Chapter 5

Lagrangian Hydrodynamics, Entropy and Dissipation

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Additional information is available at the end of the chapter

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1. Introduction

In this Chapter we deal with a formalism that is both fundamental and an apparently "niche product": I myself confess to have taken the chance of Editors' kind invitation to go deeper into this subject. The subject I am speaking about is the *Lagrangian Formulation* (LF) of Fluid Dynamics, in particular its theoretical, brutally analytical application to the case of fluids with *dissipation*.

The reasons due to which treating Fluid Dynamics via LF is not "so popular" are that it involves equations of motion which are very complicated (apparently, "uselessly complicated"), and that squeezing practical results out of it seems to be desperately hard. Moreover, performing measurements from the "Lagrangian point of view" is not common, and requires to leave a probe be transported by the flow: it is often easier to realize a station instead, so to put everything in the other way we know to describe fluids, the "Eulerian point of view".

The Eulerian Formulation (EF), also indicated as local formulation because it treats the bulk properties of a fluid as properties of the space itself, appears to be much more popular than its "first brother" the LF. The EF is "more popular" because its equations of motion appear much simpler, and because their implementation on a calculator that moves its imagination on a 1D, 2D or 3D grid sounds much easier (still, the celebrated Navier-Stokes Equation, its key statement, is one of the most controversial in the community as far as "the knowledge of all its possible solutions" is concerned).

It is really complicated to find textbooks in which *non-ideal fluids* are described in the LF, and this is because if for dissipation free fluids the LF may sound cumbersome, when it comes to dissipation complications grow even harder.

The two formulations quoted have specific applications, and the point is that traditionally the applications at best represented in EF have been prevailing. As we are going to point out rigorously few lines below, the LF is following the material flow, while EF is best describing what is seen by an observer as she sits at a point $\vec{x}$ in space, and makes measurements letting the fluid matter passing by.

Different kinds of motion can be optimally studied in each different one of the two formalisms, depending on the actual nature of the motion at hand. In particular, material regions of fluid showing a highly correlated motion, namely *coherent structures* with all their particles "going together" (see Figure 1), are best described by following them as they move throughout the space: for this kind of modes the LF may have great value.

Figure 1. Clouds in the sky indicate a local steep gradient of refraction index of the air, due to the presence of water. As a cloud appears like a "body", i.e. a complex of matter moving together coherently, its motion, or the motion of the region of air containing it, may be best represented in the Lagrangian Formalism. Unfortunately, the non-linear essence of those coherent structures (as also vortices, or current sheets in plasmas) has always been a serious difficulty to study them analytically.

Instead, those perturbations that are travelling across the fluid without any real matter transfer, i.e. mechanical *waves*, are well studied as a classical field theory, in the EF (see Figure 2).

As the scientific literature witnesses, while an entire zoology of waves, both linear and non-linear, has been developed for fluids and plasmas, quite much less has been done for coherent structures (Chang 1999), and this is accompanied by the much larger diffusion EF studies than LF ones.

In the courses of dynamics of continua taught at the university, fluid systems are first described in the LF, or at least from the Lagrangian viewpoint (Rai Choudhuri, 1998); then, as soon as

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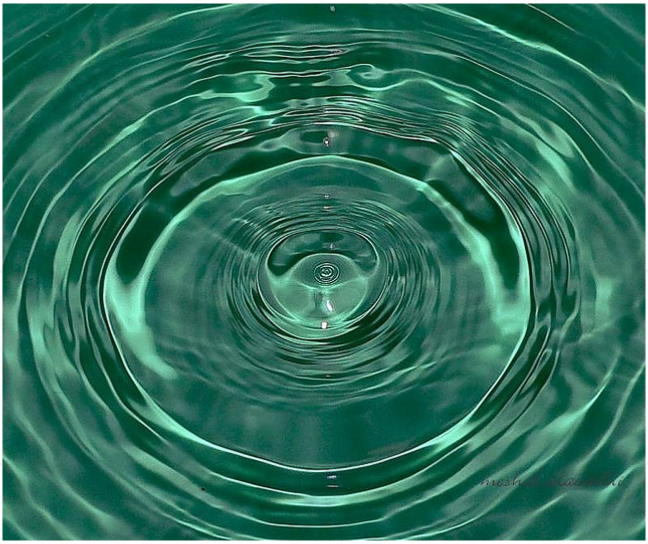

Figure 2. Some mechanical waves propagating along the surface of a fluid. The description of such modes of the continuum is traditionally (and better) done in the Eulerian Formalism, where the physical quantities locally describing the motion of a fluid are conceived as local space properties, i.e. classical fields. In waves, indeed, matter does not propagate itself, and prevalently linear terms appear in the equations of motion. Picture by "hamad M", on Flickr, at the webpage https://www.flickr.com/photos/meshal/.

"serious equations" appear, the use of the EF prevails, because of the aforementioned difficulties. However, the fact that the language of the LF is chosen to initiate students to dynamics of continua is not an oddity, rather it reflects how this framework portrays continuous systems more understandably for people having been dealing with discrete systems up to that moment.

Continua described in the LF appear as very straightforward generalizations of systems formed by material points, and their behaviour sounds much more understandable at first glance keeping in mind that the discrete system dynamics looks like. This property of the LF has not a "didactic" value only, but rather can render highly transparent physical relationships between various scales of the theory, the interaction among which is the essence of dissipation. This is, in extremely few words, why I have chosen to treat this subject in the present Chapter.

In this Chapter a special care is devoted to examining how the results about dissipation, usually presented in the Eulerian framework, appear in the LF. Processes involved in dissipation are the ones keeping trace of the granular nature of matter, and a very natural way of seeing this is to describe the fluid in the Lagrangian Formulation. Actually, the postulates of the LF must be criticized right in vision of the granularity of matter, as reported in § 6. For the moment being, let's just consider the LF as non-in-conflict with matter granularity.

