq^{\mathbb{S}_A(\mathcal{F})} [w_2]_{\rightleftharpoons}$, we also have $[w_2]_{\rightleftharpoons} \xrightarrow{\mathbf{d}} [w_2']_{\rightleftharpoons}$ for some $[w_2']_{\rightleftharpoons} \simeq^{\mathbb{S}_A(\mathcal{F})} [w_1']_{\rightleftharpoons}$. From $[w_2]_{\rightleftharpoons} \xrightarrow{\mathbf{d}} [w_2']_{\rightleftharpoons}$, by Rule (Down), we know there are $w_3 \in [w_2]_{\rightleftharpoons}$ and $w_3' \in [w_2']_{\rightleftharpoons}$ such that $w_3 \succeq w_3'$ and, by Rule (DWN) we know that $w_3 \xrightarrow{\mathbf{d}} w_3'$. Now, since $w_3 \in [w_2]_{\rightleftharpoons}$, by definition of $\rightleftharpoons$ and by construction of $\mathbb{S}_C(\mathcal{F})$ we know there are $s_0, \ldots s_n \in W$ with $s_0 = w_2$, $s_n = w_3$ such that $s_i \xrightarrow{\tau} s_{i+1}$ and $s_{i+1} \xrightarrow{\tau} s_i$, for all $i \in [0; n)$. We note that $B_C(w_1, s_i)$ for all $i \in [0; n]$. In fact for each $i \in [0; n]$ we have that $[s_i]_{\rightleftharpoons} = [w_2]_{\rightleftharpoons}$ by definition of $\rightleftharpoons$ and we also know that $[w_2]_{\rightleftharpoons} \simeq^{\mathbb{S}_A(\mathcal{F})} [w_1]_{\rightleftharpoons}$, since $B_C(w_1, w_2)$ by hypothesis. Thus we get $[s_i]_{\rightleftharpoons} \simeq^{\mathbb{S}_A(\mathcal{F})} [w_1]_{\rightleftharpoons}$, i.e. $B_C(w_1, s_i)$. Furthermore, we also note that $B_C(w_1', w_3')$. In fact $[w_3']_{\rightleftharpoons} = [w_2']_{\rightleftharpoons}$, since $w_3' \in [w_2']_{\rightleftharpoons}$. Furthermore, $[w_2']_{\rightleftharpoons} \simeq^{\mathbb{S}_A(\mathcal{F})} [w_1']_{\rightleftharpoons}$ (see above). So, we get $[w_3']_{\rightleftharpoons} \simeq^{\mathbb{S}_A(\mathcal{F})} [w_1']_{\rightleftharpoons}$, i.e. $B_C(w_1', w_3')$. In conclusion, we have that if $w_1 \xrightarrow{\mathbf{d}} w_1'$ for some $w_1' \in W$, then $w_2 = s_0 \xrightarrow{\tau} s_1 \xrightarrow{\tau} \ldots \xrightarrow{\tau} s_n = w_3 \xrightarrow{\mathbf{d}} w_3'$ with $B_C(w_1', w_3')$ and $B_C(w_1, s_i)$ for all $i \in [0; n]$.

We now prove that if $w_1 \leftrightarroweq_b^{\mathbb{S}_C(\mathcal{F})} w_2$, then $[w_1]_{\rightleftharpoons} \simeq^{\mathbb{S}_A(\mathcal{F})} [w_2]_{\rightleftharpoons}$. We show that the following relation is a strong bisimulation:

$$B_A = \{(s_1, s_2) \in S \times S \,|\, \text{there are } w_1 \in s_1, w_2 \in s_2 \text{ such that } w_1 \leftrightarroweq_b^{\mathbb{S}_C(\mathcal{F})} w_2\}.$$

Let, without loss of generality, $s_1 = [w_1]_{\rightleftharpoons}$ and $s_2 = [w_2]_{\rightleftharpoons}$ for some $w_1, w_2 \in W$ with $w_1 \leftrightarroweq_b^{\mathbb{S}_C(\mathcal{F})} w_2$, and suppose $B_A([w_1]_{\rightleftharpoons}, [w_2]_{\rightleftharpoons})$. We distinguish three cases:

