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Relaxation of quasi-stationary states in long range interacting systems and a classification of the range of pair interactions

Research Article

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Abstract: Systems of particles interacting with long range interactions present generically "quasi-stationary states" (QSS), which are approximately time-independent out of equilibrium states. In this proceedings, we explore the generalization of the formation of such QSS and their relaxation from the much studied case of gravity to a generic pair interaction with the asymptotic form of the potential $v(r) \sim 1/r^{\gamma}$ with $\gamma > 0$ in d dimensions. We compute analytic estimations of the relaxation time calculating the rate of two body collisionality in a virialized system approximated as homogeneous. We show that for $\gamma < (d-1)/2$, the collision integral is dominated by the size of the system, while for $\gamma > (d-1/2)$, it is dominated by small impact parameters. In addition, the lifetime of QSS increases with the number of particles if $\gamma < d-1$ (*i.e.* the force is not integrable) and decreases if $\gamma > d-1$. Using numerical simulations we confirm our analytic results. A corollary of our work gives a "dynamical" classification of interactions: the dynamical properties of the system depend on whether the pair force is integrable or not.

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Keywords: long range interacting systems • Vlasov equation • collisional relaxation

1. Introduction

Long range interacting particle systems are usually defined (see e.g. Ref. [1]) such that the interparticle potential $v(r)$ is not integrable because of its large scale

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behavior, *i.e.*, in d dimensions

$$\lim_{R\to\infty}\int^R d^d r\, v(r) \to \infty. \quad (1)$$

It implies that asymptotically $v(r)$ behaves as

$$v(r\to\infty) \sim \frac{1}{r^\gamma}, \quad (2)$$

with $\gamma < d$. There are many such systems in nature, like gravitational ones, two-dimensional vortices, cold atoms, *etc*. Using an appropriate $N \to \infty$ prescription (see Section 2 for a discussion about the different prescriptions to perform this limit), it is possible to compute the thermal equilibrium properties of this systems [2]. They present unusual features compared to short range systems, like ensemble inequivalence, negative specific heat in the microcanonical ensemble, non-homogeneous spatial distributions, *etc*.

However, on the timescales of typical interest (e.g. the formation and evolution of galaxies in astrophysics), the system has usually not reached thermal equilibrium [3]. Actually, the scenario of their evolution is peculiar compared to short range systems: in a characteristic time $\tau_{\rm mf}$, there is the generic formation of long lived states which are not in thermal equilibrium, called *Quasi Stationary States* (hereafter QSS), and in a much longer timescale $\tau_{\rm coll}$, the relaxation towards thermal equilibrium. For example, for a gravitational system in $d = 3$ dimensions, if we choose our units in such a way that $\tau_{\rm mf} \sim 1$, then $\tau_{\rm coll} \sim N/\log(N)$. Indeed, in the $N \to \infty$ limit, the system becomes *mean field* and it remains trapped in the QSS (which becomes stable). Finite N effects are totally suppressed, which are the mechanism (as in short range systems) which lead the system to thermal equilibrium [3].

The appropriate equation to describe such systems is the Boltzmann equation

$$\frac{\partial f}{\partial t} + \mathbf{v}(\mathbf{r},t)\cdot\frac{\partial f}{\partial \mathbf{r}} + \mathbf{F}[f]\cdot\frac{\partial f}{\partial \mathbf{v}} = C_N[f], \quad (3)$$

where $f(\mathbf{r},\mathbf{v},t)$ is the *one-point distribution function*, *i.e.*, the probability to find a particle at the position $\mathbf{r}$ with velocity $\mathbf{v}$ at time t. The term $\mathbf{F}[f]$ is the *mean field force* which, for an interaction potential of the form (2), is

$$\mathbf{F}[f] = g\int \frac{f(\mathbf{r}',\mathbf{v},t)}{|\mathbf{r}-\mathbf{r}'|^{\gamma+2}}(\mathbf{r}-\mathbf{r}')d\mathbf{r}'d\mathbf{v}. \quad (4)$$

In order that a mean-field (or collisionless limit) exist — which leads to the scenario described above — the r.h.s. of Eq. (3) should satisfy

$$\lim_{N\to\infty} C_N[f] = 0, \quad (5)$$

where the prescription to perform the limit will be described in Section 2. Under this hypothesis, the l.h.s. of Eq. (3) describes the $N \to \infty$ dynamics of the systems (the mean-field part), while the r.h.s. finite N effects (which is commonly called "collision term"). In this limit, the Boltzmann equation is called Vlasov equation. It becomes apparent at this stage that the limits $t \to \infty$ and $N \to \infty$ do not commute: the limit $t \to \infty, N \to \infty$ leads the system to thermal equilibrium, while $N \to \infty, t \to \infty$ maintains the system in the QSS forever.