In § 2 the fundamental tools of the LF are proposed, together with the physical sense of the idea of fluid parcel, and the relationship of this with the particles of matters forming the continuum. In this § the fluid geometry, kinematics and mass conservation are sketched.

The application of the LF tools to fluids without dissipation is then given in § 3: in the absence of friction the fluid is treated as a Hamiltonian system, and here the dynamics appears in LF as inherited from that of point particles. The parcel variables of the LF are regarded as a centre-of-mass versus relative variables decomposition of the discrete analytical mechanics. As an aside of this interpretation, § 4 is used to clarify the differential algebraic properties of the entropy of the fluid encoding the degrees of freedom of the microscopic particles forming the parcel.

The granular nature of matter will enter in dynamics only in § 5, where dissipative fluids are treated in the LF: the dissipative terms completing the ideal equations of motion found in the Hamiltonian limit are written, and the Poisson algebra describing the non-dissipative fluid is completed with the introduction of a metric bracket, giving rise to the metriplectic algebra for dissipative fluid in LF.

Conclusions given in § 6 do concern the concept of parcel, and of postulates given in § 2, with respect to turbulence and matter granularity; then, applications and possible developments of the LF are traced.

2. Parcels versus particles

The LF describes a fluid by representing its motion as the evolution, throughout space, of the physical domain occupied by its matter (Bennett, 2006).

At a certain time t, let the locus of points in R^3 occupied by the fluid matter be indicated as $D(t)$; if the description of the motion starts at the time t_0, so that this is the initial time, let the initial material domain occupied by the fluid be indicated as D_0. The key question is: how to assign the fluid configuration at the time t ? Not only one has to assign the set $D(t) \subseteq R^3$ of the fluid domain: also, the distribution of matter within it must be given, and the instantaneous velocity of each material point of it. This is a big job, that is rendered possible by choosing to work at the scale at which the system appears as a continuum, and the "individual" motions of the particles composing it may be disregarded as important.

What is done in the LF is to imagine the fluid domain subdivided into "an infinite" number of "infinitesimal" portions, referred to as *parcels* that are still formed by a thermodynamic number of particles. The size of the parcel is chosen so that the *bulk macroscopic quantities* with which the fluid is described (velocity, mass density, tension…) are constant within it, but a sufficient number of elementary particles of the fluid is included, so that no extreme fluctuations (i.e. no sub-fluid effects as those described in Materassi et al. (2012)) can be appreciated. Parcels sized in this way are indicated as *macroscopic infinitesimal*, indeed, all the extensive quantities pertaining to the parcel (mass, energy, momentum) will be indicated with differentials.

At the initial time, this subdivision is snapshot stating that we have one parcel, of volume d^3a, at each point $\vec{a} \in D_0$. Those parcels are imagined to move, as the fluid evolves, with some constraints that are the foundations of the continuous approximation itself, and restrict very precisely the realm of application of the whole scheme. In particular, the following requirements are met:

1. *parcel identity conservation*: parcels move without mixing their matter with each other. They can be deformed, stretched, their volume does change during the evolution of the fluid, but they do not exchange matter, they do not fragment, they do not crumble into smaller entities;

2. the connectedness and smoothness of D_0 does not change as it evolves into $D(t)$, at no time: not only the set of continuous matter remains "all one" without being cut, also no cusps, spikes, "hairy" non-differentiable regions appear in it. In particular, in topological terms, one should state

$$\dim D(t) = \dim D_0, \tag{1}$$

being dim the Hausdorff dimension of fluid matter.

Of course, it is necessary to remind that this LF can be improved, on the side of treatment of irregular motions, by adding noise terms to its otherwise deterministic and C^{∞} equations (Materassi & Consolini, 2008). Nevertheless, here we are only interested in giving the general directions of the deterministic LF dynamics.

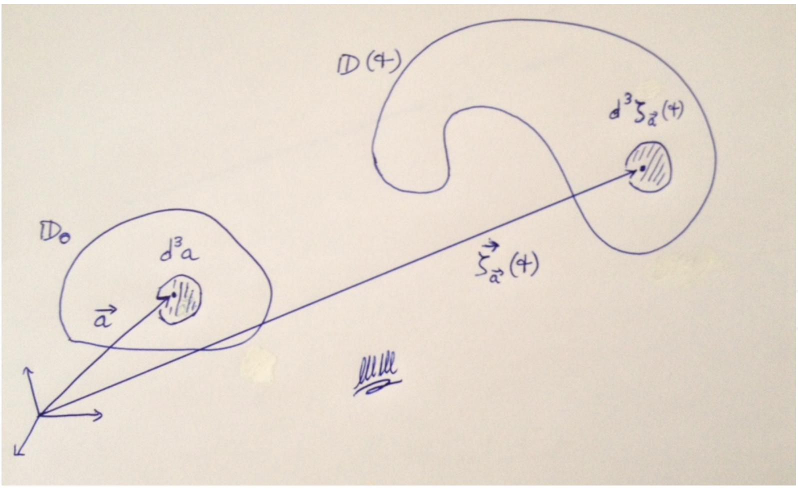

Figure 3. The transformation from the initial configuration of the fluid D_0 into the one assumed at the generic time $D(t)$. The parcel, initially located at the position $\vec{a}$ assumes position $\vec{\zeta}_{\vec{a}}(t)$, while its initial infinitesimal volume d^3a is transformed into the evolved $d^3\zeta_{\vec{a}}(t)$ (note the use of $\vec{\zeta}(\vec{a},\ t)$ instead of $\vec{\zeta}_{\vec{a}}(t)$ in the text).

