Opuscula Math. 40, no. 6 (2020), 685–702
https://doi.org/10.7494/OpMath.2020.40.6.685 Opuscula Mathematica

A NOTE ON ATTRACTIVITY FOR THE INTERSECTION OF TWO DISCONTINUITY MANIFOLDS

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Communicated by Paweł Przybyłowicz

Abstract. In piecewise smooth dynamical systems, a co-dimension 2 discontinuity manifold can be attractive either through partial sliding or by spiraling. In this work we prove that both attractivity regimes can be analyzed by means of the moments solution, a spiraling bifurcation parameter and a novel attractivity parameter, which changes sign when attractivity switches from sliding to spiraling attractivity or vice-versa. We also study what happens at what we call attractivity transition points, showing that the spiraling bifurcation parameter is always zero at those points.

Keywords: piecewise smooth systems, sliding motion, co-dimension 2, discontinuity manifold, attractivity.

Mathematics Subject Classification: 34A36.

1. INTRODUCTION

Piecewise smooth dynamical systems play an important role in physics, engineering and biological applications (e.g., see [1–3,5,6,16,20]), in particular when solution trajectories approach a discontinuity manifold Σ. The case when Σ shows some *attractivity* features is of major interest, meaning that nearby solutions are attracted, in forward time, by Σ, and solution trajectories starting on Σ are forced to stay there, providing what is called *sliding motion*. What happens in co-dimension 1 is well known (see [15]), and there are extensive results about what happens on the intersection of two co-dimension 1 discontinuity manifolds, both from a theoretical (see, e.g., [7, 8, 10, 17, 18]) and from a numerical (see, e.g., [13, 14, 19, 21]) point of view. In particular, we will focus our attention on attractivity regime (see Definition 1.2 and Definition 1.5). As already clarified in [12], this phenomenon could occur in two different ways: by attractivity through sliding and attractivity through spiraling. Our aim here is to analyze what happens on Σ when attractivity conditions switch from sliding regime to spiraling regime and vice-versa, and to fully characterize these scenarios by a single parameter, depending on the dynamics projected on Σ.

The paper structure is as follows. In the Introduction we recall basic Filippov first-order theory in co-dimension 1 and 2 and give definitions of attractivity through sliding and through spiraling. In Section 2 we introduce a parameter, which we prove can characterize the two kinds of attractivity by its sign, and prove the main results of this paper; then, in Section 3 we exemplify our results through numerical simulations. Finally, in Section 4 we propose future research directions.

1.1. THE PROBLEM

Let us consider a piecewise smooth differential system of the following type:

$$\dot{x}(t) = f(x(t)),\ f(x(t)) = f_i(x(t)), \quad x \in R_i,\ i = 1, \ldots, 4,\ t \in [0, T]. \tag{1.1}$$

Here, the $R_i \subseteq \mathbb{R}^n$ are open, disjoint and connected sets, so that (locally) $\mathbb{R}^n = \overline{\bigcup R_i}$, and on each region R_i the function f is given by a smooth vector field f_i, which is assumed to be well defined on $\overline{R}_i$. Further, the regions R_i's are separated by manifolds defined as 0-sets of smooth (at least $\mathscr{C}^2$) scalar functions h_i:

$$\Sigma_i := \{x \in \mathbb{R}^n \ : \ h_i(x) = 0\}, \quad i = 1, 2.$$

1.2. CO-DIMENSION 1 CASE

In this scenario, we are concerned with two regions separated by a manifold Σ defined as the 0-set of a smooth scalar valued function h. One has the following system:

$$\begin{aligned} &\dot{x} = f_1(x), \ \ x \in R_1, \quad \text{and} \quad \dot{x} = f_2(x), \ \ x \in R_2, \\ &\Sigma := \{x \in \mathbb{R}^n \ : \ h(x) = 0\}, \quad h : \ \mathbb{R}^n \to \mathbb{R}\,, \end{aligned} \tag{1.2}$$

where h is a $\mathscr{C}^k$ function, with $k \geq 2$, ∇h is bounded away from 0 for all $x \in \Sigma$, hence near Σ, and (without loss of generality) we label R_1 such that $h(x) < 0$ for $x \in R_1$, and R_2 such that $h(x) > 0$ for $x \in R_2$.

The interesting case is when trajectories reach Σ from R_1 (or R_2), and one has to decide what happens next. To answer this question, it is useful to look at the components of the two vector fields $f_{1,2}$ orthogonal to Σ:

$$w_1(x) := \nabla h(x)^\top f_1(x), \ \ w_2(x) := \nabla h(x)^\top f_2(x), \ \ x \in \Sigma\,. \tag{1.3}$$

Here, Σ is called *attractive in finite time* if for some positive constant c, we have

$$w_1(x) \geq c > 0 \ \text{ and } \ w_2(x) \leq -c < 0, \tag{1.4}$$

for $x \in \Sigma$ and in a neighborhood of Σ. In this case, trajectories starting near Σ must reach it, transversally, and remain there, giving rise to so-called *sliding motion.* A vector field associated to sliding motion is called *sliding vector field.* Filippov proposal is to take as sliding vector field on Σ a convex combination of f_1 and f_2, namely

$$f_\Sigma := (1 - \alpha) f_1 + \alpha f_2, \tag{1.5}$$

with α chosen so that $f_\Sigma \in T_\Sigma$ (f_Σ is tangent to Σ at each $x \in \Sigma$):

$$\dot{x} = (1-\alpha)f_1 + \alpha f_2, \quad \alpha = \frac{\nabla h(x)^\top f_1(x)}{\nabla h(x)^\top \big(f_1(x) - f_2(x)\big)}\,. \tag{1.6}$$

At the same time, Filippov theory also provides *first order exit conditions*: whenever $\alpha = 0$, respectively $\alpha = 1$, one should expect to leave Σ and enter R_1 with vector field f_1, respectively enter R_2 with vector field f_2.

It could also happen that $w_1(x) \geq 0$ and $w_2(x) \geq 0$, or $w_1(x) \leq 0$ and $w_2(x) \leq 0$, situations which are referred to as *crossing*; or, lastly, it could be that $w_1(x) \leq -c < 0$ and $w_2(x) \geq c > 0$, for some positive constant c, which is referred to as *repulsive sliding motion*.