In this proceedings we explore the conditions under which the condition (5) is satisfied (and, more specifically, how it depends on the exponent γ of the power law of the interaction potential (2)). It would depend on the collisional dynamics described by the term $C_N[f]$, which is in general a very complicated function of the N-point particle density. It is assumed, (e.g. Ref. [3]), however, that its dominant contribution comes from *two body collisions*. We will use this assumption when estimating the collisional relaxation rate in Section 3.

At this point, we would remark that in long range interacting systems, QSS exist because $\tau_{\rm mf} \ll \tau_{\rm coll}$, which is a *dynamical* property of the system. However, the definition of long range interacting systems (1) is based on the integrability property of the interaction potential, which is not directly related to the dynamics. For instance, the integrability condition for the force is $\gamma > d-1$ in Eq. (2), which is not the same than the one for the potential. As the dynamics is related to the force, we can suspect that the existence of the QSS would be conditioned by the *non-integrability* of the force. We will provide evidence in this proceedings that it is indeed the case, which will lead to a *dynamical* classification of interactions.

In this proceedings we present an extension of Ref. [4]. It is structured as follows: in the next section we will describe the appropriate $N \to \infty$ limit for long range interacting systems. In the next section we will describe a generalization of the Chandrasekhar approach to estimate the collisional relaxation rate of particle systems interacting with the potential (2). Using the scaling dependence of this estimate, we will present a *dynamical* classification of interacting systems, and show numerical evidence of our results. We will then give some conclusions and directions which we would like to investigate in future.

2. The $N \to \infty$ limit in long range systems

In order to study the thermal equilibrium properties of a system of interacting particles, the important quantity to keep constant performing the infinite N limit (up to finite

N effects), is the (potential) energy of the particles, in order that the *total* energy is extensive. Let us compute the potential energy ϕ per particle of a particle located inside a sphere of radius R with typical density n_0 and interaction potential (2):

$$\phi \sim \int_\ell^R d^d r n_0 \frac{g}{r^\gamma} \sim g n_0 \left(R^{d-\gamma} - \ell^{d-\gamma}\right), \qquad (6)$$

where we have approximated a sum over all the particles by an integral, and ℓ is some small characteristic scale. For short range systems ($\gamma > d$), Eq. (6) is dominated by the second term of its r.h.s. In this case, the limit $N \to \infty$ is performed taking $V \to \infty$ and keeping $N/V =$ constant, where V is the volume of the system. This is an appropriate manner to perform the limit because the scale ℓ is kept invariant performing the limit, and hence the potential energy ϕ remains constant (up to finite N effects). For long range systems, the integral is dominated by the first term of the r.h.s. If the limit is performed in the same way[1], the potential energy ϕ diverges with N, as we can explicitly see using that $R \propto N^{1/d}$ in Eq. (6). Then:

$$\phi \sim g n_0 N^{1-\gamma/d}. \qquad (7)$$

The usual way to overcome this problem (see e.g Ref. [1]) consists in scaling the coupling constant with an appropriate power of the volume or, equivalently, of the number of particles:

$$g \to g n_0 N^{\gamma/d-1}. \qquad (8)$$

This is an extension of the Kac prescription [6], and sometimes also called the *dilute limit*. The energy becomes then *extensive*. However, if we are interested in the dynamics of the system, the *force* should be independent of N (up to finite size effects). The typical force f is:

$$f \sim \int_\ell^R d^d r n_0 \frac{g}{r^{\gamma+1}} \sim g n_0 \left(R^{d-\gamma-1} - \ell^{d-\gamma-1}\right). \qquad (9)$$

If we use the rescaling of the coupling constant (8), the force (9) becomes

$$f \sim n_0 N^{-1/d}, \qquad (10)$$

and then in the infinite N limit,

$$\lim_{N\to\infty} f = 0, \qquad (11)$$

which means that the dynamics disappears. If one decides to scale the coupling constant with N in such a way that the force (9) becomes independent of N, then the potential would be badly defined, and the system would relax to a badly defined configuration. The conclusion of this analysis is that it is not possible to find an appropriate scaling of the coupling constant which makes the potential and the force independent on N *at the same time*, when we scale the system with the procedure given above[2].