The two requirements stated before are translated into a conceptually very simple statement, in practice giving rise to complicated equations: *the evolution map $D_0 \mapsto D(t)$ is represented by a diffeomorphism at every time* t. In this way, each neighbourhood of every point $\vec{a} \in D_0$ is mapped in an invertible way into a unique neighbourhood of the point $\vec{\zeta}(\vec{a}, t) \in D(t)$, and the identity of parcels is conserved. Not only, being the diffeomorphism a C^∞ map, the motion of the fluid is also smooth, as required.

The geometry and kinematic of the fluid are all contained in the diffeomorphism $\vec{a} \mapsto \vec{\zeta}(\vec{a}, t)$. It is natural to use $\vec{\zeta}(\vec{a}, t)$ itself as the dynamical variable of the theory, so that the LF is essentially a the field theory of $\vec{\zeta}(\vec{a}, t)$ on the 3D domain D_0 : the vector $\vec{\zeta}(\vec{a}, t)$ is the position of the $\vec{a}$-th parcel in the physical space R^3, while $\vec{a}$ is the label of the parcel, but it represents its initial position too:

$$\vec{\zeta}\left(\vec{a}, t_0\right) = \vec{a} \tag{2}$$

(see Figure 3). Even if (2) underlines how the vectors $\vec{\zeta}$ and the vectors $\vec{a}$ do “live” in the same physical space, it is useful to indicate their components with different labels: we will use Greek indices for $\vec{\zeta}$, writing its components as ζ^α, while Latin indices will label the components of $\vec{a}$, as a^i. Summation convention over repeated indices in contravariant position will be understood, so that scalar products will be written as:

$$\vec{\zeta} \cdot \vec{\xi} = \zeta^\alpha \xi_\alpha, \quad \vec{a} \cdot \vec{b} = a^i b_i. \tag{3}$$

The velocity of the $\vec{a}$-th parcel will simply be $\partial_t \vec{\zeta}(\vec{a}, t)$, even if this notation might be misleading: this is *a total derivative with respect to time*, because the “index” $\vec{a}$ in $\vec{\zeta}(\vec{a}, t)$ does not move at all, but is just “the name” of the parcel.

Another geometrical-kinematical object to be defined in LF is the *Jacobian matrix of the diffeomorphism $\vec{a} \mapsto \vec{\zeta}(\vec{a}, t)$*, indicated as $\mathcal{J}(\vec{a}, t) = \partial \vec{\zeta}(\vec{a}, t) / \partial \vec{a}$. Its components, and the components of its inverse matrix, read

$$\mathcal{J}_i^\alpha = \frac{\partial \zeta^\alpha}{\partial a^i}, \quad \left(\mathcal{J}^{-1}\right)_\alpha^i = \frac{\partial a^i}{\partial \zeta^\alpha}. \tag{4}$$

The determinant of the Jacobian is indicated as:

$$J = \det \mathcal{J}. \tag{5}$$

Clearly, $J(\vec{a}, t_0) = 1 \quad \forall \quad \vec{a} \in D_0$. Last, but not least, some *time-semigroup property* of the diffeomorphism must hold: $\vec{\zeta}(\vec{a}, t_1 + t_2) = \vec{\zeta}(\vec{\zeta}(\vec{a}, t_1), t_2)$.

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The use of $J(\vec{a},\ t)$ renders it straightforward, in practice built-in, the *mass conservation* in LF: due to the parcel identity conservation *the parcel mass will be conserved itself*, so that, if $\rho_0(\vec{a})$ is the initial mass density of the $\vec{a}$-th parcel, one may write the initial mass of the parcel as $dm(\vec{a},\ t_0)=\rho_0(\vec{a})\,d^3a$. On the other hand, at time t the same $\vec{a}$-th infinitesimal mass will be written as $dm(\vec{a},\ t)=\rho(\vec{a},\ t)\,d^3\zeta(\vec{a},\ t)$, being $\rho(\vec{a},\ t)$ the $\vec{a}$-th mass density at a generic time, and $d^3\zeta(\vec{a},\ t)$ the infinitesimal volume of the parcel at the same time. The parcel mass conservation hence reads

$$\rho_0\left(\vec{a}\right)d^3a=\rho\left(\vec{a},t\right)d^3\zeta\left(\vec{a},t\right)\ , \tag{6}$$

and since the relationship between d^3a and $d^3\zeta(\vec{a},\ t)$ is given by $d^3\zeta(\vec{a},\ t)=J(\vec{a},\ t)\,d^3a$, so that one has:

$$\rho\left(\vec{a},t\right)=\frac{\rho_0\left(\vec{a}\right)}{J\left(\vec{a},t\right)}. \tag{7}$$

The relationships (6) and (7) are all that is needed, in the LF, to let the mass conservation be adequately represented (those relationships are turned into the continuity equation of the EF).

Once the position of the parcel in $\boldsymbol{R}^3$ and its mass density have been described, the reality of the parcel as composed by a thermodynamic number of *microscopic particles* must be considered. Indeed, even if the parcel identity conservation and the smoothness of the fluid evolution render $\vec{\zeta}(\vec{a},\ t)$ sufficient to describe the configuration of the fluid at any time, while $\rho(\vec{a},\ t)$, related to the initial mass density $\rho_0(\vec{a})$ as equation (7) prescribes, completes also the mass geometry and mass conservation description, there are *dynamical effects*, that will be examined in §§ 3, 4 and 5, through which the granular nature of matter do come into the play.