1.3. CO-DIMENSION 2 CASE

Here, we are concerned with (1.1) where now the R_i's are (locally) separated by two intersecting smooth manifolds of co-dimension 1. That is, we have

$$\begin{gathered}\Sigma_1 = \{x:\ h_1(x) = 0\}, \quad \Sigma_2 = \{x:\ h_2(x) = 0\},\\ h_1,\, h_2 : \mathbb{R}^n \to \mathbb{R}, \quad \Sigma = \Sigma_1 \cap \Sigma_2,\end{gathered} \tag{1.7}$$

and we will also use the following notation

$$\Sigma_1^{\pm} = \{x:\ h_1(x) = 0,\ \ h_2(x) \gtrless 0\}, \quad \Sigma_2^{\pm} = \{x:\ h_2(x) = 0,\ \ h_1(x) \gtrless 0\}. \tag{1.8}$$

We will always assume that h_1, h_2 are $\mathscr{C}^k$ functions, with $k \geq 2$, that $\nabla h_1(x) \neq 0$, $x \in \Sigma_1$, $\nabla h_2(x) \neq 0, x \in \Sigma_2$, and further that $\nabla h_1(x)$ and $\nabla h_2(x)$ are linearly independent for x on (and in a neighborhood of) Σ; also, without loss of generality, let us assume that ∇h_1 and ∇h_2 always have unit 2-norm.

So, we have four different regions R_1, R_2, R_3 and R_4 with the four different smooth vector fields f_i, $i = 1, \ldots, 4$, in these regions:

$$\dot{x}(t) = f_i(x(t)),\ \ x \in R_i,\ \ i = 1, \ldots, 4, \tag{1.9}$$

and f_i is assumed to be well defined on $\overline{R}_i$, for $i = 1, 2, 3, 4$.

Without loss of generality, we will label these regions as follows (see Figure 1 for a visualization of the proposed setting):

$$\begin{aligned}&R_1:\ f_1 \quad \text{when} \quad h_1 < 0, h_2 < 0, \quad R_2:\ f_2 \quad \text{when} \quad h_1 < 0, h_2 > 0,\\ &R_3:\ f_3 \quad \text{when} \quad h_1 > 0, h_2 < 0, \quad R_4:\ f_4 \quad \text{when} \quad h_1 > 0, h_2 > 0.\end{aligned} \tag{1.10}$$

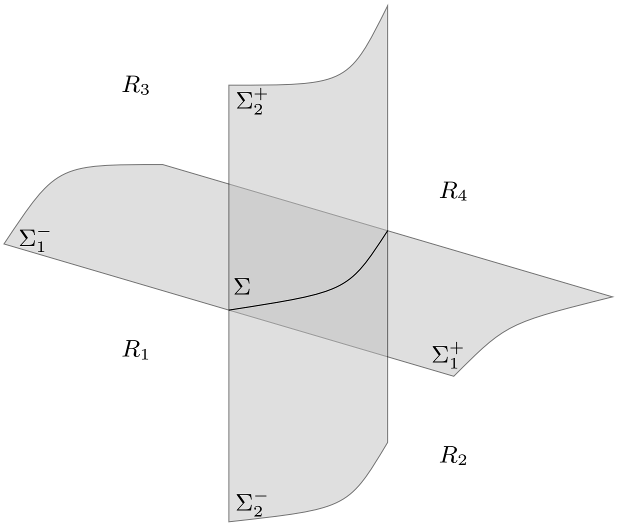

Fig. 1. Problem setting

Moreover, we set

$$w_i^j := \nabla h_j^\top f_i, \quad i = 1, 2, 3, 4,\ j = 1, 2.$$

Let us also set

$$W := \begin{bmatrix} w_1^1 & w_2^1 & w_3^1 & w_4^1 \\ w_1^2 & w_2^2 & w_3^2 & w_4^2 \end{bmatrix}, \tag{1.11}$$

and, for $i = 1, 2, 3, 4$,

$$w_i := \begin{bmatrix} w_i^1 \\ w_i^2 \end{bmatrix}. \tag{1.12}$$

We highlight that, on each $\Sigma_i^\pm$, $i = 1, 2$, we can define the so-called *sub-sliding vector fields* $f_{\Sigma_i^\pm}$ in an analogous way as in (1.5), (1.6). More specifically, we define

$$\begin{aligned}
f_{\Sigma_1^+} &:= (1 - \alpha_{\Sigma_1^+}) f_2 + \alpha_{\Sigma_1^+} f_4, \quad \alpha_{\Sigma_1^+}(x) := \frac{\nabla h_1(x)^\top f_2(x)}{\nabla h_1(x)^\top (f_2(x) - f_4(x))},\ x \in \Sigma_1^+, \\
f_{\Sigma_1^-} &:= (1 - \alpha_{\Sigma_1^-}) f_1 + \alpha_{\Sigma_1^-} f_3, \quad \alpha_{\Sigma_1^-}(x) := \frac{\nabla h_1(x)^\top f_1(x)}{\nabla h_1(x)^\top (f_1(x) - f_3(x))},\ x \in \Sigma_1^-, \\
f_{\Sigma_2^+} &:= (1 - \alpha_{\Sigma_2^+}) f_3 + \alpha_{\Sigma_2^+} f_4, \quad \alpha_{\Sigma_2^+}(x) := \frac{\nabla h_2(x)^\top f_3(x)}{\nabla h_2(x)^\top (f_3(x) - f_4(x))},\ x \in \Sigma_2^+, \\
f_{\Sigma_2^-} &:= (1 - \alpha_{\Sigma_2^-}) f_1 + \alpha_{\Sigma_2^-} f_2, \quad \alpha_{\Sigma_2^-}(x) := \frac{\nabla h_2(x)^\top f_1(x)}{\nabla h_2(x)^\top (f_1(x) - f_2(x))},\ x \in \Sigma_2^-.
\end{aligned}$$

For a more extensive treatise of the subject and a deeper insight on how to take over the natural ambiguity in defining a dynamics on Σ, e.g. see [9–11].

We will focus on the co-dimension 2 case, specifically when Σ attracts nearby dynamics. Reasonable conditions, when dealing with attractivity of a co-dimension 2 discontinuity manifolds, require that projected vector fields w_i as in (1.12) do not point away from the sub-manifolds $\Sigma^{\pm}_{1,2}$ in their respective regions of interest R_i, $i = 1, 2, 3, 4$. We are going to resort to sign pattern[1)] of suitable matrices, which is still denoted by sgn.