In this work, we will consider the so-called Vlasov limit, in which the $N \to \infty$ limit is taken keeping R constant. It corresponds exactly to the limit in which, for gravitational systems, the relation (5) holds, maintaining constant the mean-field dynamical time $\tau_{\rm mf}$. In this limit, it is possible to rescale g in such a way to maintain *both* the typical potential and force independent of N, for any value of γ. The physical meaning of this limiting procedure is the following: imagine a QSS with fixed typical size R (e.g. a galaxy) which we would like to describe more and more precisely, with larger and larger number of particles. Let us compute the typical energy and force over a particle i:

$$\phi_i = g \sum_{j\neq i}^{N} \frac{1}{|\mathbf{r}_i - \mathbf{r}_j|^\alpha} \propto gN, \qquad (12a)$$

$$\mathbf{f}_i = g \sum_{j\neq i}^{N} \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^{\alpha+2}} \propto gN. \qquad (12b)$$

Because the size of the system is fixed, both sums are proportional to the number of terms in the sum. In order to *both* the potential *and* the force be independent of N it suffices to scale the coupling constant using

$$g \to \frac{g}{N}. \qquad (13)$$

[1] *In the case of an infinite system, the problem is more subtle. One has to subtract the contribution to the force and to the potential of the average distribution (the so-called Jeans swindle). For a detailed analysis of the properties of the convergence of the force, see Ref. [5].*

[2] *Note that this is not the case for systems of spins on a lattice, such as the so-called $\alpha - HMF$ and related models [1], in which the potential and the force decay with the same power of the distance, by construction of the model.*

3. A generalization of the Chandrasekhar estimate of the relaxation rate

Chandrasekhar introduced a very successful way to estimate the relaxation rate in gravitational systems [7]. We will summarize below the main ingredients, including also an improvement which takes into account the boundaries of the system, similar to the one due to Hénon [8]. Consider a homogeneous system of particles, in which a test particles crosses the system in a typical time $\tau_{\rm mf}$. Let us begin our analysis studying the scattering of the test particle with another particle with impact factor b, for the well-known case $\gamma = 1$ (which corresponds to Rutherford scattering). The change in velocity of the test particle because of multiple scattering will give rise to a randomization of the velocities, which is assumed to be the dominant relaxation process towards thermal equilibrium. It has been shown (see e.g. Ref. [3]), that the main contribution to the change of the velocity is the *perpendicular* component of the *relative* velocity $\mathbf{V}_0$ between the two particles. Assuming that the particles have the same mass m, a simple calculation gives (e.g. Ref. [3]):

$$|\Delta \mathbf{v}_\perp| = \frac{2mbV_0^3/g}{1+\left(mbV_0^2/g\right)^2}, \qquad (14)$$

where $V_0 = |\mathbf{V}_0|$. There is a *natural cutoff* for small impact factors at

$$b_{\rm min} = \frac{g}{mV_0^2}. \qquad (15)$$

A very good approximation to Eq. (14) consists in using the so-called *weak collision approximation*

$$|\Delta \mathbf{v}_\perp| \simeq \frac{2g}{mbV_0^2} \qquad (16)$$

and neglecting the collisions with impact factor $b < b_{\rm min}$. It is not possible to calculate analytically a generalization of Eq. (14) for any $\gamma \neq 1$, but it is possible to obtain a generalization of the Chandrasekhar estimate using the *weak collision approximation* and a cutoff for the minimal impact factor. Such a generalization is computed assuming that the collision is sufficiently weak in order the deflection of the test particle to be negligible. The component of the force perpendicular to the trajectory can be calculated as

$$F_\perp = \frac{\gamma g b}{(b^2+x^2)^{\gamma/2+1}} = \frac{\gamma g}{b^{\gamma+1}}\left[1+\left(\frac{vt}{b}\right)^2\right]^{-(\gamma/2+1)}, \qquad (17)$$

where we have assumed in the last equality that the particle velocity is constant. By Newton's law, we have