Let's hence suppose that the $\vec{a}$-th parcel is formed by $N(\vec{a})$ *particles*, each of them located in the space by a position vector $\vec{r}_{I=1,\dots,N(\vec{a})}$ that belongs to the neighbourhood centred in $\vec{\zeta}(\vec{a},\ t)$ and of measure $J(\vec{a},\ t)\,d^3a$. When these particles are considered, it is no surprise to state that the vector $\vec{\zeta}(\vec{a},\ t)$, understood as "the parcel position", may be defined as the position of the centre-of-mass of the $N(\vec{a})$ particles in the parcel (Arnold, 1989), see Figure 4:

$$\vec{\zeta}\left(\vec{a},t\right)=\frac{1}{N\left(\vec{a}\right)}\sum_{I=1}^{N(\vec{a})}\vec{r}_I. \tag{8}$$

In the foregoing formula one supposes that all the particles have the same mass, so that a mere arithmetical average of their positions can be taken to define the centre-of-mass. In order to complete the set of centre-of-mass variables, let's consider there also will be a parcel momentum density

$$\vec{\pi}\left(\vec{a},t\right)=\rho_0\partial_t\vec{\zeta}\left(\vec{a},t\right)=\frac{1}{d^3a}\sum_{I=1}^{N(\vec{a})}\vec{p}_I, \tag{9}$$

that is defined more rigorously in § 3, equation (18) (in (9) $\vec{p}_I$is the linear momentum of the I-th particle in the $\vec{a}$-th parcel).

By considering the definitions (8) and (9), out of the $6N(\vec{a})$ dynamical variables $(\vec{r}_I, \vec{p}_I)$ that analytical mechanics would attribute to the $N(\vec{a})$ particles, $6(N(\vec{a})-1)$remain, those that would be referred to as *relative-to-the-centre-of-mass*, or simply *relative variables*. Due to how huge $N(\vec{a})$ is, and considering $N(\vec{a})-1\cong N(\vec{a})$, instead of defining relative independent variables in subtle complicated ways (Lusanna & Materassi, 2000), one decides to juxtapose to the variables $(\vec{\zeta}, \vec{\pi})$ the *Equilibrium Statistical Mechanics* of $N(\vec{a})$ particles of the species forming the fluid. In a sense, this is as treating the relative variables as *microscopic degrees of freedom treated statistically* (Materassi et al., 2012). As it happens for ideal gases, those namely $6(N(\vec{a})-1)$ dynamical variables may be effectively described via *two state functions* that should be the poor man's version of the Statistical Mechanics of the ponderous $6N(\vec{a})$-dimensional phase space. In the undergraduate student's thermodynamics, a gas can be described, for instance, via its *density* ρ and its *entropy* S, and so will be for the pointlike masses forming our parcel: the parcel density has already been assigned through (7), what remains is to equip the field configuration of the fluid in the LF by some entropic field, i.e. the *mass-specific entropy density* $s(\vec{a}, t)$, defined such that the parcel's infinitesimal entropy reads:

$$dS\left(\vec{a},t\right)=\rho\left(\vec{a},t\right)s\left(\vec{a},t\right)d^3\zeta\left(\vec{a},t\right)=\rho_0\left(\vec{a}\right)s\left(\vec{a},t\right)d^3a. \tag{10}$$

The set of microscopic particles forming the $\vec{a}$-th parcel is hence a macroscopically infinitesimal cloud of position $\vec{\zeta}(\vec{a}, t)$ and total momentum $\vec{\pi}(\vec{a}, t)d^3a$.

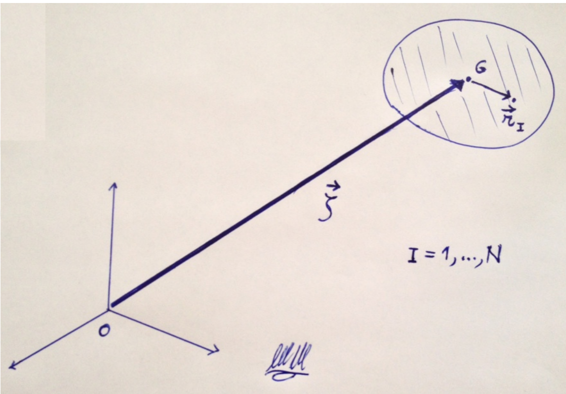

Figure 4. The parcel of position $\vec{\zeta}$ and the N particles forming it, with position relative-to-the-parcel-centre-of-mass indicated with $\vec{r}_I$. The centre-of-mass of the parcel is the point G, of position $\vec{\zeta}$ in the space.

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The "statistical buzz" of those $N(\vec{a})$ pointlike masses is encoded in the thermodynamics of $J(\vec{a}, t)$ and $s(\vec{a}, t)$. Since $J(\vec{a}, t)$ is already encoded in $\vec{\zeta}(\vec{a}, t)$, one concludes that the functionally independent fields $\vec{\zeta}(\vec{a}, t)$, $\vec{\pi}(\vec{a}, t)$and $s(\vec{a}, t)$ describe completely the field configuration, and they will be referred to as *parcel variables*.

3. Ideal fluids in Lagrangian formalism

All in all, we have established that the fluid geometry, kinematics and mass geometry may be represented in parcel variables through the parcel variables $\vec{\zeta}(\vec{a}, t)$, $\vec{\pi}(\vec{a}, t)$and $s(\vec{a}, t)$. Now it is time to describe the interactions to which those quantities undergo, determining the dynamics of the fluid as represented in parcel variables.

The dynamics of the fields $\vec{\zeta}(\vec{a}, t)$ and $\vec{\pi}(\vec{a}, t)$ describe the "collective" evolution of the matter forming the $\vec{a}$-th parcel: this will be determined by the forces external to the parcel (Feynman, 1963) acting on it. One may consider forces due to potentials that are not due to the fluid, for instance gravitation; there will be also the forces exerted on the parcel by the nearby parcels. The latter are subdivided into the "conservative" force due to the elasticity of the continuum, that will be encoded in *pressure*, and the "dissipative" force that parcels exert on the nearby ones by rubbing each other, i.e. *friction*. Both are expected to depend on gradients with respect to the label, because these are all due to the nearby parcels.

