

Experimental lower bounds to the classical capacity of quantum channels

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We show an experimental procedure to certify the classical capacity for noisy qubit channels. The method makes use of a fixed bipartite entangled state, where the system qubit is sent to the channel input and the set of local measurements $\sigma_x \otimes \sigma_x$, $\sigma_y \otimes \sigma_y$ and $\sigma_z \otimes \sigma_z$ is performed at the channel output and the ancilla qubit, thus without resorting to full quantum process tomography. The witness to the classical capacity is then achieved by reconstructing sets of conditional probabilities, noise deconvolution, and classical optimization of the pertaining mutual information. The performance of the method to provide lower bounds to the classical capacity is tested by a two-photon polarization entangled state in Pauli channels and amplitude damping channels. The measured lower bounds to the channels are in high agreement with the simulated data, which take into account both the experimental entanglement fidelity $F = 0.979 \pm 0.011$ of the input state and the systematic experimental imperfections.

I. INTRODUCTION

The complete characterization of quantum channels by quantum process tomography [1–11] is demanding in terms of state preparation and/or measurement settings, since for increasing dimension d of the system Hilbert space it scales as d^4 . When one is interested in certifying specific properties of a quantum channels, more affordable procedures can be devised without resorting to complete quantum process tomography. This is the case of the detection of entanglement-breaking properties [12, 13] or non-Markovianity [14] of quantum channels, or the certification of lower bounds to the quantum capacity of noisy quantum channels [15–18]. Typically, these direct methods have also the advantage of being more precise with respect to complete process tomography, which has the drawback of involving larger statistical errors due to error propagation.

One of the most relevant property of quantum channels is the classical capacity [19–21] for its operational importance in the quantification of the classical information that can be reliably transmitted. For the purpose of detecting lower bounds to the classical capacity, an efficient and versatile procedure has been recently proposed

in Ref. [22]. The method allows to experimentally detect lower bounds to the classical capacity of completely unknown quantum channels just by means of few local measurements, even for high-dimensional systems [23].

The gist of the procedure is to efficiently reconstruct a number of probability transition matrices for suitable input states (playing the role of “encoding”) and matched output projective measurements (the corresponding “decoding”). The method is accompanied by the optimization of the prior distribution for the single-letter encoding pertaining to each input/output transition matrix. In this way, the mutual information for different communication settings is recovered and the resulting values are compared. Hence, a lower bound to the Holevo capacity and then a certification of the minimum reliable transmission capacity is achieved. Similarly to the method of certification for the quantum capacity [15–17], here each of the conditional probabilities corresponding to a communication setting can be obtained by preparing just an initial fixed bipartite state, where only one party enters the quantum channel, while local measurements are performed at the input and output of the channel.

In this paper we present an experimental demonstration for the above certification of classical capacity in noisy qubit channels. We implement the method by using highly-pure polarization entangled photons pairs, where the encoding/decoding settings are achieved by exploiting complementary observables of the polarization state. Moreover, a faithful deconvolution of noise is performed over the experimental data, which allows to optimize the reconstruction of the probability transition matrices.

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II. THEORETICAL MODEL

Let us briefly review the method proposed in Ref. [22] for bounding from below the classical capacity, with specific attention to the case of single-qubit quantum channels.

The classical capacity C of a noisy quantum channel \mathcal{E} quantifies the maximum number of bits that can be reliably transmitted per channel use, and it is given by the regularized expression [19–21] $C = \lim_{N \rightarrow \infty} \chi(\mathcal{E}^{\otimes N})/N$, with N as the number of uses, in terms of the Holevo capacity

$$\chi(\Phi) = \max_{\{p_i, \rho_i\}} \{S[\Phi(\sum_i p_i \rho_i)] - \sum_i p_i S[\Phi(\rho_i)]\}, \quad (1)$$

where $S(\rho) = -\text{Tr}[\rho \log_2 \rho]$ denotes the Von Neumann entropy. The Holevo capacity $\chi(\mathcal{E}) \equiv C_1$, also known as one-shot classical capacity, is a lower bound for the ultimate channel capacity C , and also an upper bound for the mutual information [24–26]

$$I(X; Y) = \sum_{x, y} p_x Q(y|x) \log_2 \frac{Q(y|x)}{\sum_{x'} p_{x'} Q(y|x')}, \quad (2)$$

where any transition matrix $Q(y|x)$ corresponds to the conditional probability for outcome y in an arbitrary measurement at the output of a single use of the channel with input ρ_x , and p_x denotes an arbitrary prior probability, which corresponds to the distribution of the encoded alphabet on the quantum states $\{\rho_x\}$.

As depicted in Fig. 1, in order to obtain a witness for the classical capacity without resorting to complete process tomography one can proceed as follows. Prepare a bipartite maximally entangled state $|\phi^+\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} |k_s\rangle |k_a\rangle$ between a system and ancillary spaces, both with dimension d ; send $|\phi^+\rangle$ through the unknown channel by keeping the action of \mathcal{E} on the system alone, namely via the map $\mathcal{E} \otimes \mathbb{I}_a$; finally, measure locally a number of observables of the form $X_\alpha \otimes X_\alpha^\tau$, where τ denotes the transposition w.r.t. to the fixed basis defined by $|\phi^+\rangle$.