Definition 1.1. Let $N \subseteq \mathbb{R}^n$ be an open set such that $\Sigma \cap N \neq \emptyset$. The discontinuity manifold Σ satisfies the *sign pattern conditions* in N if

$$\operatorname{sgn} w_i(x) \neq \operatorname{sgn} \begin{bmatrix} h_1(x) \\ h_2(x) \end{bmatrix}, \text{ for } x \in R_i \cap N,\ i = 1, 2, 3, 4. \tag{1.13}$$

Attractivity regime for a discontinuity manifold Σ of co-dimension 2 can occur in two distinct ways (see [12]): through sliding or through spiraling.

Definition 1.2 (Partial Nodal Attractivity, [12])**.** The discontinuity manifold Σ is *partially nodally attractive*, or *attractive through sliding* at $x_0 \in \Sigma$ if there exists an open neighborhood N of x_0 such that:

(a) Σ satisfies sign pattern conditions (1.13) in N;
(b) at least one of the following conditions is satisfied for all $x \in \Sigma \cap N$:
   (1^+) $w^1_4(x) < 0 < w^1_2(x)$ together with (1^+_a): $\det \begin{bmatrix} w_2(x) & w_4(x) \end{bmatrix} < 0$;
   (1^-) $w^1_3(x) < 0 < w^1_1(x)$ together with (1^-_a): $\det \begin{bmatrix} w_3(x) & w_1(x) \end{bmatrix} < 0$;
   (2^+) $w^2_4(x) < 0 < w^2_3(x)$ together with (2^+_a): $\det \begin{bmatrix} w_4(x) & w_3(x) \end{bmatrix} < 0$;
   (2^-) $w^2_2(x) < 0 < w^2_1(x)$ together with (2^-_a): $\det \begin{bmatrix} w_1(x) & w_2(x) \end{bmatrix} < 0$;
(c) if any of ($1^\pm$) or ($2^\pm$) is satisfied, then ($1^\pm_a$) or ($2^\pm_a$) must be satisfied as well.

We stress that, throughout this work, exit conditions from sliding on Σ are always assumed to be first order and unambiguous (see [12]). This implies that, at potential exit points, none of the f_i's put itself tangent to Σ, and also that one, and only one, of the Filippov sub-sliding vector fields $f_{\Sigma^{\pm}_{1,2}}$ can also be tangent to Σ. We highlight this in the following.

Assumption 1.3. For all $x \in \Sigma$, one and only one sub-sliding vector field $f_{\Sigma^{\pm}_{1,2}}$ on $\Sigma^{\pm}_{1,2}$ is directed outward with respect to Σ.

Example 1.4. Assumption 1.3 says that if, at some $x \in \Sigma$,

$$\nabla h_2(x)^\top f_{\Sigma^-_1}(x) < 0,$$

1) The *sign pattern* of a matrix is obtained by replacing each entry by its sign; see [4] for a complete exposition of the subject.

then it has to necessarily hold that

$$\nabla h_2(x)^\top f_{\Sigma_1^+}(x) < 0,\ \nabla h_1(x)^\top f_{\Sigma_2^-}(x) > 0,\ \nabla h_1(x)^\top f_{\Sigma_2^+}(x) < 0,$$

provided that all the vector fields above exist and are well defined, as given in (1.5) (see also (1.6)). Analogous relations have to be valid if any other dynamics off Σ is taking place on some sub-manifold different from Σ_1^-.

Definition 1.5 (Spiral Attractivity, [7])**.** The discontinuity manifold Σ is said to be *clockwise attractive through spiraling*, or *clockwise spiraling attractive*, (see Figure 2 (a)) at $x_0 \in \Sigma$ if there exists an open neighborhood N of x_0 where the signs of Table 1 hold and, letting

$$\mu_{\text{CW}}(x_0) := \frac{w_1^2(x_0)w_4^2(x_0)w_2^1(x_0)w_3^1(x_0)}{w_1^1(x_0)w_4^1(x_0)w_2^2(x_0)w_3^2(x_0)}, \tag{1.14}$$

we have

$$\mu_{\text{CW}}(x_0) < 1.$$

Similarly, we say that the sliding regime is *counterclockwise attractive through spiraling*, or *counterclockwise spiraling attractive* (see Figure 2 (b)) at $x_0 \in \Sigma$ if there exists an open neighborhood N of x_0 where the signs of Table 2 hold and we have

$$\mu_{\text{CCW}}(x_0) < 1,$$

with $\mu_{\text{CCW}}(x_0) := \frac{1}{\mu_{\text{CW}}(x_0)}$.

Table 1
Signs of matrix W in clockwise spiraling around Σ

Component	$i=1$	$i=2$	$i=3$	$i=4$
w_i^1	$w_1^1(\Sigma_1^-) > 0$	$w_2^1(\Sigma_1^+) < 0$	$w_3^1(\Sigma_1^-) > 0$	$w_4^1(\Sigma_1^+) < 0$
w_i^2	$w_1^2(\Sigma_2^-) < 0$	$w_2^2(\Sigma_2^-) < 0$	$w_3^2(\Sigma_2^+) > 0$	$w_4^2(\Sigma_2^+) > 0$

Table 2
Signs of matrix W in counterclockwise spiraling around Σ

Component	$i=1$	$i=2$	$i=3$	$i=4$
w_i^1	$w_1^1(\Sigma_1^-) < 0$	$w_2^1(\Sigma_1^+) > 0$	$w_3^1(\Sigma_1^-) < 0$	$w_4^1(\Sigma_1^+) > 0$
w_i^2	$w_1^2(\Sigma_2^-) > 0$	$w_2^2(\Sigma_2^-) > 0$	$w_3^2(\Sigma_2^+) < 0$	$w_4^2(\Sigma_2^+) < 0$

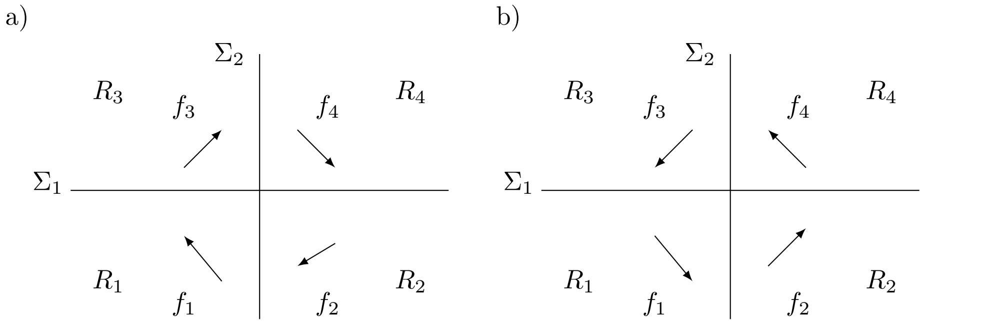

Fig. 2. Clockwise (a) and counterclockwise (b)