While external potential forces are exquisitely pertaining the collective position $\vec{\zeta}(\vec{a}, t)$, forces exchanged with the nearby parcels are encoded in the thermodynamics of the particles forming the parcel: they may be expressed through the internal energy density $\rho_0 U\left(\frac{\rho_0}{J}, s\right)$ of the fluid.

In this § the dissipation free limit will be studied, excluding for the moment the friction between nearby parcels, treated in § 5.

As elegantly suggested by Goldstein (2002), when dissipation is ruled out a mechanical system is expected to undergo an *Action Principle*, and this will be the case for the field theory of parcel variables. In order to write down the mechanical action of the fluid in LF, a resume is done of the forms of energy attributed to the parcel.

The parcel has a collective motion throughout the space, conferring it an amount of kinetic energy

$$dE_{\text{kin}} = \frac{\rho_0 \dot{\zeta}^2}{2} d^3 a, \tag{11}$$

where $\dot{\zeta}^2 = |\partial_t \vec{\zeta}|^2$ is intended. Then, assuming there exist potentials external to the fluid, these are written as:

$$dV = \rho_0 \phi\left(\vec{\zeta}\right) d^3 a. \tag{12}$$

The parcel possesses also the internal thermodynamic energy of the particles forming it, that we have already mentioned before and that here we indicate as:

$$dE_{\text{therm}} = \rho_0 U\left(\frac{\rho_0}{J}, s\right) d^3 a. \tag{13}$$

When equations (11), (12) and (13) are put all together, one may attribute *an infinitesimal Lagrangian* to the $\vec{a}$-th parcel, that reads:

$$dL = \rho_0 \left[\frac{\dot{\zeta}^2}{2} - \phi\left(\vec{\zeta}\right) - U\left(\frac{\rho_0}{J}, s\right)\right] d^3 a, \tag{14}$$

so that the Lagrangian of the whole system is simply the dL in (14) integrated on the initial volume D_0:

$$L\left[\vec{\zeta}, s\right] = \int_{D_0} \rho_0 \left[\frac{\dot{\zeta}^2}{2} - \phi\left(\vec{\zeta}\right) - U\left(\frac{\rho_0}{J}, s\right)\right] d^3 a. \tag{15}$$

The mechanical action for the fluid in LF is the aforementioned functional integrated in time along the interval $[t_0, t_f]$ along which we are interested in studying the system:

$$A\left[\vec{\zeta}, s\right] = \int_{t_0}^{t_f} dt' \int_{D_0} \rho_0 \left[\frac{\dot{\zeta}^2}{2} - \phi\left(\vec{\zeta}\right) - U\left(\frac{\rho_0}{J}, s\right)\right] d^3 a. \tag{16}$$

In Padhye (1998), those expressions are used to find the Euler-Lagrange equations for the natural motions of the system, that read:

$$\ddot{\zeta}_\alpha = -\frac{\partial \phi}{\partial \zeta^\alpha} + \frac{A^i_\alpha}{\rho_0} \frac{\partial}{\partial a^i}\left(\rho_0 \frac{\partial U}{\partial J}\right), \quad A^i_\alpha = \frac{1}{2} \varepsilon_{\alpha\kappa\lambda} \varepsilon^{imn} \frac{\partial \zeta^\kappa}{\partial a^m} \frac{\partial \zeta^\lambda}{\partial a^n},$$
$$\dot{s} = 0. \tag{17}$$

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In (17) the term $A^i_\alpha \frac{\partial}{\partial a^i}\left(\rho_0 \frac{\partial U}{\partial J}\right)$ is exactly the infinitesimal force insisting on the parcel due to the *adiabatic pressure* exerted on it by its nearby fellows: in particular, one should recognize the pressure undergone by the parcel as $p = -\rho_0 \frac{\partial U}{\partial J}$.

The second differential equation in (17) deserves some more explanation.

Varying the action A with respect to s, the effect of the absence of any derivative of it in dL appears in the final statement according to which it must be constant. This corresponds to the result that, in the dissipation free limit in which the Action Principle is applicable, the mass-specific entropy density of the $\vec{a}$-th parcel does not change. Actually, in this case the whole entropy of the parcel remains constant, as it can be seen from $dS(\vec{a}, t) = \rho_0(\vec{a}) s(\vec{a}, t) d^3 a$. The quantity $s(\vec{a}, t)$ turns out to be *a passively advected scalar*, driven throughout the space by the parcel that carries it.

The constancy of $s(\vec{a}, t)$ means adiabaticity (isoentropicity) of the parcel motion for the ideal fluid because the density at hand is supposed to play the role of mass-specific entropy density in the definition of the amount of internal energy dE_{therm} in (13).

Once the Lagrangian dynamics is defined through (14) and (15) it is possible to turn it into a Hamiltonian canonical framework through the Legendre transformation of $L[\vec{\zeta}, s]$ into the Hamiltonian $H[\vec{\zeta}, \vec{\pi}, s]$ (Goldstein, 2002).

First of all, a rigorous definition of the kinetic momentum $\vec{\pi}(\vec{a}, t)$ must be given as:

$$\pi_\beta(\vec{a}) = \frac{\delta L}{\delta \dot{\zeta}^\beta(\vec{a})}. \tag{18}$$

By calculating the functional derivative indicated in (18), the result (9) is obtained. Then, the calculation of the Hamiltonian functional of the ideal fluid in LF is straightforward:

$$H[\vec{\zeta}, \vec{\pi}, s] = \int_{D_0} d^3 a \left[\frac{\pi^2}{2\rho_0} + \rho_0 \phi(\vec{\zeta}) + \rho_0 U\left(\frac{\rho_0}{\mathbf{J}}, s\right)\right]. \tag{19}$$

The Hamilton Equations of motion of the system have a canonical form $\zeta^\alpha = \delta H / \delta \pi_\alpha$ and $\dot{\pi}_\beta = -\delta H / \delta \zeta^\beta$, that is:

$$\begin{aligned} &\dot{\zeta}^\alpha = \frac{\pi^\alpha}{\rho_0}, \quad \dot{\pi}_\alpha = -\rho_0 \frac{\partial \phi}{\partial \zeta^\alpha} + A^i_\alpha \frac{\partial}{\partial a^i}\left(\rho_0 \frac{\partial U}{\partial \mathbf{J}}\right), \\ &\dot{s} = 0, \end{aligned} \tag{20}$$

with the same definition for the symbol A_α^i as given in (17).