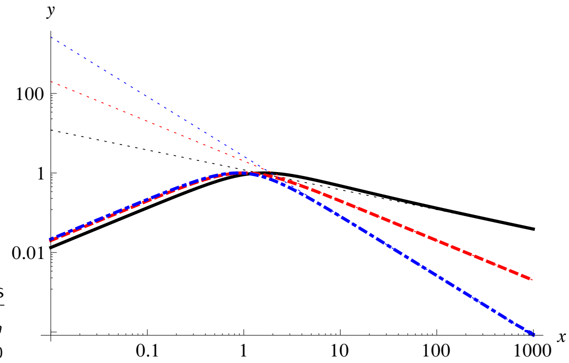

Figure 1. Numerical computation of $|\Delta \mathbf{v}_\perp|$ after a collision, for $\gamma = 1/2$ (plain curve), $\gamma = 1$ (dashed curve) and $\gamma = 3/2$ (dashed-dotted curve). The weak collision approximation for each case is represented with dotted lines, which coincides perfectly with the exact calculations for $b \gg b_{\rm min}$.

$$|\Delta \mathbf{v}_\perp| = \int_{-\infty}^{\infty} dt \frac{F_\perp}{m} = \int_{-\infty}^{\infty} dt \frac{\gamma g}{mb^{\gamma+1}}\left[1+\left(\frac{vt}{b}\right)^2\right]^{-(\gamma/2+1)} = \frac{\gamma g}{mb^\gamma v}\int_{-\infty}^{\infty} ds(1+s^2)^{-(\gamma/2+1)}. \qquad (18)$$

In Eq. (18) the integration limits are approximated to $s = vt/b = \pm\infty$ (instead of being $s \simeq \pm R/b$, where R is the size of the system), in order to be able to compute the integral analytically. This is a very good approximation if $b \ll R$ (which is typically the case), because the integral converges rapidly. Performing the integration we obtain:

$$|\Delta \mathbf{v}_\perp| \simeq \frac{A_\gamma g}{mvb^\gamma}, \qquad (19)$$

with

$$A_\gamma = \gamma\sqrt{\pi}\frac{\Gamma\left(\frac{1+\gamma}{2}\right)}{\Gamma\left(1+\frac{\gamma}{2}\right)}, \qquad (20)$$

if $\gamma \neq 1$. The cutoff is now at

$$b_{\rm min} = \left(\frac{mV_0^2}{g}\right)^{1/\gamma}. \qquad (21)$$

We checked numerically that Eqs. (19), (20) and (21) are a very good approximation, comparing an exact calculation of $|\Delta \mathbf{v}_\perp|$ with the weak collision approximation, as shown in Fig. 3.

The equation (19) gives an expression for the change in velocity due to *one* collision. We have now to integrate over all the possible collisions, *i.e.*, over all the impact

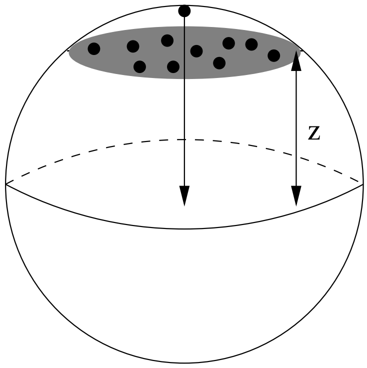

Figure 2. The QSS is approximated by a perfectly spherical distribution of particles with radius R.

parameters b. In order to be able to obtain an analytical expression we will do further approximations: *(i)* the system is a perfect d dimensional homogeneous sphere of radius R, *(ii)* we consider a test particle with zero angular momentum, and *(iii)* because we do not know the velocity distribution of the QSS, we approximate the relative velocity of the particles with the average velocity over all the particles of the system. The mean change in velocity is zero because the perturbations are randomly oriented, whereas the mean square velocity is non-zero (it is a random walk in velocity space). We will then calculate the average of the change of the square velocity. In order to perform the integration over all the impact parameters we divide the system in disks of thickness dz, sketched in Fig. (3). We perform the integration over all the impact parameters in each disk, and then the integration over the variable z. We can estimate that crossing a disk of thickness dz the particle suffers

$$\delta n = \frac{B_d N}{R^d} b^{d-2}\, db\, dz \tag{22}$$

encounters with impact parameter between b and $b+db$, where B_d is a factor which depends on the dimension d (e.g. $B_2 = 2/\pi$, $B_3 = 3/2$). Therefore, the average change in square velocity is, in $d=2$ or $d=3$:

$$\frac{\langle |\Delta \mathbf{v}^2| \rangle}{|v^2|} \simeq N A_\gamma^2 B_d \left(\frac{g}{mv^2R^\gamma}\right)^2 \int_{-1}^{1} dy \int_{b_{\min}/R}^{\sqrt{1-y}} \frac{dx}{x^{2\gamma-d+2}}. \tag{23}$$