Case A: $[w_1]_{\rightleftharpoons} \xrightarrow{\alpha} [w_1']_{\rightleftharpoons}$ with $\alpha \in \mathbf{2}^{\mathtt{PL}}$:
By Rule (PL), if $[w_1]_{\rightleftharpoons} \xrightarrow{\alpha} [w_1']_{\rightleftharpoons}$ for $\alpha \in \mathbf{2}^{\mathtt{PL}}$ and $w_1' \in W$, then $[w_1']_{\rightleftharpoons} = [w_1]_{\rightleftharpoons}$ and $\alpha = \mathcal{V}^{-1}(\{w_1\})$. On the one hand, if $p \in \alpha$ then $w_1 \xrightarrow{p} w_1$ by rule (PLC). Since $w_2 \leftrightarroweq_b^{\mathbb{S}_C(\mathcal{F})} w_1$ it follows that $w_2 \xrightarrow{\tau} \ldots \xrightarrow{\tau} \bar{w}_2 \xrightarrow{p} w_2'$ for $\bar{w}_2, w_2' \in W$ such that $p \in \mathcal{V}^{-1}(\{\bar{w}_2\})$, $\bar{w}_2 \leftrightarroweq_b^{\mathbb{S}_C(\mathcal{F})} w_1$, and $w_2' \leftrightarroweq_b^{\mathbb{S}_C(\mathcal{F})} w_1$. By rule (TAU), $p \in \mathcal{V}^{-1}(\{w_2\})$. Thus, $\alpha \subseteq \mathcal{V}^{-1}(\{w_2\})$. On the other hand, if $p \in \mathcal{V}^{-1}(\{w_2\})$ then $w_2 \xrightarrow{p} w_2$ by rule (PLC). Since $w_1 \leftrightarroweq_b^{\mathbb{S}_C(\mathcal{F})} w_2$ we have that $w_1 \xrightarrow{\tau} \ldots \xrightarrow{\tau} \bar{w}_1 \xrightarrow{p} w_1'$ for $\bar{w}_1, w_1' \in W$ such that $p \in \mathcal{V}^{-1}(\{\bar{w}_1\})$, $\bar{w}_1 \leftrightarroweq_b^{\mathbb{S}_C(\mathcal{F})} w_2$, $w_1' \leftrightarroweq_b^{\mathbb{S}_C(\mathcal{F})} w_2$. By rule (TAU) we obtain that $p \in \mathcal{V}^{-1}(\{\bar{w}_1\})$. Thus, $p \in \alpha$. Hence, $\mathcal{V}^{-1}(\{w_2\}) \subseteq \alpha$. So, $\mathcal{V}^{-1}(\{w_2\}) = \alpha$. Therefore, $[w_2]_{\rightleftharpoons} \xrightarrow{\alpha} [w_2]_{\rightleftharpoons}$ by rule (PL). By assumption, $B_A([w_1]_{\rightleftharpoons}, [w_2]_{\rightleftharpoons})$ for target states $[w_1]_{\rightleftharpoons}$ and $[w_2]_{\rightleftharpoons}$ as required.

Case B: $[w_1]_{\rightleftharpoons} \xrightarrow{\mathbf{d}} [w_1']_{\rightleftharpoons}$
If $[w_1]_{\rightleftharpoons} \xrightarrow{\mathbf{d}} [w_1']_{\rightleftharpoons}$ for some $w_1' \in W$, then, by Rule (Down), we know that there are $w_3 \in [w_1]_{\rightleftharpoons}$ and $w_3' \in [w_1']_{\rightleftharpoons}$ such that $w_3 \succeq w_3'$. This implies, by Rule (DWN), that $w_3 \xrightarrow{\mathbf{d}} w_3'$. By definition of $\rightleftharpoons$ and by construction of $\mathbb{S}_C(\mathcal{F})$ we know that there are $m \geq 0$ and $t_0, \ldots, t_m \in W$ with $t_0 = w_1$, $t_m = w_3$