Equations (20) may be put in the form of a *Poisson algebra*, by defining the Poisson bracket

$$\{f,g\}_{\mathrm{LF}} = \int_{D_0} d^3a \left[\frac{\delta f}{\delta\zeta^\alpha(\vec{a})} \frac{\delta g}{\delta\pi_\alpha(\vec{a})} - \frac{\delta g}{\delta\zeta^\alpha(\vec{a})} \frac{\delta f}{\delta\pi_\alpha(\vec{a})} \right]; \tag{21}$$

then, the dynamics of any physical variable of the system is assigned as

$$\dot{f} = \{f,H\}_{\mathrm{LF}}. \tag{22}$$

In particular, if the Poisson brackets $\{\vec{\zeta}, H\}_{\mathrm{LF}}$, $\{\vec{\pi}, H\}_{\mathrm{LF}}$and $\{s, H\}_{\mathrm{LF}}$ are calculated, it is easy to obtain the expressions (20).

4. The Casimir $S[s]$

An important observation should be done on the definition of $\{.,.\}_{\mathrm{LF}}$ in (21): indeed, despite the fact that the full field configuration of the system is given by the collection of three fields $(\vec{\zeta}, \vec{\pi}, s)$, the Poisson bracket only involves derivative with respect to $\vec{\zeta}$and $\vec{\pi}$ only, so that for sure one has:

$$\{C,F\}_{\mathrm{LF}} = 0 \quad \forall \quad C[s], \quad F[\vec{\zeta},\vec{\pi},s], \tag{23}$$

i.e. *any quantity* $C[s]$ *depending only on the mass-specific entropy density is in involution with any quantity at all*. Quantities which are in involution "with anything" are referred to as *Casimir invariants*: in the Poisson algebra $(O_{\mathrm{fluid}}^{\mathrm{LF}}, \{.,.\}_{\mathrm{LF}})$ of the fluid dynamics observables $O_{\mathrm{fluid}}^{\mathrm{LF}}$ in the LF composed by the Poisson bracket $\{.,.\}_{\mathrm{LF}}$ in (21), all the functionals of $s(\vec{a}, t)$ are Casimir invariants. In particular, the *total entropy of the fluid*

$$S[s] = \int_{D_0} \rho_0(\vec{a}) s(\vec{a},t) d^3a \tag{24}$$

is the most important one of them, as it will be stressed in § 5.

When an algebra $(O, \{.,.\})$ admits Casimir invariants it means that the Poisson bracket is *singular*, i.e. it is a bilinear application on observables giving a null result also for some non zero argument. Typically, Poisson brackets of this kind are not in the form (21): indeed, apparently that is the same expression $\{f(q,\ p),\ g(q,\ p)\}=\frac{\partial f}{\partial q}\cdot\frac{\partial g}{\partial p}-\frac{\partial g}{\partial q}\cdot\frac{\partial f}{\partial p}$ used in the canonical Hamiltonian systems with discrete variables. From Mathematical Physics we know that canonical brackets cannot have Casimirs, so there's something "missing" in here. The point is that, as far as the set $O_{\text{fluid}}^{\text{LF}}$ is the one of (smooth) functionals depending on the whole field configuration $(\vec{\zeta},\ \vec{\pi},\ s)$, *indeed* $\{.,\ .\}_{LF}$ *is not, strictly speaking, a canonical bracket*, due to the lack of $\frac{\delta}{\delta s(\vec{a})}$ derivatives. What can be said is that the Poisson bracket $\{.,\ .\}_{\text{LF}}$ is canonical if restricted to the subset of $O_{\text{fluid}}^{\text{LF}}$ of functionals depending only on the centre-of-mass variables $\vec{\zeta}$ and $\vec{\pi}$, while this properties is lost if the microscopic degrees of freedom "within the parcel" are considered too.

When Casimir invariants appear it is generally because a Poisson algebra $(\boldsymbol{M},\ \{.,\ .\})$ undergoes a *process of reduction*.

Assume to have in the set $\boldsymbol{M}$ a Hamiltonian such that $\dot{f}=\{f,\ H\}$ for any element $f\in\boldsymbol{M}$, and then observe H to be invariant under a certain group $\boldsymbol{G}$ of transformations, realized through its Lie algebra $\boldsymbol{g}$ on $(\boldsymbol{M},\ \{.,\ .\})$, so that $\{g,\ H\}=0$ for any $g\in\boldsymbol{g}$. One defines the $\boldsymbol{G}$-reduced algebra $(\boldsymbol{M}_{\text{red}},\ \{.,\ .\}_{\text{red}})$ as the set

$$\boldsymbol{M}_{\text{red}}:\left\{f\in\boldsymbol{M}\quad/\quad\left\{g,f\right\}=0\quad\forall\quad g\in\boldsymbol{g}\right\}\ . \tag{25}$$

This sub-algebra $(\boldsymbol{M}_{\text{red}},\ \{.,\ .\}_{\text{red}})$ of all the $\boldsymbol{G}$-invariant elements of $\boldsymbol{M}$ is still a Poisson bracket algebra, where $\{.,\ .\}_{\text{red}}$ is the restriction of $\{.,\ .\}$ to $\boldsymbol{M}_{\text{red}}$. By the definition of $\boldsymbol{M}_{\text{red}}$ itself, it's clear that any $g\in\boldsymbol{g}$ realized on $\boldsymbol{M}$ is a Casimir for $\{.,\ .\}_{\text{red}}$:

$$\left\{f,g\right\}_{\text{red}}=0\quad\forall\quad f\in\boldsymbol{M}_{\text{red}},\quad g\in\boldsymbol{g}. \tag{26}$$

It may happen that, while $\{.,\ .\}$ is a canonical Poisson bracket, $\{.,\ .\}_{\text{red}}$ is non-canonical (still, the system $\dot{f}=\{f,\ H\}_{\text{red}}$ is Hamiltonian).

