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Lagrangian evolution of field gradient tensor invariants in magneto-hydrodynamic theory

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ABSTRACT

In 1982 in a series of works Vielliefosse [1, 2] discussed a nonlinear homogeneous evolution equation for the velocity gradient tensor in fluid dynamics. Later Cantwell [3] extended this formalism to the non-homogeneous case including the effects of viscous diffusion and cross derivatives of pressure field. Here, we derive the evolution equations of the geometrical invariants of the magnetic and velocity field gradient tensors in the case of magneto-hydrodynamics for both non-homogeneous and homogeneous cases, i.e., considering or neglecting viscous effects and source terms. The inclusion of dissipation effects and higher-order gradient terms introduces a non trivial evolution of invariants, which can be treated as a stochastic evolution equation. Conversely, in the homogeneous case, the magnetic field invariants do not evolve, i.e., the magnetic field line topology is conserved, while the corresponding velocity invariants are affected by magnetic forces. By writing the equations of the velocity field invariants as a dynamical system we can identify the role of the different terms in the evolution equations. In detail, in the homogenous case we show that the term associated with the current density drives transitions between hyperbolic and elliptical structures. Evolution equations are also discussed in the perspective of an application to the analysis of magneto-hydrodynamic turbulence.

1. Introduction

Turbulence in fluids and magnetized plasmas is one of the most debated topics in physics [4–6]. Indeed, in spite of its wide spreading in nature, turbulence still presents some features that are poorly understood as, for instance, the phenomenon of intermittency. Another specific issue showing unknown features is the generation of structures and their evolution in time, as depending on the observational spatial scale. Indeed, interactions between different topological structures in space plasmas have been recognized to play a fundamental role in the energy transfer across scales and in the dissipation mechanisms which would involve the occurrence of reconnection processes at the sub-ion scales [7,8].

Additional information on the topology of multiscale structures and their evolution in fluid and magnetohydrodynamic turbulence can be inferred from the study of the statistics geometrical invariants of field gradient tensors and their evolution from a Lagrangian point of view. Indeed, the study of coarse-grained gradient tensor of velocity and magnetic fields can provide information on several physical processes occurring in the inertial range such as vortex stretching, dissipative structures, etc [7–11].

The first attempt in fluid dynamics was by Vieillefosse [1,2] that investigated the dynamics of the gradient tensor of the fluid velocity, $A_{ij} = \partial_i u_j$, in the approximation of an inviscid fluid. Later, Cantwell [3] moving from a restricted Euler equation derived an equation for the Lagrangian evolution of geometrical invariants associated with the velocity gradient tensor. These invariants in the case of incompressible fluids are the coefficients of the characteristic equation of the velocity gradient tensor,

$$\lambda^3 + P\lambda^2 + Q\lambda + R = 0, \tag{1}$$

where $P = -A_{ii} = 0$ for incompressible fluids, while $Q = -\frac{1}{2}A_{ij}A_{ji}$ and $R = -\frac{1}{3}A_{ik}A_{kj}A_{ji}$ (here repeated indices are meant to be summed on) are related to the trace of products of the velocity gradient tensor. In detail, these quantities are invariant under SO(3)-group. The associated dynamical equations of these quanti-

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ties are

$$\frac{d}{dt}Q + 3R = -A_{ik}H_{ki} \tag{2}$$

$$\frac{d}{dt}R - \frac{2}{3}Q^2 = -A_{ij}A_{jk}H_{ki} \tag{3}$$

where H_{ij} contains the source and the dissipative terms of the Navier-Stokes equation for the velocity gradient tensor. The solutions of the evolution equations of the invariants in the limit of $H_{ij} = 0$ have been widely studied in Cantwell [3,12].

Later, in the framework of fluid turbulence studies Chertkov et al. [9] developed a phenomenological stochastic model for the coarse-grained velocity gradient capable of providing a prediction for the probability distribution functions of Q and R invariants. This model allows us to get several information on the nonlinear dynamics related to energy transfer and vortex stretching along the inertial range. Direct numerical simulations (DNS), as well as, shell-models for the evolution of the invariants statistics $p(R, Q)$ of the coarse-grained velocity gradient tensor in the inertial domain were able to reproduce the topological and geometrical features of real turbulent flows, including the alignment between vorticity and the strain-rate and the typical teardrop shape of the joint probability density $p(R, Q)$ [13–16]. The relevance of the approach based on the analysis of the features of the velocity gradient tensor and its invariants were also outlined by experimental studies [17].

In the framework of astrophysical and space plasma, turbulence plays an important role in many processes, such as mass transport/diffusion, plasma heating, stochastic acceleration, magnetic reconnection, etc. [5,18]. In such a situation the dynamics of magnetofluids is governed by magnetohydrodynamic turbulence, which shows a higher degree of complexity with respect to the ordinary fluid turbulence due to the role of the magnetic field. Indeed, due to the highly non-Gaussian and strongly long-range correlated character of magnetic and velocity field fluctuations, understanding and modeling these fluctuations up to the typical scales of the largest eddies/structures are of a general interest. Furthermore, in the framework of space plasmas the dynamics of magnetic and velocity fields is greatly controlled by the formation of multiscale structures, whose characterization and evolution are still lacking. Till now, the largest part of the studies done on interplanetary and space plasma turbulence is devoted to the analysis of the spectral properties, intermittency, Alfvénic versus non-Alfvénic features of the magnetic and velocity field fluctuations, coherence, etc. [6], mainly using single-point measurements. Although these approaches provided some advances in the comprehension of turbulence in magnetized space plasmas [5,6], less is known on the topology of structures involved in the nonlinear dynamics of the energy transfer across the inertial range and in the dissipation mechanisms. Indeed, the characterization of turbulent structures and their evolution play a fundamental role in understanding and characterizing the magnetofluid dynamics. In this framework, theoretical and observational studies on the magnetic and velocity field gradient tensors and their invariants could provide additional information.

Recently, Consolini et al. [11,19] and Quattrociocchi et al. [20] attempted an analysis of coarse-grained velocity and magnetic field gradient tensors to characterize the features of the magnetic and velocity field structures via the gradient tensor SO(3) geometrical invariants using in-situ satellite measurements from the ESA Cluster mission. The results provided a clear similarity of the joint probability distribution $p(R, Q)$ of the velocity gradient geometric invariants to that observed in the low end of the inertial range of fluid turbulence [11]. In particular, it was found evidence for a pronounced increase in the joint statistics along the so-called *Vieillefosse tail*, which support the occurrence of dissipation/dissipation-production due to vortex stretching. On the other hand, Quattrociocchi et al. [20], analysing the joint statistics of the topological invariants of the coarse-grained magnetic field gradient tensor, evidenced how in the inertial range this is a function of the distance from the proton inertial length scale and it is compatible with a change of the fluctuation field dimensionality at the smallest scales. Furthermore, they found the evidence of an increasing role of the ingoing spiral saddle and current-associated dissipation structures at small scales, where dissipation is expected to occur. In particular, the analysis of the topological invariants suggested that, although tube-like and sheet-like topologies are present, the magnetic field lines are mainly elliptic and heating is mainly due to dissipation in current layers and current-associated topologies. Similar results were found also in the case of numerical simulations [10].

Later, Bandyopadhyay et al. [7] used the symmetric/anti-symmetric parts of the magnetic and velocity field gradient tensors to quantify the kinetic dissipation in turbulent space plasmas at sub-ion scales in the Earth's magnetosheath using MMS mission data.They showed how dissipation is clearly localized near strong current sheets. More recently, Hnat et al. [8] analysed the magnetic topology of convected structures in the solar wind studying the joint statistics $p(R, Q)$ of invariants using multipoint measurements from the ESA-Cluster mission. Their results evidenced the existence of different types of structures, plasmoids, flux ropes and X-points, which provide the evidence of the role of turbulence in solar wind dissipation and heating mechanisms.

These set of observational studies along with some numerical simulations provided a significant evidence of the potentials of gradient tensor invariants' studies in classifying the relevant structures involved in the evolution of turbulent space plasmas, as well as, in identifying the main structures involved in plasma heating and dissipation processes [7,8].

In spite of these observational and numerical studies on topological invariants of magnetic and velocity field gradient tensors a mathematical derivation of equations governing the Lagrangian evolution of these geometrical invariant quantities is to our knowledge still lacking. Indeed, we have not found in the literature studies similar to those by Cantwell [3,12] in the case of magnetohydrodynamics.

In this theoretical work, we derive the evolution equations of the geometrical invariant quantities (SO(3)-scalars) of the magnetic and velocity field gradient tensors in the framework of magnetohydrodynamic theory. We remark that the relevance of this work is to provide additional elements to characterize the topological properties of structures over different dynamical regimes in MHD turbulence. Indeed, the evolution equations of the geometrical invariants contain some additional terms with respect to the fluid case, whose relevance stands in characterizing different dynamical situations. In other words, the estimation of the relevance of these additional terms could help to better characterize turbulent plasma regions that show similar spectral/scaling features.

This work is the natural extension to MHD of previous works by Vieillefosse [1,2] and Cantwell [3,12].

2. On SO(3) topological invariants of gradient tensor: Definition and meaning

In the magnetohydrodynamic theory the relevant quantities are the magnetic and the velocity field (**B** and **v**), and the evolution equations describing the dynamics of these quantities are the well-known Navier-Stokes equation and the dynamo equation, i.e.,

$$\frac{\partial}{\partial t}\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \frac{1}{c\rho}\mathbf{J}\times\mathbf{B} + \eta\nabla^2\mathbf{v}, \tag{4}$$

and

$$\frac{\partial}{\partial t}\mathbf{B} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \chi \nabla^2 \mathbf{B}, \tag{5}$$

where ρ is the plasma mass density, p is the kinetic pressure, η and χ are the viscosity and the magnetic diffusivity, $\mathbf{J} = \frac{c}{4\pi}\nabla \times \mathbf{B}$ is the current density, and c is the speed of light. In the framework of noncollisional plasmas the viscosity η is negligible, i.e., $\eta = 0$, and in the case of incompressible plasmas the mass density ρ is taken to be constant along a Lagrangian path, being $\nabla \cdot \mathbf{v} = 0$.