Remark 1.6. The quantity μ_{CW} (resp. μ_{CCW}) is a bifurcation parameter for clockwise (resp. counterclockwise) spiral attractivity. In fact, letting $x(t)$ be solution to (1.9) with some suitable initial condition, as long as for $t > 0$ we have $x(t) \in \Sigma$ and

$$\mu_{\text{CW}}(x(t)) < 1 \text{ (resp. } \mu_{\text{CCW}}(x(t)) < 1), \tag{1.15}$$

then sliding motion on Σ persists; when instead a time $\bar{t} > 0$ is reached and, at $\overline{x} := x(\bar{t}) \in \Sigma$ we have

$$\mu_{\text{CW}}(\overline{x}) = 1 \text{ (resp. } \mu_{\text{CCW}}(\overline{x}) = 1), \tag{1.16}$$

then Σ could cease to be attractive at the so-called *potential exit point* $\overline{x}$, and sliding motion on it, despite it would still exist, could become repulsive, and thus ill-posed if μ_{CW} passes 1.

Now, for $t > \bar{t}$, two cases could occur[2)]: either $\mu_{\text{CW}}(x(t)) < 1$ (resp. $\mu_{\text{CCW}}(x(t)) < 1$), and then sliding motion turns back to be well-posed; or $\mu_{\text{CW}}(x(t)) > 1$ (resp. $\mu_{\text{CCW}}(x(t)) > 1$) and then, as observed in [7], dynamics would leave Σ and would proceed in one of the R_i's, with no qualitatively difference in the resulting dynamics, following an outward spiraling regime for $t - \bar{t}$ sufficiently small.

Remark 1.7. Let us stress that only one attractive regime could occur on Σ; so, if Σ is attractive through sliding at some point x_0, then it cannot be attractive through spiraling there, and vice-versa, as it can be deduced from definitions above.

Hereafter and before presenting the main results of the paper, without loss of generality we assume the following.

Assumption 1.8. If $N \subseteq \mathbb{R}^n$ is an open set, with $\Sigma \cap N \neq \emptyset$, such that sign pattern conditions (1.13) and attractivity, either through sliding or through spiraling, hold, then every $w \in \operatorname{conv}\{w_1, w_2, w_3, w_4\}$[3)] does not have zeros in N. In particular, both $w_i(x)$ and $\begin{bmatrix} h_1(x) \\ h_2(x) \end{bmatrix}$ are different from $0_{\mathbb{R}^2}$ whenever $x \in R_i \cap N$, $i = 1, 2, 3, 4$.

2) The possibility that $\mu_{\text{CW}}(x(t)) = 1$ (resp. $\mu_{\text{CCW}}(x(t)) = 1$) in some small right neighborhood of $\bar{t} > 0$ is a trivial configuration.

3) This set is defined as the *convex hull* of the four vectors w_i, $i = 1, 2, 3, 4$.

Remark 1.9. Assumption 1.8 guarantees that condition (b) in Definition 1.2 is satisfied and parameters μ_{CW}, μ_{CCW} from (1.14) in Definition 1.5 are always well defined whenever x belongs to a neighborhood of Σ where sign pattern conditions (1.13) and attractivity hold.

2. ANALYSIS OF TRANSITION FROM ATTRACTION THROUGH SLIDING TO ATTRACTION THROUGH SPIRALING, OR VICE-VERSA

Our interest in this section is to study what happens when attractivity switches from a sliding regime to spiraling regime, and vice-versa, a situation in which sliding motion remains well-defined as long as attractivity holds. It is well known (see [10, 11]) that the moments Filippov sliding vector field automatically detects first order exit points in attractivity through sliding, and it remains well defined also during attractivity through spiraling. We recall that it is defined as

$$f_{\mathrm{M}}(x) = \sum_{i=1}^{4} \lambda_{M,i}(x) f_i(x), \quad x \in \Sigma,$$

where $\lambda_M(x)$ is the unique solution to

$$\begin{bmatrix} W(x) \\ \mathbb{1}^\top \\ d(x)^\top \end{bmatrix} \lambda = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad x \in \Sigma, \tag{2.1}$$

where $\mathbb{1} := \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^\top$ and

$$d(x) := \begin{bmatrix} \|w_1(x)\| & -\|w_2(x)\| & -\|w_3(x)\| & \|w_4(x)\| \end{bmatrix}^\top, \quad x \in \Sigma.$$

The vector $\lambda_M(x)$ is said to be *admissible* (resp. *strictly admissible*) if $\lambda_{M,i}(x) \geq 0$ (resp. $\lambda_{M,i}(x) > 0$), $i = 1, 2, 3, 4$, and $\lambda_M(x)$ depends smoothly on $x \in \Sigma$.

Let us also recall that, differently from exit points in sliding attractivity regime, in spiraling attractivity exits from Σ are not tangential. This implies that we cannot expect (2.1) to provide any criterion to detect exit points when spiraling attractivity ceases, as it happens in the sliding attractivity case. In fact, since at an exit point dynamics could follow any of the f_i's, $i = 1, 2, 3, 4$ (see [7]), then there must exist a unique $i = 1, 2, 3, 4$ such that $\lambda_i = 1$, while $\lambda_j = 0$ for $j \neq i$. This in turn implies, from (2.1), that $w_i = 0$, which does not occur.

We introduce a concept useful for studying what happens when attractivity passes from sliding regime to spiraling regime or vice-versa. In what follows, for $r > 0$ and $x_0 \in \mathbb{R}^n$, let $B_r(x_0)$ be the ball centered at x_0 of radius r.

Definition 2.1. We say that $x_0 \in \Sigma$ is an *attractivity transition point* if and only if for any $\varepsilon > 0$ there exist x_1, $x_2 \in B_\varepsilon(x_0) \cap \Sigma \setminus \{x_0\}$ such that Σ is attractive through sliding at x_1 and attractive through spiraling at x_2.

In order to analyze what happens at attractivity transition points, we define, for all $x \in \mathbb{R}^n$ where the w_i's are well-defined,

$$\phi(x) := \min\{w_1^2(x)w_2^2(x), w_2^1(x)w_4^1(x), w_4^2(x)w_3^2(x), w_3^1(x)w_1^1(x)\}. \qquad (2.2)$$

Let us note that $\phi(x)$ is continuous on Σ.

We are now ready to prove the main results of this paper.