The map from the Lagrangian quantities in $O_{\text{fluid}}^{\text{LF}}$ to their correspondent Eulerian fields may be regarded as a reduction from $\left(O_{\text{fluid}}^{\text{LF}},\ \{.,\ .\}_{\text{LF}}\right)$ to the non-canonical Poisson algebra $\left(O_{\text{fluid}}^{\text{EF}},\ \{.,\ .\}_{\text{EF}}\right)$ (Morrison & Greene, 1980). $\mathcal{S}[s]$ is then regarded as a Casimir invariant due to the reduction $\left(O_{\text{fluid}}^{\text{LF}},\ \{.,\ .\}_{\text{LF}}\right)\mapsto\left(O_{\text{fluid}}^{\text{EF}},\ \{.,\ .\}_{\text{EF}}\right)$, where the transformations under which H is invariant is that of *relabeling transformations* (the Hamiltonian in (19) remains the same if the parcels are smoothly relabelled (Padhye & Morrison, 1996 a,b)). The original claim here is that this is not really the case: rather $\mathcal{S}[s]$ is *already a Casimir* of $\{.,\ .\}_{\text{LF}}$, as pointed before, and of course *it remains a Casimir* for $\{.,\ .\}_{\text{EF}}$ too (Materassi, 2014). The author's opinion is that yes, the entropy comes out as a Casimir from a reduction process, of which, however, the algebra $\boldsymbol{M}_{\text{red}}$ is already $O_{\text{fluid}}^{\text{LF}}$: in fact, $\{.,\ .\}_{\text{LF}}$ is *already non-canonical*.

The "original algebra" $(O_{\text{fluid}}, \{.,.\})$that should be reduced to get $\left(O_{\text{fluid}}^{\text{LF}}, \{.,.\}_{\text{LF}}\right)$ must be an algebra where the canonical couple $(\vec{\zeta}(\vec{a}), \vec{\pi}(\vec{a}))$ is completed with $6(N(\vec{a})-1)$ canonical variables describing the motion of the $N(\vec{a})$ particles of the $\vec{a}$-th parcel relative-to-the-centre-of-mass $\vec{\zeta}(\vec{a})$. The act-of-motion of the $\vec{a}$-th parcel in $\boldsymbol{R}^3$, described by $(\vec{\zeta}(\vec{a}), \vec{\pi}(\vec{a}))$, is *invariant under some class of transformation of the microscopic state of the parcel's particles*, and this should drive the reduction $(O_{\text{fluid}}, \{.,.\}) \mapsto \left(O_{\text{fluid}}^{\text{LF}}, \{.,.\}_{\text{LF}}\right)$.

Even if the picture is rather clear, from a logical point of view, making $s(\vec{a})$ arise from microscopic reshuffling transformations appear out of reach yet.

5. Dissipative fluids in Lagrangian formalism

In the absence of dissipation, the "decoupling" of the mass-specific entropy density $s(\vec{a}, t)$ from the other parcel variable in Equations (17) means the decoupling of the microscopic degrees of freedom of the particles forming the parcel, encoded only in $s(\vec{a}, t)$. That could be expected, since *dissipation should be intended as the dynamical shortcut between the macroscopic and microscopic level of the theory in terms of force and energy exchange*: so, excluding these exchanges means decoupling of $s(\vec{a}, t)$ from the centre-of-mass variable $\vec{\zeta}(\vec{a}, t)$. One may also state that the *one-way direction* of this exchange is an aspect of the *Second Law of Thermodynamics* (Dewar et al., 2014).

When dissipation is not present, the microscopic degrees of freedom animating the particles *within the parcel* are as frozen, and their world participates to the motion of the nearby parcels only through the compression-dilatation effect recorded by the term $A_\alpha^i \frac{\partial}{\partial a^i}\left(\rho_0 \frac{\partial U}{\partial J}\right)$. When dissipation is at work, then two important facts take place:

1. the energy transfer from the macroscopic degrees of freedom $(\vec{\zeta}, \vec{\pi})$ to the microscopic ones encoded in s will render $\dot{s} \neq 0$ and, according to the Second Principle, make it grow;
2. the same interaction will impede the "conservative motion" of the parcel centre-of-mass $(\vec{\zeta}, \vec{\pi})$ via "friction terms" rising due to the particle action.

All in all, both the centre-of-mass motion of the parcel and its entropy will be influenced by the presence of dissipation.

The equations of motion of fluids in the LF in the presence of dissipative terms are not such a widespread subject to be treated, let me address the reader to the book by Bennett "Lagrangian Fluid Dynamics", where the equation of motion of a viscous incompressible fluid is obtained in parcel variables (Bennett, 2006).

Looking at the equations (20), one way of finding out how to modify them in order to introduce dissipation is to examine the corresponding equations in EF (the dissipative Navier-Stokes equations, see for instance Rai Choudhuri (1998)) and then work on how the various quantities change when they undergo the inverse map $\vec{\zeta} \mapsto \vec{a}$. This is the way in which the LF equations of motion with dissipation were obtained in Materassi (2014).