Let us now consider the gradient tensors of the magnetic and velocity fields, $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{Z}}$, which are defined as follows,

$$A_{ij} = \partial_j v_i, \tag{6}$$

$$Z_{ij} = \partial_j B_i, \tag{7}$$

respectively.

An early attempt to get the evolution equations for the gradient tensors was done by Materassi and Consolini [21], that obtained

$$\frac{D}{Dt}\tilde{\mathbf{A}} = -\frac{1}{\rho}\Pi + \frac{1}{\rho}\tilde{\mathbf{Z}}^2 - \tilde{\mathbf{A}}^2 + \frac{1}{\rho}(\mathbf{B}\cdot\nabla)\tilde{\mathbf{A}} + \ldots, \tag{8}$$

$$\frac{D}{Dt}\tilde{\mathbf{Z}} = \left[\tilde{\mathbf{Z}}, \tilde{\mathbf{A}}\right] + (\mathbf{B}\cdot\nabla)\tilde{\mathbf{A}} + \ldots, \tag{9}$$

where $\frac{D}{Dt} = \frac{\partial}{\partial t} + (\mathbf{v}\cdot\nabla)$ is the Lagrangian derivative, $\Pi = \nabla\nabla P$ with $P = p + B^2/8\pi$ is the total pressure and $[.,..]$ stands for commutator. The ellipses refer to the dissipative terms in the case of a non-ideal MHD. Furthermore the two quadratic terms stand for $\left(\tilde{\mathbf{A}}^2\right)_{ij} = A_{ik}A_{kj}$ and $\left(\tilde{\mathbf{Z}}^2\right)_{ij} = Z_{ik}Z_{kj}$, respectively.

Moving from the definition of the gradient tensors of the two fields, it is possible to introduce for each of the two gradient tensors a set of geometrical scalar invariants under SO(3) that can be defined starting by the characteristic equation for the eigenvalues (λ) of the gradient tensor, i.e.,

$$\|\, \tilde{\mathbf{M}} - \lambda\tilde{\mathbf{I}} \,\| = 0, \tag{10}$$

where $\tilde{\mathbf{M}} = \tilde{\mathbf{A}}$ or $\tilde{\mathbf{Z}}$ in our case. Eq. 10 defines the characteristic polynomial of the gradient tensor $\tilde{\mathbf{X}}$, which, for the Cayley-Hamilton theorem, is conserved under the SO(3) group. In detail, for each of the two fields one can write

$$\|\, \tilde{\mathbf{A}} - \lambda\tilde{\mathbf{I}} \,\| = \lambda^3 + P\lambda^2 + Q\lambda + R = 0, \tag{11}$$

$$\|\, \tilde{\mathbf{Z}} - \lambda\tilde{\mathbf{I}} \,\| = \lambda^3 + W\lambda^2 + X\lambda + Y = 0. \tag{12}$$

Here, the scalars (P, Q, R) and (W, X, Y) are the geometrical invariants for the velocity field gradient tensor and the magnetic field one, respectively. These quantities are related to the trace of $\tilde{\mathbf{M}}^n$, with $n = 1, 2$ and 3. However, because we consider the case of an incompressible plasma medium, $\nabla\cdot\mathbf{v} = 0$, and since $\nabla\cdot\mathbf{B} = 0$, the first two geometrical invariants, P and W, are identically zero, while the other invariants are,

$$Q = -\frac{1}{2}\mathrm{Tr}(\tilde{\mathbf{A}}^2), \quad X = -\frac{1}{2}\mathrm{Tr}(\tilde{\mathbf{Z}}^2), \tag{13}$$

and

$$R = -\frac{1}{3}\mathrm{Tr}(\tilde{\mathbf{A}}^3), \quad Y = -\frac{1}{3}\mathrm{Tr}(\tilde{\mathbf{Z}}^3). \tag{14}$$

These geometrical invariants have a topological meaning, in terms of velocity and magnetic field lines, as explained hereafter: hence, we refer to them as topological invariants. The solutions of the characteristic polynomials, Eqs. 10, give information on the streamlines of the magnetic and velocity fields. These allow to identify and classify the topology of plasma structures. Indeed, the set of the solutions of the characteristic polynomials (R^*, Q^*) and (Y^*, X^*) identifies the elliptic or hyperbolic character of the flow/field lines. As a matter of fact, the zero discriminant lines,

$$\Delta_{R,Q} = 4Q^3 + 27R^2 = 0, \tag{15}$$

$$\Delta_{Y,X} = 4X^3 + 27Y^2 = 0, \tag{16}$$

define in the (R, Q) and (Y, X) planes two regions depending on Δ. In particular, if $\Delta > 0$ we deal with elliptic field lines (ingoing/outgoing spiral saddle), while if $\Delta < 0$ the field lines are hyperbolic (tube/sheet-like structures). Fig. 1 shows the invariants' plane ((R, Q) or (Y, X)) and the expected typical topology of the fields lines. A more detailed and general discussion of the flow topologies can be found in Chong et al. [22].

3. Lagrangian evolution equations of MHD gradient tensors

The evolution equations of the topological invariants of the gradient tensors in the case of visco-resistive MHD can be derived starting from the Eqs. 4 - 5.

Let us start by rewriting the Navier-Stokes and the dynamo equations in a more explicit way. Thus, consider the magnetic field force density term in the Navier-Stokes equation,

$$\frac{\mathbf{j}\times\mathbf{B}}{\rho}$$

and express it by evidencing the magnetic field pressure term, i.e.,

$$\begin{aligned}\frac{1}{\rho}(\mathbf{j}\times\mathbf{B})^i &= \frac{1}{\rho}\epsilon^{ilm}J_l B_m \\ &= \frac{B_m}{\rho}\epsilon^{ilm}\epsilon_{ljk}\partial^j B^k \\ &= -\frac{1}{2\rho}\partial^i B^2 + \frac{1}{\rho}B^n\partial_n B^i.\end{aligned}$$

Here, we set c and the term $c/4\pi$ equal to the unit for simplicity. Furthermore, to avoid any ambiguity with the indices of the vector/tensor components we will make use of the superscript/subscript notation used in quantum relativistic field theory, having in mind that A_{ij} and A^{ij} are equivalent notations for the tensor components (the same is for the vector components). Using this notation the contraction between two indices is only done between superscript and subscript indices, e.g., $\nabla\cdot\mathbf{v} = \partial^i v_i$.

Let us now consider the set of the equations of the visco-resistive MHD under the assumption of an incompressible fluid, $\nabla\cdot\mathbf{v} = 0$,

$$\begin{cases}\partial_t v^i = -v^l\partial_l v^i - \frac{1}{2\rho}\partial^i B^2 + \frac{1}{\rho}B^l\partial_l B^i - \frac{1}{\rho}\partial^i p + \frac{1}{\rho}\partial_k\sigma^{ik} \\ \partial_t B^i = -v^l\partial_l B^i + B^l\partial_l v^i + \chi\partial^2 B^i\end{cases} \tag{17}$$

where

$$\sigma^{ik} = \left[\eta\left(\delta^{ni}\delta^{mk} + \delta^{nk}\delta^{mi} - \frac{2}{3}\delta^{ik}\delta^{mn}\right) + \nu\delta^{ik}\delta^{mn}\right]\partial_m v_n. \tag{18}$$

Eqs. 17 can be written in a Lagrangian form considering the relationship between the Eulerian and Lagrangian time derivatives:

$$\begin{cases}\dot{v}^i = -\frac{1}{2\rho}\partial^i B^2 + \frac{1}{\rho}B^l\partial_l B^i - \frac{1}{\rho}\partial^i p + \frac{1}{\rho}\partial_k\sigma^{ik} \\ \dot{B}^i = B^l\partial_l v^i + \chi\partial^2 B^i\end{cases} \tag{19}$$

Furthermore, to simplify our computations we introduce a constant tensor related to the definition of σ^{ik}, i.e., we write

$$\Sigma^{ikmn} = \eta\left(\delta^{ni}\delta^{mk} + \delta^{nk}\delta^{mi} - \frac{2}{3}\delta^{ik}\delta^{mn}\right) + \nu\delta^{ik}\delta^{mn}, \tag{20}$$

such that

$$\sigma^{ik} = \Sigma^{ikmn}A_{nm} \tag{21}$$

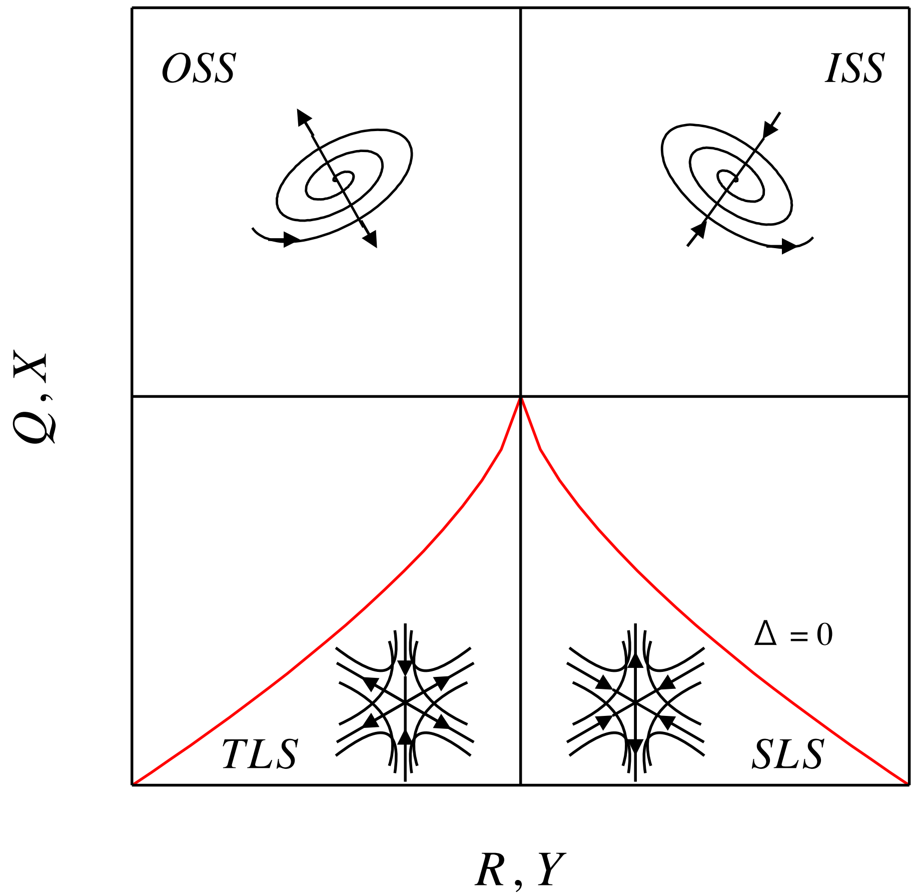

Fig. 1. The invariants' plane and the expected typical topology of the field lines. The red line is the discriminant line $\Delta = 0$. *OSS*, *ISS*, *TLS* and *SLS* stand for outgoing/ingoing spiral saddles and tube/sheet-like structures. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