The behavior of the integral depends on the value of γ and d, giving for $d > 1$

$$\frac{\langle |\Delta \mathbf{v}^2| \rangle}{|v^2|} \simeq N A_\gamma^2 B_d \left(\frac{g}{mv^2R^\gamma}\right)^2 \left[C_{\gamma,d} \left(\frac{R}{b_{\min}}\right)^{2\gamma-d+1} + D_{\gamma,d} \right], \tag{24}$$

where $C_{\gamma,d}$ and $D_{\gamma,d}$ are two constants of order unity (which depends on γ and d), which can be calculated analytically. We see therefore that if $\gamma < (d-1)/2$, the integral (23) is dominated by the size of the system, whereas if $\gamma > (d-1)/2$, it is dominated by the minimum impact parameter. We can simplify Eq. (24) making use of the *virial theorem*, which states that, for a stationary distribution

$$2K + \gamma U = 1, \tag{25}$$

where K is the total kinetic energy and U the total potential energy. Hence, by dimensional analysis:

$$\frac{g}{mv^2R^\gamma} \sim \frac{1}{N}\frac{gN^2}{(mNv^2)R^\gamma} \sim \frac{1}{N}\frac{U}{K} \sim \frac{1}{N}. \tag{26}$$

4. Estimation of the relaxation rate

We can simply estimate the relaxation rate Γ by computing the normalized change in square velocity due to collisions, in units of τ_{mf}:

$$\Gamma \tau_{\text{mf}} \simeq \frac{\langle |\Delta \mathbf{v}_\perp|^2 \rangle}{v^2}, \tag{27}$$

which gives, using Eq. (24) and (26) the following scaling:

$$\Gamma\, \tau_{\text{mf}} \sim \begin{cases} N^{-1} & \text{if } \gamma < (d-1)/2 \\ N^{-1}\left(\frac{R}{b_{\min}}\right)^{2\gamma-d+1} & \text{if } \gamma > (d-1)/2, \end{cases} \tag{28}$$

where we have dropped all the factors of order unity. This is the first main result we present in this proceedings: for $\gamma < (d-1)/2$, the relaxation is dominated by the maximum impact parameter (which is of the order of the size of the system), whereas for $\gamma > (d-1)/2$ it is dominated by the minimum impact parameter. We can control the relaxation rate in the latter case by putting a softening in the potential, obtaining in this case:

$$\Gamma \tau_{\text{mf}} \sim N^{-1} \left(\frac{R}{\epsilon}\right)^{2\gamma-d+1}. \tag{29}$$

In this case, and for any $\gamma < d$, we obtain

$$\lim_{N\to\infty} \Gamma\, \tau_{\text{mf}} = 0, \tag{30}$$

which signifies that, for any value γ, the QSS may last an infinite amount of time in this limit, provided that a sufficiently large softening $\epsilon > b_{\min}$ is used for $\gamma > d-1$. However, considering unsoftened potentials, we can see from Eq. (21) and the virial relation (25) that the minimum impact parameter depends indeed on the number of particles N. We obtain therefore:

$$\Gamma\,\tau_{\rm mf} \sim \begin{cases} N^{-1} & \text{if } \gamma < (d-1)/2 \\ N^{-(d-1-\gamma)/\gamma} & \text{if } \gamma > (d-1)/2. \end{cases} \quad (31)$$

This is the second main result presented in this proceedings: studying the zero of the exponent of the relaxation rate for $\gamma > (d-1)/2$, another threshold appears at $\gamma = d-1$. The relaxation rate becomes *divergent* for $\gamma > d-1$, which corresponds to an *integrable force*, and not an integrable potential.
We can therefore give the following *dynamical* classification of interactions, as a function of the existence of QSS in the infinite N limit:

1. **Dynamically long range systems**, in which the pair force is non integrable, *i.e.*, $\gamma < d-1$, for which $\lim_{N\to\infty} \Gamma\tau_{\rm mf} = 0$ for unsoftened potentials, *i.e.*, the QSS becomes stable in this limit. For finite N systems, the relaxation rate *decreases* increasing N. Furthermore, we have that
   - If $\gamma < (d-1)/2$ (*i.e.* more long range than gravity in $d=3$), the relaxation is dominated by the maximum impact parameter (*i.e.* the size of the system), and, for unsoftened potential, $\Gamma\tau_{\rm mf} \sim N^{-1}$.
   - If $(d-1)/2 < \gamma < d-1$, (*i.e.* less long range than gravity in $d=3$), the relaxation is dominated by the minimum impact parameter $b_{\rm min}$, and, for unsoftened potential, $\Gamma\tau_{\rm mf} \sim N^{-(d-1-\gamma)/\gamma}$.
2. **Dynamically short range systems**, in which the pair force is integrable, *i.e.*, $\gamma > d-1$, for which $\lim_{N\to\infty} \Gamma\tau_{\rm mf} = \infty$ for unsoftened potentials, *i.e.*, the QSS is immediately destroyed, the dynamics being dominated by collisions. For finite N systems, the relaxation rate *increases* increasing N.

5. Comparison with numerical simulations

We carefully compare the theoretical scalings with molecular dynamics simulations in $d=3$. To do so, we use a modification of the publicly available code GADGET2 [9], performing the appropriate modifications to integrate any power-law force. The potential was softened at the scale ϵ using a repulsive soft core, which does not modify the potential for $r > \epsilon$ and goes smoothly to $v(0) = 0$ for $r < \epsilon$.

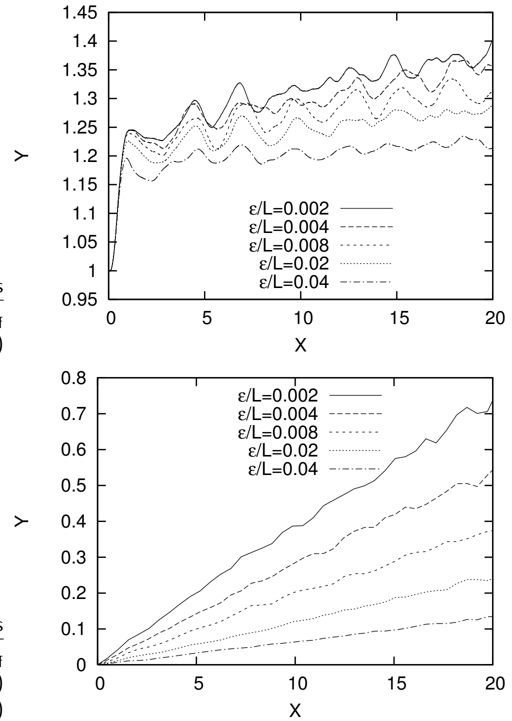

Figure 3. Upper: evolution of the total kinetic energy K for simulations with $N = 8000$ particles and different softening length ϵ in the potential $v(r)$ with $\gamma = 5/4$. Lower: evolution of the estimator $R(t)$ for the simulations.

We use soft reflecting boundary conditions in cubic box of size L, *i.e.*, when the i^{th} spatial coordinate of a particle lies outside the simulation box, we invert the i^{th} component of the velocity. The initial condition is a uniform spherical distribution of particles with random velocities normalized in order to satisfy the virial relation (25). This choice of normalization implies that the initial condition is close to a QSS, as it can be seen in the simulations. We show an example of the evolution of the total kinetic energy $K(t)$ for a set of simulations with $N = 8000$ particles, $\gamma = 5/4$ and different values of the softening parameter ϵ, in the left panel of Fig. 3. For the first few $\tau_{\rm dyn}$, the evolution of $K(t)$ is independent of ϵ, which corresponds to the mean-field evolution (violent relaxation). Then, $K(t)$ exhibits a "plateau", whose slope depends on the value of ϵ. This is a manifestation of the collisional relaxation, which depends on ϵ, as predicted in Eq. (28).