In fact, by denoting the d eigenvectors of X_α as $\{|\phi_i^{(\alpha)}\rangle\}$, from the identity [27]

$$\text{Tr}[(A \otimes B^\tau)(\mathcal{E} \otimes \mathbb{I}_a)|\phi^+\rangle\langle\phi^+|] = \frac{1}{d} \text{Tr}[A\mathcal{E}(B)], \quad (3)$$

with arbitrary operators A and B , the detection scheme allows to reconstruct the set of conditional probabilities

$$Q^{(\alpha)}(j|i) = \langle\phi_j^{(\alpha)}|\mathcal{E}(|\phi_i^{(\alpha)}\rangle\langle\phi_i^{(\alpha)}|)|\phi_j^{(\alpha)}\rangle. \quad (4)$$

For each encoding-decoding scheme α characterized by the choice of X_α , we can write the corresponding optimal mutual information, namely the Shannon capacity

$$C^{(\alpha)} = \max_{\{p_i^{(\alpha)}\}} \sum_{i, j} p_i^{(\alpha)} Q^{(\alpha)}(j|i) \log_2 \frac{Q^{(\alpha)}(j|i)}{\sum_l p_l^{(\alpha)} Q^{(\alpha)}(j|l)}. \quad (5)$$

Then, we have the chain of inequalities

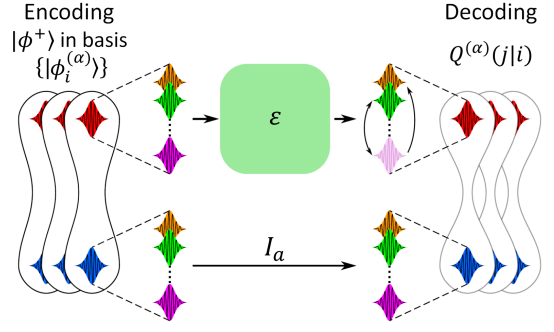


FIG. 1. Protocol's concept: The s -qudit of a maximally entangled bipartite quantum state $|\phi^+\rangle$ is sent through a noisy quantum channel \mathcal{E} , while the a -qudit propagates freely. Encoding-decoding of type α is represented by layers and is achieved by measuring locally the basis $\{|\phi_i^{(\alpha)}\rangle\}$. The conditional probability $Q^{(\alpha)}(j|i)$ of measuring $|\phi_j^{(\alpha)}\rangle$ for an input $|\phi_i^{(\alpha)}\rangle$ allows to evaluate the Shannon capacity $C^{(\alpha)}$. The witness C_D of the classical capacity is the largest of the values $\{C^{(\alpha)}\}$ among all tested types α .

$$C \geq C_1 \geq C_D \equiv \max_{\alpha} \{C^{(\alpha)}\}, \quad (6)$$

where C_D is the experimentally accessible witness that depends on the chosen set of measured observables labeled by α , and provides a lower bound to the classical capacity of the unknown channel.

In the present scenario we consider single-qubit channels and the information settings correspond to the choice of the three local observables $\sigma_x \otimes \sigma_x$, $\sigma_y \otimes \sigma_y$ and $\sigma_z \otimes \sigma_z$. Hence, each of the three conditional probabilities $Q^{(\alpha)}(j|i)$, with $\alpha = x, y, z$, is a 2×2 transition matrix that corresponds to a binary classical channel, for which the optimal prior probability $\{p_i^{(\alpha)}\}$ can be theoretically evaluated [22]. In fact, without loss of generality, for each transition matrix we can fix the labeling of logical zeros and ones such that $0 \leq \epsilon_0 \leq \frac{1}{2}$, $\epsilon_0 \leq \epsilon_1$, and $\epsilon_0 \leq 1 - \epsilon_1$, where ϵ_0 denotes the error probability of receiving 1 for input 0, and ϵ_1 denotes the error probability of receiving 0 for input 1. Then, for each $\alpha = x, y, z$ the mutual information as in Eq. (5) is maximized by a prior probability $\{p_0, p_1 = 1 - p_0\}$, with

$$p_0 = \frac{1 - \epsilon_1(1 + z)}{(1 - \epsilon_0 - \epsilon_1)(1 + z)}, \quad (7)$$

where $z = 2 \frac{H[\epsilon_0] - H[\epsilon_1]}{1 - \epsilon_0 - \epsilon_1}$, and $H(p) = -p \log_2 p - (1 - p) \log_2 (1 - p)$ denotes the binary Shannon entropy. The corresponding capacity is given by [22]

$$C_B(\epsilon_0, \epsilon_1) = \log_2 \left[1 + 2 \frac{H[\epsilon_0] - H[\epsilon_1]}{1 - \epsilon_0 - \epsilon_1} \right] + \frac{\epsilon_0}{1 - \epsilon_0 - \epsilon_1} H[\epsilon_1] - \frac{1 - \epsilon_1}{1 - \epsilon_0 - \epsilon_1} H[\epsilon_0]. \quad (8)$$

Notice that for $\epsilon_0 = \epsilon_1 = \epsilon$ one recovers the classical capacity for the binary symmetric channel

$$C_B(\epsilon, \epsilon) = 1 - H[\epsilon], \quad (9)$$

with uniform optimal prior $\{1/2, 1/2\}$, whereas for $\epsilon_0 = 0$ (i.e., when only input 1 is affected by error) one obtains the capacity of the so-called Z -channel

$$C_B(0, \epsilon) = \log_2[1 + (1 - \epsilon)\epsilon^{\frac{\epsilon}{1-\epsilon}}]. \quad (10)$$

In Appendix A we summarize the expected theoretical results for the qubit channels that we implemented experimentally.

In order to consider the experimental imperfections in our state preparation, we assume a Werner form

$$\rho_F = \frac{4F-1}{3} |\Phi^+\rangle\langle\Phi^+| + \frac{1-F}{3} \mathbb{I}_s \otimes \mathbb{I}_a, \quad (11)$$

where $F = \langle\Phi^+|\rho_F|\Phi^+\rangle$ denotes the fidelity with respect to the ideal maximally entangled state $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle)$. Hence, we can replace Eq. (3) with

$$\text{Tr}[A\mathcal{E}(B)] = \frac{2}{4F-1} \times \quad (12)$$

$$\text{Tr}\{[A \otimes (3B^T - 2(1-F)\text{Tr}[B]\mathbb{I}_a)](\mathcal{E} \otimes \mathbb{I}_a)\rho_F\}.$$

Equation (12) allows one to deconvolve the noise as long as $F \neq 1/4$, since the output state $(\mathcal{E} \otimes \mathbb{I}_a)\rho_F$ faithfully represents the unknown channel \mathcal{E} .

