Check for updates

Received: 19 August 2019 | Revised: 8 October 2019 | Accepted: 16 October 2019

DOI: 10.1002/mana.201900361

MATHEMATISCHE NACHRICHTEN

ORIGINAL PAPER

On a transmission problem for equation and dynamic boundary condition of Cahn–Hilliard type with nonsmooth potentials

Pierluigi Colli[1,2] | **Takeshi Fukao**[3] | **Hao Wu**[4,5]

[1] Dipartimento di Matematica, Università degli Studi di Pavia, Via Ferrata 5, Pavia 27100, Italy

[2] IMATI – C.N.R. Pavia, Via Ferrata 5, Pavia 27100, Italy

[3] Department of Mathematics, Faculty of Education, Kyoto University of Education, 1 Fujinomori, Fukakusa, Fushimi-ku, Kyoto 612-8522, Japan

[4] School of Mathematical Sciences and Shanghai Key Laboratory for Contemporary Applied Mathematics, Fudan University, Han Dan Road 220, Shanghai 200433, China

[5] Key Laboratory of Mathematics for Nonlinear Science (Fudan University), Ministry of Education, Han Dan Road 220, Shanghai 200433, China

Correspondence
Takeshi Fukao, Department of Mathematics, Faculty of Education, Kyoto University of Education, 1 Fujinomori, Fukakusa, Fushimi-ku, Kyoto 612-8522, Japan.
Email: fukao@kyokyo-u.ac.jp

Funding information
MIUR; University of Pavia; NNSFC, Grant/Award Number: 11631011; Fudan University, Grant/Award Numbers: 17K05321, JSPS KAKENHI

Abstract

This paper is concerned with well-posedness of the Cahn–Hilliard equation subject to a class of new dynamic boundary conditions. The system was recently derived in Liu–Wu (Arch. Ration. Mech. Anal. **233** (2019), 167–247) via an energetic variational approach and it naturally fulfils three physical constraints such as mass conservation, energy dissipation and force balance. The target problem examined in this paper can be viewed as a transmission problem that consists of Cahn–Hilliard type equations both in the bulk and on the boundary. In our approach, we are able to deal with a general class of potentials with double-well structure, including the physically relevant logarithmic potential and the nonsmooth double-obstacle potential. Existence, uniqueness and continuous dependence of global weak solutions are established. The proof is based on a novel time-discretization scheme for the approximation of the continuous problem. Besides, a regularity result is shown with the aim of obtaining a strong solution to the system.

KEYWORDS
Cahn–Hilliard system, dynamic boundary condition, nonsmooth potential, transmission problem

MSC (2010)
35K25, 35K61, 74N20, 80A22

1 | INTRODUCTION

In this paper, we consider the following initial boundary value problem for a Cahn–Hilliard equation subject to a dynamic boundary condition that is also of Cahn–Hilliard type. Let $0 < T < \infty$ be some fixed time and let $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, be a bounded domain with smooth boundary $\Gamma := \partial\Omega$. We aim to find four unknown functions $\phi, \mu : Q := (0, T) \times \Omega \to \mathbb{R}$

This is an open access article under the terms of the Creative Commons Attribution License, which permits use, distribution and reproduction in any medium, provided the original work is properly cited.
© 2020 The Authors. *Mathematische Nachrichten* published by Wiley-VCH GmbH

and $\psi, w : \Sigma := (0,T)\times\Gamma \to \mathbb{R}$ satisfying

$$\partial_t \phi - \Delta \mu = 0 \qquad \text{in } Q, \tag{1.1}$$

$$\mu = -\Delta \phi + \mathcal{W}'(\phi) \qquad \text{in } Q, \tag{1.2}$$

$$\partial_\nu \mu = 0 \qquad \text{on } \Sigma, \tag{1.3}$$

$$\phi|_\Gamma = \psi \qquad \text{on } \Sigma, \tag{1.4}$$

$$\partial_t \psi - \Delta_\Gamma w = 0 \qquad \text{on } \Sigma, \tag{1.5}$$

$$w = \partial_\nu \phi - \Delta_\Gamma \psi + \mathcal{W}_\Gamma'(\psi) \qquad \text{on } \Sigma, \tag{1.6}$$

where ∂_t and ∂_ν denote the partial time derivative and the outward normal derivative on Γ, respectively; Δ denotes the Laplacian and Δ_Γ denotes the Laplace–Beltrami operator on Γ (see, e.g., [22, Ch. 3]); $\phi|_\Gamma$ standards for the trace of ϕ on the boundary Γ. In view of (1.4), system (1.1)–(1.6) is a sort of transmission problem between the Cahn–Hilliard equation in the bulk Ω and the Cahn–Hilliard equation on the boundary Γ. The nonlinear functions $\mathcal{W}$ and $\mathcal{W}_\Gamma$ are usually referred as the double-well potentials, with two minima and a local unstable maximum in between. Typical and physically significant examples of such potentials are the so-called *classical potential*, the *logarithmic potential*, and the *double obstacle potential*, which are given, in this order, by

$$\mathcal{W}_{reg}(r) := \frac{1}{4}\left(r^2-1\right)^2, \quad r \in \mathbb{R}, \tag{1.7}$$

$$\mathcal{W}_{log}(r) := (1+r)\ln(1+r) + (1-r)\ln(1-r) - c_1 r^2, \quad r \in (-1,1), \tag{1.8}$$

$$\mathcal{W}_{2obs}(r) := c_2\left(1-r^2\right) \quad \text{if } |r| \le 1 \quad \text{and} \quad \mathcal{W}_{2obs}(r) := +\infty \quad \text{if } |r| > 1, \tag{1.9}$$

where the constants in (1.8) and (1.9) satisfy $c_1 > 1$ and $c_2 > 0$, so that $\mathcal{W}_{log}$ and $\mathcal{W}_{2obs}$ are nonconvex. The nonlinear terms $\mathcal{W}'(\phi)$ in (1.2) and $\mathcal{W}_\Gamma'(\psi)$ in (1.6) characterize the dynamics of the Cahn–Hilliard system. In cases like (1.7) and (1.8), $\mathcal{W}'$ and $\mathcal{W}_\Gamma'$ denote simply the derivatives of the related potentials; while non-smooth potentials like (1.9) are considered, then $\mathcal{W}'$ and $\mathcal{W}_\Gamma'$ denote the subdifferential of the convex part plus the derivative of the smooth concave contribution, i.e., for (1.9) it is

$$s \in \mathcal{W}_{2obs}'(r) \quad \text{if } r \in [-1,1], \quad s + 2c_2 r \begin{cases} \in (-\infty, 0] & \text{if } r = -1, \\ = 0 & \text{if } r \in (-1,1), \\ \in [0, +\infty) & \text{if } r = 1. \end{cases}$$

Of course, in this case one should replace the equalities in (1.2) and (1.6) by inclusions. In this paper, we are able to handle completely general potentials $\mathcal{W}$ and $\mathcal{W}_\Gamma$ including all the three cases (1.7)–(1.9) mentioned above.

The system (1.1)–(1.6) was first derived by Liu and Wu [29] in a more general form (see also [33]) on the basis of an energetic variational approach. It describes effective short-range interactions between the binary mixture and the solid wall (boundary), furthermore, it has the feature that the related model naturally fulfills important physical constraints such as conservation of mass, dissipation of energy and force balance relations. In its current formulation, we see that Equations (1.1) and (1.2) yield a Cahn–Hilliard system subject to a no-flux boundary condition (1.3) together with a non-homogeneous Dirichlet boundary condition (1.4), while the dynamic boundary condition (1.5) and Equation (1.6) provide an evolution system of Cahn–Hilliard type on the boundary Γ. These two Cahn–Hilliard systems in the bulk and on the boundary are coupled through the trace condition (1.4) and the normal derivative term $\partial_\nu \phi$ in (1.6).

The total energy functional for system (1.1)–(1.6) given by

$$E(\phi,\psi) := \int_\Omega \left(\frac{1}{2}|\nabla\phi|^2 + \mathcal{W}(\phi)\right) dx + \int_\Gamma \left(\frac{1}{2}|\nabla_\Gamma \psi|^2 + \mathcal{W}_\Gamma(\psi)\right) d\Gamma \tag{1.10}$$

is decreasing in time (see [29]) and furthermore, the system (1.1)–(1.6) can be interpreted as a gradient flow of $E(\phi,\psi)$ in a suitable dual space (see [18]). In light of (1.1), (1.3) and (1.5), we easily deduce that the following properties on mass

15222616, 2020, 11, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/mana.201900361 by Cnr Roma, Wiley Online Library on [27/02/2025]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

conservation:

$$\int_\Omega \phi(t)\,dx = \int_\Omega \phi(0)\,dx, \quad \int_\Gamma \psi(t)d\Gamma = \int_\Gamma \psi(0)\,d\Gamma \quad \text{for all } t \in [0,T]. \tag{1.11}$$

In this paper, we study the well-posedness of the system (1.1)–(1.6) for a weak solution subject to the following initial data

$$\phi(0) = \phi_0 \quad \text{in } \Omega, \quad \psi(0) = \psi_0 \quad \text{on } \Gamma. \tag{1.12}$$

Moreover, we also establish a regularity theory in order to obtain a strong solution. In particular, we are able to treat the initial value problem for system (1.1)–(1.6) in a wider class of nonlinearities $\mathcal{W}$ and $\mathcal{W}_\Gamma$. Indeed, in the previous contributions, the well-posedness was investigated only in the case of smooth potentials like (1.7) (cf. [29, Rem. 3.2] and [18, Rem. 2.1]): this is the point of emphasis of our present paper.

We would like to mention some related problems in the literature. In 2011, Goldstein, Miranville and Schimperna [21] studied a different type of transmission problem between the Cahn–Hilliard system in the bulk and on the boundary with non-permeable walls (cf. a previous work Gal [15] for the case with permeable walls). Their system can be derived from the same energy functional (1.10) by a variational method, however, the corresponding boundary conditions turn out to be different from (1.3) and (1.5). This also leads to a different property on the mass conservation comparing with (1.11) such that the total (bulk plus boundary) mass is conserved. We refer to [29] for more detailed information on the comparison between these models. In addition, we mention the contributions [6, 8, 15, 21] related to the well-posedness, [9, 12, 13, 16, 17, 20] for the study of long time behavior and the optimal control problems, [7, 14] for numerical analysis and [24] for the maximal regularity theory. Comparing the large number of known results on the previous model [15, 21], we are only aware of the recent papers [18, 29] that analyze the well-posedness of the system (1.1)–(1.6) with (1.12).

Let us now describe the contents of the present paper. In Section 2, we state the main well-posedness result for global weak solutions. We consider the problem within a general framework by setting $\mathcal{W}' := \beta + \pi$ and $\mathcal{W}'_\Gamma := \beta_\Gamma + \pi_\Gamma$, where β and β_Γ are maximal monotone graphs with $0 \in \beta(0)$ and $0 \in \beta_\Gamma(0)$, while π and π_Γ yield the anti-monotone terms that are Lipschitz continuous functions. The main theorems are concerned with the existence of a global weak solution (Theorem 2.3) and the continuous dependence on the given data (Theorem 2.4), which implies the uniqueness.

In Section 3, we study the time-discrete approximate problem for (1.1)–(1.6) with (1.12). We start from the viscous Cahn–Hilliard system by inserting two additional terms, $\tau\partial_t\phi$ and $\sigma\partial_t\psi$ in the right hand sides of (1.2) and (1.6), respectively, with the parameters $\tau, \sigma > 0$. Moreover, we take the Yosida approximations β_ε and $\beta_{\Gamma,\varepsilon}$ in place of the maximal monotone graphs β and β_Γ and in terms of the parameter $\varepsilon > 0$. Then we apply a time discretization scheme using the approach in [10, 11]. We can show the existence of a discrete solution taking advantage of the general maximal monotone theory. After that, we proceed to derive a sequence of uniform estimates. For this purpose, we apply the technique of [5] in order to treat different potentials in the bulk and on the boundary. In the subsequent iterations, we prove the existence results by performing the limiting procedures, with respect to the time step first, then as $\varepsilon \to 0$, finally taking the limit as either $\tau \to 0$ or $\sigma \to 0$, or both $\tau, \sigma \to 0$, in order to obtain a partially viscous Cahn–Hilliard system or a pure Cahn–Hilliard system in the limit. The continuous dependence result is then proved by using the energy method.

In Section 4, we discuss the regularity for weak solutions. Returning to the time discrete approximation, we gain some necessary higher order estimates at all the different levels up to the final limits. Thus, we are able to obtain enough regularity as to guarantee a strong solution for the pure Cahn–Hilliard system as well (see Theorem 4.1).

2 | MAIN RESULTS

We now formulate our target problem (1.1)–(1.6) and (1.12) as follows:

$$\partial_t\phi - \Delta\mu = 0 \qquad \text{a.e. in } Q, \tag{2.1}$$

$$\mu = -\Delta\phi + \xi + \pi(\phi) - f, \quad \xi \in \beta(\phi) \qquad \text{a.e. in } Q, \tag{2.2}$$

$$\partial_\nu\mu = 0 \qquad \text{a.e. on } \Sigma, \tag{2.3}$$

$$\phi|_\Gamma = \psi \qquad \text{a.e. on } \Sigma, \tag{2.4}$$

15222616, 2020, 11, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/mana.201900361 by Cnr Roma, Wiley Online Library on [27/02/2025]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

$$\partial_t \psi - \Delta_\Gamma w = 0 \qquad \text{a.e. on } \Sigma, \tag{2.5}$$

$$w = \partial_\nu \phi - \Delta_\Gamma \psi + \zeta + \pi_\Gamma(\psi) - g, \quad \zeta \in \beta_\Gamma(\psi) \quad \text{a.e. on } \Sigma, \tag{2.6}$$

$$\phi(0) = \phi_0 \quad \text{a.e. in } \Omega, \quad \psi(0) = \psi_0 \quad \text{a.e. on } \Gamma, \tag{2.7}$$

where $f : Q \to \mathbb{R}$, $g : \Sigma \to \mathbb{R}$, $\phi_0 : \Omega \to \mathbb{R}$, $\psi_0 : \Gamma \to \mathbb{R}$ are given functions. Moreover, β stands for the subdifferential of the convex part $\widehat{\beta}$ and π stands for the derivative of the concave perturbation $\widehat{\pi}$ of a double well potential $\mathcal{W}(r) = \widehat{\beta}(r) + \widehat{\pi}(r)$. The same setting holds for β_Γ and π_Γ. Typical examples of β, π are given by (cf. (1.7)–(1.9)):

- $\beta(r) = r^3$, $\pi(r) = -r$, $r \in \mathbb{R}$, for the prototype potential $\mathcal{W}_{reg}(r)$;
- $\beta(r) = \ln((1+r)/(1-r))$, $\pi(r) = -2c_1 r$, with $r \in (-1,1)$ for the logarithmic potential $\mathcal{W}_{log}(r)$;
- $\beta(r) = \partial I_{[-1,1]}(r)$, $\pi(r) = -2c_2 r$, with $r \in [-1,1]$, for the nonsmooth potential $\mathcal{W}_{2obs}(r)$.

Same considerations apply to β_Γ, π_Γ and $\mathcal{W}_\Gamma$. Since the bulk and boundary potentials are allowed to be different, in order to handle the nontrivial bulk-boundary interaction of the transmission problem, an assumption for the relationship between β and β_Γ will be needed. We shall present it later.

Hereafter, we use the spaces

$$H := L^2(\Omega), \quad H_\Gamma := L^2(\Gamma), \quad V := H^1(\Omega), \quad V_\Gamma := H^1(\Gamma)$$

with their dual spaces V^* and V_Γ^* of V and V_Γ, respectively; and

$$W := \left\{ z \in H^2(\Omega) : \partial_\nu z = 0 \text{ a.e. on } \Gamma \right\}$$

equipped with the usual norms and inner products, denote them by $|\cdot|_H$ and $(\cdot,\cdot)_H$, and so on.

Now, we define the weak solution of problem (2.1)–(2.7):

Definition 2.1. The sextuplet $(\phi, \mu, \xi, \psi, w, \zeta)$ is called a *weak solution* of problem (2.1)–(2.7), if

$$\phi \in H^1(0,T;V^*) \cap L^\infty(0,T;V) \cap L^2\big(0,T;H^2(\Omega)\big),$$

$$\mu \in L^2(0,T;V), \quad \xi \in L^2(0,T;H),$$

$$\psi \in H^1\big(0,T;V_\Gamma^*\big) \cap L^\infty\big(0,T;V_\Gamma\big) \cap L^2\big(0,T;H^2(\Gamma)\big),$$

$$w \in L^2\big(0,T;V_\Gamma\big), \quad \zeta \in L^2\big(0,T;H_\Gamma\big)$$

and $\phi, \mu, \xi, \psi, w, \zeta$ satisfy

$$\langle \partial_t \phi, z \rangle_{V^*,V} + \int_\Omega \nabla \mu \cdot \nabla z \, dx = 0 \quad \text{for all } z \in V, \qquad \text{a.e. in } (0,T), \tag{2.8}$$

$$\mu = -\Delta \phi + \xi + \pi(\phi) - f, \quad \xi \in \beta(\phi) \qquad \text{a.e. in } Q, \tag{2.9}$$

$$\phi|_\Gamma = \psi \qquad \text{a.e. on } \Sigma, \tag{2.10}$$

$$\langle \partial_t \psi, z_\Gamma \rangle_{V_\Gamma^*,V_\Gamma} + \int_\Gamma \nabla_\Gamma w \cdot \nabla_\Gamma z_\Gamma \, d\Gamma = 0 \quad \text{for all } z_\Gamma \in V_\Gamma, \qquad \text{a.e. in } (0,T), \tag{2.11}$$

$$w = \partial_\nu \phi - \Delta_\Gamma \psi + \zeta + \pi_\Gamma(\psi) - g, \quad \zeta \in \beta_\Gamma(\psi), \qquad \text{a.e. on } \Sigma, \tag{2.12}$$

$$\phi(0) = \phi_0 \quad \text{a.e. in } \Omega, \quad \psi(0) = \psi_0 \quad \text{a.e. on } \Gamma. \tag{2.13}$$

We note that, due to the lack of the regularities of time derivatives, the equations (2.1) and (2.5) are replaced by the variational formulations (2.8) and (2.11), respectively. Moreover, the boundary condition (2.3) is hidden in the weak form (2.8).

Next, we define the strong solution of problem (2.1)–(2.7).

15222616, 2020, 11, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/mana.201900361 by Cnr Roma, Wiley Online Library on [27/02/2025]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Definition 2.2. The sextuplet $(\phi, \mu, \xi, \psi, w, \zeta)$ is called a *strong solution* of problem (2.1)–(2.7) if

$$\phi \in W^{1,\infty}(0,T;V^*) \cap H^1(0,T;V) \cap L^\infty\big(0,T;H^2(\Omega)\big),$$
$$\mu \in L^\infty(0,T;V) \cap L^2\big(0,T;W \cap H^3(\Omega)\big), \quad \xi \in L^\infty(0,T;H),$$
$$\psi \in W^{1,\infty}\big(0,T;V_\Gamma^*\big) \cap H^1\big(0,T;V_\Gamma\big) \cap L^\infty\big(0,T;H^2(\Gamma)\big),$$
$$w \in L^\infty\big(0,T;V_\Gamma\big) \cap L^2\big(0,T;H^3(\Gamma)\big), \quad \zeta \in L^\infty\big(0,T;H_\Gamma\big)$$

and they satisfy (2.1)–(2.7).

Before we state our main theorems, we recall the structure of mass conservation of problem (2.1)–(2.7). Taking $z = 1$ in (2.8) and integrating from 0 to t with the help of (2.13), we obtain the first equality in (1.11). Analogously, from (2.11) and (2.13) we obtain the second condition in (1.11). Therefore, it is useful to define the following mean value functions:

$$m_\Omega(z) := \frac{1}{|\Omega|}\int_\Omega z\,dx, \quad |\Omega| := \int_\Omega 1\,dx, \tag{2.14}$$

$$m_\Gamma(z_\Gamma) := \frac{1}{|\Gamma|}\int_\Gamma z_\Gamma\,d\Gamma, \quad |\Gamma| := \int_\Gamma 1\,d\Gamma, \tag{2.15}$$

for any $z \in L^1(\Omega)$ and $z_\Gamma \in L^1(\Gamma)$.