Theorem 2.2. *Let $x_0 \in \Sigma$ and let $N \subseteq \mathbb{R}^n$ be an open neighborhood of x_0 such that Σ satisfies sign pattern conditions* (1.13) *in N. Then the following characterizations hold:*

(a) *Σ is clockwise attractive through spiraling at x_0 if and only if $\phi(x_0) > 0$ and $\mu_{CW}(x_0) < 1$;*
(b) *Σ is counterclockwise attractive through spiraling at x_0 if and only if $\phi(x_0) > 0$ and $\mu_{CCW}(x_0) < 1$;*
(c) *Σ is attractive through sliding at x_0 if and only if $\phi(x_0) \leq 0$ and $\lambda_M(x_0)$ is strictly admissible.*

Proof. (a) If Σ is attractive through spiraling, then simple computations from Table 1 and Table 2 show that each product in right-hand side of (2.2) is strictly positive, implying that $\phi(x_0) > 0$ and $\mu_{\mathrm{CW}}(x_0) < 1$.

If $\phi(x_0) > 0$ and $\mu_{\mathrm{CW}}(x_0) < 1$, then, without loss of generality, let us assume that minimum in (2.2) is attained by $w_1^2(x_0)w_2^2(x_0)$. Let us examine the case

$$w_1^2(x_0) > 0, \quad w_2^2(x_0) > 0;$$

the other one is analogous.

Let us first notice that it is necessary to have

$$w_2^1(x_0) > 0,$$

otherwise condition (1.13) would be violated. As a consequence

$$w_4^1(x_0) > 0,$$

and thus it must also be

$$w_4^2(x_0) < 0,$$

otherwise condition (1.13) would not hold.

Now, if by contradiction $w_1^1(x_0) > 0$, therefore $w_3^1(x_0) > 0$, since by definition it holds that $w_3^1(x_0)w_1^1(x_0) \geq \phi(x_0) > 0$. The case $w_3^2(x_0) < 0$ is ruled out by condition (1.13), and therefore $w_3^2(x_0) > 0$, implying $w_4^2(x_0) > 0$. Again, $w_4^1(x_0) > 0$ would violate condition (1.13) and so it must be $w_4^1(x_0) < 0$; but then $w_2^1(x_0)w_4^1(x_0) < 0$, which is not the case since $\phi(x_0) > 0$. Then, it follows that

$$w_1^1(x_0) < 0,$$

from which

$$w_3^1(x_0) < 0.$$

Thus, it must also be

$$w_3^2(x_0) < 0,$$

otherwise $w_3^2(x)_0 w_4^2(x_0) < 0$, which is not allowed since $\phi(x_0) > 0$.

Looking at Table 2, we conclude that $\phi(x_0) > 0$ implies counterclockwise spiraling attractivity for Σ.

If $w_1^2(x_0) < 0, w_2^2(x_0) < 0$, then with similar arguments clockwise spiraling attractivity would be inferred.

All other cases are completely analogous.

(b) Proof goes analogously as in previous point.

(c) If Σ is attractive through sliding at x_0, then it is not attractive through spiraling at x_0, as observed in Remark 1.7, so that $\phi(x_0) \leq 0$ from previous point.

On the other hand, if $\phi(x_0) \leq 0$ and $\lambda_M(x_0)$ is strictly admissible, then let us assume minimum is attained at $w_1^2(x_0)w_2^2(x_0)$. We analyze separately the two cases $w_1^2(x_0)w_2^2(x_0) < 0$ and $w_1^2(x_0)w_2^2(x_0) = 0$.

If $w_1^2(x_0)w_2^2(x_0) < 0$, then two scenarios can occur: the first is $w_2^2(x_0) < 0 < w_1^2(x_0)$, the other $w_1^2(x_0) < 0 < w_2^2(x_0)$.

In the first scenario sliding motion on Σ_2^- occurs: if $\nabla h_1(x_0)^\top f_{\Sigma_2^-}(x_0) < 0$, then x_0 is an exit point from Σ, and the third and fourth components of $\lambda_M(x_0)$ are negative (see [10], Theorem 2.4), which is not the case. Therefore, it must be $\nabla h_1(x_0)^\top f_{\Sigma_2^-}(x_0) > 0$, which is equivalent, after some computations, to (2_a^-) in Definition 1.2. Similar arguments hold for $w_3(x_0)$ and $w_4(x_0)$, so that attractivity through sliding is proven on Σ at (x_0).

In the second scenario, repulsive sliding motion is taking place on Σ_2^-, so let us look at $w_3(x_0)$ and $w_4(x_0)$. On the account on Assumption 1.3, it cannot be that both $\nabla h_2(x_0)^\top f_{\Sigma_1^-}(x_0) < 0$ and $\nabla h_2(x_0)^\top f_{\Sigma_1^+}(x_0) > 0$, so that either one holds or none of them. In the first case, x_0 would represent a first-order exit point from Σ, and again $\lambda_M(x_0)$ would have its second and fourth components negative, against its strict admissibility. In the second case, condition (1_a^-) would hold, providing attractivity through sliding.

If $w_1^2(x_0)w_2^2(x_0) = 0$, then let us assume $w_1^2(x_0) = 0$. Therefore, by (1.13), $w_1^1(x_0) > 0$, and then, by strict admissibility of $\lambda_M(x_0)$, it follows $w_3^2(x_0) > 0$. Now, if $w_3^1(x_0) < 0$, straightforward computations provide (1_a^-), which proves attractivity through sliding of Σ at x_0; otherwise, there would be crossing on Σ_1^-, and we need to look at what happens on Σ_2^+. If attractive sliding motion occurs on it, then (2_a^+) would hold, and the claim would be proven. If not, only crossing would be allowed by strict admissibility of $\lambda_M(x_0)$. Same could be said about Σ_1^+: either (1_a^+) is fulfilled, or there is crossing on it. In this last case, again resorting to strict admissibility of $\lambda_M(x_0)$, the only possibility is that (2_a^-) holds, thus proving the claim. □

Corollary 2.3. *Let $x_0 \in \Sigma$ and let $N \subseteq \mathbb{R}^n$ be an open neighborhood of x_0 such that Σ satisfies sign pattern conditions* (1.13) *in N. If $x_0 \in \Sigma$ is an attractivity transition point then $\phi(x_0) = 0$; in particular, Σ is attractive through sliding at x_0.*

Proof. The fact that $\phi(x_0) = 0$ is a simple consequence of Bolzano's theorem (see [22]); thus, resorting to Theorem 2.2 yields that Σ is attractive through sliding at x_0, and the claim is proven. □

Corollary 2.4. *Let $x_0 \in \Sigma$ and let $N \subseteq \mathbb{R}^n$ be an open neighborhood of x_0 such that Σ satisfies sign pattern conditions* (1.13) *in N. If x_0 is an attractivity transition point then either $\mu_{CW}(x_0) = 0$ or $\mu_{CCW}(x_0) = 0$.*

Proof. From Definition 2.1, given an arbitrary $\varepsilon > 0$ there exist $x_1, x_2 \in B_\varepsilon(x_0) \cap \Sigma \setminus \{x_0\}$ such that Σ is attractive through sliding at x_1 and attractive through spiraling at x_2. Without loss of generality, let us assume Σ to be clockwise spirally attractive at x_2; the counterclockwise case will go completely analogous.