The normalized conditional probabilities for the three binary classical channels corresponding to the information settings $\alpha = x, y, z$ where coding and decoding are performed using the eigenstates of the three Pauli matrices $\sigma_x, \sigma_y, \sigma_z$ can be obtained by the ratio $\frac{\text{Tr}[A\mathcal{E}(B)]}{\text{Tr}[\mathcal{E}(B)]}$, when replacing A and B with the pertaining eigenvectors. In practical terms, these quantities are measured from the bipartite qubit detections, which in optical channels corresponds to two-photon coincidence detections $C(ji) = \text{Tr}[(|j\rangle\langle j|_S \otimes |i\rangle\langle i|_A)(\mathcal{E} \otimes \mathbb{I}_a)\rho_F]$ on the output state. We provide in Appendix B the explicit expressions for the conditional probabilities in terms of the measured coincidences. For each information setting $\alpha = x, y, z$, the logical bits and their pertaining transition error probabilities $\epsilon_0^{(\alpha)}$ and $\epsilon_1^{(\alpha)}$ introduced before Eq. (7) are identified from the conditional probabilities as follows

$$\epsilon_0^{(\alpha)} = \min_{\{i,j\}} Q^{(\alpha)}(i|j) \equiv Q^{(\alpha)}(1|0); \quad \epsilon_1^{(\alpha)} \equiv Q^{(\alpha)}(0|1). \quad (13)$$

Using Eq. (8), from the experimental data we then obtain $C^{(\alpha)} = C_B(\epsilon_0^{(\alpha)}, \epsilon_1^{(\alpha)})$, and hence the detected classical capacity as

$$C_D = \max_{\alpha=x,y,z} C_B(\epsilon_0^{(\alpha)}, \epsilon_1^{(\alpha)}). \quad (14)$$

We remark that the effect of a fidelity value $F < 1$ (except the case $F = \frac{1}{4}$) is not detrimental for accessing the capacity witness C_D , and only the statistical noise is expected to increase for decreasing value of F . This can be argued from the identity (12) which, for any couple of bases $\{|i\rangle_A\}_{i=0,1}$ and $\{|j\rangle_S\}_{j=0,1}$, provides an unbiased estimation of the joint probabilities $p_F(j, i) = \text{Tr}[(|j\rangle\langle j|_S \otimes |i\rangle\langle i|_A)(\mathcal{E} \otimes \mathbb{I}_a)\rho_F]$ pertaining to

the case of ideal fidelity $F = 1$ in terms of the measured probabilities as

$$p_{F=1}(j, i) = \frac{1+2F}{4F-1} p_F(j, i) - \frac{2(1-F)}{4F-1} p_F(j, i \oplus 1). \quad (15)$$

By denoting with σ the typical order of magnitude of the standard deviation for the measured joint probabilities with $F < 1$, then the noise deconvolution provides the correct probabilities with standard deviation

$$\begin{aligned} \sigma_F &= \sigma \sqrt{\left(\frac{1+2F}{4F-1}\right)^2 + 4\left(\frac{1-F}{4F-1}\right)^2} \\ &= \sigma \frac{\sqrt{8F^2 - 4F + 5}}{|4F-1|}. \end{aligned} \quad (16)$$

III. EXPERIMENTAL IMPLEMENTATION

A single-mode continuous-wave laser at 405nm was used to pump a Type-II ppKTP crystal within a Sagnac interferometer (SI) in order to produce polarization-entangled photon pairs at 810nm via spontaneous parametric down conversion, as implemented in [17]. The two output modes were labeled as s -qubit and a -qubit, so the Bell state $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|H_s\rangle|H_a\rangle + |V_s\rangle|V_a\rangle)$ was generated. As shown in the full scheme of Fig. 2, tube lenses with $F_1 = 300\text{mm}$ collimate both output modes, while objective lenses with $F_2 = 11\text{mm}$ couple them into single-mode-fibers (SMFs).

The fidelity of the experimental state ρ_F defined in Eq. (11) and measured by quantum tomography techniques [28, 29] was $F = \langle\Phi^+|\rho_F|\Phi^+\rangle = 0.979 \pm 0.011$. Hence, we use Eq. (12) for noise deconvolution, and the standard deviation of the joint probabilities are increased just by 1.45% with respect to the ideal case $F = 1$, according to Eq. (16).

The observables σ_α , for $\alpha = x, y, z$, are measured with rotations of a quarter wave-plate (QWP) and a half wave-plate (HWP) before a polarizing beam splitter (PBS) for each qubit. Remaining photon pairs after these projections are measured with synchronized single-photon avalanche detectors (SPADs) within a time window of 5ns for an integration time of 10s/measurement.

By referring to the parametrization of channels given in Appendix A, within the family of Pauli channels we have chosen two special cases as implemented in [17, 30]. The first one is the phase damping (PD) channel (described by $q_z = q$ and $q_x = q_y = 0$ in Appendix A). The second one is the depolarizing (D) channel (described by $q_x = q_y = q_z = q$). Their experimental simulations were implemented by q_i -weighted combinations of independent Pauli operations, achieved by setting a HWP at 0° -degrees (45° -degrees) for σ_z (σ_x) and a liquid crystal (LC) at minimum (maximum) voltage for σ_z (\mathbb{I}) [see Fig. 2b]. Accordingly, the simultaneous operation $\sigma_y = i\sigma_x\sigma_z$ give us the last Pauli operation.

The experimental simulation of an amplitude damping (AD) channel was achieved by using a square SI