Throughout this paper, we make the following assumptions:

(A1) β, β_Γ are maximal monotone graphs in $\mathbb{R}\times\mathbb{R}$, that is, they are the subdifferentials

$$\beta = \partial\widehat{\beta}, \quad \beta_\Gamma = \partial\widehat{\beta}_\Gamma$$

of some proper lower semicontinuous and convex functions $\widehat{\beta}$ and $\widehat{\beta}_\Gamma : \mathbb{R} \to [0,\infty]$ satisfying $\widehat{\beta}(0) = \widehat{\beta}_\Gamma(0) = 0$ with the corresponding effective domains denoted by $D(\beta)$ and $D(\beta_\Gamma)$, respectively;

(A2) $\phi_0 \in V$, $\widehat{\beta}(\phi_0) \in L^1(\Omega)$, $\psi_0 \in V_\Gamma$, $\widehat{\beta}_\Gamma(\psi_0) \in L^1(\Gamma)$, and $\phi_0|_\Gamma = \psi_0$. Moreover, $m_0 := m_\Omega(\phi_0) \in \operatorname{int} D(\beta)$, $m_{\Gamma 0} := m_\Gamma(\psi_0) \in \operatorname{int} D(\beta_\Gamma)$;

(A3) $f \in L^2(0,T;V)$, $g \in L^2\big(0,T;V_\Gamma\big)$;

(A4) $\pi, \pi_\Gamma : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous functions with Lipschitz constants L and L_Γ, respectively;

(A5) $D\big(\beta_\Gamma\big) \subseteq D(\beta)$ and there exist positive constants $\varrho, c_0 > 0$ such that

$$\big|\beta^\circ(r)\big| \le \varrho\big|\beta_\Gamma^\circ(r)\big| + c_0 \quad \text{for all } r \in D\big(\beta_\Gamma\big), \tag{2.16}$$

where β° and β_Γ° denote the minimal sections of β and β_Γ.

The assumption (A1) implies that $0 \in \beta(0)$ and $0 \in \beta_\Gamma(0)$. Moreover, the minimal section β° of β is defined by $\beta^\circ(r) := \big\{r^* \in \beta(r) : |r^*| = \min_{s\in\beta(r)}|s|\big\}$ and same definition applies to β_Γ° in (A5). These assumptions are the same as in [5, 8], in particular, the compatibility condition (A5) is essential to treat different potentials β in the bulk and β_Γ on the boundary. Of course, if one chooses $\beta = \beta_\Gamma$ to be the same potential, then (A5) holds automatically.

Our first result is related to the existence of global weak solutions.

Theorem 2.3. *Under the assumptions (A1)–(A5), there exists a global weak solution of problem (2.1)–(2.7) in the sense of Definition 2.1.*

The existence of strong solutions will be discussed in Section 4 (see Theorem 4.1).

Our second result is the continuous dependence on the initial data and external sources, which immediately yields the uniqueness of weak solutions:

15222616, 2020, 11, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/mana.201900361 by Cnr Roma, Wiley Online Library on [27/02/2025]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Theorem 2.4. *Assume that (A1) and (A4) hold. Moreover, let* $f^{(1)}, f^{(2)} \in L^2(0,T;V^*)$, $g^{(1)}, g^{(2)} \in L^2\big(0,T;\ V_\Gamma^*\big)$, $\phi_0^{(1)}, \phi_0^{(2)} \in V^*$, $\psi_0^{(1)}, \psi_0^{(2)} \in V_\Gamma^*$ *and*

$$\big\langle \phi_0^{(1)}, 1 \big\rangle_{V^*,V} = \big\langle \phi_0^{(2)}, 1 \big\rangle_{V^*,V} = m_0|\Omega|, \tag{2.17}$$

$$\big\langle \psi_0^{(1)}, 1 \big\rangle_{V_\Gamma^*,V_\Gamma} = \big\langle \psi_0^{(2)}, 1 \big\rangle_{V_\Gamma^*,V_\Gamma} = m_{\Gamma 0}|\Gamma|. \tag{2.18}$$

Let sextuplets of functions $\big(\phi^{(i)}, \mu^{(i)}, \xi^{(i)}, \psi^{(i)}, w^{(i)}, \zeta^{(i)}\big)$ *be weak solutions of problem (2.1)–(2.7) corresponding to the given data* $f^{(i)}, g^{(i)}, \phi_0^{(i)}$ *and* $\psi_0^{(i)}$ *for* $i = 1, 2$. *Then, there exists a positive constant* $C > 0$, *depending on* L, L_Γ *and* T, *such that*

$$\begin{aligned}
&\big|\phi^{(1)} - \phi^{(2)}\big|_{C([0,T];V^*)} + \big|\psi^{(1)} - \psi^{(2)}\big|_{C([0,T];V_\Gamma^*)} + \big|\phi^{(1)} - \phi^{(2)}\big|_{L^2(0,T;V)} + \big|\psi^{(1)} - \psi^{(2)}\big|_{L^2(0,T;V_\Gamma)} \\
&\le C\left(\big|\phi_0^{(1)} - \phi_0^{(2)}\big|_{V^*} + \big|\psi_0^{(1)} - \psi_0^{(2)}\big|_{V_\Gamma^*} + \big|f^{(1)} - f^{(2)}\big|_{L^2(0,T;V^*)} + \big|g^{(1)} - g^{(2)}\big|_{L^2(0,T;V_\Gamma^*)} \right).
\end{aligned}$$

In order to prove Theorem 2.3, we quote the abstract framework as in [25, 26] and we also prepare the following function spaces:

$$V_0 := \big\{ z \in V : m_\Omega(z) = 0 \big\}, \qquad V_{0*} := \big\{ z^* \in V^* : \langle z^*, 1 \rangle_{V^*,V} = 0 \big\},$$
$$V_{\Gamma,0} := \big\{ z_\Gamma \in V_\Gamma : m_\Gamma(z_\Gamma) = 0 \big\}, \quad V_{\Gamma,0*} := \big\{ z_\Gamma^* \in V_\Gamma^* : \langle z_\Gamma^*, 1 \rangle_{V_\Gamma^*,V_\Gamma} = 0 \big\}.$$

From the Poincaré–Wirtinger inequalities (see, e.g., [23]), we see that there exists a positive constant C_{P} such that

$$|z|_V^2 \le C_{\mathrm{P}} |z|_{V_0}^2, \qquad |z|_{V_0} := \left(\int_\Omega |\nabla z|^2 \, dx \right)^{1/2} \quad \text{for all } z \in V_0,$$

$$|z_\Gamma|_{V_\Gamma}^2 \le C_{\mathrm{P}} |z_\Gamma|_{V_{\Gamma,0}}^2, \quad |z_\Gamma|_{V_{\Gamma,0}} := \left(\int_\Gamma |\nabla_\Gamma z_\Gamma|^2 \, d\Gamma \right)^{1/2} \quad \text{for all } z_\Gamma \in V_{0,\Gamma}.$$

Then, based on the Lax–Milgram theorem, we introduce the operator $\mathcal{N}_\Omega : V_{0*} \to V_0$ by: $u = \mathcal{N}_\Omega v$ if and only if $m_\Omega(u) = 0$ and

$$\int_\Omega \nabla u \cdot \nabla z \, dx = \langle v, z \rangle_{V^*,V} \quad \text{for all } z \in V; \tag{2.19}$$

analogously, we define $\mathcal{N}_\Gamma : V_{\Gamma,0*} \to V_{\Gamma,0}$ by: $u_\Gamma = \mathcal{N}_\Gamma v_\Gamma$ if and only if $m_\Gamma(u_\Gamma) = 0$ and

$$\int_\Gamma \nabla_\Gamma u_\Gamma \cdot \nabla_\Gamma z_\Gamma \, d\Gamma = \langle v_\Gamma, z_\Gamma \rangle_{V_\Gamma^*,V_\Gamma} \quad \text{for all } z_\Gamma \in V_\Gamma. \tag{2.20}$$

By virtue of these definitions, we can also introduce the norms

$$|z|_{V_{0*}} := \left(\int_\Omega |\nabla \mathcal{N}_\Omega z|^2 \, dx \right)^{1/2} \quad \text{for all } z \in V_{0*},$$

equivalent to the usual norm $|\cdot|_{V^*}$, for the elements of V_{0*}; and

$$|z_\Gamma|_{V_{\Gamma,0*}} := \left(\int_\Gamma |\nabla_\Gamma \mathcal{N}_\Gamma z_\Gamma|^2 \, d\Gamma \right)^{1/2} \quad \text{for all } z_\Gamma \in V_{\Gamma,0*},$$

equivalent to the usual norm $|\cdot|_{V_\Gamma^*}$, for the elements of $V_{\Gamma,0*}$, respectively.

15222616, 2020, 11, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/mana.201900361 by Cnr Roma, Wiley Online Library on [27/02/2025]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

3 | WELL-POSEDNESS

In this section, we prove the existence of global weak solutions and the continuous dependence with respect to given data. To do so, we introduce an approximate problem for problem (2.1)–(2.7). The idea is based on a time-discretization scheme, the Moreau–Yosida regularization, together with a viscous Cahn–Hilliard approach.

Let $N \in \mathbb{N}$ and put $h := T/N$, the time step of discretization. Moreover, $\tau, \sigma \in (0,1]$ stand for viscosity coefficients; $\varepsilon \in (0,1]$ is used as a parameter of Moreau–Yosida regularization for maximal monotone graphs. We consider the following equations and conditions for $n = 0, 1, \dots, N-1$:

$$\frac{\phi_{n+1} - \phi_n}{h} + \mu_{n+1} - \mu_n - \Delta\mu_{n+1} = 0 \qquad \text{a.e. in } \Omega, \tag{3.1}$$

$$\mu_{n+1} = \tau\frac{\phi_{n+1} - \phi_n}{h} - \Delta\phi_{n+1} + \beta_\varepsilon(\phi_{n+1}) + \pi(\phi_{n+1}) - f_n \qquad \text{a.e. in } \Omega, \tag{3.2}$$

$$\partial_\nu \mu_{n+1} = 0 \qquad \text{a.e. on } \Gamma, \tag{3.3}$$

$$(\phi_{n+1})|_\Gamma = \psi_{n+1} \qquad \text{a.e. on } \Gamma, \tag{3.4}$$

$$\frac{\psi_{n+1} - \psi_n}{h} + w_{n+1} - w_n - \Delta_\Gamma w_{n+1} = 0 \qquad \text{a.e. on } \Gamma, \tag{3.5}$$

$$w_{n+1} = \partial_\nu\phi_{n+1} + \sigma\frac{\psi_{n+1} - \psi_n}{h} - \Delta_\Gamma\psi_{n+1} + \beta_{\Gamma,\varepsilon}(\psi_{n+1}) + \pi_\Gamma(\psi_{n+1}) - g_n \qquad \text{a.e. on } \Gamma. \tag{3.6}$$

Note that ϕ_0 and ψ_0 are known and, in order to solve the system (3.1)–(3.6), we need to prepare initial data μ_0 and w_0, respectively. In the level of time-discrete approximation, we set up as follows:

$$\mu_0 := 0, \quad w_0 := 0. \tag{3.7}$$

Indeed, the terms $\mu_{n+1} - \mu_n$ in Equation (3.1) and $w_{n+1} - w_n$ in Equation (3.5) play a role of viscosities with the parameter h. In (3.2) and (3.6), f_n and g_n are known too, defined by

$$f_n := \frac{1}{h}\int_{nh}^{(n+1)h} f(s)\,ds, \quad g_n := \frac{1}{h}\int_{nh}^{(n+1)h} g(s)\,ds \quad \text{for } n = 0, 1, \dots, N-1.$$

In order to approximate the maximal monotone graphs, we recall the Moreau–Yosida regularization (see, e.g., [1, 2]). For each $\varepsilon \in (0,1]$, we define $\beta_\varepsilon, \beta_{\Gamma,\varepsilon} : \mathbb{R} \to \mathbb{R}$, along with the associated resolvent operators $J_\varepsilon, J_{\Gamma,\varepsilon} : \mathbb{R} \to \mathbb{R}$ given by

$$\beta_\varepsilon(r) := \frac{1}{\varepsilon}\big(r - J_\varepsilon(r)\big), \qquad J_\varepsilon(r) := (I + \varepsilon\beta)^{-1}(r),$$

$$\beta_{\Gamma,\varepsilon}(r) := \frac{1}{\varepsilon\varrho}\big(r - J_{\Gamma,\varepsilon}(r)\big), \quad J_{\Gamma,\varepsilon}(r) := \big(I + \varepsilon\varrho\beta_\Gamma\big)^{-1}(r),$$

for all $r \in \mathbb{R}$, where $\varrho > 0$ is same as in the condition (2.16). As a remark, the above two definitions are not symmetric, more precisely, the parameter of approximation is not directly ε but $\varepsilon\varrho$ in the definition of $\beta_{\Gamma,\varepsilon}$ and $J_{\Gamma,\varepsilon}$. This is important in order to apply [5, Lemma 4.4], which ensures that

$$\big|\beta_\varepsilon(r)\big| \le \varrho\big|\beta_{\Gamma,\varepsilon}(r)\big| + c_0 \quad \text{for all } r \in \mathbb{R}, \tag{3.8}$$

for all $\varepsilon \in (0,1]$ with the same constants ϱ and c_0 as in (2.16). We also have $\beta_\varepsilon(0) = \beta_{\Gamma,\varepsilon}(0) = 0$. Moreover, the related Moreau–Yosida regularizations $\widehat{\beta}_\varepsilon, \widehat{\beta}_{\Gamma,\varepsilon}$ of $\widehat{\beta}, \widehat{\beta}_\Gamma : \mathbb{R} \to \mathbb{R}$ fulfill

$$\widehat{\beta}_\varepsilon(r) := \inf_{s\in\mathbb{R}}\left\{\frac{1}{2\varepsilon}|r-s|^2 + \widehat{\beta}(s)\right\} = \frac{1}{2\varepsilon}\big|r - J_\varepsilon(r)\big|^2 + \widehat{\beta}\big(J_\varepsilon(r)\big) = \int_0^r \beta_\varepsilon(s)\,ds,$$

15222616, 2020, 11, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/mana.201900361 by Cnr Roma, Wiley Online Library on [27/02/2025]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

$$\widehat{\beta}_{\Gamma,\varepsilon}(r) := \inf_{s\in\mathbb{R}} \left\{ \frac{1}{2\varepsilon\varrho}|r-s|^2 + \widehat{\beta}_\Gamma(s) \right\} = \int_0^r \beta_{\Gamma,\varepsilon}(s)\,ds,$$

for all $r \in \mathbb{R}$. Then, we see that β_ε and $\beta_{\Gamma,\varepsilon}$ are Lipschitz continuous with constants $1/\varepsilon$ and $1/(\varepsilon\varrho)$, respectively. Additionally, we also use the following facts:

$$\left|\beta_\varepsilon(r)\right| \le \left|\beta^\circ(r)\right|, \quad \left|\beta_{\Gamma,\varepsilon}(r)\right| \le \left|\beta_\Gamma^\circ(r)\right|, \quad 0 \le \widehat{\beta}_\varepsilon(r) \le \widehat{\beta}(r), \quad 0 \le \widehat{\beta}_{\Gamma,\varepsilon}(r) \le \widehat{\beta}_\Gamma(r), \tag{3.9}$$

for all $r \in \mathbb{R}$.

3.1 | Time-discrete approximate solution

In this subsection, firstly we discuss the existence of solutions to the time-discrete approximate problem (3.1)–(3.6) for all $n = 0, 1, \dots, N-1$, for arbitrary but fixed parameters $\tau, \sigma \in (0, 1]$. Secondly, by introducing the piecewise linear and constant interpolants, we construct the approximate problem of a viscous Cahn–Hilliard system.

Proposition 3.1. *There is a value $h^* \in (0, 1]$, depending on τ and σ, such that for every $h \in (0, h^*]$, there exists a unique quadruplet $(\phi_{n+1}, \mu_{n+1}, \psi_{n+1}, w_{n+1})$ with $\phi_{n+1} \in H^2(\Omega)$, $\mu_{n+1} \in W$, $\psi_{n+1}, w_{n+1} \in H^2(\Gamma)$ such that (3.1)–(3.6) holds for all $n = 0, 1, 2, \dots, N-1$.*

Proof. Define $\Delta_{\rm N} : W \to H$ be the Laplace operator, subject to the homogeneous Neumann boundary condition. From (3.1) and (3.3), we infer that

$$\mu_{n+1} = \left(I - \Delta_{\rm N}\right)^{-1}\left(\mu_n - \frac{\phi_{n+1} - \phi_n}{h}\right) \quad \text{in } H, \tag{3.10}$$

where $I - \Delta_{\rm N}$ is a linear operator from its domain $W \subset H$ to H. At the same time, from (3.5) we obtain

$$w_{n+1} = \left(I - \Delta_\Gamma\right)^{-1}\left(w_n - \frac{\psi_{n+1} - \psi_n}{h}\right) \quad \text{in } H_\Gamma, \tag{3.11}$$

where $I - \Delta_\Gamma$ is a linear operator from $H^2(\Gamma) \subset H_\Gamma$ to H_Γ. As a consequence, Equation (3.2) can be rewritten as

$$\begin{aligned} &\left(I - \Delta_{\rm N}\right)^{-1}\phi_{n+1} + \tau\phi_{n+1} - h\Delta\phi_{n+1} + h\beta_\varepsilon(\phi_{n+1}) + h\pi(\phi_{n+1}) \\ &\quad = \left(I - \Delta_{\rm N}\right)^{-1}\phi_n + h\left(I - \Delta_{\rm N}\right)^{-1}\mu_n + \tau\phi_n + hf_n \quad \text{in } H \end{aligned} \tag{3.12}$$

and the condition (3.6) becomes

$$\begin{aligned} &h\partial_\nu\phi_{n+1} + \left(I - \Delta_\Gamma\right)^{-1}\psi_{n+1} + \sigma\psi_{n+1} - h\Delta_\Gamma\psi_{n+1} + h\beta_{\Gamma,\varepsilon}(\psi_{n+1}) + h\pi_\Gamma(\psi_{n+1}) \\ &\quad = \left(I - \Delta_\Gamma\right)^{-1}\psi_n + h\left(I - \Delta_\Gamma\right)^{-1}w_n + \sigma\psi_n + hg_n \quad \text{in } H_\Gamma. \end{aligned} \tag{3.13}$$

Now, the map

$$(z, z_\Gamma) \mapsto \left(-h\Delta z, h\partial_\nu z - h\Delta_\Gamma z_\Gamma\right)$$

gives a maximal monotone operator A from its domain

$$D(A) := \left\{(z, z_\Gamma) \in H^2(\Omega) \times H^2(\Gamma) : z|_\Gamma = z_\Gamma \text{ a.e. on } \Gamma\right\}$$

to $\boldsymbol{H} := H \times H_\Gamma$, with reference to [1, p. 47, Thm. 2.8]. Indeed, it coincides with the subdifferential of the proper, lower semicontinuous and convex functional $J : \boldsymbol{H} \to [0, \infty)$ defined by

15222616, 2020, 11, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/mana.201900361 by Cnr Roma, Wiley Online Library on [27/02/2025]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

$$J(z,z_\Gamma) := \begin{cases} \dfrac{h}{2}\displaystyle\int_\Omega |\nabla z|^2\,dx + \dfrac{h}{2}\int_\Gamma |\nabla z_\Gamma|^2\,d\Gamma & \text{if } (z,z_\Gamma)\in \boldsymbol{V}, \\ +\infty \quad \text{otherwise,} \end{cases}$$

where $\boldsymbol{V} := \{(z,z_\Gamma)\in V\times V_\Gamma : z|_\Gamma = z_\Gamma \text{ a.e. on } \Gamma\}$. This also implies that the subdifferential of J in $\boldsymbol{H}$ at (z,z_Γ) coincides with $A(z,z_\Gamma) = (-h\Delta z, h\partial_\nu z - h\Delta_\Gamma z_\Gamma)$. Next, we define another operator $B : \boldsymbol{H}\to\boldsymbol{H}$ by

$$B(z,z_\Gamma) := \Big((I-\Delta_N)^{-1} z + \tau z + h\beta_\varepsilon(z) + h\pi(z), (I-\Delta_\Gamma)^{-1} z_\Gamma + \sigma z_\Gamma + h\beta_{\Gamma,\varepsilon}(z_\Gamma) + h\pi_\Gamma(z_\Gamma)\Big)$$

with its domain $D(B)=\boldsymbol{H}$. Then, we see that B is Lipschitz continuous and monotone provided that h is sufficiently small compared to τ and σ, namely $h\in(0,h^*]$ where $h^*L<\tau/2$ and $h^*L_\Gamma<\sigma/2$:

$$\big(B(z_1,z_{\Gamma,1})-B(z_2,z_{\Gamma,2}),(z_1,z_{\Gamma,1})-(z_2,z_{\Gamma,2})\big)_{\boldsymbol{H}} \ge \frac{\tau}{2}|z_1-z_2|_H^2 + \frac{\sigma}{2}|z_{\Gamma,1}-z_{\Gamma,2}|_{H_\Gamma}^2 \quad \text{for all } (z_i,z_{\Gamma,i})\in\boldsymbol{H},\ i=1,2,$$

of course, B is also coercive.