A particle moving in a mean-field potential (e.g. the one correspondent to the mean-field force (4)) conserves its total energy. A common way to estimate the collisional relaxation consists then in measuring the change in energy of the individuals particles with time. We use the

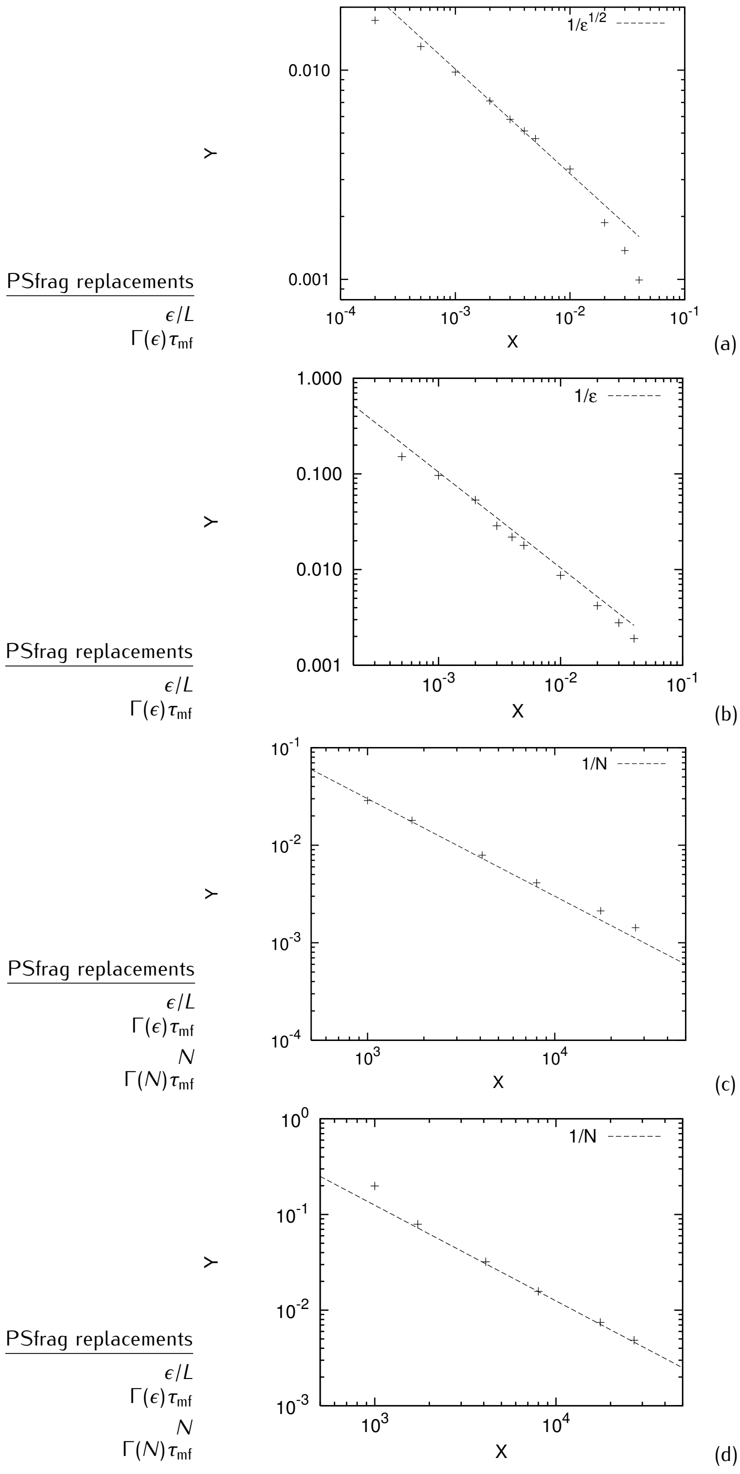

Figure 4. (a) Relaxation rate estimated using the estimator (32) for $N = 8000$, $\gamma = 5/4$ and different values of the softening ϵ. (b) The same plot but for $\gamma = 3/2$. (c) Relaxation rate for $\epsilon/L = 0.01$, $\gamma = 5/4$ and different number of particles N. (d) The same plot but for $\gamma = 3/2$.