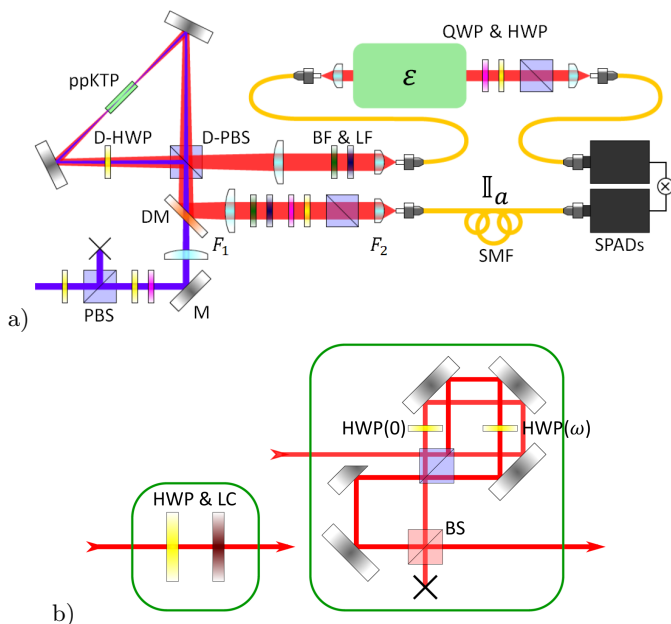


FIG. 2. a) Full scheme with quantum source, propagation channels, projection and measurement devices. PBS: polarizing beam splitter; M: mirror; BS: non-polarizing 50:50 beam splitter; QWP: quarter wave-plate; HWP: half wave-plate; DM: dichroic mirror; ppKTP: periodically poled non-linear crystal; BF: band-pass filter; LF: long-pass filter; F_k : lenses; SPADs: single photon avalanche detectors; SMF: single mode fiber; LC: liquid crystal element. Additional D-label for compatibility with both pump and generated wavelength. b) Left: Setup for phase-damping and depolarizing channels. Right: Setup for amplitude damping channel.

with displaced trajectories as implemented in [17, 31], where a HWP at 0° -degrees was placed in the H-polarized counter-clockwise trajectory, while another HWP at $\omega = \frac{\arccos(-\sqrt{1-\eta})}{2}$ degrees was placed in the V-polarized clockwise trajectory. The induced rotation of a single polarization (or damping) sends this light to the long arm of an unbalanced Mach-Zehnder interferometer (MZI), which then re-combines it out of coherence in a non-polarizing beam splitter (BS).

IV. RESULTS

In Fig. 3 we show the measured Shannon capacities $C^{(\alpha)}$ in all three bases for the experimental simulation of three different environments, namely a PD-channel, a D-channel, and an AD-channel. This is compared to the predicted behaviour for the real input state, which is fidelity-dependent from Eq. (12) and in detail explained in the Appendix B. See Appendix C for details on the noise propagation and Appendix D for details on the experimental raw data.

For a PD-channel the witness C_D of the channel capacity is obtained for $\alpha = z$ and can be compared with the expected theoretical value $C_D = 1$. The presented results

confirm the behavior in which phase-damping processes do not affect binary data transmission in the logical basis, but only in complementary superposition bases. The systematic offset below unity value is due to the propagation of error by replacing experimental negative values of $\epsilon_0^{(z)}$ with their absolute values. We observe that the customary constrained maximum-likelihood technique can solve the possible issue of negative reconstructed probabilities and the resulting bias in the capacity bounds.

For a D-channel the theoretical value $C_D = 1 - H(2q)$ is equally achieved by any of the three bases. Here, we find the best agreement with respect to this prediction, because small differences of the experimental state or channel from the ideal ones are rapidly compensated during the decoherence process by the simultaneous action of all three Pauli operations. Accordingly, the balance increase (decrease) in the diagonal (off-diagonal) terms of the density matrix reduces bias projections and final data dispersion.

For an AD-channel the experimental implementation requires control on multiple optical paths, where systematic error can have unbalanced contribution among different bases. Such an issue, combined with the non-linear dependency of η with respect to ω and an angular uncertainty of 0.5° -degrees, gives considerable propagated errors. The expected capacity witness $C_D = 1 - H\left(\frac{1-\sqrt{1-\eta}}{2}\right)$ is achieved equivalently by the bases x or y . Our method still succeeds in showing the general behavior and providing a sensible lower bound to the classical channel capacity.

In summary, by comparison with the theoretical predictions, the method proved to be very effective in providing a witness for the classical capacity of noisy quantum channels, even if the channel implementation has critical and systematic imperfections. Moreover, by means of the noise deconvolution, we have shown that the method is robust with respect to an imperfect probe-state preparation and provides an unbiased estimation of the theoretical lower bounds.

V. CONCLUSIONS

The experimental coincidence counts allow to directly reconstruct the sets of conditional probabilities for three different information settings. For each setting $\alpha = x, y, z$, by Eq. (13) the logical bits and the values of $\epsilon_0^{(\alpha)}$ and $\epsilon_1^{(\alpha)}$ are identified. From these values one recovers three values of C_B as in Eq. (8), which correspond to the Shannon capacity, namely the optimized mutual information for coding/decoding on the eigenstates of σ_x , σ_y , and σ_z . The highest among such three values is selected as a classical capacity witness, namely it certifies a lower bound to the ultimate classical capacity of the unknown quantum channel. For each information setting, by Eq. (7) the identified values of $\epsilon_0^{(\alpha)}$ and $\epsilon_1^{(\alpha)}$ also allow one to obtain the optimal weight for the coding of the

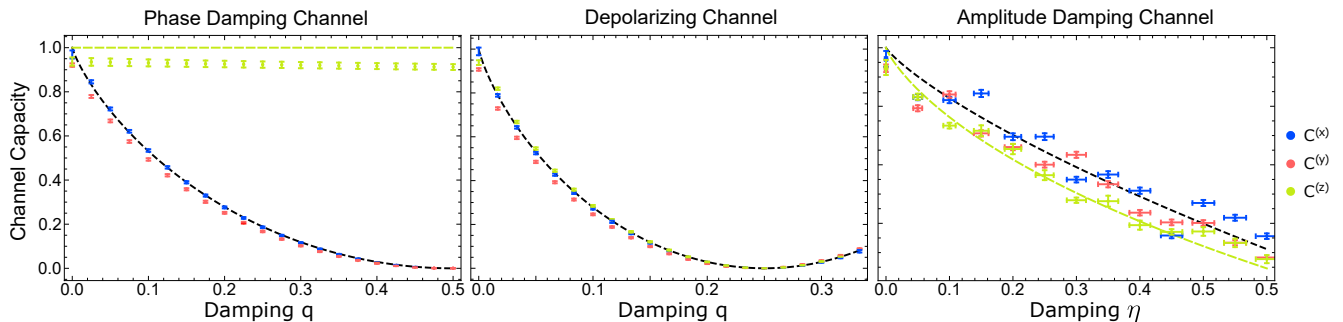


FIG. 3. **Experimental Channel Capacity.** Left: $C^{(\alpha)}$ for PD-channel. Center: $C^{(\alpha)}$ for D-channel. Right: $C^{(\alpha)}$ for AD-channel. In blue (dark grey) $C^{(x)}$, in red (grey) $C^{(y)}$, in green (light grey) $C^{(z)}$. For the predicted theoretical behavior, $C^{(z)}$ is depicted with dashed (light grey), while $C^{(x)}$ and $C^{(y)}$ with dashed black lines. All error-bars (standard deviation) are calculated from at least 8 values per point; their x -axis components originate from the propagated uncertainty in the angle of the HWPs, which dominates over other sources of error, like wrong retardance in the LC.

two logical characters.