Hence, from general theory of the maximal monotone operator [1, pp. 35–36, Cor. 2.1 and 2.2], we conclude that the range of $A+B$ is the whole space $\boldsymbol{H}$. This implies that for sufficiently small $h\in(0,h^*]$, for each $\phi_n,\mu_n\in H$ and $\psi_n,w_n\in H_\Gamma$ given by the previous step, there exists a unique pair $(\phi_{n+1},\psi_{n+1})\in\boldsymbol{V}$ solving (3.12) and (3.13), where the uniqueness is a consequence of the strict coerciveness of B. Next, we can recover $\mu_{n+1}\in W$ and $w_{n+1}\in H^2(\Gamma)$ from (3.10) and (3.11), respectively. By comparison in Equations (3.2) and (3.6), we also deduce that $\phi_{n+1}\in H^2(\Omega)$ and $\psi_{n+1}\in H^2(\Gamma)$, using the elliptic regularity theory (see, e.g., [31, Lem. A.1]). Thus, we can complete the proof of Proposition 3.1 by iterating from $n=0$ to $n=N-1$. □

According to the standard manner, we now define the following piecewise linear functions and step functions:

$$\begin{aligned}
&\hat\phi_h(t) := \phi_n + \frac{\phi_{n+1}-\phi_n}{h}(t-nh) && \text{for } t\in[nh,(n+1)h], && n=0,1,\dots,N-1,\\
&\bar\phi_h(t) := \phi_{n+1} && \text{for } t\in(nh,(n+1)h], && n=0,1,\dots,N-1,\\
&\underline{f}_h(t) := f_n && \text{for } t\in(nh,(n+1)h], && n=0,1,\dots,N-1,
\end{aligned}$$

and analogously for $\hat\mu_h,\bar\mu_n,\hat\psi_h,\bar\psi_h,\hat w_h,\bar w_h,\underline{g}_h$. Then, we have the following useful properties:

$$\big|\hat\phi_h\big|^2_{L^2(0,T;X)} \le \frac{h}{2}|\phi_0|_X^2 + \big|\bar\phi_h\big|^2_{L^2(0,T;X)}, \tag{3.14}$$

$$\big|\hat\phi_h\big|_{L^\infty(0,T;X)} = \max\Big\{|\phi_0|_X, \big|\bar\phi_h\big|_{L^\infty(0,T;X)}\Big\}, \tag{3.15}$$

$$\big|\hat\phi_h-\bar\phi_h\big|^2_{L^2(0,T;X)} = \frac{h^2}{3}\big|\partial_t\hat\phi_h\big|^2_{L^2(0,T;X)}, \tag{3.16}$$

for some suitable function space X. Indeed, (3.15) is clear from the definition, the equality (3.16) is obtained from the direct calculation as follows:

$$\begin{aligned}
\big|\hat\phi_h-\bar\phi_h\big|^2_{L^2(0,T;X)} &= \sum_{n=0}^{N-1}\left|\frac{\phi_{n+1}-\phi_n}{h}\right|_X^2\int_{nh}^{(n+1)h}\big(t-h(n+1)\big)^2\,dt\\
&= \frac{h^3}{3}\sum_{n=0}^{N-1}\left|\frac{\phi_{n+1}-\phi_n}{h}\right|_X^2\\
&= \frac{h^2}{3}\big|\partial_t\hat\phi_h\big|^2_{L^2(0,T;X)}.
\end{aligned}$$

15222616, 2020, 11, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/mana.201900361 by Cnr Roma, Wiley Online Library on [27/02/2025]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Concerning the inequality (3.14), invoking the convexity and Jensen's inequality we obtain that

$$\begin{aligned}
\left|\hat{\phi}_h\right|^2_{L^2(0,T;X)} &= \sum_{n=0}^{N-1} \int_{nh}^{(n+1)h} \left| \left(\frac{t-nh}{h}\right)\phi_{n+1} + \left[1 - \left(\frac{t-nh}{h}\right)\right]\phi_n \right|^2_X dt \\
&\le \sum_{n=0}^{N-1} \int_{nh}^{(n+1)h} \left\{ \left(\frac{t-nh}{h}\right)|\phi_{n+1}|^2_X + \left[1 - \left(\frac{t-nh}{h}\right)\right]|\phi_n|^2_X \right\} dt \\
&= \sum_{n=0}^{N-1} \left(\frac{h}{2}|\phi_{n+1}|^2_X + \frac{h}{2}|\phi_n|^2_X \right) \\
&\le \frac{h}{2}|\phi_0|^2_X + \left|\bar{\phi}_h\right|^2_{L^2(0,T;X)}.
\end{aligned}$$

Under these settings, we see from (3.1)–(3.6) that the functions

$$\hat{\phi}_h,\ \bar{\phi}_h,\ \hat{\mu}_h,\ \bar{\mu}_n,\ \hat{\psi}_h,\ \bar{\psi}_h,\ \hat{w}_h,\ \bar{w}_h$$

constructed above solve the following polygonal approximate problem of the viscous Cahn–Hilliard system:

$$\partial_t \hat{\phi}_h + h\partial_t \hat{\mu}_h - \Delta \bar{\mu}_h = 0 \qquad \text{a.e. in } Q, \tag{3.17}$$

$$\bar{\mu}_h = \tau \partial_t \hat{\phi}_h - \Delta \bar{\phi}_h + \beta_\varepsilon(\bar{\phi}_h) + \pi(\bar{\phi}_h) - \underline{f}_h \qquad \text{a.e. in } Q, \tag{3.18}$$

$$\partial_\nu \bar{\mu}_h = 0 \qquad \text{a.e. on } \Sigma, \tag{3.19}$$

$$(\bar{\phi}_h)|_\Gamma = \bar{\psi}_h \qquad \text{a.e. on } \Sigma, \tag{3.20}$$

$$\partial_t \hat{\psi}_h + h\partial_t \hat{w}_h - \Delta_\Gamma \bar{w}_h = 0 \qquad \text{a.e. on } \Sigma, \tag{3.21}$$

$$\bar{w}_h = \partial_\nu \bar{\phi}_h + \sigma \partial_t \hat{\psi}_h - \Delta_\Gamma \bar{\psi}_h + \beta_{\Gamma,\varepsilon}(\bar{\psi}_h) + \pi_\Gamma(\bar{\psi}_h) - \underline{g}_h \qquad \text{a.e. on } \Sigma, \tag{3.22}$$

$$\hat{\phi}_h(0) = \phi_0, \quad \hat{\mu}_h(0) = 0 \quad \text{a.e. in } \Omega, \quad \hat{\psi}_h(0) = \psi_0, \quad \hat{w}_h(0) = 0 \quad \text{a.e. on } \Gamma, \tag{3.23}$$

for every $h \in (0, h^*]$. By virtue of the definitions of f_n and g_n we see that $\{\underline{f}_h\}_{h>0}$ and $\{\underline{g}_h\}_{h>0}$ are bounded in $L^2(0,T;V)$ and $L^2(0,T;V_\Gamma)$, respectively. Indeed, from the Hölder inequality we infer that

$$\begin{aligned}
\int_0^T \left|\underline{f}_h(t)\right|^2_V dt &= \sum_{n=0}^{N-1} h|f_n|^2_V \\
&= \sum_{n=0}^{N-1} h \left| \frac{1}{h} \int_{nh}^{(n+1)h} f(s) ds \right|^2_V \\
&\le \frac{1}{h} \sum_{n=0}^{N-1} \left(\int_{nh}^{(n+1)h} \left|f(s)\right|^2_V ds \right) \left(\int_{nh}^{(n+1)h} 1\, ds \right) \\
&= |f|^2_{L^2(0,T;V)}, \quad \text{for all } h > 0,
\end{aligned} \tag{3.24}$$

and a similar result holds for $\{\underline{g}_h\}_{h>0}$. In the next subsection we will proceed to derive necessary uniform estimates for problem (3.17)–(3.23).

15222616, 2020, 11, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/mana.201900361 by Cnr Roma, Wiley Online Library on [27/02/2025]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

3.2 | A priori estimates and limiting procedure

Hereafter, we derive uniform estimates that are independent of $h = T/N$ for problem (3.17)–(3.23). We also take care of the dependence with respect to $\tau, \sigma, \varepsilon \in (0,1]$.

Lemma 3.2. *There exists a positive constant M_1, independent of $h \in (0, h^{**}]$, $\tau, \sigma, \varepsilon \in (0,1]$, such that*

$$\begin{aligned}
&\left|\bar{\phi}_h\right|^2_{L^\infty(0,T;V)} + \left|\bar{\psi}_h\right|^2_{L^\infty(0,T;V_\Gamma)} + \left|\partial_t\hat{\phi}_h + h\partial_t\hat{\mu}_h\right|^2_{L^2(0,T;V^*)} + \left|\partial_t\hat{\psi}_h + h\partial_t\hat{w}_h\right|^2_{L^2(0,T;V_\Gamma{}^*)} \\
&\quad + h|\bar{\mu}_h|^2_{L^\infty(0,T;H)} + h^2\left|\partial_t\hat{\mu}_h\right|^2_{L^2(0,T;H)} + h|\bar{w}_h|^2_{L^\infty(0,T;H_\Gamma)} + h^2\left|\partial_t\hat{w}_h\right|^2_{L^2(0,T;H_\Gamma)} \\
&\quad + \tau\left|\partial_t\hat{\phi}_h\right|^2_{L^2(0,T;H)} + \sigma\left|\partial_t\hat{\psi}_h\right|^2_{L^2(0,T;H_\Gamma)} + h\left|\partial_t\hat{\phi}_h\right|^2_{L^2(0,T;V)} + h\left|\partial_t\hat{\psi}_h\right|^2_{L^2(0,T;V_\Gamma)} \\
&\quad + \left|\widehat{\beta}_\varepsilon(\bar{\phi}_h)\right|_{L^\infty(0,T;L^1(\Omega))} + \left|\widehat{\beta}_{\Gamma,\varepsilon}(\bar{\psi}_h)\right|_{L^\infty(0,T;L^1(\Gamma))} \le M_1,
\end{aligned} \tag{3.25}$$

for all $h \in (0, h^{**}]$, where $h^{**} \in (0, h^*]$ is a threshold value for the step size depending on $\tau, \sigma \in (0,1]$.

Proof. By integrating (3.1) over Ω, with the help of (3.3) we deduce the following relation for the mean values defined in (2.14):

$$m_\Omega(\phi_{n+1} + h\mu_{n+1}) = m_\Omega(\phi_n + h\mu_n) = m_0, \tag{3.26}$$

for all $n = 0, 1, \dots, N-1$, where (cf. (A2) and (3.7))

$$m_0 := m_\Omega(\phi_0) = \frac{1}{|\Omega|}\int_\Omega \phi_0\,dx = \frac{1}{|\Omega|}\int_\Omega (\phi_0 + h\mu_0)\,dx.$$

In a similar manner, from (3.5) and (2.15) it follows that

$$m_\Gamma(\psi_{n+1} + hw_{n+1}) = m_\Gamma(\psi_n + hw_n) = m_{\Gamma 0}, \tag{3.27}$$

for all $n = 0, 1, \dots, N-1$, where, thanks to (A2) and (3.7),

$$m_{\Gamma 0} := m_\Gamma(\psi_0) = \frac{1}{|\Gamma|}\int_\Gamma \psi_0\,d\Gamma = \frac{1}{|\Gamma|}\int_\Gamma (\psi_0 + hw_0)\,d\Gamma.$$

Then, as (3.26) entails that

$$\langle \phi_{n+1} + h\mu_{n+1} - \phi_n - h\mu_n, 1\rangle_{V^*,V} = \int_\Omega (\phi_{n+1} + h\mu_{n+1})\,dx - \int_\Omega (\phi_n + h\mu_n)\,dx = 0,$$

that is, $\phi_{n+1} + h\mu_{n+1} - \phi_n - h\mu_n \in V_{0*}$, we can test (3.1) by

$$\mathcal{N}_\Omega(\phi_{n+1} + h\mu_{n+1} - \phi_n - h\mu_n)$$

and, using (3.3) and (2.19), we obtain

$$h\left|\frac{\phi_{n+1} + h\mu_{n+1} - \phi_n - h\mu_n}{h}\right|^2_{V_{0*}} + \int_\Omega \mu_{n+1}(\phi_{n+1} + h\mu_{n+1} - \phi_n - h\mu_n)\,dx = 0, \tag{3.28}$$

15222616, 2020, 11, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/mana.201900361 by Cnr Roma, Wiley Online Library on [27/02/2025]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

for all $n = 0, 1, 2, \dots, N-1$. Next, from (3.27) we see that

$$\langle \psi_{n+1} + hw_{n+1} - \psi_n - hw_n, 1 \rangle_{V_\Gamma^*, V_\Gamma} = \int_\Gamma (\psi_{n+1} + hw_{n+1})\, d\Gamma - \int_\Gamma (\psi_n + hw_n)\, d\Gamma = 0,$$

that is, $\psi_{n+1} + hw_{n+1} - \psi_n - hw_n \in V_{\Gamma,0*}$; therefore, testing (3.5) by

$$\mathcal{N}_\Gamma(\psi_{n+1} + hw_{n+1} - \psi_n - hw_n),$$

and using (2.20) we obtain

$$h\left|\frac{\psi_{n+1} + hw_{n+1} - \psi_n - hw_n}{h}\right|^2_{V_{\Gamma,0*}} + \int_\Gamma w_{n+1}(\psi_{n+1} + hw_{n+1} - \psi_n - hw_n)\, d\Gamma = 0, \tag{3.29}$$

for all $n = 0, 1, \dots, N-1$. Next, we add ϕ_{n+1} to both sides of (3.2), multiply the resultant by $\phi_{n+1} - \phi_n$ and use the condition (3.4) and Equation (3.6), to find out that

$$\begin{aligned}
&\int_\Omega \mu_{n+1}(\phi_{n+1} - \phi_n)\, dx + \int_\Gamma w_{n+1}(\psi_{n+1} - \psi_n)\, d\Gamma \\
&\quad \geq \tau h\left|\frac{\phi_{n+1} - \phi_n}{h}\right|^2_H + \sigma h\left|\frac{\psi_{n+1} - \psi_n}{h}\right|^2_{H_\Gamma} + \frac{1}{2}|\phi_{n+1}|_V^2 + \frac{1}{2}|\phi_{n+1} - \phi_n|_V^2 - \frac{1}{2}|\phi_n|_V^2 \\
&\qquad + \frac{1}{2}|\psi_{n+1}|_{V_\Gamma}^2 + \frac{1}{2}|\psi_{n+1} - \psi_n|_{V_\Gamma}^2 - \frac{1}{2}|\psi_n|_{V_\Gamma}^2 + \int_\Omega \widehat{\beta}_\varepsilon(\phi_{n+1})\, dx - \int_\Omega \widehat{\beta}_\varepsilon(\phi_n)\, dx \\
&\qquad + \int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}(\psi_{n+1})\, d\Gamma - \int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}(\psi_n)\, d\Gamma + \int_\Omega (\pi(\phi_{n+1}) - f_n - \phi_{n+1})(\phi_{n+1} - \phi_n)\, dx \\
&\qquad + \int_\Gamma (\pi_\Gamma(\psi_{n+1}) - g_n - \psi_{n+1})(\psi_{n+1} - \psi_n)\, d\Gamma
\end{aligned} \tag{3.30}$$

for all $n = 0, 1, \dots, N-1$, where we used the elementary inequality $r(r-s) = \big(r^2 + (r-s)^2 - s^2\big)/2$ for $r, s \in \mathbb{R}$.

Now, we collect (3.28)–(3.30), sum up for $n = 0, 1, \dots, m-1$ and apply (3.7) and (3.9), obtaining

$$\begin{aligned}
&\sum_{n=0}^{m-1} h\left|\frac{\phi_{n+1} - \phi_n}{h} + \mu_{n+1} - \mu_n\right|^2_{V_{0*}} + \sum_{n=0}^{m-1} h\left|\frac{\psi_{n+1} - \psi_n}{h} + w_{n+1} - w_n\right|^2_{V_{\Gamma,0*}} \\
&\quad + \frac{h}{2}|\mu_m|_H^2 + \sum_{n=0}^{m-1} \frac{h}{2}|\mu_{n+1} - \mu_n|_H^2 + \frac{h}{2}|w_m|_{H_\Gamma}^2 + \sum_{n=0}^{m-1} \frac{h}{2}|w_{n+1} - w_n|_{H_\Gamma}^2 \\
&\quad + \tau \sum_{n=0}^{m-1} h\left|\frac{\phi_{n+1} - \phi_n}{h}\right|^2_H + \sigma \sum_{n=0}^{m-1} h\left|\frac{\psi_{n+1} - \psi_n}{h}\right|^2_{H_\Gamma} + \frac{1}{2}|\phi_m|_V^2 + \frac{1}{2}\sum_{n=0}^{m-1} |\phi_{n+1} - \phi_n|_V^2 \\
&\quad + \frac{1}{2}|\psi_m|_{V_\Gamma}^2 + \frac{1}{2}\sum_{n=0}^{m-1} |\psi_{n+1} - \psi_n|_{V_\Gamma}^2 + \int_\Omega \widehat{\beta}_\varepsilon(\phi_m)\, dx + \int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}(\psi_m)\, d\Gamma \\
&\leq \frac{1}{2}|\phi_0|_V^2 + \frac{1}{2}|\psi_0|_{V_\Gamma}^2 + \int_\Omega \widehat{\beta}(\phi_0) dx + \int_\Gamma \widehat{\beta}_\Gamma(\psi_0)\, d\Gamma - \sum_{n=0}^{m-1} \int_\Omega (\pi(\phi_{n+1}) - f_n - \phi_{n+1})(\phi_{n+1} - \phi_n) dx \\
&\quad - \sum_{n=0}^{m-1} \int_\Gamma (\pi_\Gamma(\psi_{n+1}) - g_n - \psi_{n+1})(\psi_{n+1} - \psi_n)\, d\Gamma
\end{aligned} \tag{3.31}$$