such that $t_i \xrightarrow{\tau} t_{i+1}$ and $t_{i+1} \xrightarrow{\tau} t_i$, for all $i \in [0;m)$. This implies that $w_1 \leftrightarroweq_b^{\mathbb{S}_C(\mathcal{F})} w_3$, and consequently $w_2 \leftrightarroweq_b^{\mathbb{S}_C(\mathcal{F})} w_3$, since $w_1 \leftrightarroweq_b^{\mathbb{S}_C(\mathcal{F})} w_2$ by hypothesis. Furthermore, since $w_3 \leftrightarroweq_b^{\mathbb{S}_C(\mathcal{F})} w_2$, there are $n \geq 0$ and $v_0, \ldots, v_n, v_{n+1} \in W$ with $w_2 = v_0 \xrightarrow{\tau} \cdots \xrightarrow{\tau} v_n \xrightarrow{\mathbf{d}} v_{n+1}$, such that $w_3' \leftrightarroweq_b^{\mathbb{S}_C(\mathcal{F})} v_{n+1}$ and $w_3 \leftrightarroweq_b^{\mathbb{S}_C(\mathcal{F})} v_i$ for all $i \in [0;n]$. Moreover, by Rule (DWN), we have $v_n \succeq v_{n+1}$ which imples, by Rule (Down), that $[v_n]_{\rightleftharpoons} \xrightarrow{\mathbf{d}} [v_{n+1}]_{\rightleftharpoons}$. Note that, by construction of $\mathbb{S}_C(\mathcal{F})$ we also have $\mathcal{V}^{-1}(w_2) = \mathcal{V}^{-1}(v_0) = \ldots = \mathcal{V}^{-1}(v_n)$ and so $[v_i] = [w_2]_{\rightleftharpoons}$ for all $i \in [0;n]$. Thus, $[w_2]_{\rightleftharpoons} = [v_n]_{\rightleftharpoons} \xrightarrow{\mathbf{d}} [v_{n+1}]_{\rightleftharpoons}$. Furthermore, $B_A([w_3']_{\rightleftharpoons}, [v_{n+1}]_{\rightleftharpoons})$ holds, since $w_3' \leftrightarroweq_b^{\mathbb{S}_C(\mathcal{F})} v_{n+1}$ (see above) and, recalling that $[w_3']_{\rightleftharpoons} = [w_1']_{\rightleftharpoons}$, we also know that $B_A([w_1']_{\rightleftharpoons}, [v_{n+1}]_{\rightleftharpoons})$.

Case C: $[w_1]_{\rightleftharpoons} \xrightarrow{\mathbf{s}} [w_1']_{\rightleftharpoons}$
If $[w_1]_{\rightleftharpoons} \xrightarrow{\mathbf{s}} [w_1']_{\rightleftharpoons}$ for some $w_1' \in W$, then, by Rule (Step), we know that there are $w_3 \in [w_1]_{\rightleftharpoons}$ and $w_3' \in [w_1']_{\rightleftharpoons}$ such that $w_3 \preccurlyeq^{\pm} w_3'$. We distinguish two cases:
Case C1: $\mathcal{V}^{-1}(\{w_3\}) = \mathcal{V}^{-1}(\{w_3'\})$.
If $\mathcal{V}^{-1}(\{w_3\}) = \mathcal{V}^{-1}(\{w_3'\})$, then, by Rule (TAU), we know $w_3 \xrightarrow{\tau} w_3'$. But then, by definition of $\rightleftharpoons$, we get $[w_3]_{\rightleftharpoons} = [w_3']_{\rightleftharpoons}$ and since $[w_3]_{\rightleftharpoons} = [w_1]_{\rightleftharpoons}$ and $[w_3']_{\rightleftharpoons} = [w_1']_{\rightleftharpoons}$ (see above), we