We emphasize that our method is highly robust against experimental imperfections, even in cases where the prepared quantum state is partially mixed as the Werner state. For each of the three chosen bases, the Shannon capacity is precisely retrieved for all values of the channel parameters. Nevertheless, the most accurate values were obtained for the Pauli channels, which are effectively produced by mixing unitary operations over the input state. For mixtures of non-unitary operations as the amplitude damping channel, we find a less smooth behavior in the detected classical capacity because experimental imperfections do not act in balanced ways for both photon polarizations.

We recommend this novel method for certifying the classical capacity of quantum channels due to its high precision under different scenarios. The studied cases were not particularly designed to match with the protocol, but chosen to represent a variety of classes. No prior information is needed about the channel structure, while the measurement settings are much less demanding with respect to full process tomography.

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Appendix A: Expected theoretical results for the implemented qubit channels

For an **Amplitude Damping (AD) channel**, described by

$$\mathcal{E}^{AD}(\rho) = K_0 \rho K_0^\dagger + K_1 \rho K_1^\dagger, \quad (\text{A1})$$

where $K_0 = |0\rangle\langle 0| + \sqrt{1-\eta}|1\rangle\langle 1|$ and $K_1 = \sqrt{\eta}|0\rangle\langle 1|$, the error probabilities of the three binary channels correspond to

$$\begin{aligned} \epsilon_0^{(x)} = \epsilon_1^{(x)} = \epsilon_0^{(y)} = \epsilon_1^{(y)} &= (1 - \sqrt{1-\eta})/2; \\ \epsilon_0^{(z)} = 0; \quad \epsilon_1^{(z)} &= \eta. \end{aligned} \quad (\text{A2})$$

The detected capacity is obtained equivalently by the x or y information setting and is given by

$$C_D = C^{(x)} = C^{(y)} = 1 - H\left(\frac{1 - \sqrt{1-\eta}}{2}\right). \quad (\text{A3})$$

For any η , this result outperforms the z -setting, for which one has $C^{(z)} = C_B(0, \eta)$, according to Eq. (10). We recall that the detected capacity in Eq. (A3) is strictly lower than the Holevo capacity C_1 , which can be evaluated as [32, 33]

$$C_1 = \max_{t \in [0,1]} (H[t(1-\eta)] - H[g(\eta, t)]), \quad (\text{A4})$$

where $g(\eta, t) \equiv \frac{1}{2}[1 + \sqrt{1 - 4\eta(1-\eta)t^2}]$.

For a **Pauli (P) channel**, described by

$$\mathcal{E}^P(\rho) = q_{\mathbb{I}}\rho + q_x\sigma_x\rho\sigma_x + q_y\sigma_y\rho\sigma_y + q_z\sigma_z\rho\sigma_z, \quad (\text{A5})$$

with $q_{\mathbb{I}} = 1 - (q_x + q_y + q_z)$ and $q_x + q_y + q_z \leq 1$ (with $q_i \geq 0$), the three channels that are compared are symmetric and correspond to the error probabilities

$$\begin{aligned} \epsilon_0^{(x)} = \epsilon_1^{(x)} &= \min\{q_y + q_z, 1 - (q_y + q_z)\}; \\ \epsilon_0^{(y)} = \epsilon_1^{(y)} &= \min\{q_x + q_z, 1 - (q_x + q_z)\}; \\ \epsilon_0^{(z)} = \epsilon_1^{(z)} &= \min\{q_x + q_y, 1 - (q_x + q_y)\}. \end{aligned} \quad (\text{A6})$$

The detected capacity is then given by

$$C_D = 1 - \min\{H(q_y + q_z), H(q_x + q_z), H(q_x + q_y)\}, \quad (\text{A7})$$

where $\min\{\cdot\}$ compares the x , y , and z information settings, respectively. We recall that for Pauli channels the capacity witness C_D equals the Holevo and the classical capacity (i.e., $C = C_1 = C_D$, since the additivity hypothesis holds true for unital qubit channels [34]).

Appendix B: Probability transition matrices for the three information settings σ_z , σ_x , σ_y

From Eq. (12) the ratio $\frac{\text{Tr}[A\mathcal{E}(B)]}{\text{Tr}[\mathcal{E}(B)]}$ is given by

$$\frac{\text{Tr}[A\mathcal{E}(B)]}{\text{Tr}[\mathcal{E}(B)]} = \frac{\text{Tr}\{[A \otimes (3B^\tau - 2(1-F)\text{Tr}[B]\mathbb{I}_a)](\mathcal{E} \otimes \mathbb{I}_a)\rho_F\}}{\text{Tr}\{[\mathbb{I}_s \otimes (3B^\tau - 2(1-F)\text{Tr}[B]\mathbb{I}_a)](\mathcal{E} \otimes \mathbb{I}_a)\rho_F\}}. \quad (\text{B1})$$

When qubits are encoded in the single-photon polarization degree of freedom, the z -information setting corresponds to choice of A and B as the projectors on the basis $\{|H\rangle, |V\rangle\}$, with H (V) as the horizontal (vertical) polarization. Using Eq. (B1) the corresponding probability transition matrix is then given in terms of the measured coincidences $C(ji) = \text{Tr}[(|j\rangle\langle j|_S \otimes |i\rangle\langle i|_A)(\mathcal{E} \otimes \mathbb{I}_a)\rho_F]$ as

$$Q^{(z)}(H|H) = \frac{(1+2F)C(HH) - 2(1-F)C(HV)}{(1+2F)[C(HH) + C(VH)] - 2(1-F)[C(HV) + C(VV)]}, \quad Q^{(z)}(V|H) = 1 - Q^{(z)}(H|H), \quad (\text{B2})$$

$$Q^{(z)}(H|V) = \frac{(1+2F)C(HV) - 2(1-F)C(HH)}{(1+2F)[C(HV) + C(VV)] - 2(1-F)[C(HH) + C(VH)]}, \quad Q^{(z)}(V|V) = 1 - Q^{(z)}(H|V). \quad (\text{B3})$$