By smoothness of $W(x)$ in $B_\varepsilon(x_0)$, we could then deduce that signs of Table 1 apply to x_0, possibly with large inequalities instead of strict ones. Also, on the account of Corollary 2.3, we have $\phi(x_0) = 0$. If, without loss of generality, the minimum in (2.2) is attained by $w_1^2(x_0)w_2^2(x_0) = 0$, then let us note that it cannot be $w_2^2(x_0) = 0$, otherwise Σ would be counterclockwise spirally attractive around x_0, which is not the case; so it must be $w_1^2(x_0) = 0$. As a consequence and on the account of Assumption 1.8, it must also be $w_1^1(x_0) > 0$; moreover it cannot be $w_3^2(x_0) = 0$, so that we have $w_3^1(x_0) \geq 0$ and $w_3^2(x_0) > 0$. With completely analogous arguments, we deduce that $w_4^1(x_0) < 0$, $w_4^2(x_0) \geq 0$, $w_2^1(x_0) \leq 0$ and $w_2^2(x_0) < 0$. Therefore $\mu_{\text{CW}}(x_0)$ is well defined and $\mu_{\text{CW}}(x_0) = 0$, which proves the claim. □

Remark 2.5. Let us highlight that when a given dynamics, solution to (1.9) with an assigned initial conditions $x(0) = x_0 \in \mathbb{R}^n$, reaches Σ at some $t > 0$, that is $h_1(x(t)) = h_2(x(t)) = 0$, then Theorem 2.2 allows to leverage parameters ϕ, μ_{CW} or μ_{CCW} and the moments solution λ_M to check for attractivity on Σ, instead of checking conditions in Definition 1.2 and Definition 1.5. Thus, if moments sliding vector field is selected on Σ when attractivity holds, applying Theorem 2.2 simplifies attractivity condition checking while dynamics is integrated on Σ.

Further, let us stress that while results in [10] provide sufficient conditions regarding admissibility of moments solution, Theorem 2.2 provides also necessary conditions for it.

3. NUMERICAL SIMULATIONS

In this section, we exemplify on theoretical results obtained in previous section. We stress that examples below are not meant to analyze dynamical properties of the unique solution to (1.9) when it is assigned a specific initial condition; instead, they show the numerical behaviors of Theorem 2.2, Corollary 2.3 and Corollary 2.4 and their usefulness in easily detecting when attractivity switches from sliding regime to spiraling regime or vice-versa.

It is worth stressing that both Assumption 1.3 and Assumption 1.8 are satisfied in examples below, as well as sign pattern conditions (1.13).

Example 3.1. Let us consider (1.9) with initial condition $x(0) = x_0$, for some $x_0 \in \mathbb{R}^3$, and let $f_i(x(t))$ be such that they provide nodal attractivity through sliding at x_0 at $t = 0$, and then they suitably rotate, for $t > 0$, until partial nodal attractivity first and eventually attractivity through spiraling hold on Σ, as explained below.
First, let us assume that

$$f_1(x_0) := \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \; f_2(x_0) := \begin{bmatrix} \frac{3}{2} \\ -1 \\ 1 \end{bmatrix},$$
$$f_3(x_0) := \begin{bmatrix} -5 \\ 1 \\ 1 \end{bmatrix}, \; f_4(x_0) := \begin{bmatrix} -\frac{1}{5} \\ -1 \\ 1 \end{bmatrix},$$

so that, with Σ_1 and Σ_2 defined by $h_1(x) := x_1$, $h_2(x) := x_2$ respectively, from (1.11) it follows that

$$W(x_0) = \begin{bmatrix} 1 & \frac{3}{2} & -5 & -\frac{1}{5} \\ 1 & -1 & 1 & -1 \end{bmatrix},$$

and Σ is (nodally) attractive through sliding at x_0. Now, defining

$$R(\theta) := \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \tag{3.1}$$

we assume that for $t \in \left[0, \frac{\pi}{2}\right]$ the projected dynamics around $x(t)$ is given by

$$w_1(x(t)) = R(t)w_1(x_0), \quad w_i(x(t)) = w_i(x_0),\; i = 2, 3, 4.$$

Let us note that at $t_1 \approx 1.3$ sliding motion on Σ_1^- ceases to exist, while attractivity for Σ is preserved in the sense of Definition 1.2.

Then, for $t \in \left[\frac{\pi}{2}, \pi\right]$ we assume that

$$w_2(x(t)) = R\left(t - \frac{\pi}{2}\right) w_2(x_0), \quad w_i(x(t)) = w_i\left(x\left(\frac{\pi}{2}\right)\right),\; i = 1, 3, 4.$$

Now, at $t_2 \approx 3.5$ sliding motion on Σ_2^- disappears, but Σ is still partially nodally attractive.

For $t \in \left[\pi, \frac{3}{2}\pi\right]$ the projected dynamics is assumed to evolve according to

$$w_4(x(t)) = R\,(t - \pi)\, w_4\,(x_0)\,, \quad w_i(x(t)) = w_i(x(\pi)),\; i = 1, 2, 3.$$

At $t_3 \approx 5.4$ sliding on Σ_2^+ disappears and Σ still retains partially nodal attractivity.

For $t \in \left[\frac{3}{2}\pi, 2\pi\right]$ we assume to have

$$w_3(x(t)) = R\left(t - \frac{3}{2}\pi\right) w_3\,(x_0)\,, \quad w_i(x(t)) = w_i\left(x\left(\frac{3}{2}\pi\right)\right),\; i = 1, 2, 4.$$

At $t_4 \approx 7.8$ the projected dynamics has reached an attractivity transition point, and attractivity regime switches from sliding to spiraling: here, as depicted in Figure 3, we have $\mu(x(t_4)) = \phi(x(t_4)) = 0$, a behavior expected according to Corollary 2.3 and Corollary 2.4. In agreement with Theorem 2.2, $\phi(x(t)) > 0$ for $t > t_4$: now spiral attractivity holds around Σ, and such a regime persists until $t \approx 8.2$, when $\mu = 1$, and after which attractivity becomes ill-posed; at this point, dynamics should leave Σ following either of the f_i's, $i = 1, 2, 3, 4$.