get $[w_1']_{\rightleftharpoons} = [w_1]_{\rightleftharpoons}$. On the other hand, since, trivially, $w_2 \preccurlyeq^{\pm} w_2$, by Rule (Step), we also get that $[w_2]_{\rightleftharpoons} \xrightarrow{\mathbf{s}} [w_2]_{\rightleftharpoons}$. Moreover, since by hypothesis, we also have $B_A([w_1]_{\rightleftharpoons}, [w_2]_{\rightleftharpoons})$, we finally get that also $B_A([w_1']_{\rightleftharpoons}, [w_2]_{\rightleftharpoons})$.
Case C2: $\mathcal{V}^{-1}(\{w_3\}) \neq \mathcal{V}^{-1}(\{w_3'\})$.
If $\mathcal{V}^{-1}(\{w_3\}) \neq \mathcal{V}^{-1}(\{w_3'\})$, then, by Rule (CNG), we know $w_3 \xrightarrow{\mathbf{c}} w_3'$. By definition of $\rightleftharpoons$ and by construction of $\mathbb{S}_C(\mathcal{F})$ we know that there are $m \geq 0$ and $t_0, \ldots, t_m \in W$ with $t_0 = w_1$, $t_m = w_3$ such that $t_i \xrightarrow{\tau} t_{i+1}$ and $t_{i+1} \xrightarrow{\tau} t_i$, for all $i \in [0;m)$. This implies that $w_1 \leftrightarroweq_b^{\mathbb{S}_C(\mathcal{F})} w_3$, and consequently $w_2 \leftrightarroweq_b^{\mathbb{S}_C(\mathcal{F})} w_3$, since $w_1 \leftrightarroweq_b^{\mathbb{S}_C(\mathcal{F})} w_2$ by hypothesis. Furthermore, since $w_3 \leftrightarroweq_b^{\mathbb{S}_C(\mathcal{F})} w_2$, there are $n \geq 0$ and $v_0, \ldots, v_n, v_{n+1} \in W$ with $w_2 = v_0 \xrightarrow{\tau} \cdots \xrightarrow{\tau} v_n \xrightarrow{\mathbf{c}} v_{n+1}$, such that $w_3' \leftrightarroweq_b^{\mathbb{S}_C(\mathcal{F})} v_{n+1}$ and $w_3 \leftrightarroweq_b^{\mathbb{S}_C(\mathcal{F})} v_i$ for all $i \in [0;n]$. Moreover, by Rule (CNG), we have $v_n \preccurlyeq^{\pm} v_{n+1}$ which imples, by Rule (Step), that $[v_n]_{\rightleftharpoons} \xrightarrow{\mathbf{s}} [v_{n+1}]_{\rightleftharpoons}$. Note that, by construction of $\mathbb{S}_C(\mathcal{F})$ we also have $\mathcal{V}^{-1}(w_2) = \mathcal{V}^{-1}(v_0) = \ldots = \mathcal{V}^{-1}(v_n)$ and so $[v_i] = [w_2]_{\rightleftharpoons}$ for all $i \in [0;n]$. Thus, $[w_2]_{\rightleftharpoons} = [v_n]_{\rightleftharpoons} \xrightarrow{\mathbf{s}} [v_{n+1}]_{\rightleftharpoons}$. Furthermore, $B_A([w_3']_{\rightleftharpoons}, [v_{n+1}]_{\rightleftharpoons})$ holds, since $w_3' \leftrightarroweq_b^{\mathbb{S}_C(\mathcal{F})} v_{n+1}$ (see above) and, recalling that $[w_3']_{\rightleftharpoons} = [w_1']_{\rightleftharpoons}$, we also know that $B_A([w_1']_{\rightleftharpoons}, [v_{n+1}]_{\rightleftharpoons})$.