which is a dissipative term. Using this definition, Eqs. 19 in the Eulerian representation are

$$\begin{cases}\partial_t v^i = -v^l\partial_l v^i - \frac{1}{2\rho}\partial^i B^2 + \frac{1}{\rho}B^l\partial_l B^i - \frac{1}{\rho}\partial^i p + \frac{1}{\rho}\partial_k\left(\Sigma^{ikmn}A_{nm}\right)\\ \partial_t B^i = -v^l\partial_l B^i + B^l\partial_l v^i + \chi\,\partial^2 B^i\end{cases}. \tag{22}$$

This is the starting point to derive the evolution equation of the velocity, $A_{ij} = \partial_j v_i$, and magnetic, $Z_{ij} = \partial_j B_i$, field gradient tensors, i.e., to compute the following evolution equations for the gradient tensors,

$$\begin{cases}\partial_t A^{ij} = \partial^j\left[-v^l\partial_l v^i + \frac{1}{\rho}\partial_k\left(\Sigma^{ikmn}A_{nm}\right) - \frac{1}{\rho}\partial^i p - \frac{1}{2\rho}\partial^i B^2 + \frac{1}{\rho}B^l\partial_l B^i\right]\\ \partial_t Z^{ij} = \partial^j\left(-v^l\partial_l B^i + B^l\partial_l v^i + \chi\,\partial^2 B^i\right)\end{cases}. \tag{23}$$

To derive a compact form for these two equations we consider them separately, starting from the evolution equation of the velocity gradient tensor.

$$\begin{aligned}\partial_t A^{ij} &= -\partial^j\left(v^l\partial_l v^i\right) + \Sigma^{ikmn}\partial^j\left(\frac{1}{\rho}\partial_k A_{nm}\right) - \partial^j\left(\frac{1}{\rho}\partial^i p\right) + \\ &\quad - \partial^j\left(\frac{1}{2\rho}\partial^i B^2\right) + \partial^j\left(\frac{1}{\rho}B^l\partial_l B^i\right)\\ &= -v^l\partial_l A^{ij} - A^i{}_l A^{lj} + \frac{\Sigma^{ikmn}}{\rho}\partial^j\partial_k A_{nm} - \frac{1}{\rho}\partial^i\partial^j p + \\ &\quad - \frac{1}{\rho}\partial^i B_l\partial^j B^l - \frac{1}{\rho}B_l\partial^i\partial^j B^l + \frac{1}{\rho}\partial_l\left(B^i\partial^j B^l + B^l\partial^j B^i\right)\\ &= -v^l\partial_l A^{ij} - A^i{}_l A^{lj} + \frac{\Sigma^{ikmn}}{\rho}\partial^j\partial_k A_{nm} - \frac{1}{\rho}\partial^i\partial^j p + \\ &\quad + \frac{1}{\rho}\left(Z^i{}_l - Z_l{}^i\right)Z^{lj} + \frac{1}{\rho}B^l\left(\partial_l Z^{ij} - \partial^i Z^{lj}\right),\end{aligned}$$

where we used the definition of the magnetic field gradient tensor Z_{ij}. Thus, the evolution equation of the velocity gradient tensor is given by

$$\begin{aligned}\partial_t A^{ij} = -v^l\partial_l A^{ij} - A^i{}_l A^{lj} + \tfrac{\Sigma^{ikmn}}{\rho}\partial^j\partial_k A_{nm} - \tfrac{1}{\rho}\partial^i\partial^j p + \\ + \tfrac{1}{\rho}\left(Z^i{}_l - Z_l{}^i\right)Z^{lj} + \tfrac{1}{\rho}B^l\left(\partial_l Z^{ij} - \partial^i Z^{lj}\right).\end{aligned} \tag{24}$$

This result has been derived under the assumption of a spatially constant density ρ, which is compatible with incompressible fluid assumption.

Consider now the following quantity $\Xi^{ik} = Z^{ik} - Z^{ki}$. This can be related to the skew-symmetric part of the $\tilde{\mathbf{Z}}$. Indeed, we have

$$\begin{aligned}\Xi^{ik} &= Z^{ik} - Z^{ki}\\ &= \left(\delta^i_a\delta^k_b - \delta^i_b\delta^k_a\right)Z^{ab}\\ &= -\epsilon^{ikh}\epsilon_{hba}\partial^b B^a\\ &= -\epsilon^{ikh}(\nabla\times\mathbf{B})_h\\ &= -\epsilon^{ikh}j_h\end{aligned}$$

i.e.,

$$\Xi^{ik} = -\epsilon^{ikh}j_h \tag{25}$$

where we introduce the electric current density $\mathbf{j}$. Using the new defined quantity Ξ^{ik} and Eq. 26, the evolution equation of the velocity field gradient tensor can be re-written as

$$\begin{aligned}\partial_t A^{ij} = -v^l\partial_l A^{ij} - A^i{}_l A^{lj} + \tfrac{\Sigma^{ikmn}}{\rho}\partial^j\partial_k A_{nm} - \tfrac{1}{\rho}\partial^i\partial^j p\\ + \tfrac{1}{\rho}\epsilon^{ihl}j_h Z_l{}^j + \tfrac{1}{\rho}B^l\left(\partial_l Z^{ij} - \partial^i Z^{lj}\right).\end{aligned} \tag{26}$$

Let us now move to the evolution equation of the magnetic field gradient tensor, Z_{ij}. From Eq. 23 we have,

$$\partial_t Z^{ij} = \partial^j\left(-v^l\partial_l B^i + B^l\partial_l v^i + \chi\,\partial^2 B^i\right)$$

$$= -\partial^j v^l \partial_l B^i - v^l \partial_l \partial^j B^i + \partial^j B^l \partial_l v^i + B^l \partial^j \partial_l v^i + \chi \partial^2 \partial^j B^i$$
$$= -v^l \partial_l Z^{ij} + A^i{}_l Z^{lj} - Z^i{}_l A^{lj} + B^l \partial_l A^{ij} + \chi \partial^2 Z^{ij}.$$

The evolution equation of the magnetic field gradient tensor Z^{ij} can be, thus, written as

$$\partial_t Z^{ij} = -v^l \partial_l Z^{ij} + A^i{}_l Z^{lj} - Z^i{}_l A^{lj} + B^l \partial_l A^{ij} + \chi \partial^2 Z^{ij}. \quad (27)$$

Summarizing the equations for the evolution of the gradient tensors in a Lagrangian form are

$$\begin{cases} \dot{A}^{ij} = -A^i{}_l A^{lj} + \frac{\Sigma^{ikmn}}{\rho} \partial^j \partial_k A_{nm} - \frac{1}{\rho} \partial^i \partial^j p + \\ \qquad + \frac{1}{\rho} \Xi^{il} Z_l{}^j + \frac{1}{\rho} \left(B^l \partial_l Z^{ij} - B_l \partial^i Z^{lj} \right) \\ \dot{Z}^{ij} = A^i{}_l Z^{lj} - Z^i{}_l A^{lj} + B^l \partial_l A^{ij} + \chi \partial^2 Z^{ij} \end{cases} \quad (28)$$

The first of these two evolution equations can be written in a slightly different and more compact way by making some simple considerations on some of its terms. For instance, we can observe that the first term in the brackets on the right side can be written as

$$-\frac{1}{\rho} B_l \partial^i Z^{lj} = -\frac{1}{\rho} B_l \partial^i \partial^j B^l = -\partial^i \partial^j \left(\frac{B^2}{2\rho} \right),$$

so that, reminding that we are considering an incompressible fluid, $\nabla \cdot \mathbf{v} = 0$, we get

$$\dot{A}^{ij} = -A^i{}_l A^{lj} + \frac{\Sigma^{ikmn}}{\rho} \partial^j \partial_k A_{nm} - \partial^i \partial^j \left(\frac{p}{\rho} + \frac{B^2}{2\rho} \right) + \frac{1}{\rho} \Xi^{il} Z_l{}^j + \frac{1}{\rho} B^l \partial_l Z^{ij}. \quad (29)$$

As a consequence of the incompressibility condition since $\nabla \cdot \mathbf{v} = \partial^i v_i = A^i{}_i$, the trace of the velocity field gradient tensor, $\mathrm{Tr}\tilde{\mathbf{A}} = 0$, remains conserved along the Lagrangian evolution, i.e.,

$$\frac{d}{dt} \mathrm{Tr}\tilde{\mathbf{A}} = \dot{A}^i{}_i = 0.$$

Inserting into Eq. 29

$$0 = -Tr(A^2) + \frac{\Sigma^{lkmn}}{\rho} \partial_l \partial_k A_{nm} - \partial^2 \left(\frac{p}{\rho} + \frac{B^2}{2\rho} \right) + \frac{1}{\rho} \Xi^{kl} \left(\frac{1}{2} \Xi_{lk} + \frac{1}{2} \Psi_{lk} \right), \quad (30)$$

where $\Psi_{lk} = Z_{lk} + Z_{kl}$.