15222616, 2020, 11, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/mana.201900361 by Cnr Roma, Wiley Online Library on [27/02/2025]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

for all $m = 1, 2, \dots, N$. We know that there exists a positive constant C_1 such that $|z|^2_{V^*} \le C_1 |z|^2_{V_{0*}}$ for all $z \in V_{0*}$, as well as $|z|^2_{V^*} \le C_1 |z|^2_H$ for all $z \in H$. Therefore, in order to estimate the terms on the right hand side of (3.31) we can argue with the help of assumptions (A3) and (A4). First, we have

$$\begin{aligned}
&\left| \sum_{n=0}^{m-1} \int_\Omega \big(\pi(\phi_{n+1}) - f_n - \phi_{n+1}\big)\big(\phi_{n+1} - \phi_n\big)\, dx \right| \\
&\quad \le \delta \sum_{n=0}^{m-1} h \left| \frac{\phi_{n+1} - \phi_n}{h} \right|^2_{V^*} + C_\delta \sum_{n=0}^{m-1} h\big(1 + |\phi_{n+1}|^2_V + |f_n|^2_V\big) \\
&\quad \le 2\delta C_1 \sum_{n=0}^{m-1} h \left| \frac{\phi_{n+1} - \phi_n}{h} + \mu_{n+1} - \mu_n \right|^2_{V_{0*}} + 2\delta C_1 \sum_{n=0}^{m-1} h |\mu_{n+1} - \mu_n|^2_H \\
&\qquad + C_\delta \sum_{n=0}^{m-1} h\big(1 + |\phi_{n+1}|^2_V + |f_n|^2_V\big),
\end{aligned} \tag{3.32}$$

for all $\delta > 0$, where we also use Young's inequality with $\delta > 0$; C_δ is a positive constant such that $C_\delta \to \infty$ as $\delta \to 0$. Indeed, taking care of (3.26), we have $\big(\phi_{n+1} - \phi_n\big)/h + \mu_{n+1} - \mu_n \in V_{0*}$ for all $n = 0, 1, \dots, N-1$. From (3.27), a very similar procedure can be used to estimate the other contribution

$$\begin{aligned}
&\left| \sum_{n=0}^{m-1} \int_\Gamma \big(\pi_\Gamma(\psi_{n+1}) - g_n - \psi_{n+1}\big)\big(\psi_{n+1} - \psi_n\big)\, d\Gamma \right| \\
&\quad \le 2\delta C_2 \sum_{n=0}^{m-1} h \left| \frac{\psi_{n+1} - \psi_n}{h} + w_{n+1} - w_n \right|^2_{V_{\Gamma,0*}} + 2\delta C_2 \sum_{n=0}^{m-1} h |w_{n+1} - w_n|^2_{H_\Gamma} \\
&\qquad + C_\delta \sum_{n=0}^{m-1} h\Big(1 + |\psi_{n+1}|^2_{V_\Gamma} + |g_n|^2_{V_\Gamma}\Big),
\end{aligned} \tag{3.33}$$

where C_2 is a positive constant such that $|z_\Gamma|^2_{V^*_\Gamma} \le C_2 |z_\Gamma|^2_{V_{\Gamma,0*}}$ for all $z_\Gamma \in V_{\Gamma,0*}$ and $|z_\Gamma|^2_{V^*_\Gamma} \le C_2 |z_\Gamma|^2_{H_\Gamma}$ for all $z_\Gamma \in H_\Gamma$. Then, we can choose $\delta > 0$ in order that

$$\delta \le \min\left\{ \frac{1}{8C_1}, \frac{1}{8C_2} \right\}$$

and consequently we fix the constant C_δ in the above estimates. Next, we choose a threshold value for the step size $h^{**} \in (0, h^*]$ with the requirement that

$$C_\delta h^{**} \le \frac{1}{4}.$$

Then, collecting (3.31)–(3.33) and recalling (2.17)–(2.18), it is not difficult to obtain

$$\begin{aligned}
\frac{1}{2}|\phi_m|^2_V + \frac{1}{2}|\psi_m|^2_{V_\Gamma} \le \frac{1}{2}|\phi_0|^2_V + \frac{1}{2}|\psi_0|^2_{V_\Gamma} + \int_\Omega \widehat{\beta}(\phi_0)\, dx + \int_\Gamma \widehat{\beta}_\Gamma(\psi_0)\, d\Gamma \\
+ C_\delta \sum_{n=0}^{m-1} h\big(1 + |f_n|^2_V\big) + C_\delta \sum_{n=1}^{m-1} h |\phi_n|^2_V + C_\delta h |\phi_m|^2_V \\
+ C_\delta \sum_{n=0}^{m-1} h\Big(1 + |g_n|^2_{V_\Gamma}\Big) + C_\delta \sum_{n=1}^{m-1} h |\psi_n|^2_{V_\Gamma} + C_\delta h |\psi_m|^2_{V_\Gamma}
\end{aligned}$$

15222616, 2020, 11, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/mana.201900361 by Cnr Roma, Wiley Online Library on [27/02/2025]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

and consequently

$$\frac{1}{2}|\phi_m|_V^2+\frac{1}{2}|\psi_m|_{V_\Gamma}^2\le\frac{1}{2}|\phi_0|_V^2+\frac{1}{2}|\psi_0|_{V_\Gamma}^2+\int_\Omega\widehat{\beta}(\phi_0)\,dx+\int_\Gamma\widehat{\beta}_\Gamma(\psi_0)\,d\Gamma$$
$$+C_\delta\Big(T+|f|_{L^2(0,T;V)}^2\Big)+C_\delta\sum_{n=0}^{m-1}h|\phi_n|_V^2+\frac{1}{4}|\phi_m|_V^2$$
$$+C_\delta\Big(T+|g|_{L^2(0,T;V_\Gamma)}^2\Big)+C_\delta\sum_{n=0}^{m-1}h|\psi_n|_{V_\Gamma}^2+\frac{1}{4}|\psi_m|_{V_\Gamma}^2,$$

for all $m=1,2,\dots,N$. Therefore, applying the discrete Gronwall lemma with assumptions (A2) and (A3), we conclude that there exists a positive constant $\widetilde{M}_1$, independent of $h\in(0,h^{**}]$, $\tau,\sigma,\varepsilon\in(0,1]$, such that

$$|\phi_m|_V^2+|\psi_m|_{V_\Gamma}^2\le\widetilde{M}_1$$

for all $m=1,2,\dots,N$. Moreover, going back to (3.31)–(3.33), we plainly deduce (3.25) for some positive constant $M_1\ge\widetilde{M}_1$ independent of $h\in(0,h^{**}]$ and $\tau,\sigma,\varepsilon\in(0,1]$. □

Lemma 3.3. *There exist two functions $\Lambda_1,\Lambda_2\in L^2(0,T)$ and a positive constant M_2, independent of $h\in(0,h^{**}]$, $\tau,\sigma,\varepsilon\in(0,1]$, such that*

$$\big|\bar\mu_h(t)-m_\Omega\big(\bar\mu_h(t)\big)\big|_V+\big|\bar w_h(t)-m_\Gamma\big(\bar w_h(t)\big)\big|_{V_\Gamma}\le\Lambda_1(t),\tag{3.34}$$

$$\big|\beta_\varepsilon\big(\bar\phi_h(t)\big)\big|_{L^1(\Omega)}+\big|\beta_{\Gamma,\varepsilon}\big(\bar\psi_h(t)\big)\big|_{L^1(\Gamma)}\le M_2\Big(\Lambda_2(t)+\big|\partial_\nu\bar\phi_h(t)\big|_{H_\Gamma}\Big)\tag{3.35}$$

for a.a. $t\in(0,T)$.

Proof. Firstly, multiplying (3.17) by $\bar\mu_h-m_\Omega(\bar\mu_h)$, integrating the resultant over Ω, using the boundary condition (3.19), and applying the Young and Poincaré–Wirtinger inequalities, we obtain

$$\int_\Omega\big|\nabla\big(\bar\mu_h-m_\Omega(\bar\mu_h)\big)\big|^2\,dx\le\big|\partial_t\hat\phi_h+h\partial_t\hat\mu_h\big|_{V^*}\big|\bar\mu_h-m_\Omega(\bar\mu_h)\big|_V$$
$$\le\frac{1}{2\delta}\big|\partial_t\hat\phi_h+h\partial_t\hat\mu_h\big|_{V^*}^2+\frac{\delta}{2}\big|\bar\mu_h-m_\Omega(\bar\mu_h)\big|_V^2$$
$$\le\frac{1}{2\delta}\big|\partial_t\hat\phi_h+h\partial_t\hat\mu_h\big|_{V^*}^2+\frac{C_{\mathrm{P}}\delta}{2}\big|\bar\mu_h-m_\Omega(\bar\mu_h)\big|_{V_0}^2$$

a.e. in $(0,T)$, for all $\delta>0$. Similarly, multiplying (3.21) by $\bar w_h-m_\Gamma(\bar w_h)$ and integrating the resultant over Γ, we obtain

$$\int_\Gamma\big|\nabla_\Gamma\big(\bar w_h-m_\Gamma(\bar w_h)\big)\big|^2\,d\Gamma\le\big|\partial_t\hat\psi_h+h\partial_t\hat w_h\big|_{V_\Gamma^*}\big|\bar w_h-m_\Gamma(\bar w_h)\big|_{V_\Gamma}$$
$$\le\frac{1}{2\delta}\big|\partial_t\hat\psi_h+h\partial_t\hat w_h\big|_{V_\Gamma^*}^2+\frac{C_{\mathrm{P}}\delta}{2}\big|\bar w_h-m_\Gamma(\bar w_h)\big|_{V_{\Gamma,0}}^2$$

a.e. in $(0,T)$. Letting $\delta:=1/C_{\mathrm{P}}$ and applying the Poincaré–Wirtinger inequality again, we deduce that

$$\big|\bar\mu_h-m_\Omega(\bar\mu_h)\big|_V^2\le C_{\mathrm{P}}\big|\bar\mu_h-m_\Omega(\bar\mu_h)\big|_{V_0}^2\le C_{\mathrm{P}}^2\big|\partial_t\hat\phi_h+h\partial_t\hat\mu_h\big|_{V^*}^2,$$
$$\big|\bar w_h-m_\Gamma(\bar w_h)\big|_{V_\Gamma}^2\le C_{\mathrm{P}}^2\big|\partial_t\hat\psi_h+h\partial_t\hat w_h\big|_{V_\Gamma^*}^2,$$

15222616, 2020, 11, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/mana.201900361 by Cnr Roma, Wiley Online Library on [27/02/2025]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

a.e. in $(0,T)$. Taking (3.25) into account, we conclude the estimate (3.34) with

$$\Lambda_1(t) := \sqrt{2}C_{\mathrm{P}}\left(\left|\partial_t\hat{\phi}_h(t)+h\partial_t\hat{\mu}_h(t)\right|_{V^*}+\left|\partial_t\hat{\psi}_h(t)+h\partial_t\hat{w}_h(t)\right|_{V_\Gamma^*}\right), \tag{3.36}$$

for a.a. $t\in(0,T)$.

Secondly, recalling (3.26) and (3.27), we have

$$m_\Omega(\bar{\phi}_h+h\bar{\mu}_h)=m_0,\quad m_\Gamma(\bar{\psi}_h+h\bar{w}_h)=m_{\Gamma 0}\quad \text{a.e. on } (0,T).$$

Then, we multiply (3.18) by $\bar{\phi}_h-m_0$ and integrate the resultant over Ω. Also, we use (3.20) and exploit the argument devised in [30, Appendix, Prop. A.1] (see also [19] for a complete proof) along with (A2) to infer that there exist two positive constants $C_3,C_4>0$, independent of $h\in(0,h^{**}]$, $\tau,\sigma,\varepsilon\in(0,1]$, such that

$$\begin{aligned}
C_3\int_\Omega\left|\beta_\varepsilon(\bar{\phi}_h)\right|dx-C_4&\le\int_\Omega\beta_\varepsilon(\bar{\phi}_h)(\bar{\phi}_h-m_0)\,dx\\
&=-\int_\Omega(\tau\partial_t\hat{\phi}_h+\pi(\bar{\phi}_h)-\underline{f}_h)(\bar{\phi}_h-m_0)\,dx-\int_\Omega\left|\nabla\bar{\phi}_h\right|^2dx\\
&\quad+\int_\Omega\bar{\mu}_h\left(\bar{\phi}_h+h\bar{\mu}_h-m_0\right)dx-h\int_\Omega|\bar{\mu}_h|^2\,dx\\
&\quad+\int_\Gamma\partial_\nu\bar{\phi}_h\left(\bar{\psi}_h-m_{\Gamma 0}\right)d\Gamma+\int_\Gamma\partial_\nu\bar{\phi}_h(m_{\Gamma 0}-m_0)\,d\Gamma,
\end{aligned}\tag{3.37}$$

a.e. in $(0,T)$. Similarly, we multiply (3.22) by $\bar{\psi}_h-m_{0\Gamma}$ and integrate the resultant over Γ to infer that there exist two positive constants $C_5,C_6>0$, independent of $h\in(0,h^{**}]$, $\tau,\sigma,\varepsilon\in(0,1]$, such that

$$\begin{aligned}
C_5\int_\Gamma\left|\beta_{\Gamma,\varepsilon}(\bar{\psi}_h)\right|d\Gamma-C_6&\le\int_\Gamma\beta_{\Gamma,\varepsilon}(\bar{\psi}_h)(\bar{\psi}_h-m_{\Gamma 0})\,d\Gamma\\
&=-\int_\Gamma\partial_\nu\bar{\phi}_h(\bar{\psi}_h-m_{\Gamma 0})\,d\Gamma-\int_\Gamma(\sigma\partial_t\hat{\psi}_h+\pi_\Gamma(\bar{\psi}_h)-\underline{g}_h)(\bar{\psi}_h-m_{\Gamma 0})\,d\Gamma\\
&\quad-\int_\Gamma\left|\nabla_\Gamma\bar{\psi}_h\right|^2d\Gamma+\int_\Gamma\bar{w}_h(\bar{\psi}_h+h\bar{w}_h-m_{\Gamma 0})\,d\Gamma-h\int_\Gamma|\bar{w}_h|^2\,d\Gamma,
\end{aligned}\tag{3.38}$$

a.e. in $(0,T)$. Adding (3.37) and (3.38), we obtain that

$$\begin{aligned}
&C_3\left|\beta_\varepsilon(\bar{\phi}_h)\right|_{L^1(\Omega)}+C_5\left|\beta_{\Gamma,\varepsilon}(\bar{\psi}_h)\right|_{L^1(\Gamma)}\\
&\le C_4+C_6+\left|\tau\partial_t\hat{\phi}_h+\pi(\bar{\phi}_h)-\underline{f}_h\right|_H\left|\bar{\phi}_h-m_0\right|_H+\left|\bar{\mu}_h-m_\Omega(\bar{\mu}_h)\right|_H\left|\bar{\phi}_h+h\bar{\mu}_h-m_0\right|_H\\
&\quad+\left|\partial_\nu\bar{\phi}_h\right|_{H_\Gamma}|m_{\Gamma 0}-m_0|_{H_\Gamma}+\left|\sigma\partial_t\hat{\psi}_h+\pi_\Gamma(\bar{\psi}_h)-\underline{g}_h\right|_{H_\Gamma}\left|\bar{\psi}_h-m_{\Gamma 0}\right|_{H_\Gamma}\\
&\quad+\left|\bar{w}_h-m_\Gamma(\bar{w}_h)\right|_{H_\Gamma}\left|\bar{\psi}_h+h\bar{w}_h-m_{\Gamma 0}\right|_{H_\Gamma},
\end{aligned}\tag{3.39}$$

a.e. in $(0,T)$. Hence, from Lemma 3.2 and (3.34) it follows that there exist a function $\Lambda_2\in L^2(0,T)$ and a positive constant M_2, independent of $h\in(0,h^{**}]$, $\tau,\sigma,\varepsilon\in(0,1]$, such that (3.35) holds. □

Lemma 3.4. *There exist functions $\Lambda_3,\Lambda_4\in L^2(0,T)$ and positive constants M_3,M_4, independent of $h\in(0,h^{**}]$, $\tau,\sigma,\varepsilon\in(0,1]$, such that*

$$\left|m_\Omega(\bar{\mu}_h(t))\right|+\left|m_\Gamma(\bar{w}_h(t))\right|\le M_3\left(\Lambda_3(t)+\left|\partial_\nu\bar{\phi}_h(t)\right|_{H_\Gamma}\right), \tag{3.40}$$

15222616, 2020, 11, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/mana.201900361 by Cnr Roma, Wiley Online Library on [27/02/2025]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

$$\left|\bar{\mu}_h(t)\right|_V + \left|\bar{w}_h(t)\right|_{V_\Gamma} \le M_4\left(\Lambda_4(t) + \left|\partial_\nu\bar{\phi}_h(t)\right|_{H_\Gamma}\right), \tag{3.41}$$

for a.a. $t \in (0,T)$.

Proof. Multiplying Equation (3.18) by $1/|\Omega|$ and Equation (3.22) by $1/|\Gamma|$, using integration by parts, and adding the resultants together, we obtain

$$\begin{aligned}\left|m_\Omega(\bar{\mu}_h)\right| + \left|m_\Gamma(\bar{w}_h)\right| \le{}& \frac{1}{|\Omega|}\int_\Omega\left|\beta_\varepsilon(\bar{\phi}_h)\right|dx + \frac{1}{|\Omega|}\int_\Omega\left|\tau\partial_t\hat{\phi}_h + \pi(\bar{\phi}_h) - \underline{f}_h\right|dx + \frac{1}{|\Omega|}\int_\Gamma\left|\partial_\nu\bar{\phi}_h\right|d\Gamma\\ &+ \frac{1}{|\Gamma|}\int_\Gamma\left|\beta_{\Gamma,\varepsilon}(\bar{\psi}_h)\right|d\Gamma + \frac{1}{|\Gamma|}\int_\Gamma\left|\partial_\nu\bar{\phi}_h + \sigma\partial_t\hat{\psi}_h + \pi_\Gamma(\bar{\psi}_h) - \underline{g}_h\right|d\Gamma,\end{aligned}$$

a.e. in $(0,T)$. Therefore, using Hölder's inequality and recalling (3.25) and (3.35), we see that there exist a function $\Lambda_3 \in L^2(0,T)$ depending on M_2, Λ_2 and a positive constant M_3 depending on $M_2, L, L_\Gamma, |f|_{L^2(0,T;H)}, |g|_{L^2(0,T;H_\Gamma)}, |\Omega|$ and $|\Gamma|$, independent of $h \in (0,h^{**}]$, $\tau,\sigma,\varepsilon \in (0,1]$, such that the estimate (3.40) holds. Additionally, from (3.34) and the above estimate on mean values of $\bar{\mu}_h, \bar{w}_h$, we infer (3.41), where the function $\Lambda_4 \in L^2(0,T)$ and the positive constant M_4 are independent of $h \in (0,h^{**}]$, $\tau,\sigma,\varepsilon \in (0,1]$. □

Lemma 3.5. *There exist a function* $\Lambda_5 \in L^2(0,T)$ *and a positive constant* M_5*, independent of* $h \in (0,h^{**}]$, $\tau,\sigma,\varepsilon \in (0,1]$*, such that*

$$\left|\beta_\varepsilon(\bar{\phi}_h(t))\right|_H + \left|\beta_{\Gamma,\varepsilon}(\bar{\psi}_h(t))\right|_{H_\Gamma} \le M_5\left(\Lambda_5(t) + \left|\partial_\nu\bar{\phi}_h(t)\right|_{H_\Gamma}\right) \tag{3.42}$$

for a.a. $t \in (0,T)$.

Proof. Multiplying Equation (3.18) by $\beta_\varepsilon(\bar{\phi}_h)$, integrating over Ω, and using (3.20), we have

$$\begin{aligned}&\int_\Omega\beta_\varepsilon'(\bar{\phi}_h)\left|\nabla\bar{\phi}_h\right|^2dx + \int_\Omega\left|\beta_\varepsilon(\bar{\phi}_h)\right|^2dx\\ &\quad\le \frac{1}{2}\left|\bar{\mu}_h - \tau\partial_t\hat{\phi}_h - \pi(\bar{\phi}_h) + \underline{f}_h\right|_H^2 + \frac{1}{2}\left|\beta_\varepsilon(\bar{\phi}_h)\right|_H^2 + \int_\Gamma\partial_\nu\bar{\phi}_h\beta_\varepsilon(\bar{\psi}_h)\,d\Gamma,\end{aligned} \tag{3.43}$$

a.e. in $(0,T)$. Next, multiplying Equation (3.22) by $\beta_{\Gamma,\varepsilon}(\bar{\psi}_h)$ and integrating the resultant over Γ, we have

$$\begin{aligned}&\int_\Gamma\beta_{\Gamma,\varepsilon}'(\bar{\psi}_h)\left|\nabla_\Gamma\bar{\psi}_h\right|^2d\Gamma + \int_\Gamma\left|\beta_{\Gamma,\varepsilon}(\bar{\psi}_h)\right|^2d\Gamma\\ &\quad\le \left|\bar{w}_h - \sigma\partial_t\hat{\psi}_h - \pi_\Gamma(\bar{\psi}_h) + \underline{g}_h\right|_{H_\Gamma}^2 + \frac{1}{4}\left|\beta_{\Gamma,\varepsilon}(\bar{\psi}_h)\right|_{H_\Gamma}^2 - \int_\Gamma\partial_\nu\bar{\phi}_h\beta_{\Gamma,\varepsilon}(\bar{\psi}_h)\,d\Gamma,\end{aligned} \tag{3.44}$$

a.e. in $(0,T)$. Now, recalling (3.8), we can find a positive constant $\widetilde{M}_5$ that

$$\begin{aligned}\left|\int_\Gamma\partial_\nu\bar{\phi}_h\beta_\varepsilon(\bar{\psi}_h)\,d\Gamma - \int_\Gamma\partial_\nu\bar{\phi}_h\beta_{\Gamma,\varepsilon}(\bar{\psi}_h)\,d\Gamma\right| &\le \left|\partial_\nu\bar{\phi}_h\right|_{H_\Gamma}\left(\varrho\left|\beta_{\Gamma,\varepsilon}(\bar{\psi}_h)\right|_{H_\Gamma} + |\Gamma|^{1/2}c_0 + \left|\beta_{\Gamma,\varepsilon}(\bar{\psi}_h)\right|_{H_\Gamma}\right)\\ &\le \frac{1}{4}\left|\beta_{\Gamma,\varepsilon}(\bar{\psi}_h)\right|_{H_\Gamma}^2 + \widetilde{M}_5\left(1 + \left|\partial_\nu\bar{\phi}_h\right|_{H_\Gamma}^2\right),\end{aligned}$$

by Young's inequality. Finally, adding (3.43), (3.44), recalling the monotonicity of β_ε in (3.43) and $\beta_{\Gamma,\varepsilon}$ in (3.44), and using the above estimate, we infer that there exist a function $\Lambda_5 \in L^2(0,T)$ and a positive constant M_5, both independent of $h \in (0,h^{**}]$, $\tau,\sigma,\varepsilon \in (0,1]$ such that (3.42) holds a.e. in $(0,T)$. □

15222616, 2020, 11, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/mana.201900361 by Cnr Roma, Wiley Online Library on [27/02/2025]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Lemma 3.6. *There exists a function $\Lambda_6 \in L^2(0,T)$, independent of $h \in (0, h^{**}]$, $\tau, \sigma, \varepsilon \in (0,1]$, such that*

$$\left|\bar{\phi}_h(t)\right|_{H^2(\Omega)} + \left|\bar{\psi}_h(t)\right|_{H^2(\Gamma)} + \left|\partial_\nu \bar{\phi}_h(t)\right|_{H_\Gamma} \le \Lambda_6(t) \tag{3.45}$$

for a.a. $t \in (0,T)$.