Lemma 5. *Given finite poset model $\mathcal{F} = (W, \preceq, \mathcal{V})$. Then for all $w_1, w_2 \in W$ the following holds: if $[w_1]_{\rightleftharpoons} \simeq^{\mathbb{S}_A(\mathcal{F})} [w_2]_{\rightleftharpoons}$, then $\mathcal{V}^{-1}(\{w_1\}) = \mathcal{V}^{-1}(\{w_2\})$.*

Proof. By Rule (PL), we have $[w_1]_{\rightleftharpoons} \stackrel{\mathcal{V}^{-1}(\{w_1\})}{\longrightarrow} [w_1]_{\rightleftharpoons}$ and, by hypothesis, we also have $[w_2]_{\rightleftharpoons} \stackrel{\mathcal{V}^{-1}(\{w_1\})}{\longrightarrow} [w_2']_{\rightleftharpoons}$, for some $[w_2']_{\rightleftharpoons} \simeq [w_1]_{\rightleftharpoons}$. But then, using again Rule (PL), we get $[w_2']_{\rightleftharpoons} = [w_2]_{\rightleftharpoons}$ and $\mathcal{V}^{-1}(\{w_1\}) = \mathcal{V}^{-1}(\{w_2\})$.

B.3 Proof of Theorem 3

Theorem 3. *Given finite poset model $\mathcal{F} = (W, \preceq, \mathcal{V})$ let $\mathcal{F}_{\min}$ be defined as in Definition 4. Then, for each $w \in W$ and $\mathtt{SLCS}_\eta$ formula Φ the following holds:*

$$\mathcal{F}, w \models \Phi \text{ if and only if } \mathcal{F}_{\min}, [w]_{\equiv_\eta} \models \Phi.$$

Proof. We first prove that $\mathcal{F}, w \models \Phi$ implies $\mathcal{F}_{\min}, [w]_{\equiv_\eta} \models \Phi$. We proceed by induction on the structure of Φ and we show the proof only for $\Phi = \eta(\Phi_1, \Phi_2)$ the other cases being straightforward. Suppose $\mathcal{F}, w \models \eta(\Phi_1, \Phi_2)$. This means there is a ±-path π of some length $\ell \geq 2$ such that $\pi(0) = w$, $\mathcal{F}, \pi(\ell) \models \Phi_2$ and $\mathcal{F}, \pi(i) \models \Phi_1$ for all $i \in [0; \ell)$. Define now $\pi_{\min} : [0; \ell] \to W_{\min}$ with $\pi_{\min}(i) = [\pi(i)]$ for all $i \in [0; \ell]$. We show that $\pi_{\min}$ is a ±-path with respect to $R_{\min}$. We have that $R_{\min}(\pi_{\min}(0), \pi_{\min}(1))$ by definition of $R_{\min}$ because $\pi(0) \in [\pi(0)] = \pi_{\min}(0)$, $\pi(1) \in [\pi(1)] = \pi_{\min}(1)$ and $\pi(0) \preceq \pi(1)$ by assumption. Similarly, we have that $R^-_{\min}(\pi_{\min}(\ell-1), \pi_{\min}(\ell))$ and also that $R^\pm(\pi_{\min}(i), \pi_{\min}(i+1))$ for all $i \in (0; \ell-1)$. Furthermore, since $\mathcal{F}, \pi(\ell) \models \Phi_2$, by the Induction Hypothesis, we have that $\mathcal{F}_{\min}, \pi_{\min}(\ell) \models \Phi_2$. Similarly, we have that $\mathcal{F}_{\min}, \pi_{\min}(i) \models \Phi_1$ for all $i \in [0; \ell)$ since $\mathcal{F}, \pi(i) \models \Phi_1$. So $\mathcal{F}_{\min}, [w]_{\equiv_\eta} \models \eta(\Phi_1, \Phi_2)$.

Now we prove that $\mathcal{F}_{\min}, [w]_{\equiv_\eta} \models \Phi$ implies $\mathcal{F}, w \models \Phi$. Also in this case we proceed by induction on the structure of Φ and we show the proof only for $\Phi = \eta(\Phi_1, \Phi_2)$. Suppose $\mathcal{F}_{\min}, [w]_{\equiv_\eta} \models \eta(\Phi_1, \Phi_2)$. If $\mathcal{F}_{\min}, [w]_{\equiv_\eta} \models \eta(\Phi_1, \Phi_2)$ then there is a ±-path $\pi_{\min}$ such that $\pi_{\min}(0) = [w]_{\equiv_\eta}$, $\mathcal{F}_{\min}, \pi(\ell_{\min}) \models \Phi_2$ and $\mathcal{F}_{\min}, \pi_{\min}(i) \models \Phi_1$ for all $i \in [0; \ell_{\min})$. Since $R_{\min}$ is reflexive, using Lemma 2 on page 24, we know that there is also an $\upharpoonleft\!\downharpoonright$-path $\hat{\pi}_{\min}$ from $[w]_{\equiv_\eta}$ of some length $2k$, for $k \geq 1$, with the same starting-/ending points and the same intermediate points as $\pi_{\min}$ and that obviously witnesses $\eta(\Phi_1, \Phi_2)$ for $[w]_{\equiv_\eta}$. By induction on k, in the sequel, we show that there is a ±-path π from w witnessing $\eta(\Phi_1, \Phi_2)$.

Base case: $k = 1$.
In this case, we have that

- $\hat{\pi}_{\min}(0) = [w]_{\equiv_\eta}$,
- $\mathcal{F}_{\min}, \hat{\pi}_{\min}(0) \models \Phi_1$
- $\mathcal{F}_{\min}, \hat{\pi}_{\min}(1) \models \Phi_1$, and
- $\mathcal{F}_{\min}, \hat{\pi}_{\min}(2) \models \Phi_2$

Furthermore, since $\hat{\pi}_{\min}$ is an $\upharpoonleft\!\downharpoonright$-path with respect to $R_{\min}$, we know that

$$\hat{\pi}_{\min}(0) = [w]_{\equiv_\eta}, R_{\min}(\hat{\pi}_{\min}(0), \hat{\pi}_{\min}(1)), R^-_{\min}(\hat{\pi}_{\min}(1), \hat{\pi}_{\min}(2))$$

and, by definition of $R_{\min}$, there are $w_0 \in \hat{\pi}_{\min}(0) = [w]_{\equiv_\eta}$, $w_1', w_1'' \in \hat{\pi}_{\min}(1)$ and $w_2 \in \hat{\pi}_{\min}(2)$ such that $w_0 \preceq w_1'$ and $w_1'' \succeq w_2$. Moreover, by the Induction