For the x -information setting A and B correspond to the projectors on the diagonal basis $|+\rangle = \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle)$ and $|-\rangle = \frac{1}{\sqrt{2}}(|H\rangle - |V\rangle)$, and so one has

$$Q^{(x)}(+|+) = \frac{(1+2F)C(++) - 2(1-F)C(+-)}{(1+2F)[C(++) + C(-+)] - 2(1-F)[C(+-) + C(--)]}, \quad Q^{(x)}(-|+) = 1 - Q^{(x)}(+|+), \quad (\text{B4})$$

$$Q^{(x)}(+|-) = \frac{(1+2F)C(+-) - 2(1-F)C(++)}{(1+2F)[C(+-) + C(--)] - 2(1-F)[C(++) + C(-+)]}, \quad Q^{(x)}(-|-) = 1 - Q^{(x)}(+|-). \quad (\text{B5})$$

Finally, for the y -information setting A and B correspond to the projectors on the circular basis $|R\rangle = \frac{1}{\sqrt{2}}(|H\rangle - i|V\rangle)$ and $|L\rangle = \frac{1}{\sqrt{2}}(|H\rangle + i|V\rangle)$, and hence

$$Q^{(y)}(R|R) = \frac{(1+2F)C(RL) - 2(1-F)C(RR)}{(1+2F)[C(RL) + C(LL)] - 2(1-F)[C(RR) + C(LR)]}, \quad Q^{(y)}(L|R) = 1 - Q^{(y)}(R|R), \quad (\text{B6})$$

$$Q^{(y)}(R|L) = \frac{(1+2F)C(RR) - 2(1-F)C(RL)}{(1+2F)[C(RR) + C(LR)] - 2(1-F)[C(RL) + C(LL)]}, \quad Q^{(y)}(L|L) = 1 - Q^{(y)}(R|L). \quad (\text{B7})$$

Notice the different symmetry in the equations for the y -coding with respect to the z and x cases, due to the presence of the transposition in Eq. (B1). For each of the above binary classical channels the transition errors $\epsilon_0^{(\alpha)}$ and $\epsilon_1^{(\alpha)}$ are identified from the conditional probabilities by Eq. (13).

According to the above equations, all expectation values are obtained just by the measurement of 12 polarization projections of the state (4 by each of the 3 observables), which makes the process efficient in terms of registration and analysis of data, given the reduced number of operations compared with standard process tomography.

Appendix C: Detection efficiencies and experimental error analysis

The overall optical efficiencies for both system and ancilla modes are given by

$$\epsilon_s = \epsilon_{Opt} \cdot \epsilon_{channel} \cdot \epsilon_{SMF}^2 \cdot \epsilon_{SPAD} \quad (\text{C1})$$

$$\epsilon_a = \epsilon_{Opt} \cdot \epsilon_{SMF} \cdot \epsilon_{SPAD}, \quad (\text{C2})$$

where the single-photon transmission efficiency of the main optical components is $\epsilon_{Opt} = 0.9$, the single-mode fiber coupling efficiency is $\epsilon_{SMF} = 0.73$ and the SPADs detection efficiency is $\epsilon_{SPAD} = 0.7$. Accordingly the two-photon optical efficiency was

$$\epsilon_{s,a} = \epsilon_s \cdot \epsilon_a \approx 0.15 \cdot \epsilon_{channel} \quad (\text{C3})$$

The channel optical efficiencies $\epsilon_{channel}$ were 0.6 for the AD-channel and 0.98 for the PD-channel and D-channel. Thus, the overall coincidence efficiencies $\epsilon_{s,a}$ are 0.09 and 0.15, respectively.

The propagated standard deviation in our data was calculated from by Monte Carlo simulations of Poissonian statistics on the photon coincidence counts. In our experiment the main sources of error were:

1. The setting of the rotation angle in the LC and HWPs.
2. The setting of the LC voltage for the precise retardance.
3. The statistical propagation due to the mixing of independent Pauli operations.

The first source of error is negligible, because all waveplates contained in the channels and used for the projective measurements and the LC were calibrated to a precision of ± 0.1 degrees. For waveplates at an angle around 0° -degrees or 45° -degrees, the dependence of counts on a small angle deviation is linear, thus the overall error is strongly dominated by the Poissonian statistics on the counts. This is not valid in the AD-channel configuration, where the interference leads to a strong non-linearity between waveplate angles and coincidence counts, governed by $\omega = \frac{\arccos(-\sqrt{1-\eta})}{2}$. Accordingly, we considered a prudent error (0.5° -degrees) on the angle, leading to a considerable increase of the error on the damping parameter.

The second source of error is negligible as well, because when performing the calibration of the LC we use horizontally polarized light, the LC at 45 degrees and a PBS. Here, by scanning the voltage on the LC we verified a non-linearly retardance, which is almost flat around either the minimum and maximum voltages used in the experiment, $V_{low} = 0.5V$ and $V_{high} = 25.0V$, respec-

tively. We confirmed a visibility of 0.989 between these two voltages, which represents a much higher value than the fidelity of the state itself, meaning that any slight imperfection of \mathbb{I} and σ operations will weakly affect the data tendency compared to the noise coming from the state generation.

The third source of error is the error propagation that comes from mixing counts coming from different Pauli operations, suitably weighted. This error is already included in our Monte Carlo simulations.

Appendix D: Experimental conditional probabilities

In this section we show the tables of all two-photon conditional probabilities $Q^{(\alpha)}(A|B)$ extracted from the experimental coincidence measurements. We show the results for the AD-channel in Table I, for the PD-channel in Table II, and for the D-channel in Table III. Due to the complement rule for orthogonal input states [for instance, $Q^{(\alpha)}(\cdot|H) = 1 - Q^{(\alpha)}(\cdot|V)$], we only show half of the total data.