Another important relation due to symmetry features is that the contraction $\Xi^{kl} \Psi_{kl}$ is identically null, so that the last relation in Eq. 30 reduces to

$$-\partial^2 \left(\frac{p}{\rho} + \frac{B^2}{2\rho} \right) = Tr(A^2) - \frac{1}{2\rho} Tr(\Xi^2) - \frac{\Sigma^{lkmn}}{\rho} \partial_l \partial_k A_{nm}. \quad (31)$$

In other words, the requirement of incompressibility and its validity along the material trajectories (Lagrangian view) of the plasma imply that the total pressure $P_{MHD} = \frac{p}{\rho} + \frac{B^2}{2\rho}$ is not a free variable but it can be calculated as a dependent parameter on the two tensors A and Z. Furthermore, the last equation provides us also some information on the term

$$-\partial^i \partial^j \left(\frac{p}{\rho} + \frac{B^2}{2\rho} \right)$$

which appears in Eq. 29. Indeed, assuming that this tensor $-\partial^i \partial^j \left(\frac{p}{\rho} + \frac{B^2}{2\rho} \right)$ is isotropic, we can express it as

$$-\partial^i \partial^j \left(\frac{p}{\rho} + \frac{B^2}{2\rho} \right) = -\frac{\delta ij}{3} \left[-Tr(A^2) + \frac{1}{2\rho} Tr(\Xi^2) + \frac{\Sigma^{klmn}}{\rho} \partial_l \partial_k A_{nm} \right]$$
$$\doteq -\frac{\delta^{ij}}{3} \tau_{press}(A, \Xi),$$

where τ_{press} is a new quantity.

Let us now move to expand the form of the term containing the tensor $\tilde{\boldsymbol{\Sigma}}$ in Eq. 29

$$\frac{\Sigma^{ikmn}}{\rho} \partial^j \partial_k A_{nm} = \frac{1}{\rho} \left[\eta \left(\delta^{ni} \delta^{mk} + \delta^{nk} \delta^{mi} - \frac{2}{3} \delta^{ik} \delta^{mn} \right) + \nu \delta^{ik} \delta^{mn} \right] \partial^j \partial_k A_{nm}$$
$$= \frac{\eta}{\rho} \left(\delta^{ni} \delta^{mk} \partial^j \partial_k A_{nm} + \delta^{nk} \delta^{mi} \partial^j \partial_k A_{nm} - \frac{2}{3} \delta^{ik} \delta^{mn} \partial^j \partial_k A_{nm} \right)$$
$$= \frac{\eta}{\rho} \partial^j \partial_k \left(A^{ik} + A^{ki} \right)$$
$$= \frac{2\eta}{\rho} \partial^j \partial_k S^{ik},$$

where we have introduced the symmetric part of the tensor $\tilde{\mathbf{A}}$, i.e.,

$$\tilde{\mathbf{S}} = \frac{1}{2} \left(\tilde{\mathbf{A}} + \tilde{\mathbf{A}}^T \right).$$

Thus, we have the following identity,

$$\frac{\Sigma^{ikmn}}{\rho} \partial^j \partial_k A_{nm} = \frac{2\eta}{\rho} \partial^j \partial_k S^{ik}$$

which implies that also the following identity is valid,

$$-\frac{\Sigma^{lkmn}}{\rho} \partial_l \partial_k A_{nm} = -\frac{2\eta}{\rho} \partial_l \partial_k S^{lk} \equiv 0.$$

The symmetric tensor term, $\partial^j \partial_k S^{ik}$, can be further rearranged to get a new expression, obtaining,

$$\partial^j \partial_k S^{ik} = \frac{1}{2} \partial^j \partial_k \left(\partial^k v^i + \partial^i v^k \right)$$
$$= \frac{1}{2} \left(\partial^j \partial_k \partial^k v^i + \partial^j \partial_k \partial^i v^k \right)$$
$$= \frac{1}{2} \partial^2 \partial^j v^i + \frac{1}{2} \partial^j \partial^i \left(\vec{\nabla} \cdot \vec{v} \right)$$
$$= \frac{1}{2} \partial^2 A^{ij},$$

and, therefore,

$$\frac{\Sigma^{ikmn}}{\rho} \partial^j \partial_k A_{nm} = \frac{\eta}{\rho} \partial^2 A^{ij}. \quad (32)$$

Now, remembering the condition $A_i{}^i = 0$ Eq. 31 becomes

$$-\partial^2 \left(\frac{p}{\rho} + \frac{B^2}{2\rho} \right) = Tr(A^2) - \frac{1}{2\rho} Tr(\Xi^2). \quad (33)$$

Joining all the previous considerations we get for the evolution equations of the gradient tensors the following expression

$$\begin{cases} \dot{A}^{ij} = -A^i{}_l A^{lj} + \frac{\eta}{\rho} \partial^2 A^{ij} - \frac{1}{3} \left[\frac{1}{2\rho} Tr(\Xi^2) - Tr(A^2) \right] \delta^{ij} \\ \qquad + \frac{1}{\rho} \Xi^{il} Z_l{}^j + \frac{1}{\rho} B^l \partial_l Z^{ij} \\ \dot{Z}^{ij} = A^i{}_l Z^{lj} - Z^i{}_l A^{lj} + B^l \partial_l A^{ij} + \chi \partial^2 Z^{ij} \end{cases} \quad (34)$$

that can be joined with the following two expressions for the trace of the two gradient tensors

$$\begin{cases} \mathrm{Tr}\dot{\tilde{\mathbf{A}}} & = \frac{1}{\rho} \mathrm{Tr}\left(\tilde{\boldsymbol{\Xi}} \cdot \tilde{\mathbf{Z}}^S \right) = 0 \\ \mathrm{Tr}\dot{\tilde{\mathbf{Z}}} & = 0 \end{cases} \quad (35)$$

These two conditions on the solenoidal condition of the velocity and magnetic field can be used to write Eq. 34 in a more compact way, i.e.,

$$\begin{cases} \dot{\tilde{\mathbf{A}}} = -\tilde{\mathbf{A}}^2 + \frac{\eta}{\rho} \nabla^2 \tilde{\mathbf{A}} + \frac{1}{3} \left[-\frac{1}{2\rho} Tr\left(\tilde{\boldsymbol{\Xi}}^2 \right) + \mathrm{Tr}\left(\tilde{\mathbf{A}}^2 \right) \right] \mathbf{1} + \frac{1}{\rho} \tilde{\boldsymbol{\Xi}} \cdot \tilde{\mathbf{Z}} + \frac{1}{\rho} (\mathbf{B} \cdot \nabla) \tilde{\mathbf{Z}} \\ \dot{\tilde{\mathbf{Z}}} = \left[\tilde{\mathbf{A}}, \tilde{\mathbf{Z}} \right] + (\mathbf{B} \cdot \nabla) \tilde{\mathbf{Z}} + \chi \nabla^2 \tilde{\mathbf{Z}} \end{cases} \quad (36)$$

Now, following the same approach used to get the ODE of the gradient $\tilde{\mathbf{A}}$ in the case of hydrodynamic situations [3] we start again from Eq. 29, which is now reformulated applying some of the previous expression in the following form,

$$\dot{A}^{ij} + A^{i}_{\ l}A^{lj} - \frac{1}{\rho}\Xi^{il}Z_{l}^{\ j} = \frac{\eta}{\rho}\partial^2 A^{ij} - \partial^i\partial^j\left(\frac{p}{\rho} + \frac{B^2}{2\rho}\right) + \frac{1}{\rho}B^l\partial_l Z^{ij}. \tag{37}$$

If we now look at the right hand side term of this equation, we can notice that this term is zero if we neglect the gradients of the tensors $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{Z}}$ and the one of the total pressure $\frac{p}{\rho} + \frac{B^2}{2\rho}$. This situation is true when one considers a very small parcel of fluid in order to introduce a Lagrangian approach. Let us now take into account the condition $\mathrm{Tr}\dot{\tilde{\mathbf{A}}} = 0$ and Eq. 33. Thus, we have,

$$-\partial^2\left(\frac{p}{\rho} + \frac{B^2}{2\rho}\right)\frac{\delta^{ij}}{3} = \left[Tr\left(A^2\right) - \frac{1}{2\rho}Tr\left(\Xi^2\right)\right]\frac{\delta^{ij}}{3} \tag{38}$$

and subtracting Eq. 38 from Eq. 37 we obtain an expression for the evolution of the tensor A^{ij}

$$\dot{A}^{ij} + A^{i}_{\ l}A^{lj} - \left[Tr\left(A^2\right) - \frac{1}{2\rho}Tr\left(\Xi^2\right)\right]\frac{\delta^{ij}}{3} - \frac{1}{\rho}\Xi^{il}Z_{l}^{\ j} =$$
$$= \frac{\eta}{\rho}\partial^2 A^{ij} - \left[\partial^i\partial^j\left(\frac{P_{MHD}}{\rho}\right) - \partial^2\left(\frac{P_{MHD}}{\rho}\right)\frac{\delta^{ij}}{3}\right] + \frac{1}{\rho}B^l\partial_l Z^{ij}$$

where

$$P_{MHD} = p + \frac{B^2}{2}$$

The last expression for the evolution of the velocity field gradient tensor can be written in a more compact matrix form

$$\dot{\tilde{\mathbf{A}}} + \tilde{\mathbf{A}}^2 - \left[\mathrm{Tr}\left(\tilde{\mathbf{A}}^2\right) - \frac{\mathrm{Tr}\left(\tilde{\boldsymbol{\Xi}}^2\right)}{2\rho}\right]\frac{1}{3} - \frac{\tilde{\boldsymbol{\Xi}}\cdot\tilde{\mathbf{Z}}}{\rho} = \tilde{\mathbf{H}} \tag{39}$$

with

$$\tilde{\mathbf{H}} = \frac{\eta}{\rho}\nabla^2\tilde{\mathbf{A}} - \left(\nabla\otimes\nabla - \frac{1}{3}\nabla^2\right)\left(\frac{P_{MHD}}{\rho}\right) + \frac{1}{\rho}\mathbf{B}\cdot\nabla\tilde{\mathbf{Z}} \tag{40}$$