Let us note that $\mu(x)$ could get positive, negative and zero values during sliding attractivity. Moreover, we can not expect $\phi(x)$ to be more than continuous, as clear from Figure 3.

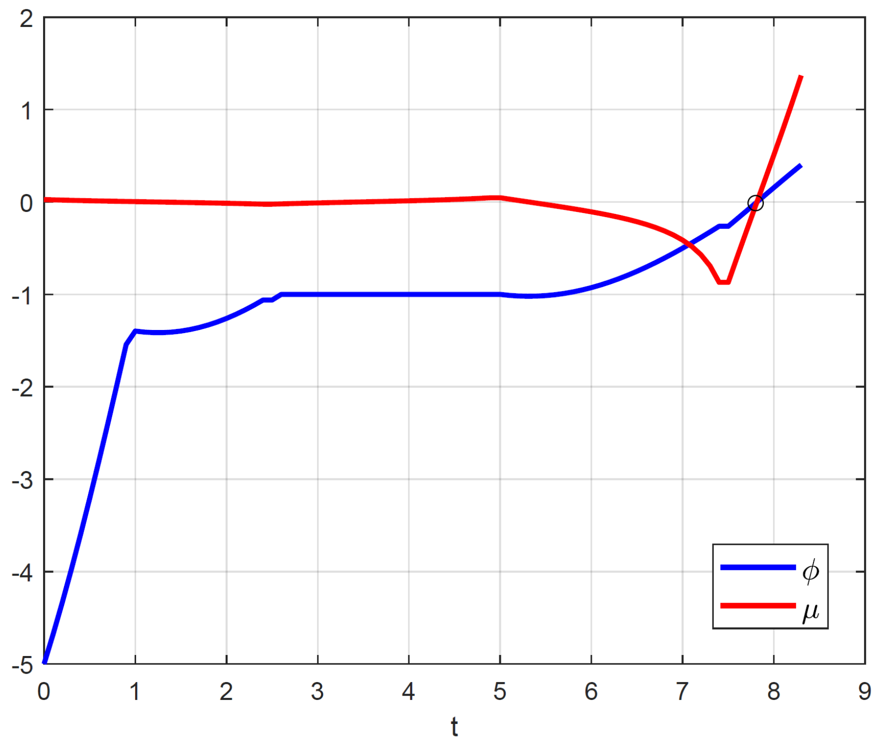

Fig. 3. Plots of $\phi(x)$ and $\mu(x)$ relative to Example 3.1. Black circle at time $t_4 \approx 7.8$ represents the time at which $\phi = \mu = 0$, that is where dynamics has reached an attractivity transition point from sliding to spiraling attractivity

Further, in Figure 4 it can be noticed how moments solution components remain well defined even where $\mu \geq 1$, that is when spiraling sliding motion ceases to be attractive. Lastly, let us observe here that, since vector fields $f_i(x)$, $i = 1, 2, 3, 4$, lie on the plane $x_3 = 1$ for all $x \in \Sigma$, then the resulting sliding vector field is $f_\Sigma = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^\top$, and it is independent on the particular convex combination chosen to determine it.

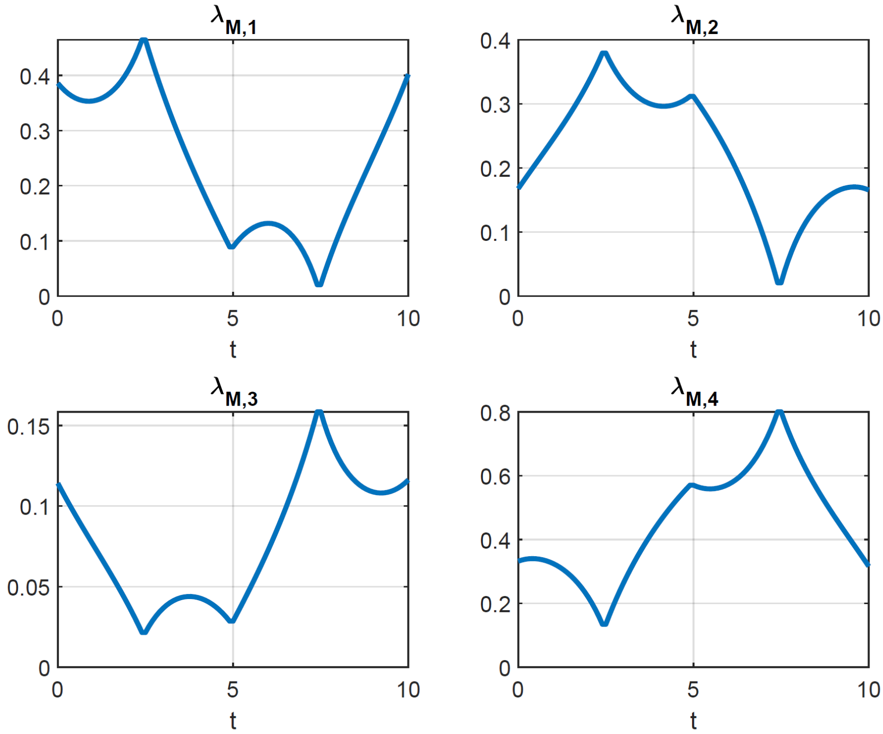

Fig. 4. Plots of moments solution components relative to Example 3.1. It can be appreciated here that, as expected, these components are as smooth as $W(x)$ and stay between 0 and 1 even for $t > 8.5$, that is when $\mu > 1$

Example 3.2. Let us again consider (1.9) with initial condition $x(0) = x_0$, for some $x_0 \in \mathbb{R}^3$, and let us again assume that the $f_i(x(t))$'s, $i = 1, 2, 3, 4$, suitably rotate from a regime of counterclockwise spiral attractivity at x_0 to sliding attractivity, until it eventually ceases to hold, as detailed below. First, let us set, at $t = 0$,

$$f_1(x_0) := \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 1 \end{bmatrix}, \ f_2(x_0) := \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

$$f_3(x_0) := \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \ f_4(x_0) := \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix},$$

so that, with Σ_1 and Σ_2 defined by $h_1(x) := x_1$, $h_2(x) := x_2$ respectively, from (1.11) it follows that

$$W(x_0) = \begin{bmatrix} -\frac{1}{2} & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix},$$

and Σ is counterclockwise spirally attractive at x_0. For $t > 0$ we assume that

$$w_2(x(t)) = R(t)w_2(x_0), \quad w_i(x(t)) = w_i(x_0),\ i = 1, 3, 4,$$

where $R(t)$ is the rotation matrix defined in (3.1). It then can be seen that at $t_1 \approx 3.7$ attractivity regime around Σ switches from spiraling attractivity to sliding attractivity, as clear from Figure 5. At this attractivity transition point the function $\phi(x(t))$, which was non-negative for $t \in [0, t_1]$, becomes zero and then changes sign together with $\mu(x(t))$.