Proof. We rewrite (3.18), (3.20), (3.22) in the following elliptic problem for $\bar{\phi}_h$ and $\bar{\psi}_h$

$$\begin{cases} -\Delta \bar{\phi}_h = \bar{\mu}_h - \tau \partial_t \hat{\phi}_h - \beta_\varepsilon(\bar{\phi}_h) - \pi(\bar{\phi}_h) + \underline{f}_h & \text{a.e. in } \Omega, \\ (\bar{\phi}_h)|_\Gamma = \bar{\psi}_h & \text{a.e. on } \Gamma, \\ -\Delta_\Gamma \bar{\psi}_h + \bar{\psi}_h + \partial_\nu \bar{\phi}_h = \bar{w}_h - \sigma \partial_t \hat{\psi}_h - \beta_{\Gamma,\varepsilon}(\bar{\psi}_h) - \pi_\Gamma(\bar{\psi}_h) + \underline{g}_h + \bar{\psi}_h & \text{a.e. on } \Gamma, \end{cases} \tag{3.46}$$

a.e. in $(0,T)$. Then it follows from [31, Lemma A.1] that the following estimate holds

$$\begin{aligned} &\left|\bar{\phi}_h\right|_{H^2(\Omega)} + \left|\bar{\psi}_h\right|_{H^2(\Gamma)} \\ &\quad \le C_{\mathrm{MZ}} \left(\left|\bar{\mu}_h - \tau \partial_t \hat{\phi}_h - \beta_\varepsilon(\bar{\phi}_h) - \pi(\bar{\phi}_h) + \underline{f}_h\right|_H + \left|\bar{w}_h - \sigma \partial_t \hat{\psi}_h - \beta_{\Gamma,\varepsilon}(\bar{\psi}_h) - \pi_\Gamma(\bar{\psi}_h) + \underline{g}_h + \bar{\psi}_h\right|_{H_\Gamma} \right) \end{aligned}$$

a.e. in $(0,T)$, where C_{MZ} is a positive constant independent of $h \in (0, h^{**}]$, $\tau, \sigma, \varepsilon \in (0,1]$. Accounting for (3.25), (3.41) and (3.42), we see that there exist a function $\widetilde{\Lambda}_6 \in L^2(0,T)$ and a positive constant $\widetilde{M}_6$, independent of $h \in (0, h^{**}]$, $\tau, \sigma, \varepsilon \in (0,1]$, such that

$$\left|\bar{\phi}_h(t)\right|_{H^2(\Omega)} + \left|\bar{\psi}_h(t)\right|_{H^2(\Gamma)} \le \widetilde{M}_6 \left(\widetilde{\Lambda}_6(t) + \left|\partial_\nu \bar{\phi}_h(t)\right|_{H_\Gamma} \right) \tag{3.47}$$

for a.a. $t \in (0,T)$. We observe that, for some $3/2 < s < 2$, there exists a positive constant C_7 such that

$$\left|\partial_\nu \bar{\phi}_h(t)\right|_{H_\Gamma} \le C_7 \left|\bar{\phi}_h(t)\right|_{H^s(\Omega)} \tag{3.48}$$

for a.a. $t \in (0,T)$ (see, e.g., [3, Thm. 2.25, p. 1.62]), and consequently, by the compactness inequality (see, e.g., [27, Lem. 5.1, p. 58] or [28, Thm. 16.4, p. 102]) as well as (3.47) and (3.48), we infer

$$\begin{aligned} \left|\bar{\phi}_h(t)\right|_{H^2(\Omega)} + \left|\bar{\psi}_h(t)\right|_{H^2(\Gamma)} + \left|\partial_\nu \bar{\phi}_h(t)\right|_{H_\Gamma} &\le \widetilde{M}_6 \left(\widetilde{\Lambda}_6(t) + C_7 \left|\bar{\phi}_h(t)\right|_{H^s(\Omega)} \right) + C_7 \left|\bar{\phi}_h(t)\right|_{H^s(\Omega)} \\ &\le \frac{1}{2} \left|\bar{\phi}_h(t)\right|_{H^2(\Omega)} + C_8 \left(\widetilde{\Lambda}_6(t) + \left|\bar{\phi}_h(t)\right|_V \right) \end{aligned}$$

for a.a. $t \in (0,T)$, whence, also by (3.25), we conclude that (3.45) holds for some function $\Lambda_6 \in L^2(0,T)$, independent of $h \in (0, h^{**}]$, $\tau, \sigma, \varepsilon \in (0,1]$. □

Now, using these uniform estimates, we can discuss the existence of weak solutions to the viscous problem for the original system (2.1)–(2.7), by taking $h \to 0$ and $\varepsilon \to 0$. The subscripts of τ and σ for functions mean the dependence on parameters $\tau, \sigma \in (0,1]$, however in the next proposition we omit them for simplicity.

Proposition 3.7. *Assume the (A1)–(A5) are satisfied. For each $\tau, \sigma \in (0,1]$, there exists a sextuplet $(\phi, \mu, \xi, \psi, w, \zeta)$ of functions*

$$\begin{aligned} &\phi := \phi_{\tau,\sigma} \in H^1(0,T;V^*) \cap L^\infty(0,T;V) \cap L^2\big(0,T;H^2(\Omega)\big), \\ &\partial_\nu \phi \in L^2\big(0,T;H_\Gamma\big), \quad \sqrt{\tau}\phi \in H^1(0,T;H), \\ &\mu := \mu_{\tau,\sigma} \in L^2(0,T;V), \quad \sqrt{\tau}\mu \in L^2(0,T;W), \quad \xi := \xi_{\tau,\sigma} \in L^2(0,T;H), \end{aligned}$$

15222616, 2020, 11, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/mana.201900361 by Cnr Roma, Wiley Online Library on [27/02/2025]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

$$\psi := \psi_{\tau,\sigma} \in H^1\big(0,T;V_\Gamma^*\big) \cap L^\infty\big(0,T;V_\Gamma\big) \cap L^2\big(0,T;H^2(\Gamma)\big), \quad \sqrt{\sigma}\psi \in H^1\big(0,T;H_\Gamma\big),$$
$$w := w_{\tau,\sigma} \in L^2\big(0,T;V_\Gamma\big), \quad \sqrt{\sigma}w \in L^2\big(0,T;H^2(\Gamma)\big), \quad \zeta := \zeta_{\tau,\sigma} \in L^2\big(0,T;H_\Gamma\big)$$

such that they satisfy the following viscous Cahn–Hilliard system:

$$\partial_t\phi - \Delta\mu = 0 \qquad \text{a.e. in } Q, \tag{3.49}$$
$$\mu = \tau\partial_t\phi - \Delta\phi + \xi + \pi(\phi) - f, \quad \xi \in \beta(\phi) \qquad \text{a.e. in } Q, \tag{3.50}$$
$$\partial_\nu\mu = 0 \qquad \text{a.e. on } \Sigma, \tag{3.51}$$
$$\phi|_\Gamma = \psi \qquad \text{a.e. on } \Sigma, \tag{3.52}$$
$$\partial_t\psi - \Delta_\Gamma w = 0 \qquad \text{a.e. on } \Sigma, \tag{3.53}$$
$$w = \sigma\partial_t\psi + \partial_\nu\phi - \Delta_\Gamma\psi + \zeta + \pi_\Gamma(\psi) - g, \quad \zeta \in \beta_\Gamma(\psi) \qquad \text{a.e. on } \Sigma, \tag{3.54}$$
$$\phi(0) = \phi_0 \quad \text{a.e. in } \Omega, \quad \psi(0) = \psi_0 \quad \text{a.e. on } \Gamma. \tag{3.55}$$

Proof. Let $\tau,\sigma \in (0,1]$. Recalling (3.25), (3.41), (3.42) and (3.45), we deduce that there exist positive constants M_6, M_7, independent of $h \in (0,h^{**}]$, $\tau,\sigma,\varepsilon \in (0,1]$, such that

$$|\bar\mu_h|_{L^2(0,T;V)} + |\bar w_h|_{L^2(0,T;V_\Gamma)} + \Big|\beta_\varepsilon\big(\bar\phi_h\big)\Big|_{L^2(0,T;H)} + \Big|\beta_{\Gamma,\varepsilon}\big(\bar\psi_h\big)\Big|_{L^2(0,T;H_\Gamma)}$$
$$+ \Big|\bar\phi_h\Big|_{L^2(0,T;H^2(\Omega))} + \Big|\bar\psi_h\Big|_{L^2(0,T;H^2(\Gamma))} + \Big|\partial_\nu\bar\phi_h\Big|_{L^2(0,T;H_\Gamma)} \le M_6, \tag{3.56}$$

$$\sqrt{\tau}|\bar\mu_h|_{L^2(0,T;W)} + \sqrt{\sigma}|\bar w_h|_{L^2(0,T;H^2(\Gamma))} \le M_7, \tag{3.57}$$

for all $h \in (0,h^{**}]$ and $\varepsilon \in (0,1]$, where also a comparison in (3.17), (3.19) and (3.21) has been used for (3.57). Next, using (3.14)–(3.16), we observe that

$$\Big|\hat\phi_h\Big|_{L^\infty(0,T;V)} \le \max\Big\{|\phi_0|_V, \Big|\bar\phi_h\Big|_{L^\infty(0,T;V)}\Big\} \le \sqrt{M_1} + |\phi_0|_V,$$
$$\tau\Big|\hat\phi_h - \bar\phi_h\Big|^2_{L^2(0,T;H)} = \frac{h^2}{3}\tau\Big|\partial_t\hat\phi_h\Big|^2_{L^2(0,T;H)} \le \frac{h^2}{3}M_1 \le \frac{1}{3}M_1,$$
$$\Big|h\partial_t\hat\mu_h\Big|_{L^2(0,T;H)} + \Big|\partial_t\hat\phi_h\Big|_{L^2(0,T;V^*)} \le 3\sqrt{M_1},$$
$$\Big|\hat\mu_h\Big|^2_{L^2(0,T;V)} \le \frac{h}{2}|\mu_0|^2_V + |\bar\mu_h|^2_{L^2(0,T;V)} \le M_6^2, \quad \Big|h\hat\mu_h\Big|_{L^2(0,T;V)} \le hM_6,$$
$$\Big|\hat\psi_h\Big|_{L^\infty(0,T;V_\Gamma)} \le \max\Big\{|\psi_0|_{V_\Gamma}, \Big|\bar\psi_h\Big|_{L^\infty(0,T;V_\Gamma)}\Big\} \le \sqrt{M_1} + |\psi_0|_{V_\Gamma},$$
$$\sigma\Big|\hat\psi_h - \bar\psi_h\Big|^2_{L^2(0,T;H_\Gamma)} = \frac{h^2}{3}\sigma\Big|\partial_t\hat\psi_h\Big|^2_{L^2(0,T;H_\Gamma)} \le \frac{h^2}{3}M_1 \le \frac{1}{3}M_1,$$
$$\Big|h\partial_t\hat w_h\Big|_{L^2(0,T;H_\Gamma)} + \Big|\partial_t\hat\psi_h\Big|_{L^2(0,T;V_\Gamma^*)} \le 3\sqrt{M_1},$$
$$\Big|\hat w_h\Big|^2_{L^2(0,T;V_\Gamma)} \le \frac{h}{2}|w_0|^2_{V_\Gamma} + |\bar w_h|^2_{L^2(0,T;V_\Gamma)} \le M_6^2, \quad \Big|h\hat w_h\Big|_{L^2(0,T;V_\Gamma)} \le hM_6,$$

for all $h \in (0,h^{**}]$ and $\varepsilon \in (0,1]$. Then, there exist functions $\phi_\varepsilon, \mu_\varepsilon, \psi_\varepsilon, w_\varepsilon$ and a subsequence $\{h_k\}_{k\in\mathbb{N}}$ of $h \to 0$ such that

$$\hat\phi_{h_k} \to \phi_\varepsilon \quad \text{weakly star in } H^1(0,T;V^*) \cap L^\infty(0,T;V) \text{ and strongly in } C\big([0,T];H\big),$$

15222616, 2020, 11, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/mana.201900361 by Cnr Roma, Wiley Online Library on [27/02/2025]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

$$\tau\hat{\phi}_{h_k} \to \tau\phi_\varepsilon \quad \text{weakly in } H^1(0,T;H),$$
$$\bar{\phi}_{h_k} \to \phi_\varepsilon \quad \text{weakly star in } L^\infty(0,T;V) \cap L^2\big(0,T;H^2(\Omega)\big) \text{ and strongly in } L^2(0,T;H),$$
$$\bar{\mu}_{h_k} \to \mu_\varepsilon \quad \text{weakly in } L^2(0,T;W),$$
$$h_k\hat{\mu}_{h_k} \to 0 \quad \text{weakly in } H^1(0,T;H) \text{ and strongly in } L^2(0,T;V),$$

analogously

$$\hat{\psi}_{h_k} \to \psi_\varepsilon \quad \text{weakly star in } H^1\big(0,T;V_\Gamma^*\big) \cap L^\infty\big(0,T;V_\Gamma\big) \text{ and strongly in } C\big([0,T];H_\Gamma\big),$$
$$\sigma\hat{\psi}_{h_k} \to \sigma\psi_\varepsilon \quad \text{weakly in } H^1\big(0,T;H_\Gamma\big),$$
$$\bar{\psi}_{h_k} \to \psi_\varepsilon \quad \text{weakly star in } L^\infty\big(0,T;V_\Gamma\big) \cap L^2\big(0,T;H^2(\Gamma)\big) \text{ and strongly in } L^2(0,T;H_\Gamma),$$
$$\bar{w}_{h_k} \to w_\varepsilon \quad \text{weakly in } L^2\big(0,T;H^2(\Gamma)\big),$$
$$h_k\hat{w}_{h_k} \to 0 \quad \text{weakly in } H^1\big(0,T;H_\Gamma\big) \text{ and strongly in } L^2\big(0,T;V_\Gamma\big),$$
$$\partial_\nu\bar{\phi}_{h_k} \to \partial_\nu\phi_\varepsilon \quad \text{weakly in } L^2\big(0,T;H_\Gamma\big)$$

as $k \to +\infty$, where we applied the compactness results [32, Sect. 8, Cor. 4] and (3.16) to obtain the strong convergences. We recall that τ and σ are positive and fixed in this limit procedure. Moreover, due to the Lipschitz continuity of β_ε, π, $\beta_{\Gamma,\varepsilon}$ and π_Γ, we have that

$$\beta_\varepsilon(\bar{\phi}_{h_k}) \to \beta_\varepsilon(\phi_\varepsilon), \qquad \pi(\bar{\phi}_{h_k}) \to \pi(\phi_\varepsilon) \qquad \text{strongly in } L^2(0,T;H),$$
$$\beta_{\Gamma,\varepsilon}(\bar{\psi}_{h_k}) \to \beta_{\Gamma,\varepsilon}(\psi_\varepsilon), \qquad \pi_\Gamma(\bar{\psi}_{h_k}) \to \pi_\Gamma(\psi_\varepsilon) \qquad \text{strongly in } L^2\big(0,T;H_\Gamma\big)$$

as $k \to +\infty$. Besides, it is not difficult to check that

$$f_h \to f \quad \text{strongly in } L^2(0,T;V),$$
$$g_h \to g \quad \text{strongly in } L^2\big(0,T;V_\Gamma\big)$$

as $h \to 0$ (in [11, Appendix] the argument is fully detailed). Based on all these convergence results, we can pass to the limit as $h_k \to 0$ in problem (3.17)–(3.23) and find that the quadruplet $(\phi_\varepsilon, \mu_\varepsilon, \psi_\varepsilon, w_\varepsilon)$ solves

$$\partial_t\phi_\varepsilon - \Delta\mu_\varepsilon = 0 \qquad \text{a.e. in } Q, \tag{3.58}$$
$$\mu_\varepsilon = \tau\partial_t\phi_\varepsilon - \Delta\phi_\varepsilon + \beta_\varepsilon(\phi_\varepsilon) + \pi(\phi_\varepsilon) - f \qquad \text{a.e. in } Q, \tag{3.59}$$
$$\partial_\nu\mu_\varepsilon = 0 \qquad \text{a.e. on } \Sigma, \tag{3.60}$$
$$(\phi_\varepsilon)|_\Gamma = \psi_\varepsilon \qquad \text{a.e. on } \Sigma, \tag{3.61}$$
$$\partial_t\psi_\varepsilon - \Delta_\Gamma w_\varepsilon = 0 \qquad \text{a.e. on } \Sigma, \tag{3.62}$$
$$w_\varepsilon = \partial_\nu\phi_\varepsilon + \sigma\partial_t\psi_\varepsilon - \Delta_\Gamma\psi_\varepsilon + \beta_{\Gamma,\varepsilon}(\psi_\varepsilon) + \pi_\Gamma(\psi_\varepsilon) - g \qquad \text{a.e. on } \Sigma, \tag{3.63}$$
$$\phi_\varepsilon(0) = \phi_0 \quad \text{a.e. in } \Omega, \qquad \psi_\varepsilon(0) = \psi_0 \quad \text{a.e. on } \Gamma. \tag{3.64}$$

Moreover, by weak or weak star lower semicontinuity of norms, we see that the following estimates hold (see (3.25), (3.56), (3.57))

$$|\partial_t\phi_\varepsilon|_{L^2(0,T;V^*)} \le \liminf_{k\to\infty} \big|\partial_t\hat{\phi}_{h_k}\big|_{L^2(0,T;V^*)} \le 3\sqrt{M_1}, \tag{3.65}$$