Eq. 39 provides a description of the evolution of the velocity gradient tensor in a magnetized fluid from the Lagrangian point of view, i.e., following a (practically pointlike) parcel of fluid. The second hand of Eq. 39, i.e., the tensor $\tilde{\mathbf{H}}$, can be considered to be negligile if the gradients of the gradient tensors, $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{Z}}$, and of the pressure term P_{MHD} are small/negligible, i.e., if the medium can be consider locally homogeneous. In such a situation, Eq. 39 reduces to the following homogeneous form

$$\dot{\tilde{\mathbf{A}}} + \tilde{\mathbf{A}}^2 - \left[\mathrm{Tr}\left(\tilde{\mathbf{A}}^2\right) - \frac{\mathrm{Tr}\left(\tilde{\boldsymbol{\Xi}}^2\right)}{2\rho}\right]\frac{1}{3} - \frac{\tilde{\boldsymbol{\Xi}}\cdot\tilde{\mathbf{Z}}}{\rho} = 0. \tag{41}$$

Alternatively, we can use the full expression of the Eq. 39 where the tensor $\tilde{\mathbf{H}}$ in the right hand can be considered as a noise term. This is for instance the case of turbulent plasma media where the quantity $\tilde{\mathbf{H}}$ may be irregular enough to be associated with a spatio-temporal noise term as already done by Cantwell [3].

Let us now move to the evolution equation of the magnetic field gradient tensor, $\tilde{\mathbf{Z}}$, and write it in a more compact form. Thus, we get

$$\dot{\tilde{\mathbf{Z}}} + \left[\tilde{\mathbf{Z}}, \tilde{\mathbf{A}}\right] = \chi\nabla^2\tilde{\mathbf{Z}} + \mathbf{B}\cdot\nabla\tilde{\mathbf{A}}, \tag{42}$$

where $[\ldots,\ldots]$ stands for a commutator. In analogy to the case previously studied, the right-hand term can be posed to be a new tensor term, $\tilde{\boldsymbol{\Theta}}$, i.e.,

$$\tilde{\boldsymbol{\Theta}} = \chi\nabla^2\tilde{\mathbf{Z}} + \mathbf{B}\cdot\nabla\tilde{\mathbf{A}},$$

which is associated with dissipation ($\chi\nabla^2\tilde{\mathbf{Z}}$) and deformation ($\mathbf{B}\cdot\nabla\tilde{\mathbf{A}}$) of the magnetic field topology in the transported parcel. In the case of a turbulent plasma, the term $\tilde{\boldsymbol{\Theta}}$ can be assimilated to a spatio-temporal noise source term as it is for $\tilde{\mathbf{H}}$. Thus, with the above definition we can write

$$\begin{cases}\dot{\tilde{\mathbf{Z}}} + \left[\tilde{\mathbf{Z}}, \tilde{\mathbf{A}}\right] = \tilde{\boldsymbol{\Theta}} \\ \tilde{\boldsymbol{\Theta}} = \chi\nabla^2\tilde{\mathbf{Z}} + \mathbf{B}\cdot\nabla\tilde{\mathbf{A}}\end{cases}. \tag{43}$$

In conclusion, the resulting evolution equations for the magnetic and velocity filed gradient tensors, $\tilde{\mathbf{Z}}$ and $\tilde{\mathbf{A}}$, reduce to the following set of equations

$$\begin{cases}\dot{\tilde{\mathbf{A}}} + \tilde{\mathbf{A}}^2 - \left[\mathrm{Tr}\left(\tilde{\mathbf{A}}^2\right) - \frac{\mathrm{Tr}\left(\tilde{\boldsymbol{\Xi}}^2\right)}{2\rho}\right]\frac{1}{3} - \frac{\tilde{\boldsymbol{\Xi}}\cdot\tilde{\mathbf{Z}}}{\rho} = \tilde{\mathbf{H}} \\ \dot{\tilde{\mathbf{Z}}} + \left[\tilde{\mathbf{Z}}, \tilde{\mathbf{A}}\right] = \tilde{\boldsymbol{\Theta}}\end{cases}, \tag{44}$$

where the two noise terms, $\tilde{\mathbf{H}}$ and $\tilde{\boldsymbol{\Theta}}$, can be neglected in the case of smooth and homogenous plasma (small gradients). In particular, if we can assume that $\tilde{\boldsymbol{\Theta}}$ is negligible, the magnetic gradient $\tilde{\mathbf{Z}}$ is transported by the operator $[., \tilde{\mathbf{A}}]$ along the parcel trajectory, so that $\tilde{\mathbf{A}}$ generates *á la Lie*, without deformation and dissipation.

Eqs. 44 are well defined *á la Langevin* and thus can be considered the starting point for the derivation of a stochastic approach of the velocity and magnetic field gradient tensors via a Fokker-Planck description, as well as, the starting point to derive the evolution equation of the associated SO(3) geometrical invariants of the two gradient tensor. The latter is the next step that we will discuss in the following Section.

4. Derivation of the evolution equations for the topological quantities

The next step is to get the set of equations describing the evolution of the SO(3) geometrical invariants associated with the gradient tensors of the velocity and magnetic fields.

As discussed in Section 2 moving from the definitions of the velocity and magnetic field gradient tensors ($\tilde{\mathbf{A}}$ and $\tilde{\mathbf{Z}}$) it is possible to introduce some invariant quantities to describe the local topology of the field lines for the velocity and the magnetic field. In the case of incompressible plasmas the set of geometrical invariant quantities reduces to four quantities associated with the traces of the gradient tensors and their powers (see Section 2), i.e.,

$$\begin{cases}Q = -\frac{1}{2}\mathrm{Tr}(\tilde{\mathbf{A}}^2) \\ X = -\frac{1}{2}\mathrm{Tr}(\tilde{\mathbf{Z}}^2) \\ R = -\frac{1}{3}\mathrm{Tr}(\tilde{\mathbf{A}}^3) \\ Y = -\frac{1}{3}\mathrm{Tr}(\tilde{\mathbf{Z}}^3)\end{cases},$$

As widely discussed in Section 2 these geometrical invariants are associated with the secular equations of the two gradient tensors. Under the requirement of zero divergence condition for the magnetic and velocity field lines the eigenvalue equations reduce to

$$\lambda_v^3 + \lambda_v Q + R = 0, \qquad \lambda_B^3 + \lambda_B X + Y = 0, \tag{45}$$

being invariant expressions under SO(3) group transformations.

Now, moving from the evolution equations of the two gradient tensors in Eqs. 44 we attempt a derivation and a discussion of the evolution equations for these topological invariants. These equations allow to descrive the Lagrangian evolution of the topology of the magnetic and velocity field lines in a transported plasma parcel, which is important as far as some paramount phenomena in plasma, i.e., the magnetic reconnection, etc., has to do with topological changes.

Let us start with the derivation of the evolution equation for the quantity Q considering the expression

$$\dot{Q} = \frac{d}{dt}\left[-\frac{1}{2}\mathrm{Tr}\left(\tilde{\mathbf{A}}^2\right)\right]$$

$$= -\frac{1}{2}\frac{d}{dt}\left(A^{ij}A_{ji}\right) =$$
$$= --\frac{1}{2}\left(\dot{A}^{ij}A_{ji} + \dot{A}^{ji}A_{ij}\right)$$
$$= -\dot{A}^{ij}A_{ji}$$

Now using the Eq. 41 in its homogeneous form, i.e., neglecting the noise term, we get

$$\dot{Q} =$$
$$= \left\{\left(\tilde{\mathbf{A}}^2\right)^{ij} - \left[\mathrm{Tr}\left(\tilde{\mathbf{A}}^2\right) - \frac{\mathrm{Tr}\left(\tilde{\Xi}^2\right)}{2\rho}\right]\frac{\delta^{ij}}{3} - \frac{\left(\tilde{\Xi}\cdot\tilde{\mathbf{Z}}\right)^{ij}}{\rho}\right\}A_{ij}$$
$$= \left(\tilde{\mathbf{A}}^2\right)^{ij}A_{ji} - \left[\mathrm{Tr}\left(\tilde{\mathbf{A}}^2\right) - \frac{\mathrm{Tr}\left(\tilde{\Xi}^2\right)}{2\rho}\right]\frac{\mathrm{Tr}\tilde{\mathbf{A}}}{3} - \frac{\left(\tilde{\Xi}\cdot\tilde{\mathbf{Z}}\right)^{ij}}{\rho}A_{ji}$$
$$= \mathrm{Tr}\left(\tilde{\mathbf{A}}^3\right) - \frac{1}{\rho}\mathrm{Tr}\left(\tilde{\Xi}\cdot\tilde{\mathbf{Z}}\cdot\tilde{\mathbf{A}}\right)$$
$$= -3R - \frac{1}{\rho}\Xi^{ij}Z_{jk}A^k_i.$$

Then in the case $\tilde{\mathbf{H}} = 0$,

$$\dot{Q} + 3R + \frac{1}{\rho}\mathrm{Tr}\left(\tilde{\Xi}\cdot\tilde{\mathbf{Z}}\cdot\tilde{\mathbf{A}}\right) = 0. \tag{46}$$

The same approach can be used to derive the evolution equation for the other geometrical invariant of the velocity field. In detail, for the R quantity we get

$$\dot{R} = \mathrm{Tr}\left(\tilde{\mathbf{A}}^4\right) - \frac{4}{3}Q^2 - \frac{\mathrm{Tr}\left(\tilde{\Xi}^2\right)}{3\rho}Q - \frac{\mathrm{Tr}\left(\tilde{\Xi}\cdot\tilde{\mathbf{Z}}\cdot\tilde{\mathbf{A}}^2\right)}{\rho}. \tag{47}$$