η	$Q^{(z)}(H H)$	$Q^{(z)}(V H)$	$Q^{(x)}(++)$	$Q^{(x)}(- +)$	$Q^{(y)}(L L)$	$Q^{(y)}(R L)$
0	0.9983 ± 0.0011	-0.01396 ± 0.00078	1.0001 ± 0.0011	-0.0018 ± 0.0011	0.0145 ± 0.0014	0.9906 ± 0.0014
0.05	0.9642 ± 0.0017	0.0195 ± 0.0015	0.9661 ± 0.0017	0.0329 ± 0.0017	0.0480 ± 0.0019	0.9551 ± 0.0019
0.1	0.9300 ± 0.0021	0.0530 ± 0.0020	0.9322 ± 0.0021	0.0674 ± 0.0021	0.0815 ± 0.0023	0.9199 ± 0.0023
0.15	0.8958 ± 0.0024	0.0866 ± 0.0023	0.8983 ± 0.0024	0.1018 ± 0.0024	0.1149 ± 0.0025	0.8849 ± 0.0026
0.2	0.8617 ± 0.0026	0.1201 ± 0.0026	0.8646 ± 0.0027	0.1360 ± 0.0027	0.1482 ± 0.0028	0.8501 ± 0.0028
0.25	0.8275 ± 0.0028	0.1537 ± 0.0028	0.8311 ± 0.0029	0.1700 ± 0.0029	0.1814 ± 0.0030	0.8156 ± 0.0030
0.3	0.7934 ± 0.0030	0.1872 ± 0.0030	0.7976 ± 0.0031	0.2038 ± 0.0031	0.2146 ± 0.0031	0.7813 ± 0.0032
0.35	0.7593 ± 0.0032	0.2208 ± 0.0032	0.7643 ± 0.0032	0.2375 ± 0.0032	0.2477 ± 0.0033	0.7472 ± 0.0033
0.4	0.7252 ± 0.0033	0.2543 ± 0.0033	0.7310 ± 0.0034	0.2709 ± 0.0034	0.2806 ± 0.0034	0.7133 ± 0.0034
0.45	0.6911 ± 0.0034	0.2879 ± 0.0034	0.6979 ± 0.0035	0.3042 ± 0.0035	0.3135 ± 0.0035	0.6796 ± 0.0035
0.5	0.6570 ± 0.0035	0.3215 ± 0.0035	0.6649 ± 0.0035	0.3373 ± 0.0035	0.3464 ± 0.0036	0.6462 ± 0.0036
0.55	0.6230 ± 0.0035	0.3552 ± 0.0036	0.6321 ± 0.0036	0.3702 ± 0.0036	0.3791 ± 0.0036	0.6130 ± 0.0037
0.6	0.5889 ± 0.0036	0.3888 ± 0.0037	0.5993 ± 0.0037	0.4030 ± 0.0036	0.4118 ± 0.0037	0.5799 ± 0.0037

TABLE I. Experimental conditional probabilities in AD-channel.

q	$Q^{(z)}(H H)$	$Q^{(z)}(V H)$	$Q^{(x)}(+ +)$	$Q^{(x)}(- +)$	$Q^{(y)}(L L)$	$Q^{(y)}(R L)$
0	1.0005 ± 0.0015	-0.0163 ± 0.0014	1.0023 ± 0.0015	-0.0040 ± 0.0015	0.0124 ± 0.0017	0.9928 ± 0.0017
0.05	1.0008 ± 0.0015	-0.0162 ± 0.0014	0.9766 ± 0.0019	0.0214 ± 0.0019	0.0375 ± 0.0020	0.9664 ± 0.0020
0.1	1.0012 ± 0.0015	-0.0161 ± 0.0014	0.9509 ± 0.0022	0.0469 ± 0.0022	0.0625 ± 0.0023	0.9402 ± 0.0023
0.15	1.0016 ± 0.0015	-0.0160 ± 0.0014	0.9252 ± 0.0024	0.0723 ± 0.0024	0.0875 ± 0.0025	0.9140 ± 0.0025
0.2	1.0019 ± 0.0015	-0.0159 ± 0.0014	0.8996 ± 0.0026	0.0977 ± 0.0026	0.1124 ± 0.0027	0.8880 ± 0.0027
0.25	1.0023 ± 0.0015	-0.0159 ± 0.0014	0.8740 ± 0.0028	0.1232 ± 0.0028	0.1374 ± 0.0028	0.8620 ± 0.0029
0.3	1.0027 ± 0.0015	-0.0158 ± 0.0014	0.8484 ± 0.0029	0.1485 ± 0.0029	0.1623 ± 0.0030	0.8361 ± 0.0030
0.35	1.0030 ± 0.0015	-0.0157 ± 0.0014	0.8228 ± 0.0030	0.1739 ± 0.0030	0.1871 ± 0.0031	0.8103 ± 0.0031
0.4	1.0034 ± 0.0015	-0.0156 ± 0.0014	0.7973 ± 0.0032	0.1993 ± 0.0032	0.2120 ± 0.0032	0.7846 ± 0.0033
0.45	1.0038 ± 0.0015	-0.0155 ± 0.0014	0.7719 ± 0.0033	0.2246 ± 0.0033	0.2368 ± 0.0033	0.7590 ± 0.0034
0.5	1.0042 ± 0.0015	-0.0154 ± 0.0014	0.7464 ± 0.0034	0.2499 ± 0.0034	0.2616 ± 0.0034	0.7334 ± 0.0034
0.55	1.0045 ± 0.0015	-0.0153 ± 0.0014	0.7210 ± 0.0034	0.2752 ± 0.0034	0.2864 ± 0.0035	0.7080 ± 0.0035
0.6	1.0049 ± 0.0015	-0.0152 ± 0.0014	0.6956 ± 0.0035	0.3005 ± 0.0035	0.3111 ± 0.0035	0.6826 ± 0.0036
0.65	1.0052 ± 0.0015	-0.0151 ± 0.0014	0.6702 ± 0.0036	0.3257 ± 0.0036	0.3359 ± 0.0036	0.6573 ± 0.0036
0.7	1.0056 ± 0.0015	-0.0150 ± 0.0014	0.6449 ± 0.0036	0.3510 ± 0.0036	0.3605 ± 0.0036	0.6321 ± 0.0037
0.75	1.0060 ± 0.0015	-0.0150 ± 0.0014	0.6196 ± 0.0037	0.3762 ± 0.0037	0.3852 ± 0.0037	0.6070 ± 0.0037
0.8	1.0063 ± 0.0015	-0.0149 ± 0.0014	0.5944 ± 0.0037	0.4013 ± 0.0037	0.4098 ± 0.0037	0.5819 ± 0.0037
0.85	1.0067 ± 0.0015	-0.0148 ± 0.0014	0.5691 ± 0.0037	0.4265 ± 0.0037	0.4345 ± 0.0037	0.5570 ± 0.0037
0.9	1.0071 ± 0.0015	-0.0147 ± 0.0014	0.5439 ± 0.0037	0.4517 ± 0.0038	0.4590 ± 0.0037	0.5321 ± 0.0038
0.95	1.0075 ± 0.0014	-0.0146 ± 0.0014	0.5187 ± 0.0037	0.4768 ± 0.0038	0.4836 ± 0.0037	0.5073 ± 0.0038
1	1.0079 ± 0.0014	-0.0145 ± 0.0014	0.4936 ± 0.0037	0.5019 ± 0.0038	0.5081 ± 0.0037	0.4826 ± 0.0038