following estimator $R(t)$ (see e.g. Ref. [10]),

$$R(t) = \frac{\langle (e(t) - e(t_0))^2 \rangle}{4K(t_0)^2}, \tag{32}$$

where $e(t)$ is the total energy of a single particle at time t, $\langle \dots \rangle$ and average over all the particles, $K(t)$ the total kinetic energy and t_0 a reference time for which the system had time to form the QSS. In the right panel of Fig. 3 we show the evolution of $R(t)$ for the same set of simulations as the one in the left panel. We estimate the relaxation rate Γ computing the slope of the linear fit of $R(t)$, for the range in which it is linear. In Fig. 4 we plot the scaling measured in a set of simulations. In the upper panel the relaxation rate Γ is plotted for simulations with different softening ϵ and constant number of particles N. There is a very good agreement with the theoretical scaling (28) for a wide range of ϵ. For the smallest value of ϵ, the relaxation rate is smaller than the one predicted by the scaling: it corresponds to values of $\epsilon < b_{\min}$. For the largest values of ϵ, the mean field dynamics is altered by the softening, giving a different relaxation rate than the one predicted by the theory. In the lower panel of Fig. 4 we see the scalings obtained from a set of simulations taking $\epsilon/L = 0.01$ and varying the number of particles, for $\gamma = 5/4$ and $\gamma = 3/2$. We see again a very good agreement with the theoretical scaling (28).

6. Conclusion

In this proceedings, we have generalized the Chandrasekhar estimate for the collisional relaxation rate for a system of particles with power-law pair interaction $v(r \to \infty) = 1/r^\gamma$ in d dimensions. The main result we have found is that the relaxation rate expressed in units of the mean field dynamical time $\tau_{\rm mf}$ converges only if the pair force is integrable, *i.e.*, $\gamma < d - 1$. This leads to the following *dynamical* classification of interactions:

1. The interactions is *dynamically* long range if $\tau_{\rm mf} \ll \tau_{\rm coll}$ for a sufficiently large number of particles, and in particular $\lim_{N\to\infty} \Gamma\tau_{\rm mf} = 0$, which occurs for $\gamma < d - 1$. In addition, we have identified that for $\gamma < (d-1)/2$, the relaxation is dominated by collisions with the maximum allowed impact parameter by the system (in our case the size of the system R), and if $\gamma > (d-1)/2$, the relaxation is dominated by collisions with minimum impact parameter $b_{\min}$, in the absence of softening.

2. The interaction is *dynamically* short range if $\tau_{\rm mf} \gg \tau_{\rm coll}$ for a sufficiently large number of particles, and

in particular $\lim_{N\to\infty} \Gamma\tau_{\rm mf} = \infty$, which occurs for $\gamma < d - 1$.

As anticipated in the introduction, this classification differs from the one according to the thermal equilibrium of the system, in which the important quantity is the integrability of the *potential*. There is therefore a range of γ, $d - 1 < \gamma < d$, in which the interaction is *dynamically* short range, but long range according to its thermal equilibrium properties. In this case, if the system has a large enough number of particles, the QSS would not form (as in short range systems), but however the thermal equilibrium state will presents the features of a long range system.
In the near future, we will present elsewhere a more detailed report of numerical simulations, including simulations with different cores (such as attractive ones) and the study of systems which are more long range than gravity in $d = 3$, such that the systems with $\gamma < 1$ in $d = 3$ or gravity in $d = 2$.

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References

[1] A. Campa, T. Dauxois, S. Ruffo, Phys. Rep. 480, 57 (2009)
[2] I. Ispolatov, E. G. D. Cohen, Phys. Rev. E 64, 056103 (2001)
[3] J. Binney, S. Tremaine, Galactic Dynamics (Princeton University Press, 1994)
[4] A. Gabrielli, M. Joyce, B. Marcos, Phys. Rev. Lett. 105, 210602 (2010)
[5] A. Gabrielli, M. Joyce, B. Marcos, F. Sicard, J. Stat. Phys. 141, 970 (2010)
[6] M. Kac, G. E. Uhlenbeck, P. C. Hemmer, J. Math. Phys. 4, 216 (1963)
[7] S. Chandrasekhar, Chicago, The University of Chicago press (1942)
[8] M. Hénon, Annales d'Astrophysique 21, 186 (1958)
[9] V. Springel, Mon. Not. R. Astron. Soc 364, 1105 (2005)
[10] J. Diemand, B. Moore, J. Stadel, S. Kazantzidis, Mon. Not. R. Astron. Soc. 348, 977 (2004)