This equation contains a 4^{th}-power term of the velocity gradient tensor, which can be simplified using the Cayley-Hamilton theorem, i.e.,

$$\mathrm{Tr}\left(\tilde{\mathbf{A}}^4\right) = \mathrm{Tr}\left(-Q\tilde{\mathbf{A}}^2 - R\tilde{\mathbf{A}}\right).$$

Thus, we have

$$\dot{R} = \frac{2}{3}Q^2 - \frac{\mathrm{Tr}\left(\tilde{\Xi}^2\right)}{3\rho}Q - \frac{\mathrm{Tr}\left(\tilde{\Xi}\cdot\tilde{\mathbf{Z}}\cdot\tilde{\mathbf{A}}^2\right)}{\rho}. \tag{48}$$

Now we proceed to derive the evolution equations of the topological invariants related to the magnetic field, X and Y, under the hypothesis of a negligible noise term, i.e., assuming that

$$\dot{\tilde{\mathbf{Z}}} = \left[\tilde{\mathbf{A}}, \tilde{\mathbf{Z}}\right].$$

In this peculiar situation we get trivial evolution equations for the magnetic field invariants. Indeed, for the X quantity we have

$$\dot{X} = \frac{d}{dt}\left(-\frac{1}{2}\mathrm{Tr}\left(\tilde{\mathbf{Z}}^2\right)\right)$$
$$= -\dot{Z}_{ij}Z^{ji}$$
$$= \left(\left[\tilde{\mathbf{Z}}, \tilde{\mathbf{A}}\right]\right)_{ij}Z^{ji}$$
$$= \mathrm{Tr}\left(\tilde{\mathbf{Z}}\tilde{\mathbf{A}}\tilde{\mathbf{Z}}\right) - \mathrm{Tr}\left(\tilde{\mathbf{Z}}\tilde{\mathbf{A}}\tilde{\mathbf{Z}}\right) = 0,$$

i.e.,

$$\dot{X} = 0. \tag{49}$$

An analogous equation can be recovered for the evolution equation of the Y quantity, i.e.,

$$\dot{Y} = 0. \tag{50}$$

We can now resume the set of the evolution equations for the geometrical invariants when the noise terms can be neglected. Indeed, in the case of describing a fluid poit-like parcel the gradients of P_{MHD}, $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{Z}}$ are negligible. Thus, the evolution equations for geometrical invariants are homogeneous, i.e.,

$$\begin{cases} \dot{Q} + 3R + \frac{1}{\rho}\mathrm{Tr}\left(\tilde{\Xi}\cdot\tilde{\mathbf{Z}}\cdot\tilde{\mathbf{A}}\right) = 0 \\ \dot{R} - \frac{2}{3}Q^2 + \frac{\mathrm{Tr}\left(\tilde{\Xi}^2\right)}{3\rho}Q + \frac{\mathrm{Tr}\left(\tilde{\Xi}\cdot\tilde{\mathbf{Z}}\cdot\tilde{\mathbf{A}}^2\right)}{\rho} = 0 \\ \dot{X} = 0 \\ \dot{Y} = 0 \end{cases} \tag{51}$$

This situation is equivalent to the case of a restricted *Euleur evolution equation* for the magnetofluid, i.e. in the absence of viscosity and dissipation. In the case of a more general situation, i.e., when noise terms are not negligible (i.e., $\tilde{\mathbf{H}} \neq 0$ and $\tilde{\mathbf{\Theta}} \neq 0$), we obtain

$$\begin{cases} \dot{Q} + 3R + \frac{1}{\rho}\mathrm{Tr}\left(\tilde{\Xi}\cdot\tilde{\mathbf{Z}}\cdot\tilde{\mathbf{A}}\right) = -\mathrm{Tr}\left(\tilde{\mathbf{H}}\cdot\tilde{\mathbf{A}}\right) \\ \dot{R} - \frac{2}{3}Q^2 + \frac{\mathrm{Tr}\left(\tilde{\Xi}^2\right)}{3\rho}Q + \frac{\mathrm{Tr}\left(\tilde{\Xi}\cdot\tilde{\mathbf{Z}}\cdot\tilde{\mathbf{A}}^2\right)}{\rho} = -\mathrm{Tr}\left(\tilde{\mathbf{H}}\cdot\tilde{\mathbf{A}}^2\right) \\ \dot{X} = -\mathrm{Tr}\left(\tilde{\mathbf{\Theta}}\cdot\tilde{\mathbf{Z}}\right) \\ \dot{Y} = -\mathrm{Tr}\left(\tilde{\mathbf{\Theta}}\cdot\tilde{\mathbf{Z}}^2\right) \end{cases}. \tag{52}$$

5. Discussion

In previous Sections we derived the evolution equations for the geometrical (topological) invariants of the gradient tensors of the velocity and magnetic fields in the framework of MHD description of plasmas. The results of our work are contained in the two sets of Eqs. 51 and 52, which deal with homogenous and inhomogeneous situations, respectively.

These equations provide the evolution of magnetized fluid (plasma) flow topologies, which experience the presence of a magnetic field. If we compare these equations with the corresponding ones in the case of a neutral fluid (see Eqs. 3), we immediately realize that in this case we have some extra terms related to the effects of the magnetic field on the fluid. The extra terms are

$$\begin{cases} \frac{1}{\rho}\mathrm{Tr}\left(\tilde{\Xi}\cdot\tilde{\mathbf{Z}}\cdot\tilde{\mathbf{A}}\right) = -\frac{1}{\rho}\epsilon^{ijh}j_h\partial_jB_l\partial^l v_i \\ \frac{\mathrm{Tr}\left(\tilde{\Xi}^2\right)}{3\rho}Q = -\frac{2}{3\rho}j^2Q \\ \frac{\mathrm{Tr}\left(\tilde{\Xi}\cdot\tilde{\mathbf{Z}}\cdot\tilde{\mathbf{A}}^2\right)}{\rho} = -\frac{1}{\rho}\epsilon^{ijh}j_h\partial_jB_l\partial^l v_k\partial^k v_i \end{cases}. \tag{53}$$

These terms are mainly related to electromagnetic force $(\mathbf{j}\times\mathbf{B})$ stretching terms and a j^2 term, which act on the fluid velocity. Now strictly speaking these terms are themselves time dependent. Indeed, we can assume that they evolve along the point-like fluid parcel according to the evolution equations of gradient tensors, i.e., Eqs. 44. As a first step in understanding the relevance of the different terms, let us assume that these terms are small, so that we can investigate the evolution of the velocity fluid structures in the $[R, Q]$ plane by means of a dynamical systems approach considering them as small perturbations. Clearly, at the present stage what follows has to be considered only a *toy model*, requiring the understanding of the different terms a deeper discussion and analysis.

Thus, under the above assumption and focusing our attention on the second term in Eq. 53 (i.e., the current term), the homogeneous evolution equations, Eqs. 51, can be written as

$$\begin{cases} \dot{Q} = -3R \\ \dot{R} = \frac{2}{3}\left[Q^2 + \beta Q\right] \end{cases}, \tag{54}$$

where β does not depend on the velocity variables Q and R.

Fig. 2 shows the evolution of the velocity gradient tensor invariants in the case of the *restricted Euler equation* and considering only the effect of the term $-\frac{2}{3\rho}j^2Q$, i.e., $\beta \geq 0$ (here $\beta = j^2/\rho$) in Eqs. 54. In detail, we notice that, in the case of trajectories above

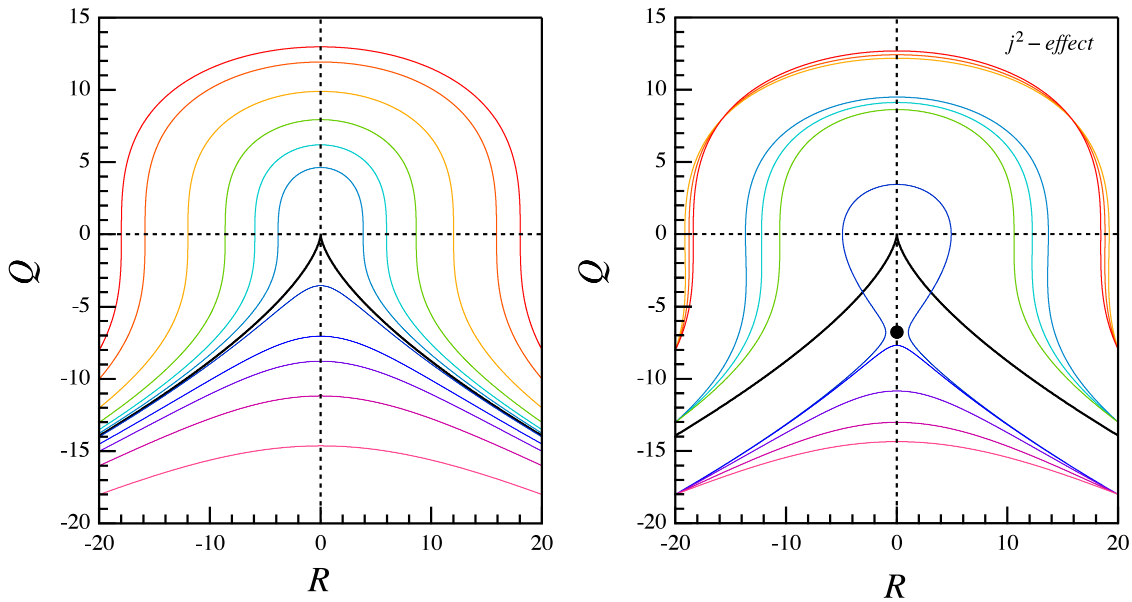

Fig. 2. Evolution of the velocity gradient tensor invariants in the case of the *restricted Euler equation* (left panel) and considering only the effect of the j^2 term (right panel) in the case of different starting values, $[R_0, Q_0] = [-20, y]$ with $y \in [-18, -8]$. For the study of the effect of j^2 term we have explored different values of $\beta = \frac{j^2}{\rho} \in [0, 6.78]$. In all cases the motion is from left to right. The black line is the discriminant line and the black circle is the critical value of β.