15222616, 2020, 11, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/mana.201900361 by Cnr Roma, Wiley Online Library on [27/02/2025]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

$$|\phi_\varepsilon|_{L^\infty(0,T;V)} \le \liminf_{k\to\infty} \left|\hat\phi_{h_k}\right|_{L^\infty(0,T;V)} \le \sqrt{M_1} + |\phi_0|_V, \tag{3.66}$$

$$|\phi_\varepsilon|_{L^2(0,T;H^2(\Omega))} \le \liminf_{k\to\infty} \left|\bar\phi_{h_k}\right|_{L^2(0,T;H^2(\Omega))} \le M_6, \tag{3.67}$$

$$\sqrt{\tau}|\partial_t\phi_\varepsilon|_{L^2(0,T;H)} \le \liminf_{k\to\infty} \sqrt{\tau}\left|\partial_t\hat\phi_{h_k}\right|_{L^2(0,T;H)} \le \sqrt{M_1}, \tag{3.68}$$

$$|\mu_\varepsilon|_{L^2(0,T;V)} \le \liminf_{k\to\infty} \left|\bar\mu_{h_k}\right|_{L^2(0,T;V)} \le M_6, \tag{3.69}$$

$$\sqrt{\tau}|\mu_\varepsilon|_{L^2(0,T;W)} \le \liminf_{k\to\infty} \sqrt{\tau}\left|\bar\mu_{h_k}\right|_{L^2(0,T;W)} \le M_7, \tag{3.70}$$

analogously

$$|\partial_t\psi_\varepsilon|_{L^2(0,T;V_\Gamma^*)} \le \liminf_{k\to\infty} \left|\partial_t\hat\psi_{h_k}\right|_{L^2(0,T;V_\Gamma^*)} \le 3\sqrt{M_1}, \tag{3.71}$$

$$|\psi_\varepsilon|_{L^\infty(0,T;V_\Gamma)} \le \liminf_{k\to\infty} \left|\hat\psi_{h_k}\right|_{L^\infty(0,T;V_\Gamma)} \le \sqrt{M_1} + |\psi_0|_{V_\Gamma}, \tag{3.72}$$

$$|\psi_\varepsilon|_{L^2(0,T;H^2(\Gamma))} \le \liminf_{k\to\infty} \left|\bar\psi_{h_k}\right|_{L^2(0,T;H^2(\Gamma))} \le M_6, \tag{3.73}$$

$$\sqrt{\sigma}|\partial_t\psi_\varepsilon|_{L^2(0,T;H_\Gamma)} \le \liminf_{k\to\infty} \sqrt{\sigma}\left|\partial_t\hat\psi_{h_k}\right|_{L^2(0,T;H_\Gamma)} \le \sqrt{M_1}, \tag{3.74}$$

$$|w_\varepsilon|_{L^2(0,T;V_\Gamma)} \le \liminf_{k\to\infty} \left|\bar w_{h_k}\right|_{L^2(0,T;V_\Gamma)} \le M_6, \tag{3.75}$$

$$\sqrt{\sigma}|w_\varepsilon|_{L^2(0,T;H^2(\Gamma))} \le \liminf_{k\to\infty} \sqrt{\sigma}\left|\bar w_{h_k}\right|_{L^2(0,T;H^2(\Gamma))} \le M_7, \tag{3.76}$$

$$|\partial_\nu\phi_\varepsilon|_{L^2(0,T;H_\Gamma)} \le \liminf_{k\to\infty} \left|\partial_\nu\bar\phi_{h_k}\right|_{L^2(0,T;H_\Gamma)} \le M_6, \tag{3.77}$$

with

$$\left|\beta_\varepsilon(\phi_\varepsilon)\right|_{L^2(0,T;H)} = \lim_{k\to\infty} \left|\beta_\varepsilon\left(\bar\phi_{h_k}\right)\right|_{L^2(0,T;H)} \le M_6, \tag{3.78}$$

$$\left|\beta_{\Gamma,\varepsilon}(\psi_\varepsilon)\right|_{L^2(0,T;H_\Gamma)} = \lim_{k\to\infty} \left|\beta_{\Gamma,\varepsilon}\left(\bar\psi_{h_k}\right)\right|_{L^2(0,T;H_\Gamma)} \le M_6. \tag{3.79}$$

Due to the uniform estimates (3.65)–(3.79), we are able to pass to the limit along a subsequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ of ε, in the problem (3.58)–(3.64) by finding elements

$$\phi := \phi_{\tau,\sigma},\ \ \mu := \mu_{\tau,\sigma},\ \ \xi := \xi_{\tau,\sigma},\ \ \psi := \psi_{\tau,\sigma},\ \ w := w_{\tau,\sigma},\ \ \zeta := \zeta_{\tau,\sigma}$$

such that

$$\begin{aligned}
\phi_{\varepsilon_k} \to \phi \quad & \text{weakly star in } H^1(0,T;V^*) \cap L^\infty(0,T;V) \cap L^2\left(0,T;H^2(\Omega)\right) \\
& \text{and strongly in } C\left([0,T];H\right) \cap L^2(0,T;V), \\
\tau\phi_{\varepsilon_k} \to \tau\phi \quad & \text{weakly in } H^1(0,T;H), \\
\mu_{\varepsilon_k} \to \mu \quad & \text{weakly in } L^2(0,T;W), \\
\beta_{\varepsilon_k}\left(\phi_{\varepsilon_k}\right) \to \xi \quad & \text{weakly in } L^2(0,T;H), \\
\psi_{\varepsilon_k} \to \psi \quad & \text{weakly star in } H^1\left(0,T;V_\Gamma^*\right) \cap L^\infty\left(0,T;V_\Gamma\right) \cap L^2\left(0,T;H^2(\Gamma)\right) \\
& \text{and strongly in } C\left([0,T];H_\Gamma\right) \cap L^2\left(0,T;V_\Gamma\right),
\end{aligned}$$

15222616, 2020, 11, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/mana.201900361 by Cnr Roma, Wiley Online Library on [27/02/2025]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

$$\sigma\psi_{\varepsilon_k} \to \sigma\psi \quad \text{weakly in } H^1(0,T;H_\Gamma),$$
$$w_{\varepsilon_k} \to w \quad \text{weakly in } L^2(0,T;H^2(\Gamma)),$$
$$\beta_{\Gamma,\varepsilon_k}(\psi_{\varepsilon_k}) \to \zeta \quad \text{weakly in } L^2(0,T;H_\Gamma),$$
$$\partial_\nu \phi_{\varepsilon_k} \to \partial_\nu \phi \quad \text{weakly in } L^2(0,T;H_\Gamma)$$

as $k \to +\infty$, due to the compactness theorems again. Now, we observe that $\xi \in \beta(\phi)$ a.e. in Q and $\zeta \in \beta_\Gamma(\psi)$ a.e. on Σ, due to the maximal monotonicity of β and β_Γ, and the weak-strong convergence for $\beta_{\varepsilon_k}(\phi_{\varepsilon_k})$ and ϕ_{ε_k} in $L^2(0,T;H)$, and for $\beta_{\Gamma,\varepsilon_k}(\psi_{\varepsilon_k})$ and ψ_{ε_k} in $L^2(0,T;H_\Gamma)$, respectively. Finally, we observe that

$$\pi(\phi_{\varepsilon_k}) \to \pi(\phi) \quad \text{strongly in } C([0,T];H),$$
$$\pi_\Gamma(\psi_{\varepsilon_k}) \to \pi_\Gamma(\psi) \quad \text{strongly in } C([0,T];H_\Gamma).$$

Thus, we can pass to the limit as $k \to \infty$ in the regularized problem (3.58)–(3.64) to obtain the viscous Cahn–Hilliard system (3.49)–(3.55). □

3.3 | From viscous to pure Cahn–Hilliard system

As a summary of the previous subsection, we can find a sextuplet $(\phi_{\tau,\sigma}, \mu_{\tau,\sigma}, \xi_{\tau,\sigma}, \psi_{\tau,\sigma}, w_{\tau,\sigma}, \zeta_{\tau,\sigma})$ of functions, depending on $\tau, \sigma \in (0,1]$, such that it satisfies the viscous Cahn–Hilliard system (3.49)–(3.55). Moreover, in (3.65)–(3.79), from weak or weak star lower semicontinuity of norms, we also know that there exists a positive constant M_8, independent of $\tau, \sigma \in (0,1]$, such that

$$\begin{aligned}
&|\phi_{\tau,\sigma}|_{H^1(0,T;V^*)} + |\phi_{\tau,\sigma}|_{L^\infty(0,T;V)} + |\phi_{\tau,\sigma}|_{L^2(0,T;H^2(\Omega))} + \sqrt{\tau}|\phi_{\tau,\sigma}|_{H^1(0,T;H)} \\
&\quad + |\mu_{\tau,\sigma}|_{L^2(0,T;V)} + \sqrt{\tau}|\mu_{\tau,\sigma}|_{L^2(0,T;W)} + |\psi_{\tau,\sigma}|_{H^1(0,T;V_\Gamma^*)} + |\psi_{\tau,\sigma}|_{L^\infty(0,T;V_\Gamma)} \\
&\quad + |\psi_{\tau,\sigma}|_{L^2(0,T;H^2(\Gamma))} + \sqrt{\sigma}|\psi_{\tau,\sigma}|_{H^1(0,T;H_\Gamma)} + |w_{\tau,\sigma}|_{L^2(0,T;V_\Gamma)} + \sqrt{\sigma}|w_{\tau,\sigma}|_{L^2(0,T;H^2(\Gamma))} \\
&\quad + |\partial_\nu \phi_{\tau,\sigma}|_{L^2(0,T;H_\Gamma)} + |\xi_{\tau,\sigma}|_{L^2(0,T;H)} + |\zeta_{\tau,\sigma}|_{L^2(0,T;H_\Gamma)} \le M_8.
\end{aligned} \tag{3.80}$$

Now we are in a position to prove Theorem 2.3.

Proof of Theorem 2.3. We obtain from (3.49), (3.51) and (3.53), the following variational formulations:

$$\langle \partial_t \phi_{\tau,\sigma}, z \rangle_{V^*,V} + \int_\Omega \nabla \mu_{\tau,\sigma} \cdot \nabla z \, dx = 0 \quad \text{for all } z \in V, \tag{3.81}$$

$$\langle \partial_t \psi_{\tau,\sigma}, z_\Gamma \rangle_{V^*,V} + \int_\Gamma \nabla_\Gamma w_{\tau,\sigma} \cdot \nabla_\Gamma z_\Gamma \, d\Gamma = 0 \quad \text{for all } z_\Gamma \in V_\Gamma, \tag{3.82}$$

a.e. in $(0,T)$. At this point, we can pass to the limit as either $\tau \to 0$ or $\sigma \to 0$, or both $\tau, \sigma \to 0$ in order to obtain a partially viscous Cahn–Hilliard system or a pure Cahn–Hilliard system at the limit. Let us detail only the last case with $(\tau,\sigma) \to (0,0)$ along a joint subsequence (τ_k, σ_k). We see that there exists a sextuplet $(\phi, \mu, \xi, \psi, w, \zeta)$ such that

$$\phi_{\tau_k,\sigma_k} \to \phi \quad \text{weakly star in } H^1(0,T;V^*) \cap L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega))$$
$$\text{and strongly in } C([0,T];H) \cap L^2(0,T;V),$$
$$\tau_k \phi_{\tau_k,\sigma_k} \to 0 \quad \text{strongly in } H^1(0,T;H),$$
$$\mu_{\tau_k,\sigma_k} \to \mu \quad \text{weakly in } L^2(0,T;V),$$

15222616, 2020, 11, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/mana.201900361 by Cnr Roma, Wiley Online Library on [27/02/2025]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

$$
\begin{aligned}
\xi_{\tau_k,\sigma_k} &\to \xi \quad \text{weakly in } L^2(0,T;H),\\
\psi_{\tau_k,\sigma_k} &\to \psi \quad \text{weakly star in } H^1\big(0,T;V_\Gamma^*\big)\cap L^\infty\big(0,T;V_\Gamma\big)\cap L^2\big(0,T;H^2(\Gamma)\big)\\
&\qquad\quad \text{and strongly in } C\big([0,T];H_\Gamma\big)\cap L^2\big(0,T;V_\Gamma\big),\\
\sigma_k\psi_{\tau_k,\sigma_k} &\to 0 \quad \text{strongly in } H^1\big(0,T;H_\Gamma\big),\\
w_{\tau_k,\sigma_k} &\to w \quad \text{weakly in } L^2\big(0,T;V_\Gamma\big),\\
\zeta_{\tau_k,\sigma_k} &\to \zeta \quad \text{weakly in } L^2\big(0,T;H_\Gamma\big),\\
\partial_\nu\phi_{\tau_k,\sigma_k} &\to \partial_\nu\phi \quad \text{weakly in } L^2\big(0,T;H_\Gamma\big)
\end{aligned}
$$

as $k\to+\infty$. Different from the previous subsection, we can pass to the limit in

$$
\xi_{\tau_k,\sigma_k}\in\beta\big(\phi_{\tau_k,\sigma_k}\big)\quad\text{a.e. in } Q,\qquad \zeta_{\tau_k,\sigma_k}\in\beta_\Gamma\big(\psi_{\tau_k,\sigma_k}\big)\quad\text{a.e. in }\Sigma
$$

just using the demi-closedness of β and β_Γ, respectively, to obtain the same inclusions at the limit. To complete this limiting procedure, we pass to the limit in (3.50), (3.52), (3.54), (3.55), (3.81), (3.82) to obtain (2.8)–(2.13). Hence, we arrive at the conclusion. □

Let us remark that, if we let only one of the parameters τ,σ go to 0, then we also have the convergence

$$
\mu_{\tau,\sigma_k}\to\mu_\tau\quad\text{weakly in } L^2(0,T;W)\quad\text{if only } \sigma_k\to 0,
$$

or the convergence

$$
w_{\tau_k,\sigma}\to w_\sigma\quad\text{weakly in } L^2\big(0,T;H^2(\Gamma)\big)\quad\text{if only } \tau_k\to 0.
$$

In these cases, we can keep the smoothness of the time derivative, more precisely, $\phi_\tau\in H^1(0,T;H)$ if $\tau>0$, or $\psi_\sigma\in H^1\big(0,T;H_\Gamma\big)$ if $\sigma>0$, respectively.

Proof of Theorem 2.4. We now prove a continuous dependence estimates with two weak solutions

$$
\big(\phi^{(i)},\mu^{(i)},\xi^{(i)},\psi^{(i)},w^{(i)},\zeta^{(i)}\big)\quad\text{for } i=1,2,
$$

corresponding to the initial data

$$
\big(\phi_0^{(i)},\psi_0^{(i)}\big)\in V^*\times V_\Gamma^*\quad\text{for } i=1,2,
$$

satisfying (2.17), (2.18), and the sources

$$
\big(f^{(i)},g^{(i)}\big)\in L^2(0,T;V^*)\times L^2\big(0,T;V_\Gamma^*\big)\quad\text{for } i=1,2.
$$

We take the difference of (2.8) and choose $z:=1$ to obtain that

$$
\Big\langle\partial_t\big(\phi^{(1)}-\phi^{(2)}\big),1\Big\rangle_{V^*,V}=0
$$

a.e. in $(0,T)$, whence

$$
\Big\langle\phi^{(1)}(t)-\phi^{(2)}(t),1\Big\rangle_{V^*,V}=\Big\langle\phi_0^{(1)}-\phi_0^{(2)},1\Big\rangle_{V^*,V}=0
$$

15222616, 2020, 11, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/mana.201900361 by Cnr Roma, Wiley Online Library on [27/02/2025]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

for all $t \in [0,T]$. Then, we can take $z := \mathcal{N}_\Omega(\phi^{(1)} - \phi^{(2)})$ as a test function in the difference of (2.8) and obtain

$$\frac{1}{2}\left|\phi^{(1)}(t) - \phi^{(2)}(t)\right|^2_{V_{0*}} + \int_0^t \int_\Omega (\mu^{(1)} - \mu^{(2)})(\phi^{(1)} - \phi^{(2)})\, dx\, ds = \frac{1}{2}\left|\phi_0^{(1)} - \phi_0^{(2)}\right|^2_{V_{0*}} \tag{3.83}$$

for all $t \in [0,T]$. By operating on the difference of (2.11) in the same way, that is, $z_\Gamma := 1$ first and $z_\Gamma := \mathcal{N}_\Gamma(\psi^{(1)} - \psi^{(2)})$ second, we obtain the similar formula

$$\frac{1}{2}\left|\psi^{(1)}(t) - \psi^{(2)}(t)\right|^2_{V_{\Gamma,0*}} + \int_0^t \int_\Gamma (w^{(1)} - w^{(2)})(\psi^{(1)} - \psi^{(2)})\, d\Gamma\, ds = \frac{1}{2}\left|\psi_0^{(1)} - \psi_0^{(2)}\right|^2_{V_{\Gamma,0*}} \tag{3.84}$$

for all $t \in [0,T]$. Next, we multiply the difference of the equalities in (2.9) by $\phi^{(1)} - \phi^{(2)}$ and integrate the resultant with respect to space and time. Using (2.10) and (2.12), we infer that

$$\begin{aligned}
&\int_0^t \int_\Omega (\mu^{(1)} - \mu^{(2)})(\phi^{(1)} - \phi^{(2)})\, dx\, ds + \int_0^t \int_\Gamma (w^{(1)} - w^{(2)})(\psi^{(1)} - \psi^{(2)})\, d\Gamma\, ds \\
&= \int_0^t \int_\Omega \left|\nabla(\phi^{(1)} - \phi^{(2)})\right|^2 dx\, ds + \int_0^t \int_\Gamma \left|\nabla_\Gamma(\psi^{(1)} - \psi^{(2)})\right|^2 d\Gamma\, ds \\
&\quad + \int_0^t \int_\Omega (\xi^{(1)} - \xi^{(2)})(\phi^{(1)} - \phi^{(2)})\, dx\, ds + \int_0^t \int_\Gamma (\zeta^{(1)} - \zeta^{(2)})(\psi^{(1)} - \psi^{(2)})\, d\Gamma\, ds \\
&\quad + \int_0^t \int_\Omega (\pi(\phi^{(1)}) - \pi(\phi^{(2)}))(\phi^{(1)} - \phi^{(2)})\, dx\, ds - \int_0^t \langle f^{(1)} - f^{(2)}, \phi^{(1)} - \phi^{(2)} \rangle_{V^*,V}\, ds \\
&\quad + \int_0^t \int_\Gamma (\pi_\Gamma(\psi^{(1)}) - \pi_\Gamma(\psi^{(2)}))(\psi^{(1)} - \psi^{(2)})\, d\Gamma\, ds - \int_0^t \langle g^{(1)} - g^{(2)}, \psi^{(1)} - \psi^{(2)} \rangle_{V_\Gamma^*,V_\Gamma}\, ds,
\end{aligned}$$

for all $t \in [0,T]$. Then, we take the sum of (3.83), (3.84) and combine with the above equality. Thanks to the monotonicity of β, β_Γ, the Lipschitz continuity of π, π_Γ, and the Poincaré–Wirtinger inequality, we deduce that

$$\begin{aligned}
&\left|\phi^{(1)}(t) - \phi^{(2)}(t)\right|^2_{V_{0*}} + \left|\psi^{(1)}(t) - \psi^{(2)}(t)\right|^2_{V_{\Gamma,0*}} + 2\int_0^t \int_\Omega \left|\nabla(\phi^{(1)} - \phi^{(2)})\right|^2 dx\, ds + 2\int_0^t \int_\Gamma \left|\nabla_\Gamma(\psi^{(1)} - \psi^{(2)})\right|^2 d\Gamma\, ds \\
&\le \left|\phi_0^{(1)} - \phi_0^{(2)}\right|^2_{V_{0*}} + \left|\psi_0^{(1)} - \psi_0^{(2)}\right|^2_{V_{\Gamma,0*}} + 2L\int_0^t \int_\Omega \left|\phi^{(1)} - \phi^{(2)}\right|^2 dx\, ds \\
&\quad + 2L_\Gamma \int_0^t \int_\Gamma \left|\psi^{(1)} - \psi^{(2)}\right|^2 d\Gamma\, ds + \frac{1}{2}\int_0^t \left|\phi^{(1)} - \phi^{(2)}\right|^2_{V_0} ds + \frac{C_P}{2}\left|f^{(1)} - f^{(2)}\right|^2_{L^2(0,T;V^*)} \\
&\quad + \frac{1}{2}\int_0^t \left|\psi^{(1)} - \psi^{(2)}\right|^2_{V_{\Gamma,0}} ds + \frac{C_P}{2}\left|g^{(1)} - g^{(2)}\right|^2_{L^2(0,T;V_\Gamma^*)},
\end{aligned}$$

for all $t \in [0,T]$. Here, we observe that

$$\begin{aligned}
2L\int_0^t \int_\Omega \left|\phi^{(1)} - \phi^{(2)}\right|^2 dx\, ds &= 2L\int_0^t \int_\Omega \nabla\mathcal{N}_\Omega(\phi^{(1)} - \phi^{(2)}) \cdot \nabla(\phi^{(1)} - \phi^{(2)})\, dx\, ds \\
&\le \frac{1}{2}\int_0^t \int_\Omega \left|\nabla(\phi^{(1)} - \phi^{(2)})\right|^2 dx\, ds + 2L^2 \int_0^t \int_\Omega \left|\nabla\mathcal{N}_\Omega(\phi^{(1)} - \phi^{(2)})\right|^2 dx\, ds \\
&= \frac{1}{2}\int_0^t \left|\phi^{(1)} - \phi^{(2)}\right|^2_{V_0} ds + 2L^2 \int_0^t \left|\phi^{(1)} - \phi^{(2)}\right|^2_{V_{0*}} ds,
\end{aligned}$$