For $t > t_1$, sliding motion on Σ takes place, which remains well defined until $t_2 \approx 9$, when the moments solution components $\lambda_{M,3}$ and $\lambda_{M,4}$ become negative, providing a smooth exit point on Σ (see [10]), with a sliding motion on Σ_2^-.

Let us stress that we have again chosen vector fields lying on the plane $x_3 = 1$ for all $x \in \Sigma$, so that the resulting sliding vector field is independent on the convex combination of the four f_i's, $i = 1, 2, 3, 4$. In fact, what matters here is the behavior of $\lambda_M(x)$, as $x \in \Sigma$, rather than the selected Filippov sliding vector field on Σ, in order to analyze and check which kind of attractivity is occurring during sliding motion.

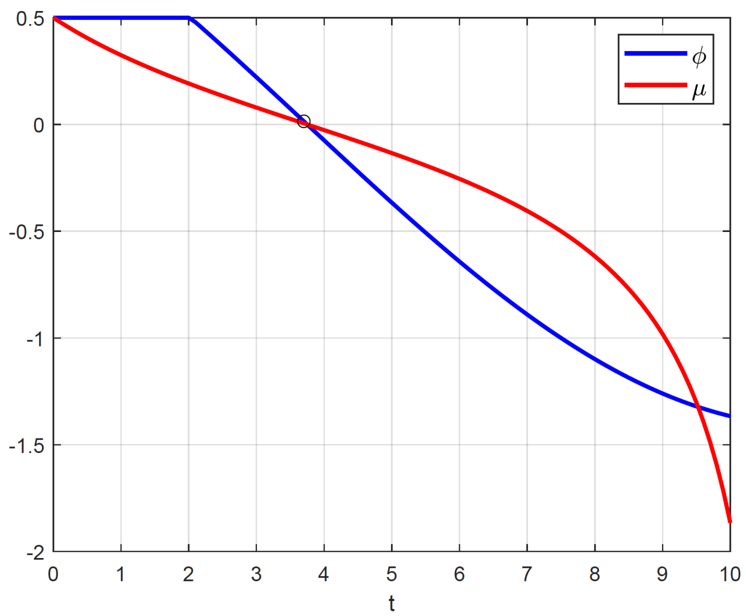

Fig. 5. Plots of $\phi(x)$ and $\mu(x)$ relative to Example 3.1. Black circle at time $t_1 \approx 3.7$ represents the time at which $\phi = \mu = 0$, that is where dynamics has reached an attractivity transition point from spiraling to sliding attractivity

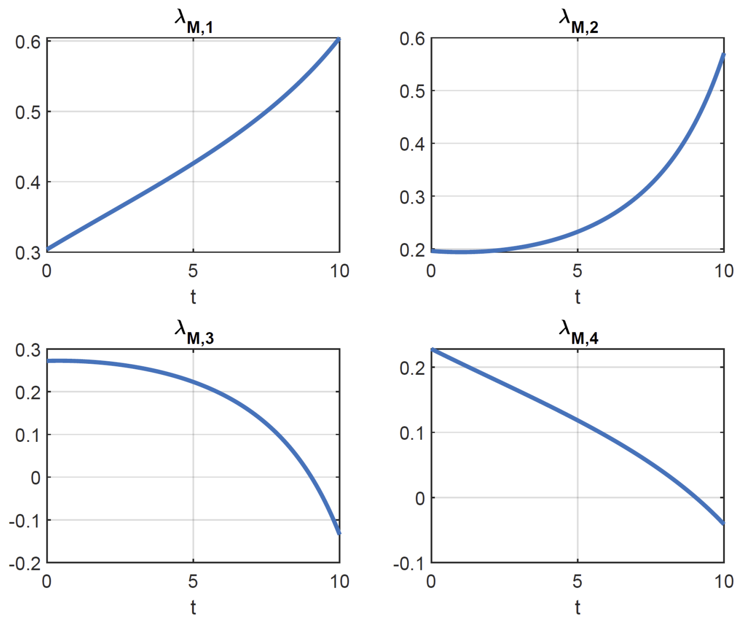

Fig. 6. Plots of moments solution components relative to Example 3.2. These components are as smooth as $W(x)$ and stay between 0 and 1 for $t \in [0, t_2]$; at $t = t_2$ dynamics reaches an exit point on Σ, and keeps evolving according to the unique sliding motion on Σ_2^- given by $f_{\Sigma_2^-}$, since third and fourth components turn negative at t_2

4. CONCLUSIONS AND FUTURE WORKS

We have introduced a new parameter which is useful, together with moments vector λ_M solution of (2.1) and μ_{CW}, or μ_{CCW}, as in (1.14), to characterize attractivity for a co-dimension 2 discontinuity manifold in piecewise smooth differential systems: this parameter is non-positive whenever Σ is attractive through sliding and is positive when it is attractive through spiraling, becoming continuously zero when attractivity regime is at an attractivity transition point. Moreover, we have proven that, at attractivity transition points, the spiraling bifurcation parameter introduced in [7] becomes zero as well. We have then exemplified our construction with some examples, corroborating results and definitions proposed in the paper.

As future research directions, we want to leverage the parameter ϕ to explore attractivity in co-dimension higher than 2, and prove that moments solution still remains well defined in these settings, extending results is [11] to cover all possible cases under attractivity conditions. Further, since Theorem 2.2 is easier to verify than Definition 1.2 and Definition 1.5 for checking attractivity on a co-dimension 2 manifold, it can be used within numerical solvers for discontinuous ODEs in order to simplify their implementation.

Acknowledgements
This work has been supported by the OP RDE (OP VVV) funded project reg. no: CZ.02.1.01/0.0/0.0/16_019/0000765 - "The Research Center for Informatics".

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Received: September 18, 2020.
Revised: October 3, 2020.
Accepted: October 5, 2020.