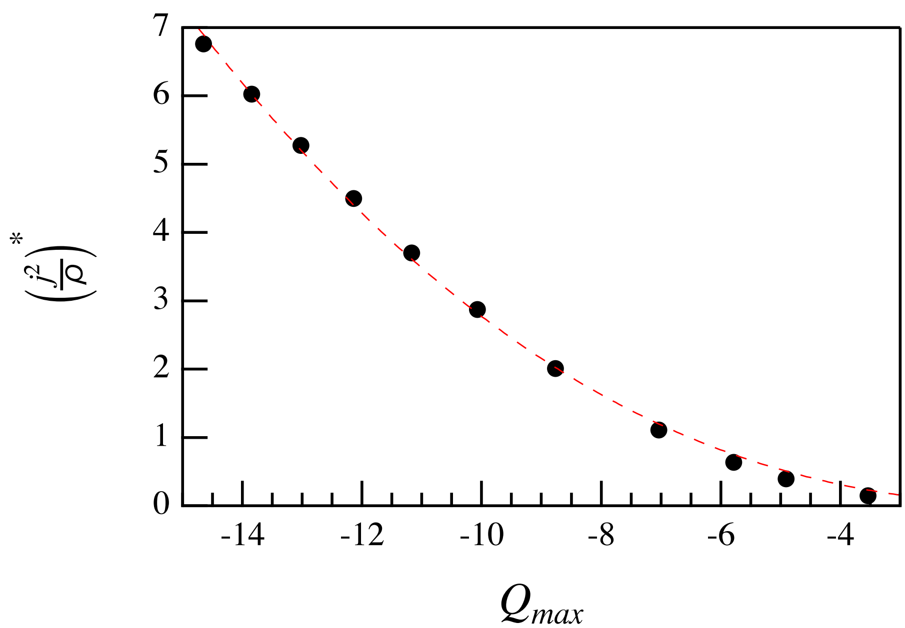

Fig. 3. Dependence of the critical value of j^2 term ($\frac{j^2}{\rho}$) for which we observe this hyperbolic/elliptic/hyperbolic transition as a function of the maximum value Q_{max} of the trajectory for zero j^2 term. The dashed red line is a power law fit. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

the discriminant line, the effect of the j^2 term is only to shift the trajectories upward and make them clusterizing differently, thus without changing the character of the flow lines which remain spiral saddles. Conversely, in the case of trajectories below the discriminant line we find that there is a critical value of the j^2 term effect for which the topology of the flow lines is changed from a tube/sheet like structure to ingoing/outgoing spiral saddles in a certain interval of R-values during the evolution. This critical value of β^* corresponds to the presence of an unstable saddle point in the correspondence of $[R^*, Q^*] = [0, -\beta^*]$ which can be simply derived by a stability analysis of Eq. 54 (see Appendix A for more details).

Figure 3 shows the dependence of the critical value of j^2 term effect ($\frac{j^2}{\rho}$) for which we observe this hyperbolic/helliptic/hyperbolic transition in the Lagrangian evolution of the structures in the $[R, Q]$ plane as a function of the maximum value Q_{max} of the trajectory for zero j^2 term. Thus, j^2 term effect is capable of modifying hyperbolic solutions of the invariants generating elliptical structures and vice versa. Furthermore the critical value of the j^2 term, $\left(j^2/\rho\right)^*$ depends on the Q_{max} of the trajectory in the case of no-j^2 term according to a power law.

The situation is less clear if we consider the effect of the other two terms, $\mathrm{Tr}\left(\tilde{\boldsymbol{\Xi}} \cdot \tilde{\mathbf{Z}} \cdot \tilde{\mathbf{A}}\right)/\rho$ and $\mathrm{Tr}\left(\tilde{\boldsymbol{\Xi}} \cdot \tilde{\mathbf{Z}} \cdot \tilde{\mathbf{A}}^2\right)/\rho$, neglecting the j^2 term effect. In this case and still assuming the homogeneous situ-

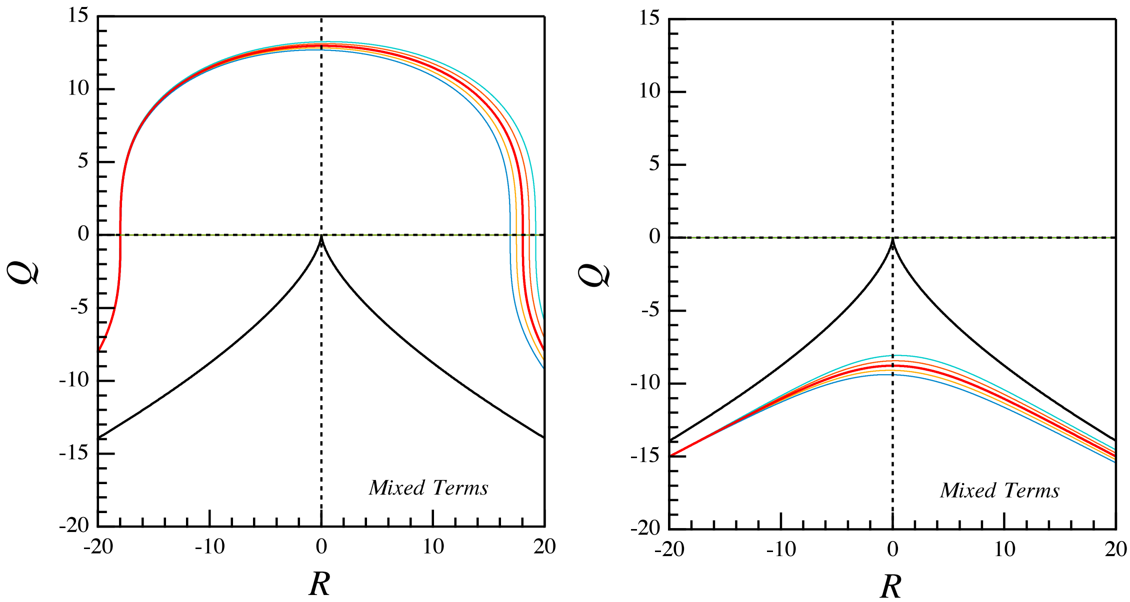

Fig. 4. Effect of a set of values of the two terms, $\mathrm{Tr}\left(\tilde{\boldsymbol{\Xi}}\cdot\tilde{\mathbf{Z}}\cdot\tilde{\mathbf{A}}\right)/\rho$ and $\mathrm{Tr}\left(\tilde{\boldsymbol{\Xi}}\cdot\tilde{\mathbf{Z}}\cdot\tilde{\mathbf{A}}^2\right)/\rho$, on the trajectories for elliptical (left panel) and hyperbolic (right panel) structures neglecting the j^2 term, $\mathrm{Tr}\left(\tilde{\boldsymbol{\Xi}}^2\right)=0$. The black line is the discriminant line.

ation the $\mathrm{Tr}\left(\tilde{\boldsymbol{\Xi}}\cdot\tilde{\mathbf{Z}}\cdot\tilde{\mathbf{A}}\right)/\rho$ is expected to be of the same order of $\tilde{\mathbf{A}}$ and that Q is of the order of $\left(\tilde{\mathbf{A}}^2\right)$. However, this has to be considered a very crude approximation, whose validity has to be investigated by studying the trace terms of Eq. 53 by means of simulations or real situations. Thus, Eq. 54 becomes

$$\begin{cases}\dot{Q}=-3R+\alpha\sqrt{|Q|}\\ \dot{R}=\frac{2}{3}\left[Q^2+\beta Q\right]+\gamma Q\end{cases}, \tag{55}$$

where we set $\beta=j^2/\rho=0$, i.e., assuming a zero conductivity medium, and α and γ are two $Q-$ and $R-$independent values associated with $\mathrm{Tr}\left(\tilde{\boldsymbol{\Xi}}\cdot\tilde{\mathbf{Z}}\cdot\tilde{\mathbf{A}}\right)/\rho$ and $\mathrm{Tr}\left(\tilde{\boldsymbol{\Xi}}\cdot\tilde{\mathbf{Z}}\cdot\tilde{\mathbf{A}}^2\right)/\rho$, respectively.

Fig. 4 shows the effect of a set of values of the other two terms for elliptical and hyperbolic structures. There is a tendency to deform the original trajectories and depending on the distance from the discriminant line and from the values of these two terms we can assist to transition between the two main classes of solutions, i.e., elliptic and hyperbolic solutions. Thus, in general, we have that differently from the case of the restricted Euler equation for fluid [3] the action of the magnetic field on the velocity field can allow a significative change of the topology due to the forcing related to the electromagnetic force density and the j^2 term that appear in the evolution equations of Q and R. Conversely, in the homogeneous case (Eq. 51) the magnetic field invariants, X and Y, are preserved inside a plasma parcel being rigidly transported along the plasma flow lines.

In the inhomogeneous case the situation is less simple. Indeed, the presence of the two noise terms, associated with the tensors $\tilde{\mathbf{H}}$ and $\tilde{\boldsymbol{\Theta}}$ introduces a non trivial evolution of both the velocity and magnetic field invariants. We may note that for non-collisional and non-resistive plasmas the expression of the two noise terms is essentially related to the total pressure and the gradient of the velocity and magnetic field gradient tensors, i.e.,

$$\begin{cases}\tilde{\mathbf{H}}=-\left(\nabla\otimes\nabla-\frac{1}{3}\nabla^2\right)\left(\frac{P_{MHD}}{\rho}\right)+\frac{1}{\rho}\mathbf{B}\cdot\nabla\tilde{\mathbf{Z}}\\ \tilde{\boldsymbol{\Theta}}=\ \mathbf{B}\cdot\nabla\tilde{\mathbf{A}}\end{cases}. \tag{56}$$

Just to show the possible effect on the Lagrangian evolution of the topologies under the action of these noise terms in the case of the velocity invariants, we have attempted an integration of the evolution equation that have written in terms of stochastic Langevin equations, i.e.,

$$\begin{cases}\dot{Q}=-3R+\sqrt{\alpha'|Q|}\zeta^B(t)+\sqrt{\eta_1}\zeta^a(t)\\ \dot{R}=\frac{2}{3}Q^2+\beta Q+\sqrt{\gamma' Q}\zeta^B(t)+\sqrt{\eta_2}\zeta^b(t)\end{cases}, \tag{57}$$

where

- α', β and γ' are the same quantities reported in Eq. 55 but that now assumes the role of a noise variance,
- $\sqrt{\eta_1}\zeta^a(t)$ and $\sqrt{\eta_2}\zeta^b(t)$ are noise terms,
- $\eta=(\eta_1,\eta_2)$ are noise variances and
- ζ^i are delta-correlated unit variance noises.