15222616, 2020, 11, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/mana.201900361 by Cnr Roma, Wiley Online Library on [27/02/2025]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

and similarly,

$$2L_\Gamma \int_0^t \int_\Gamma \left|\psi^{(1)} - \psi^{(2)}\right|^2 d\Gamma\, ds \le \frac{1}{2}\int_0^t \left|\psi^{(1)} - \psi^{(2)}\right|^2_{V_{\Gamma,0}} ds + 2L_\Gamma^2 \int_0^t \left|\psi^{(1)} - \psi^{(2)}\right|^2_{V_{\Gamma,0*}} ds,$$

for all $t \in [0,T]$. Therefore, applying the Gronwall lemma and invoking the equivalences of norms, we conclude the proof of Theorem 2.4. As an immediate sequence, the continuous dependence implies the uniqueness of the weak solution obtained in Theorem 2.3. □

The continuous dependence estimate can be extended to the viscous or partially viscous cases, with the following modification: there exists a constant C such that the inequality

$$\begin{aligned}
&\left|\phi^{(1)} - \phi^{(2)}\right|_{C([0,T];V^*)} + \left|\psi^{(1)} - \psi^{(2)}\right|_{C([0,T];V_\Gamma^*)} + \left|\phi^{(1)} - \phi^{(2)}\right|_{L^2(0,T;V)} \\
&\quad + \left|\psi^{(1)} - \psi^{(2)}\right|_{L^2(0,T;V_\Gamma)} + \sqrt{\tau}\left|\phi^{(1)} - \phi^{(2)}\right|_{C([0,T];H)} + \sqrt{\sigma}\left|\psi^{(1)} - \psi^{(2)}\right|_{C([0,T];H_\Gamma)} \\
&\quad \le C\Big\{ \left|\phi_0^{(1)} - \phi_0^{(2)}\right|_{V^*} + \left|\psi_0^{(1)} - \psi_0^{(2)}\right|_{V_\Gamma^*} + \left|f^{(1)} - f^{(2)}\right|_{L^2(0,T;V^*)} + \left|g^{(1)} - g^{(2)}\right|_{L^2(0,T;V_\Gamma^*)} \\
&\qquad + \sqrt{\tau}\left|\phi_0^{(1)} - \phi_0^{(2)}\right|_H + \sqrt{\sigma}\left|\psi_0^{(1)} - \psi_0^{(2)}\right|_{H_\Gamma} \Big\}
\end{aligned}$$

holds for $\tau \ge 0$ and $\sigma \ge 0$.

4 | EXISTENCE OF STRONG SOLUTION

In this section, we establish a regularity result, which leads to the existence of a strong solution in the case of the pure Cahn–Hilliard system (2.1)–(2.7).

Now, we point out the additional assumptions we need on the given data:

(A6) $f \in H^1(0,T;H)$ and $g \in H^1(0,T;H_\Gamma)$;
(A7) $-\Delta\phi_0 + \beta_\varepsilon(\phi_0) + \pi(\phi_0) - f(0)$ remains bounded in V as $\varepsilon \to 0$, and $\partial_\nu\phi_0 - \Delta_\Gamma\psi_0 + \beta_{\Gamma,\varepsilon}(\psi_0) + \pi_\Gamma(\psi_0) - g(0)$ remains bounded in V_Γ as $\varepsilon \to 0$.

Our third result of this paper is related to the existence of strong solutions:

Theorem 4.1. *Under assumptions (A1)–(A7), the unique weak solution of problem (2.1)–(2.7) obtained in Theorem 2.1 is a strong one in the sense of Definition 2.2.*

Proof. Let us take the difference of Equations (3.1) written for n and $n-1$. Then multiplying the resultant by

$$z := \mathcal{N}_\Omega\big(\phi_{n+1} + h\mu_{n+1} - \phi_n - h\mu_n\big)$$

and integrating the resultant over Ω, we obtain

$$\frac{1}{h}\int_\Omega \big(\phi_{n+1} + h\mu_{n+1} - 2\phi_n - 2h\mu_n + \phi_{n-1} + h\mu_{n-1}\big) z\, dx + \int_\Omega \nabla(\mu_{n+1} - \mu_n)\cdot\nabla z\, dx = 0,$$

whence

$$\begin{aligned}
&\frac{1}{2h}|\phi_{n+1} + h\mu_{n+1} - \phi_n - h\mu_n|^2_{V_{0*}} - \frac{1}{2h}|\phi_n + h\mu_n - \phi_{n-1} - h\mu_{n-1}|^2_{V_{0*}} \\
&\quad + \frac{1}{2h}|\phi_{n+1} + h\mu_{n+1} - 2\phi_n - 2h\mu_n + \phi_{n-1} + h\mu_{n-1}|^2_{V_{0*}} + \int_\Omega (\mu_{n+1} - \mu_n)(\phi_{n+1} + h\mu_{n+1} - \phi_n - h\mu_n)\, dx = 0, \qquad (4.1)
\end{aligned}$$

15222616, 2020, 11, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/mana.201900361 by Cnr Roma, Wiley Online Library on [27/02/2025]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

for all $n = 1, \dots, N-1$. Next, we use (3.2) to derive that

$$\begin{aligned}
&\int_\Omega (\mu_{n+1} - \mu_n)(\phi_{n+1} + h\mu_{n+1} - \phi_n - h\mu_n)\, dx \\
&= \frac{\tau}{2h}|\phi_{n+1} - \phi_n|_H^2 - \frac{\tau}{2h}|\phi_n - \phi_{n-1}|_H^2 + \frac{\tau}{2h}|\phi_{n+1} - 2\phi_n + \phi_{n-1}|_H^2 \\
&\quad + \int_\Omega \left|\nabla(\phi_{n+1} - \phi_n)\right|^2 dx - \int_\Gamma \partial_\nu(\phi_{n+1} - \phi_n)(\psi_{n+1} - \psi_n)\, d\Gamma \\
&\quad + \int_\Omega (\beta_\varepsilon(\phi_{n+1}) - \beta_\varepsilon(\phi_n))(\phi_{n+1} - \phi_n)\, dx + \int_\Omega (\pi(\phi_{n+1}) - \pi(\phi_n))(\phi_{n+1} - \phi_n)\, dx \\
&\quad - \int_\Omega (f_n - f_{n-1})(\phi_{n+1} - \phi_n)\, dx + h\int_\Omega |\mu_{n+1} - \mu_n|^2\, dx,
\end{aligned} \tag{4.2}$$

for all $n = 0, 1, \dots, N-1$. Next, in order to treat the fifth term on the right hand side of (4.2), we recall Equation (3.6) and infer that

$$\begin{aligned}
&-\int_\Gamma \partial_\nu(\phi_{n+1} - \phi_n)(\psi_{n+1} - \psi_n)\, d\Gamma \\
&= -\int_\Gamma (w_{n+1} - w_n)(\psi_{n+1} - \psi_n)\, d\Gamma + \frac{\sigma}{2h}|\psi_{n+1} - \psi_n|_{H_\Gamma}^2 - \frac{\sigma}{2h}|\psi_n - \psi_{n-1}|_{H_\Gamma}^2 \\
&\quad + \frac{\sigma}{2h}|\psi_{n+1} - 2\psi_n + \psi_{n-1}|_{H_\Gamma}^2 + \int_\Gamma \left|\nabla_\Gamma(\psi_{n+1} - \psi_n)\right|^2 d\Gamma \\
&\quad + \int_\Gamma (\beta_{\Gamma,\varepsilon}(\psi_{n+1}) - \beta_{\Gamma,\varepsilon}(\psi_n))(\psi_{n+1} - \psi_n)\, d\Gamma + \int_\Gamma (\pi_\Gamma(\psi_{n+1}) - \pi_\Gamma(\psi_n))(\psi_{n+1} - \psi_n)\, d\Gamma \\
&\quad - \int_\Gamma (g_n - g_{n-1})(\psi_{n+1} - \psi_n)\, d\Gamma,
\end{aligned} \tag{4.3}$$

for all $n = 0, 1, \dots, N-1$. We now exploit (3.5) to discuss the first term on the right hand side of (4.3). Taking the test function

$$z_\Gamma := \mathcal{N}_\Gamma(\psi_{n+1} + hw_{n+1} - \psi_n - hw_n),$$

we have that

$$\begin{aligned}
&\frac{1}{h}\int_\Gamma (\psi_{n+1} + hw_{n+1} - \psi_n - hw_n - (\psi_n + hw_n - \psi_{n-1} - hw_{n-1}))z_\Gamma\, d\Gamma \\
&\quad = (\Delta_\Gamma(w_{n+1} - w_n), z_\Gamma)_{H_\Gamma} \\
&\quad = -\int_\Gamma \nabla_\Gamma(w_{n+1} - w_n)\cdot\nabla_\Gamma z_\Gamma\, d\Gamma \\
&\quad = -\int_\Gamma (w_{n+1} - w_n)(\psi_{n+1} - \psi_n)\, d\Gamma - h\int_\Gamma |w_{n+1} - w_n|^2\, d\Gamma,
\end{aligned} \tag{4.4}$$

for all $n = 1, \dots, N-1$.

Now, by collecting the identities (4.1)–(4.4) and using (A4), we deduce from Young's inequality that

$$\begin{aligned}
&\frac{1}{2h}|\phi_{n+1} + h\mu_{n+1} - \phi_n - h\mu_n|_{V_{0*}}^2 + \frac{1}{2h}|\psi_{n+1} + hw_{n+1} - \psi_n - hw_n|_{V_{\Gamma,0*}}^2 \\
&\quad - \frac{1}{2h}|\phi_n + h\mu_n - \phi_{n-1} - h\mu_{n-1}|_{V_{0*}}^2 - \frac{1}{2h}|\psi_n + hw_n - \psi_{n-1} - hw_{n-1}|_{V_{\Gamma,0*}}^2
\end{aligned}$$

15222616, 2020, 11, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/mana.201900361 by Cnr Roma, Wiley Online Library on [27/02/2025]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

$$
\begin{aligned}
&+\frac{1}{2h}|\phi_{n+1}+h\mu_{n+1}-2\phi_n-2h\mu_n+\phi_{n-1}+h\mu_{n-1}|^2_{V_{0*}}\\
&+\frac{1}{2h}|\psi_{n+1}+hw_{n+1}-2\psi_n-2hw_n+\psi_{n-1}+hw_{n-1}|^2_{V_{\Gamma,0*}}\\
&+\frac{\tau}{2h}|\phi_{n+1}-\phi_n|^2_H+\frac{\sigma}{2h}|\psi_{n+1}-\psi_n|^2_{H_\Gamma}-\frac{\tau}{2h}|\phi_n-\phi_{n-1}|^2_H-\frac{\sigma}{2h}|\psi_n-\psi_{n-1}|^2_{H_\Gamma}\\
&+\frac{\tau}{2h}|\phi_{n+1}-2\phi_n+\phi_{n-1}|^2_H+\frac{\sigma}{2h}|\psi_{n+1}-2\psi_n+\psi_{n-1}|^2_{H_\Gamma}+\int_\Omega\left|\nabla(\phi_{n+1}-\phi_n)\right|^2dx\\
&+\int_\Gamma\left|\nabla_\Gamma(\psi_{n+1}-\psi_n)\right|^2d\Gamma+h\int_\Omega|\mu_{n+1}-\mu_n|^2\,dx+h\int_\Gamma|w_{n+1}-w_n|^2\,d\Gamma\\
&\le(L+1)\int_\Omega|\phi_{n+1}-\phi_n|^2\,dx+\frac14\int_\Omega|f_n-f_{n-1}|^2\,dx+(L_\Gamma+1)\int_\Gamma|\psi_{n+1}-\psi_n|^2\,d\Gamma+\frac14\int_\Gamma|g_n-g_{n-1}|^2\,d\Gamma
\end{aligned}
$$

for $n=1,2,\dots,N-1$. We divide the above inequality by h and sum up for $n=1$ to $n=m$, by finding that

$$
\begin{aligned}
&\frac12\left|\frac{\phi_{m+1}-\phi_m}{h}+\mu_{m+1}-\mu_m\right|^2_{V_{0*}}+\frac12\left|\frac{\psi_{m+1}-\psi_m}{h}+w_{m+1}-w_m\right|^2_{V_{\Gamma,0*}}\\
&\quad+\frac{\tau}{2}\left|\frac{\phi_{m+1}-\phi_m}{h}\right|^2_H+\frac{\sigma}{2}\left|\frac{\psi_{m+1}-\psi_m}{h}\right|^2_{H_\Gamma}+\sum_{n=1}^m h\int_\Omega\left|\nabla\left(\frac{\phi_{n+1}-\phi_n}{h}\right)\right|^2dx\\
&\quad+\sum_{n=1}^m h\int_\Gamma\left|\nabla_\Gamma\left(\frac{\psi_{n+1}-\psi_n}{h}\right)\right|^2d\Gamma+\sum_{n=1}^m|\mu_{n+1}-\mu_n|^2_H+\sum_{n=1}^m|w_{n+1}-w_n|^2_{H_\Gamma}\\
&\le\frac12\left|\frac{\phi_1-\phi_0}{h}+\mu_1-\mu_0\right|^2_{V_{0*}}+\frac12\left|\frac{\psi_1-\psi_0}{h}+w_1-w_0\right|^2_{V_{\Gamma,0*}}+\frac{\tau}{2}\left|\frac{\phi_1-\phi_0}{h}\right|^2_H\\
&\quad+\frac{\sigma}{2}\left|\frac{\psi_1-\psi_0}{h}\right|^2_{H_\Gamma}+(L+1)\sum_{n=1}^m h\left|\frac{\phi_{n+1}-\phi_n}{h}\right|^2_H+\frac14\sum_{n=1}^m h\left|\frac{f_n-f_{n-1}}{h}\right|^2_H\\
&\quad+(L_\Gamma+1)\sum_{n=1}^m h\left|\frac{\psi_{n+1}-\psi_n}{h}\right|^2_{H_\Gamma}+\frac14\sum_{n=1}^m h\left|\frac{g_n-g_{n-1}}{h}\right|^2_{H_\Gamma},
\end{aligned}
\tag{4.5}
$$

for all $m=1,2,\dots,N-1$. In order to estimate the first four terms on the right hand side of (4.5), we multiply (3.1) at $n=0$ by

$$
\frac1h\mathcal{N}_\Omega(\phi_1+h\mu_1-\phi_0-h\mu_0)
$$

and obtain

$$
\left|\frac{\phi_1-\phi_0}{h}+\mu_1-\mu_0\right|^2_{V_{0*}}+\int_\Omega\mu_1\left(\frac{\phi_1-\phi_0}{h}+\mu_1-\mu_0\right)dx=0.
\tag{4.6}
$$

From (3.2) and (3.7), it follows that

$$
\begin{aligned}
&\int_\Omega\mu_1\left(\frac{\phi_1-\phi_0}{h}+\mu_1-\mu_0\right)dx\\
&\quad=\tau\int_\Omega\left|\frac{\phi_1-\phi_0}{h}\right|^2dx+\int_\Omega\nabla\phi_1\cdot\nabla\left(\frac{\phi_1-\phi_0}{h}\right)dx-\int_\Gamma\partial_\nu\phi_1\left(\frac{\psi_1-\psi_0}{h}\right)d\Gamma
\end{aligned}
$$

15222616, 2020, 11, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/mana.201900361 by Cnr Roma, Wiley Online Library on [27/02/2025]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

$$+\int_\Omega (\beta_\varepsilon(\phi_1)-\beta_\varepsilon(\phi_0))\left(\frac{\phi_1-\phi_0}{h}\right)dx+\int_\Omega \beta_\varepsilon(\phi_0)\left(\frac{\phi_1-\phi_0}{h}\right)dx$$
$$+\int_\Omega (\pi(\phi_1)-\pi(\phi_0))\left(\frac{\phi_1-\phi_0}{h}\right)dx+\int_\Omega \pi(\phi_0)\left(\frac{\phi_1-\phi_0}{h}\right)dx$$
$$-\int_\Omega f_0\left(\frac{\phi_1-\phi_0}{h}\right)dx+\int_\Omega |\mu_1-\mu_0|^2dx. \tag{4.7}$$

Next, we multiply (3.6) at $n=0$ by $(\psi_1-\psi_0)/h$ to obtain

$$-\int_\Gamma \partial_\nu\phi_1\left(\frac{\psi_1-\psi_0}{h}\right)d\Gamma$$
$$=-\int_\Gamma w_1\left(\frac{\psi_1-\psi_0}{h}\right)d\Gamma+\sigma\int_\Gamma\left|\frac{\psi_1-\psi_0}{h}\right|^2d\Gamma+\int_\Gamma \nabla_\Gamma\psi_1\cdot\nabla_\Gamma\left(\frac{\psi_1-\psi_0}{h}\right)d\Gamma$$
$$+\int_\Gamma (\beta_{\Gamma,\varepsilon}(\psi_1)-\beta_{\Gamma,\varepsilon}(\psi_0))\left(\frac{\psi_1-\psi_0}{h}\right)d\Gamma+\int_\Gamma \beta_{\Gamma,\varepsilon}(\psi_0)\left(\frac{\psi_1-\psi_0}{h}\right)d\Gamma$$
$$+\int_\Gamma (\pi_\Gamma(\psi_1)-\pi_\Gamma(\psi_0))\left(\frac{\psi_1-\psi_0}{h}\right)d\Gamma+\int_\Gamma \pi_\Gamma(\psi_0)\left(\frac{\psi_1-\psi_0}{h}\right)d\Gamma-\int_\Gamma g_0\left(\frac{\psi_1-\psi_0}{h}\right)d\Gamma. \tag{4.8}$$

Besides, we go back to (3.5) at $n=0$, multiply it by

$$\frac{1}{h}\mathcal{N}_\Gamma(\psi_1+hw_1-\psi_0-hw_0),$$

and integrate the resultant over Γ, to obtain

$$\left|\frac{\psi_1-\psi_0}{h}+w_1-w_0\right|^2_{V_{\Gamma,0*}}+\int_\Gamma w_1\left(\frac{\psi_1-\psi_0}{h}+w_1-w_0\right)d\Gamma=0. \tag{4.9}$$

Collecting (4.6)–(4.9) and applying the assumption (A7) along with the monotonicity of β_ε, $\beta_{\Gamma,\varepsilon}$, the Lipschitz continuity of π, π_Γ and Young's inequality, we deduce that there exists a positive constant $\widetilde{M}_9$, independent of $h\in(0,h^{**}]$, $\tau,\sigma,\varepsilon\in(0,1]$, such that

$$\left|\frac{\phi_1-\phi_0}{h}+\mu_1-\mu_0\right|^2_{V_{0*}}+\left|\frac{\psi_1-\psi_0}{h}+w_1-w_0\right|^2_{V_{\Gamma,0*}}+\tau\int_\Omega\left|\frac{\phi_1-\phi_0}{h}\right|^2dx+\sigma\int_\Gamma\left|\frac{\psi_1-\psi_0}{h}\right|^2d\Gamma$$
$$+h\int_\Omega\left|\nabla\left(\frac{\phi_1-\phi_0}{h}\right)\right|^2dx+h\int_\Gamma\left|\nabla_\Gamma\left(\frac{\psi_1-\psi_0}{h}\right)\right|^2d\Gamma+\int_\Omega|\mu_1-\mu_0|^2dx+\int_\Gamma|w_1-w_0|^2d\Gamma$$
$$\le-\int_\Omega(-\Delta\phi_0+\beta_\varepsilon(\phi_0)+\pi(\phi_0)-f_0)\left(\frac{\phi_1-\phi_0}{h}\right)dx+Lh\left|\frac{\phi_1-\phi_0}{h}\right|^2_H$$
$$-\int_\Gamma(\partial_\nu\phi_0-\Delta_\Gamma\psi_0+\beta_{\Gamma,\varepsilon}(\psi_0)+\pi_\Gamma(\psi_0)-g_0)\left(\frac{\psi_1-\psi_0}{h}\right)d\Gamma+L_\Gamma h\left|\frac{\psi_1-\psi_0}{h}\right|^2_{H_\Gamma}$$
$$\le\widetilde{M}_9+\frac{1}{2}\left|\frac{\phi_1-\phi_0}{h}+\mu_1-\mu_0\right|^2_{V_{0*}}+\frac{1}{2}|\mu_1-\mu_0|^2_H+Lh\left|\frac{\phi_1-\phi_0}{h}\right|^2_H$$
$$+\frac{1}{2}\left|\frac{\psi_1-\psi_0}{h}+w_1-w_0\right|^2_{V_{\Gamma,0*}}+\frac{1}{2}|w_1-w_0|^2_{H_\Gamma}+L_\Gamma h\left|\frac{\psi_1-\psi_0}{h}\right|^2_{H_\Gamma}. \tag{4.10}$$