Hypothesis with respect to the structure of formulas, we have that $\mathcal{F}, w_0 \models \Phi_1$, $\mathcal{F}, w_1' \models \Phi_1$, $\mathcal{F}, w_1'' \models \Phi_1$, and $\mathcal{F}, w_2 \models \Phi_2$. Note that $\mathcal{F}, w_1'' \models \eta(\Phi_1, \Phi_2)$, witnessed by the following $\pm$-path: (w_1'', w_1'', w_2). But then we have that also $\mathcal{F}, w_1' \models \eta(\Phi_1, \Phi_2)$ holds since $w_1' \equiv_\eta w_1''$, recalling that $w_1', w_1'' \in \hat{\pi}_{\min}(1) \in W_{/\equiv_\eta}$. There is then a $\pm$-path $\pi' : [0; \ell'] \to W$ from w_1' of some length ℓ' such that $\mathcal{F}, \pi'(\ell') \models \Phi_2$ and $\mathcal{F}, \pi'(i) \models \Phi_1$ for all $i \in [0; \ell')$. Furthermore, $w_0 \preceq w_1'$ by hypothesis and so $\pi = (w_0, w_1') \cdot \pi' : [0; \ell' + 1] \to W$ is a $\pm$-path from w_0 witnessing $\mathcal{F}, w_0 \models \eta(\Phi_1, \Phi_2)$. Finally, recalling that $w, w_0 \in \hat{\pi}_{\min}(0) \in W_{/\equiv_\eta}$, we know that $w \equiv_\eta w_0$ and so we have proven the assertion $\mathcal{F}, w \models \eta(\Phi_1, \Phi_2)$.

Induction step: $k = n + 1$ assuming the assertion holds for $k = n$, for $n > 0$. Since $k > 1$, we know that $\mathcal{F}_{\min}, \hat{\pi}_{\min}(1) \models \Phi_1$ and $\mathcal{F}_{\min}, \hat{\pi}_{\min}(2) \models \Phi_1 \wedge \neg\Phi_2$. Furthermore,

$$\hat{\pi}_{\min}(0) = [w]_{\equiv_\eta}, R_{\min}(\hat{\pi}_{\min}(0), \hat{\pi}_{\min}(1)), R^-_{\min}(\hat{\pi}_{\min}(1), \hat{\pi}_{\min}(2))$$