We remark that in this case also the first and the third term which depend on $\tilde{\mathbf{Z}}$, may acquire a stochastic character (see Eqs. 44). Fig. 5 shows some examples of the stochastic evolution that we can get for different values of the noise variances. The observed stochastic patterns evidenced how the presence of noise terms allows transitions between the different structures. This issue is extremely relevant in the case of turbulent plasma media, where the Lagrangian evolution of structures can imply substantial changes in the topology of the flow and magnetic field lines. Clearly, in this framework one of the main issues is to understand the relative relevance of the terms in Eqs. 51 and 52 and, in particular, the statistics of the noise terms.

6. Summary and conclusions

Here, we provide a theoretical derivation of the evolution equations of the geometrical invariants of the gradient tensors in the framework of MHD for both the homogeneous and the inhomogeneous scenario, i.e., considering or neglecting dissipative effects and source terms. Then, we provide a description of the Lagrangian dynamics of the velocity gradient tensor invariants in the $[R,Q]$ plane using a dynamical system approach. In the case of inhomogeneous case the evolution equations have been described in terms of stochastic Langevin processes.

One of the main results of our study is the evidence that, differently from the restricted Euler dynamics, also in absence of noise terms, i.e., dissipative and source terms, a dynamical transition be-

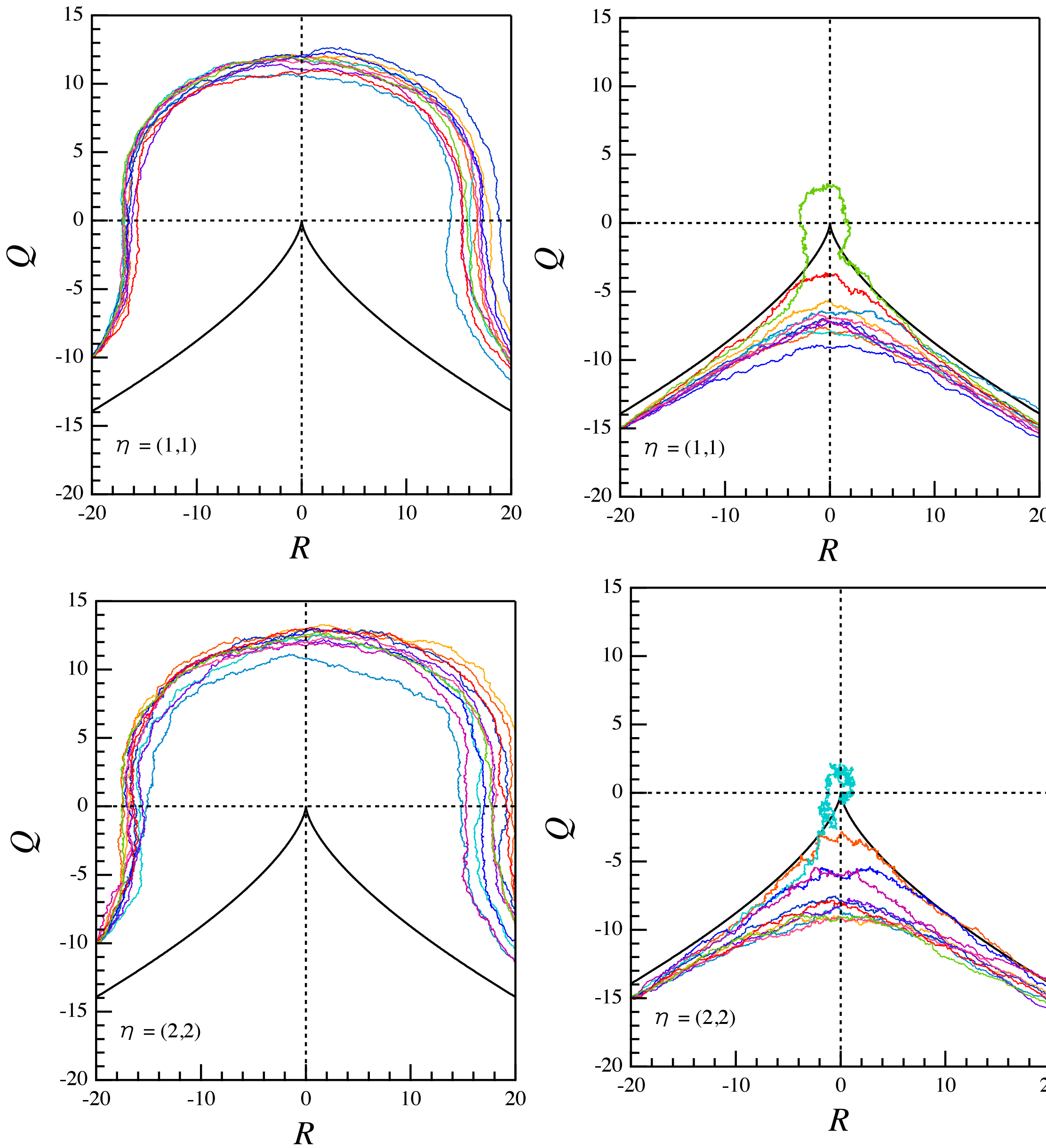

Fig. 5. Effect of the noise terms in the evolution of invariants for different values of the variances $\eta = (\eta_1, \eta_2)$. Here, we set $\alpha' = \gamma' = 0.025$ and $\beta = 1$. The black line is the discriminant line.

tween elliptic and hyperbolic structures, governed by the density of the current term j^2, is still possible. Furthermore, understanding the role of the different terms in the evolution equations provides a way of deducing information on the evolution of interplanetary topological structures (e.g., coronal mass ejection (CME)), in their propagation through the heliosphere, on structures involved in reconnection processes, on topological structures related to the cascade process across the inertial range, and so on.

However, it is important to underline that the obtained theoretical evolution equations provide additional information on the main features of turbulent media in a complementary way to standard approaches (spectral features and scaling properties). For instance, an estimation of the terms reported in Eq. 53 from real observations, such as, in-situ multi-satellite measurements, can give insights on the relevance of the effect of magnetic forces on plasma velocity structures, helping in a better classification of the dynamical features of the observed turbulence in space plasmas showing similar spectral and/or scaling features.

Some possible approaches to understand the relative role/weight of the different terms in the topological invariants could be the following:

i) to study the relevance of the terms by means of numerical simulations,
ii) to study some real situations, such as, turbulence in the solar wind, in the Earth's magnetosheath, in the Earth's central plasma sheet, etc.

Once we have a proper estimation of the different terms, we can attempt to integrate the above equations to get the evolution of the geometrical invariants and the associated topologies of the magnetic and velocity flow field lines. This is the target of a future work that will be finalized in a further paper.

Clearly, the theoretical study presented in this work needs to be expanded including a correct estimation of the different terms in the evolution equations of the gradient tensor invariants. These topics will be devoted to a forthcoming work.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

CRediT authorship contribution statement

Virgilio Quattrociocchi: Conceptualization, Methodology, Investigation, Writing – original draft, Writing – review & editing. **Giuseppe Consolini:** Methodology, Writing – review & editing, Supervision. **Massimo Materassi:** Methodology, Writing – review & editing. **Tommaso Alberti:** Methodology, Writing – review & editing. **Ermanno Pietropaolo:** Supervision, Writing – review & editing.

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Appendix A. Stability analysis of Eq. 54

In this Appendix we discuss the stability analysis of Eq. 54. We start by the computation of the fixed points of Eq. 54 by posing

$$\begin{cases} \dot{Q} = 0 \\ \dot{R} = 0 \end{cases}, \tag{A.1}$$

i.e., we look for the solutions of the following system of equations,

$$\begin{cases} R = 0 \\ Q^2 + \beta Q = 0 \end{cases}. \tag{A.2}$$

This system has two simple solutions,

$$\begin{cases} P_0 \equiv [R_0^*, Q_0^*] = [0, 0] \\ P_1 \equiv [R_1^*, Q_1^*] = [0, -\beta] \end{cases}, \tag{A.3}$$

which are the two fixed points associated with the evolution equations of R and Q.

We now proceed by computing the stability of the two fixed points. Thus, we compute the associated Jacobian matrix,

$$J = \begin{bmatrix} \frac{\partial \dot{R}}{\partial R} & \frac{\partial \dot{R}}{\partial Q} \\ \frac{\partial \dot{Q}}{\partial R} & \frac{\partial \dot{Q}}{\partial Q}. \end{bmatrix} \tag{A.4}$$

Now, we compute the determinant of the Jacobian matrix, $||J||$, the trace $\mathrm{Tr}(J)$, the determinant Δ and the corresponding eigenvalues, $\lambda_{1,2}$, for the two fixed points obtaining, respectively,

i) for P_0, $||J|| = +2\beta$, $\mathrm{Tr}(J) = 0$, $\Delta = -8\beta$ and $\lambda_{1,2} = \pm i\sqrt{2\beta}$,
ii) for P_1, $||J|| = -2\beta$, $\mathrm{Tr}(J) = 0$, $\Delta = 8\beta$ and $\lambda_{1,2} = \pm\sqrt{2\beta}$.

Thus, because $\beta = \frac{j^2}{\rho} \geq 0$ we obtain that

i) $P_0 = [0, 0]$ is a center with marginal stability and
ii) for $P_1 = [0, -\beta]$ is an unstable saddle point.

This explains why in correspondence of P_1 we can observe a transition between hyperbolic and elliptic solutions.

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