15222616, 2020, 11, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/mana.201900361 by Cnr Roma, Wiley Online Library on [27/02/2025]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Then, we can add (4.5) and (4.10) to obtain that

$$\begin{aligned}
&\frac{1}{2}\left|\frac{\phi_{m+1}-\phi_m}{h}+\mu_{m+1}-\mu_m\right|^2_{V_{0*}}+\frac{1}{2}\left|\frac{\psi_{m+1}-\psi_m}{h}+w_{m+1}-w_m\right|^2_{V_{\Gamma,0*}}\\
&\quad+\frac{\tau}{2}\left|\frac{\phi_{m+1}-\phi_m}{h}\right|^2_H+\frac{\sigma}{2}\left|\frac{\psi_{m+1}-\psi_m}{h}\right|^2_{H_\Gamma}+\sum_{n=0}^m h\int_\Omega\left|\nabla\left(\frac{\phi_{n+1}-\phi_n}{h}\right)\right|^2dx\\
&\quad+\sum_{n=0}^m h\int_\Gamma\left|\nabla_\Gamma\left(\frac{\psi_{n+1}-\psi_n}{h}\right)\right|^2d\Gamma+\frac{1}{2}\sum_{n=0}^m|\mu_{n+1}-\mu_n|^2_H+\frac{1}{2}\sum_{n=0}^m|w_{n+1}-w_n|^2_{H_\Gamma}\\
&\le M_9+(L+1)\sum_{n=0}^m h\left|\frac{\phi_{n+1}-\phi_n}{h}\right|^2_H+(L_\Gamma+1)\sum_{n=0}^m h\left|\frac{\psi_{n+1}-\psi_n}{h}\right|^2_{H_\Gamma},
\end{aligned}$$

for all $m=1,2,\dots,N-1$, where M_9 is a positive constant depending on $|\partial_t f|_{L^2(0,T;H)}$, $|\partial_t g|_{L^2(0,T;H_\Gamma)}$ and $\widetilde{M}_9$. Finally, we apply the compactness inequality on the right hand side of that above inequality as follows:

$$\begin{aligned}
&(L+1)\sum_{n=0}^m h\left|\frac{\phi_{n+1}-\phi_n}{h}\right|^2_H+(L_\Gamma+1)\sum_{n=0}^m h\left|\frac{\psi_{n+1}-\psi_n}{h}\right|^2_{H_\Gamma}\\
&\le\delta\sum_{n=0}^m h\int_\Omega\left|\nabla\left(\frac{\phi_{n+1}-\phi_n}{h}\right)\right|^2dx+\delta\sum_{n=0}^m h\int_\Gamma\left|\nabla_\Gamma\left(\frac{\psi_{n+1}-\psi_n}{h}\right)\right|^2d\Gamma\\
&\quad+C_\delta\sum_{n=0}^m h\left|\frac{\phi_{n+1}-\phi_n}{h}\right|^2_{V^*}+C_\delta\sum_{n=0}^m h\left|\frac{\psi_{n+1}-\psi_n}{h}\right|^2_{V_\Gamma^*},
\end{aligned}$$

for all $\delta>0$. Observe now that by taking $\delta:=1/2$, the last two terms are already bounded due to (3.25). Thus, we conclude that there exists a positive constant M_{10}, independent of $h\in(0,h^{**}]$, $\tau,\sigma,\varepsilon\in(0,1]$, such that

$$\begin{aligned}
&\left|\partial_t\hat\phi_h+h\partial_t\hat\mu_h\right|^2_{L^\infty(0,T;V^*)}+\left|\partial_t\hat\psi_h+h\partial_t\hat w_h\right|^2_{L^\infty(0,T;V_\Gamma{}^*)}+\tau\left|\partial_t\hat\phi_h\right|^2_{L^\infty(0,T;H)}\\
&\quad+\sigma\left|\partial_t\hat\psi_h\right|^2_{L^\infty(0,T;H_\Gamma)}+\left|\partial_t\hat\phi_h\right|^2_{L^2(0,T;V)}+\left|\partial_t\hat\psi_h\right|^2_{L^2(0,T;V_\Gamma)}+h\left|\partial_t\hat\mu_h\right|^2_{L^2(0,T;H)}\\
&\quad+h\left|\partial_t\hat w_h\right|^2_{L^2(0,T;H_\Gamma)}+\left|h\partial_t\hat\mu_h\right|^2_{L^\infty(0,T;H)}+\left|h\partial_t\hat w_h\right|^2_{L^\infty(0,T;H_\Gamma)}\le M_{10},
\end{aligned}\tag{4.11}$$

for all $h\in(0,h^{**}]$, where we used

$$|h\partial_t\hat\mu_h|^2_{L^\infty(0,T;H)}=\max_{n=0,1,\dots,N-1}|\mu_{n+1}-\mu_n|^2_H\le\sum_{n=0}^{N-1}|\mu_{n+1}-\mu_n|^2_H$$

for the estimate of $h\partial_t\hat\mu_h$ in $L^\infty(0,T;H)$. Similar holds for $h\partial_t\hat w_h$. The subsequent estimates repeat the previous ones, that is, from Lemmas 3.3 to 3.6 and Proposition 3.7 as follows:

- ▷ From (3.34), using (4.11) we infer that

$$|\bar\mu_h-m_\Omega(\bar\mu_h)|_{L^\infty(0,T;V)}+|\bar w_h-m_\Gamma(\bar w_h)|_{L^\infty(0,T;V_\Gamma)}\le\Lambda_1(t),$$

  for all $h\in(0,h^{**}]$ and $\varepsilon\in(0,1]$, with Λ_1 being defined by (3.36). Now Λ_1 is bounded in $L^\infty(0,T)$;
- ▷ arguing as in (3.37)–(3.39) and checking the right hand side of (3.39), using (3.25) and (4.11) we arrive at (3.35) with $\Lambda_2\in L^\infty(0,T)$;

15222616, 2020, 11, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/mana.201900361 by Cnr Roma, Wiley Online Library on [27/02/2025]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

- we can repeat the estimates in Lemmas 3.4 and 3.5, so that we arrive at (3.40)–(3.42) with $\Lambda_3, \Lambda_4, \Lambda_5 \in L^\infty(0,T)$;
- now, we consider the same elliptic system (3.46) and observe that we can derive (3.45) with $\Lambda_6 \in L^\infty(0,T)$;
- instead of (3.56) and (3.57), here we derive from the above modifications the final estimates

$$
\begin{aligned}
&|\bar\mu_h|_{L^\infty(0,T;V)} + |\bar w_h|_{L^\infty(0,T;V_\Gamma)} + \left|\beta_\varepsilon(\bar\phi_h)\right|_{L^\infty(0,T;H)} + \left|\beta_{\Gamma,\varepsilon}(\bar\psi_h)\right|_{L^\infty(0,T;H_\Gamma)} \\
&\quad + \left|\bar\phi_h\right|_{L^\infty(0,T;H^2(\Omega))} + \left|\bar\psi_h\right|_{L^\infty(0,T;H^2(\Gamma))} + \left|\partial_\nu\bar\phi_h\right|_{L^\infty(0,T;H_\Gamma)} \le M_{11}, \\
&\sqrt{\tau}|\bar\mu_h|_{L^\infty(0,T;W)} + \sqrt{\sigma}|\bar w_h|_{L^\infty(0,T;H^2(\Gamma))} \le M_{11},
\end{aligned}
$$

for all $h \in (0, h^{**}]$ and $\varepsilon \in (0,1]$, where M_{11} is a positive constant independent of $h \in (0, h^{**}]$, $\tau, \sigma, \varepsilon \in (0,1]$.

Thus, we have obtained sufficient additional estimates that can be extended to the limit functions as $h_k \to 0$, by weak or weak star lower semicontinuity of norms. In particular, the approximate solution $(\phi_\varepsilon, \mu_\varepsilon, \psi_\varepsilon, w_\varepsilon)$ to problem (3.58)–(3.64), which is unique due to the continuous dependence estimate stated in Theorem 2.4, additionally satisfies (cf. (3.65)–(3.79))

$$
\begin{aligned}
&|\phi_\varepsilon|_{W^{1,\infty}(0,T;V^*)} + |\phi_\varepsilon|_{H^1(0,T;V)} + |\phi_\varepsilon|_{L^\infty(0,T;H^2(\Omega))} + \sqrt{\tau}|\phi_\varepsilon|_{W^{1,\infty}(0,T;H)} \\
&\quad + |\mu_\varepsilon|_{L^\infty(0,T;V)} + |\mu_\varepsilon|_{L^2(0,T;W\cap H^3(\Omega))} + \sqrt{\tau}|\mu_\varepsilon|_{L^\infty(0,T;W)} \\
&\quad + |\psi_\varepsilon|_{W^{1,\infty}(0,T;V_\Gamma^*)} + |\psi_\varepsilon|_{H^1(0,T;V_\Gamma)} + |\psi_\varepsilon|_{L^\infty(0,T;H^2(\Gamma))} + \sqrt{\sigma}|\psi_\varepsilon|_{W^{1,\infty}(0,T;H_\Gamma)} \\
&\quad + |w_\varepsilon|_{L^\infty(0,T;V_\Gamma)} + |w_\varepsilon|_{L^2(0,T;H^3(\Gamma))} + \sqrt{\sigma}|w_\varepsilon|_{L^\infty(0,T;H^2(\Gamma))} \\
&\quad + |\partial_\nu\phi_\varepsilon|_{L^\infty(0,T;H_\Gamma)} + \left|\beta_\varepsilon(\phi_\varepsilon)\right|_{L^\infty(0,T;H)} + \left|\beta_{\Gamma,\varepsilon}(\psi_\varepsilon)\right|_{L^\infty(0,T;H_\Gamma)} \le M_{12},
\end{aligned}
$$

for all $\varepsilon \in (0,1]$, where M_{12} is a positive constant independent of $\tau, \sigma, \varepsilon \in (0,1]$. Let us point out that, in order to obtain the above estimate, we have to make comparison of terms in (3.58) with (3.60) and (3.62), as well as to apply the elliptic regularity results. In this framework, the final regularity of the sextuplet $(\phi, \mu, \xi, \psi, w, \zeta)$ solving problem (2.1)–(2.7) is

$$
\begin{aligned}
&\phi \in W^{1,\infty}(0,T;V^*) \cap H^1(0,T;V) \cap L^\infty\big(0,T;H^2(\Omega)\big), \\
&\mu \in L^\infty(0,T;V) \cap L^2\big(0,T;W\cap H^3(\Omega)\big), \quad \xi \in L^\infty(0,T;H), \\
&\psi \in W^{1,\infty}\big(0,T;V_\Gamma^*\big) \cap H^1(0,T;V_\Gamma) \cap L^\infty\big(0,T;H^2(\Gamma)\big), \\
&w \in L^\infty(0,T;V_\Gamma) \cap L^2\big(0,T;H^3(\Gamma)\big), \quad \zeta \in L^\infty\big(0,T;H_\Gamma\big)
\end{aligned}
$$

and under these regularities, (2.8) and (2.10) can be replaced by

$$
\begin{aligned}
&\partial_t\phi - \Delta\mu = 0 \qquad \text{a.e. in } Q, \\
&\partial_\nu\mu = 0 \qquad \text{a.e. on } \Sigma, \\
&\partial_t\psi - \Delta_\Gamma w = 0 \qquad \text{a.e. on } \Sigma.
\end{aligned}
$$

Moreover, in order to obtain the regularities for μ and w stated above, we simply use the weak star lower semicontinuity property. Thus, we complete the proof of Theorem 4.1. □

ACKNOWLEDGEMENTS

This work was started during a visit of P. Colli and T. Fukao at the School of Mathematical Sciences of Fudan University: the contributed support and warm hospitality of the School are very gratefully acknowledged. P. Colli also acknowledges other support from the Italian Ministry of Education, University and Research (MIUR): Dipartimenti di Eccellenza Program (2018–2022) – Dept. of Mathematics "F. Casorati", University of Pavia, and the GNAMPA (Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni) of INdAM (Istituto Nazionale di Alta Matematica). T. Fukao acknowl-

15222616, 2020, 11, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/mana.201900361 by Cnr Roma, Wiley Online Library on [27/02/2025]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

edges the supports from the JSPS KAKENHI Grant-in-Aid for Scientific Research(C), Japan Grant Number 17K05321 and the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University. H. Wu acknowledges the support from the NNSFC Grant No. 11631011 and the Shanghai Center for Mathematical Sciences at Fudan University.

ORCID

Pierluigi Colli https://orcid.org/0000-0002-7921-5041
Takeshi Fukao https://orcid.org/0000-0003-4899-6890
Hao Wu https://orcid.org/0000-0003-0342-4709

REFERENCES

[1] V. Barbu, *Nonlinear differential equations of monotone types in Banach spaces*, Springer, London, 2010.
[2] H. Brézis, *Opérateurs maximaux monotones et semi-groupes de contractions dans les especes de Hilbert*, North-Holland, Amsterdam, 1973 (French).
[3] F. Brezzi and G. Gilardi, *Finite element handbook, Chapters 1–3*, (H. Kardestuncer and D. H. Norrie, eds.), McGraw–Hill Book Co., New York, 1987.
[4] J. W. Cahn and J. E. Hilliard, *Free energy of a nonuniform system I. Interfacial free energy*, J. Chem. Phys **2** (1958), 258–267.
[5] L. Calatroni and P. Colli, *Global solution to the Allen–Cahn equation with singular potentials and dynamic boundary conditions*, Nonlinear Anal. **79** (2013), 12–27.
[6] L. Cherfils, S. Gatti, and A. Miranville, *A variational approach to a Cahn–Hilliard model in a domain with nonpermeable walls*, J. Math. Sci. (N.Y.) **189** (2013), 604–636.
[7] L. Cherfils and M. Petcu, *A numerical analysis of the Cahn–Hilliard equation with non-permeable walls*, Numer. Math. **128** (2014), 518–549.
[8] P. Colli and T. Fukao, *Equation and dynamic boundary condition of Cahn–Hilliard type with singular potentials*, Nonlinear Anal. **127** (2015), 413–433.
[9] P. Colli, G. Gilardi, and J. Sprekels, *On the longtime behavior of a viscous Cahn–Hilliard system with convection and dynamic boundary conditions*, J. Elliptic Parabol. Equ. **4** (2018), 327–347.
[10] P. Colli, G. Gilardi, and J. Sprekels, *Well-posedness and regularity for a generalized fractional Cahn–Hilliard system*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. **30** (2019), 437–478.
[11] P. Colli and S. Kurima, *Global existence for a phase separation system deduced from the entropy balance*, Nonlinear Anal. **190** (2020), 111613.
[12] T. Fukao and H. Wu, *Separation property and convergence to equilibrium for the equation and dynamic boundary condition of Cahn–Hilliard type with singular potential*, Asymptot. Anal. (to appear).
[13] T. Fukao and N. Yamazaki, *A boundary control problem for the equation and dynamic boundary condition of Cahn–Hilliard type*, Solvability, Regularity, Optimal Control of Boundary Value Problems for PDEs, Springer INdAM Ser., vol. 22, Springer, 2017, pp. 255–280.
[14] T. Fukao, S. Yoshikawa, and S. Wada, *Structure-preserving finite difference schemes for the Cahn–Hilliard equation with dynamic boundary conditions in the one-dimensional case*, Commun. Pure Appl. Anal. **16** (2017), 1915–1938.
[15] C. Gal, *A Cahn–Hilliard model in bounded domains with permeable walls*, Math. Methods Appl. Sci. **29** (2006), 2009–2036.
[16] C. Gal, *Robust family of exponential attractors for a conserved Cahn–Hilliard model with singularly perturbed boundary conditions*, Commun. Pure Appl. Anal. **7** (2008), 819–836.
[17] C. Gal and H. Wu, *Asymptotic behavior of a Cahn–Hilliard equation with Wentzell boundary conditions and mass conservation*, Discrete Contin. Dyn. Syst. **22** (2008), 1041–1063.
[18] H. Garcke and P. Knopf, *Weak solutions of the Cahn–Hilliard system with dynamic boundary conditions: a gradient flow approach*, SIAM J. Math. Anal. **52** (2020), 340–369.
[19] G. Gilardi, A. Miranville, and G. Schimperna, *On the Cahn–Hilliard equation with irregular potentials and dynamic boundary conditions*, Commun. Pure Appl. Anal. **8** (2009), 881–912.
[20] G. Gilardi and J. Sprekels, *Asymptotic limits and optimal control for the Cahn–Hilliard system with convection and dynamic boundary conditions*, Nonlinear Anal. **178** (2019), 1–31.
[21] G. R. Goldstein, A. Miranville, and G. Schimperna, *A Cahn–Hilliard model in a domain with non-permeable walls*, Phys. D **240** (2011), 754–766.
[22] A. Grigor'yan, *Heat kernel and analysis on manifolds*, Amer. Math. Soc., International Press, Boston, 2009.
[23] E. Hebey, *Sobolev spaces on Riemannian manifolds*, Lecture Notes in Math., vol. 1635, Springer, Berlin, 1996.
[24] N. Kajiwara, *Global well-posedness for a Cahn–Hilliard equation on bounded domains with permeable and non-permeable walls in maximal regularity spaces*, Adv. Math. Sci. Appl. **27** (2018), 277–298.
[25] N. Kenmochi, M. Niezgódka, and I. Pawłow, *Subdifferential operator approach to the Cahn–Hilliard equation with constraint*, J. Differential Equations **117** (1995), 320–354.
[26] M. Kubo, *The Cahn–Hilliard equation with time-dependent constraint*, Nonlinear Anal. **75** (2012), 5672–5685.
[27] J. L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod Gauthier-Villas, Paris, 1968 (French).
[28] J. L. Lions and E. Magenes, *Non-homogeneous boundary value problems and applications I*, Springer-Verlag, New York–Heidelberg, 1972.

15222616, 2020, 11, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/mana.201900361 by Cnr Roma, Wiley Online Library on [27/02/2025]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

[29] C. Liu and H. Wu, *An energetic variational approach for the Cahn–Hilliard equation with dynamic boundary conditions: model derivation and mathematical analysis*, Arch. Ration. Mech. Anal. **233** (2019), 167–247.
[30] A. Miranville and S. Zelik, *Robust exponential attractors for Cahn–Hilliard type equations with singular potentials*, Math. Methods Appl. Sci. **27** (2004), 545–582.
[31] A. Miranville and S. Zelik, *Exponential attractors for the Cahn–Hilliard equation with dynamical boundary conditions*, Math. Meth. Appl. Sci. **28** (2005), 709–735.
[32] J. Simon, *Compact sets in the spaces* $L^p(0,T;B)$, Ann. Mat. Pura. Appl. (4) **146** (1987), 65–96.
[33] H. Wu, *The Cahn–Hilliard equation with a new class of dynamic boundary conditions*, Theory of Evolution Equation and Mathematical Analysis of Nonlinear Phenomena, RIMS Kôkyûroku, vol. 2090, Kyoto University, 2018, pp. 117–131.

How to cite this article: Colli P, Fukao T, Wu H. On a transmission problem for equation and dynamic boundary condition of Cahn–Hilliard type with nonsmooth potentials. *Mathematische Nachrichten*. 2020;293:2051–2081. https://doi.org/10.1002/mana.201900361

15222616, 2020, 11, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/mana.201900361 by Cnr Roma, Wiley Online Library on [27/02/2025]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License