because $\hat{\pi}_{\min}$ is an $\updownarrow$-path. By definition of $R_{\min}$, there are $w_0 \in \hat{\pi}_{\min}(0) = [w]_{\equiv_\eta}$, $w_1', w_1'' \in \hat{\pi}_{\min}(1)$ and $w_2 \in \hat{\pi}_{\min}(2)$ such that $w_0 \preceq w_1'$ and $w_1'' \succeq w_2$. By the Induction Hypothesis with respect to the structure of the formula, we get that $\mathcal{F}, w_0 \models \Phi_1$, $\mathcal{F}, w_1' \models \Phi_1$, $\mathcal{F}, w_1'' \models \Phi_1$, and $\mathcal{F}, w_2 \models \Phi_1 \wedge \neg\Phi_2$. We consider now the $\updownarrow$-path $\hat{\pi}_{\min} \uparrow 2$ from $\hat{\pi}_{\min}(2)$ of length $2n$, noting that it witnesses $\eta(\Phi_1, \Phi_2)$, since so does $\hat{\pi}_{\min}$ and $k > 1$. In other words, we have that $\mathcal{F}_{\min}, \hat{\pi}_{\min}(2) \models \eta(\Phi_1, \Phi_2)$ with $w_2 \in \hat{\pi}_{\min}(2)$. By the Induction Hypothesis with respect to k, we then have that $\mathcal{F}, w_2 \models \eta(\Phi_1, \Phi_2)$. So there is a $\updownarrow$-path $\pi_2 : [0; \ell_2] \to W$ from w_2 of some length ℓ_2 such that $\mathcal{F}, \pi_2(\ell_2) \models \Phi_2$ and $\mathcal{F}, \pi_2(i) \models \Phi_1$ for $i \in [0; \ell_2)$. Note that $\mathcal{F}, \pi_2(0) \models \Phi_1$ as well, since $\pi_2(0) = w_2$ and $\mathcal{F}, w_2 \models \Phi_1 \wedge \neg\Phi_2$ (see above). Let us consider now the path $\pi'' = (w_1'', w_1'', w_2) \cdot \pi_2$. Such a path is an $\updownarrow$-path since so is π_2, and $w_1'' \succeq w_2$ by hypothesis. Note that $\updownarrow$-path π'' witnesses $\mathcal{F}, w_1'' \models \eta(\Phi_1, \Phi_2)$. But then we have that also $\mathcal{F}, w_1' \models \eta(\Phi_1, \Phi_2)$ holds since $w_1' \equiv_\eta w_1''$, recalling that $w_1', w_1'' \in \hat{\pi}_{\min}(1) \in W_{/\equiv_\eta}$. Thus, we have that the following holds: $\mathcal{F}, w_1' \models \Phi_1 \wedge \eta(\Phi_1, \Phi_2)$. There is then a $\pm$-path $\pi' : [0; \ell'] \to W$ from w_1' of some length ℓ' such that $\mathcal{F}, \pi'(\ell') \models \Phi_2$ and $\mathcal{F}, \pi'(i) \models \Phi_1$ for all $i \in [0; \ell')$. Furthermore, $w_0 \preceq w_1'$ by hypothesis and so $\pi = (w_0, w_1') \cdot \pi' : [0; \ell' + 1] \to W$ is a $\pm$-path from w_0 witnessing $\mathcal{F}, w_0 \models \eta(\Phi_1, \Phi_2)$. Finally, recalling that $w, w_0 \in \hat{\pi}_{\min}(0) \in W_{/\equiv_\eta}$, we know that $w \equiv_\eta w_0$ and so we have proven the assertion $\mathcal{F}, w \models \eta(\Phi_1, \Phi_2)$.

B.4 Proof of Theorem 4

Theorem 4. *For any poset model $\mathcal{F} = (W, \preceq, \mathcal{V})$ and $\mathcal{F}_{\min}$ as of Def. 4 and for all $\alpha_1, \alpha_2 \in W_{\min}$, it holds that $R_{min}(\alpha_1, \alpha_2)$ if and only if $\alpha_2 \xrightarrow{\mathbf{d}} \alpha_1$ is a transition of the minimal LTS obtained from $\mathbb{S}_C(\mathcal{F})$ via branching equivalence.*

Proof. In the sequel, we let $\mathbb{S}_C(\mathcal{F})_{/\underline{\leftrightarrow}_b}$ denote the minimal LTS obtained from $\mathbb{S}_C(\mathcal{F})$ via branching equivalence. First of all, by Corollary 1, $W_{\min}$ coincides with the quotient of the set of states W of $\mathbb{S}_C(\mathcal{F})$ modulo branching equivalence. Now, suppose that $\alpha_2 \xrightarrow{\mathbf{d}} \alpha_1$ is a transition of $\mathbb{S}_C(\mathcal{F})_{/\underline{\leftrightarrow}_b}$. By standard construction of the minimal LTS modulo an equivalence on its state set, we know that $w_1 \in \alpha_1$ and $w_2 \in \alpha_2$ exist such that $w_2 \xrightarrow{\mathbf{d}} w_1$ is a transition of $\mathbb{S}_C(\mathcal{F})$. But then, by Rule (DWN), we get that $w_1 \preceq w_2$ and so, by definition of $\mathcal{F}_{\min}$, we finally get $R(\alpha_1, \alpha_2)$. If, on the other hand, $R(\alpha_1, \alpha_2)$ holds, then we know that there exist $w_1 \in \alpha_1$ and $w_2 \in \alpha_2$ such that $w_1 \preceq w_2$, by definition of $\mathcal{F}_{\min}$. But then, by Rule (DWN), we get that $w_2 \xrightarrow{\mathbf{d}} w_1$ is a transition of $\mathbb{S}_C(\mathcal{F})$. Again, by standard construction of the minimal LTS modulo an equivalence on its state set, we know that $\alpha_2 \xrightarrow{\mathbf{d}} \alpha_1$ is a transition of $\mathbb{S}_C(\mathcal{F})_{/\underline{\leftrightarrow}_b}$.