TABLE II. Experimental conditional probabilities in PD-channel, from the family of P-channels.

q	$Q^{(z)}(H H)$	$Q^{(z)}(V H)$	$Q^{(x)}(+ +)$	$Q^{(x)}(- +)$	$Q^{(y)}(L L)$	$Q^{(y)}(R L)$
0	0.9983 ± 0.0011	-0.01396 ± 0.00078	1.0001 ± 0.0011	-0.0018 ± 0.0011	0.0145 ± 0.0014	0.9906 ± 0.0014
0.017	0.9642 ± 0.0017	0.0195 ± 0.0015	0.9661 ± 0.0017	0.0329 ± 0.0017	0.0480 ± 0.0019	0.9551 ± 0.0019
0.033	0.9300 ± 0.0021	0.0530 ± 0.0020	0.9322 ± 0.0021	0.0674 ± 0.0021	0.0815 ± 0.0023	0.9199 ± 0.0023
0.05	0.8958 ± 0.0024	0.0866 ± 0.0023	0.8983 ± 0.0024	0.1018 ± 0.0024	0.1149 ± 0.0025	0.8849 ± 0.0026
0.067	0.8617 ± 0.0026	0.1201 ± 0.0026	0.8646 ± 0.0027	0.1360 ± 0.0027	0.1482 ± 0.0028	0.8501 ± 0.0028
0.083	0.8275 ± 0.0028	0.1537 ± 0.0028	0.8311 ± 0.0029	0.1700 ± 0.0029	0.1814 ± 0.0030	0.8156 ± 0.0030
0.1	0.7934 ± 0.0030	0.1872 ± 0.0030	0.7976 ± 0.0031	0.2038 ± 0.0031	0.2146 ± 0.0031	0.7813 ± 0.0032
0.117	0.7593 ± 0.0032	0.2208 ± 0.0032	0.7643 ± 0.0032	0.2375 ± 0.0032	0.2477 ± 0.0033	0.7472 ± 0.0033
0.133	0.7252 ± 0.0033	0.2543 ± 0.0033	0.7310 ± 0.0034	0.2709 ± 0.0034	0.2806 ± 0.0034	0.7133 ± 0.0034
0.15	0.6911 ± 0.0034	0.2879 ± 0.0034	0.6979 ± 0.0035	0.3042 ± 0.0035	0.3135 ± 0.0035	0.6796 ± 0.0035
0.167	0.6570 ± 0.0035	0.3215 ± 0.0035	0.6649 ± 0.0035	0.3373 ± 0.0035	0.3464 ± 0.0036	0.6462 ± 0.0036
0.183	0.6230 ± 0.0035	0.3552 ± 0.0036	0.6321 ± 0.0036	0.3702 ± 0.0036	0.3791 ± 0.0036	0.6130 ± 0.0037
0.2	0.5889 ± 0.0036	0.3888 ± 0.0037	0.5993 ± 0.0037	0.4030 ± 0.0036	0.4118 ± 0.0037	0.5799 ± 0.0037
0.217	0.5549 ± 0.0036	0.4224 ± 0.0037	0.5666 ± 0.0037	0.4356 ± 0.0037	0.4444 ± 0.0037	0.5472 ± 0.0037
0.233	0.5209 ± 0.0036	0.4561 ± 0.0037	0.5341 ± 0.0037	0.4680 ± 0.0037	0.4769 ± 0.0037	0.5145 ± 0.0037
0.25	0.4869 ± 0.0036	0.4898 ± 0.0038	0.5016 ± 0.0037	0.5003 ± 0.0037	0.5093 ± 0.0037	0.4822 ± 0.0037
0.267	0.4529 ± 0.0036	0.5234 ± 0.0038	0.4693 ± 0.0037	0.5324 ± 0.0037	0.5417 ± 0.0037	0.4500 ± 0.0037
0.283	0.4189 ± 0.0036	0.5571 ± 0.0037	0.4371 ± 0.0037	0.5643 ± 0.0037	0.5740 ± 0.0037	0.4180 ± 0.0037
0.3	0.3849 ± 0.0036	0.5908 ± 0.0037	0.4050 ± 0.0036	0.5961 ± 0.0036	0.6062 ± 0.0036	0.3862 ± 0.0036
0.317	0.3509 ± 0.0035	0.6245 ± 0.0037	0.3730 ± 0.0036	0.6276 ± 0.0036	0.6383 ± 0.0036	0.3546 ± 0.0036
0.333	0.3169 ± 0.0034	0.6583 ± 0.0036	0.3411 ± 0.0035	0.6591 ± 0.0035	0.6703 ± 0.0035	0.3232 ± 0.0035

TABLE III. Experimental conditional probabilities in D-channel, from the family of P-channels.

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