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The relationship between $\vec{\zeta}$ and $\vec{\pi}$ remain the same as in (20) when dissipation is introduced, that is $\zeta^\alpha = \frac{\pi^\alpha}{\rho_0}$.

The equation of motion for the momentum $\vec{\pi}$ changes because *now friction forces appear*: these pertain to the relative motion of the parcel with respect to its neighbours, so that one expects (as in the viscous force between pointlike particles and a medium) these will depend on the velocity difference between neighbouring parcels. Not only: due to the smoothness of the map $\vec{a} \mapsto \vec{\zeta}$, it is clear that neighbouring parcels do correspond to neighbouring labels, and so the friction forces will depend on the gradients of velocities, i.e. on $\vec{a}$-gradients of the field $\vec{\pi}(\vec{a}, t)$.

It is possible to show that, provided the stress tensor of the fluid appears in the EF as reported in Materassi et al. (2012), it is possible to write the equation of motion for π_α in the presence of dissipation as:

$$\dot{\pi}_\alpha = -\rho_0 \frac{\partial \phi}{\partial \zeta^\alpha} + A_\alpha^i \frac{\partial}{\partial a^i}\left(\rho_0 \frac{\partial U}{\partial J}\right) + J \Lambda_{\alpha\beta\gamma\delta} \nabla^\beta \nabla^\gamma \left(\frac{\pi^\delta}{\rho_0}\right). \tag{27}$$

In (27) the tensor $\Lambda_{\alpha\beta\gamma\delta}$ is constant, and reads:

$$\Lambda_{\alpha\beta\gamma\delta} = \eta\left(\delta_{\alpha\delta}\delta_{\beta\gamma} + \delta_{\beta\delta}\delta_{\alpha\gamma} - \frac{2}{3}\delta_{\gamma\delta}\delta_{\alpha\beta}\right) + \nu \delta_{\gamma\delta}\delta_{\alpha\beta}. \tag{28}$$

The coefficients η and ν are the two kinds of viscosity (compressible and incompressible, respectively) already quoted in Materassi & Tassi (2012 a) and in Materassi et al. (2012).

The derivative indicated as ∇^β is a differential operator acting on fields that depend on the label $\vec{a}$, and is constructed *through the chain rule* as:

$$\nabla^\beta = \frac{\partial a^i}{\partial \zeta_\beta}\frac{\partial}{\partial a^i} = \left(J^{-1}\right)_\beta^i \frac{\partial}{\partial a^i}, \tag{29}$$

being the matrix J^{-1} the Jacobian of the inverse map $\vec{\zeta} \mapsto \vec{a}$.

The last equation in (20) must be modified in order to include the dissipative terms in the entropy evolution.

In particular, in order to understand which terms should be included, it is of some use to consider that the entropy variation always corresponds, within quasi-equilibrium processes (Zemansky & Dittman, 1996), to *the ratio between the heat transferred and the temperature at which this transfer takes place*. Then, one has just to think about which are the processes transferring heat to the parcel.

Two sources of heat must be considered: there is a contribution given by the work dissipated through viscosity along the surface of the parcel, that should correspond to the force terms $J\Lambda_{\alpha\beta\gamma\delta}\nabla^{\beta}\nabla^{\gamma}\left(\frac{\pi^{\delta}}{\rho_0}\right)$; then, there is the heat flowing into the parcel, still across its surface, due to the finiteness of the thermal conductivity κ of the fluid.

The temperature at which these heat transfers take place is the temperature of the $\vec{a}$-th parcel at time t, that may as well be obtained by the internal energy U thanks to the relationship $T=\frac{\partial U}{\partial s}$.

In Bennett (2006) and in Materassi (2014) both the heat transfer terms have been obtained from the corresponding terms in the EF equations (as these are reported, e.g., in Morrison (1984) and Materassi & Tassi (2012 a)) through the inverse map $\vec{\zeta}\mapsto\vec{a}$:

$$\delta Q_{work} = \frac{\Lambda_{\alpha\beta\delta\gamma}\partial^{\alpha}v^{\beta}\partial^{\gamma}v^{\delta}}{\rho T}dt, \quad \delta Q_{conduction} = \frac{\kappa}{\rho T}\partial^2 T dt. \tag{30}$$

In (30) all the quantities are local fields, i.e. Eulerian variables in $O^{\mathrm{EF}}_{\mathrm{fluid}}$, while $\vec{\partial}$ is the gradient with respect to the space position, of components ∂^{α}, and $\vec{v}$ is the Eulerian field of velocities, of components v^{β}. The convention $\partial^{\alpha}\partial_{\alpha}=\partial^2$ was used.

After some relatively easy, but lengthy, algebra on (30), one ends up with the following equation of motion of the mass-specific entropy density:

$$\dot{s} = \frac{J}{\rho_0 T}\Lambda_{\alpha\beta\delta\gamma}\nabla^{\alpha}\left(\frac{\pi^{\beta}}{\rho_0}\right)\nabla^{\gamma}\left(\frac{\pi^{\delta}}{\rho_0}\right) + \frac{\kappa J}{\rho_0 T}\nabla^{\eta}\nabla_{\eta}T. \tag{31}$$

We are now in the position to provide the reader with the system of equations generalizing (20) in the case of viscous fluids:

$$\begin{aligned} &\dot{\zeta}^{\alpha} = \frac{\pi^{\alpha}}{\rho_0}, \quad \dot{\pi}_{\alpha} = -\rho_0\frac{\partial\phi}{\partial\zeta^{\alpha}} + A^i_{\alpha}\frac{\partial}{\partial a^i}\left(\rho_0\frac{\partial U}{\partial J}\right) + J\Lambda_{\alpha\beta\gamma\delta}\nabla^{\beta}\nabla^{\gamma}\left(\frac{\pi^{\delta}}{\rho_0}\right), \\ &\dot{s} = \frac{J}{\rho_0 T}\Lambda_{\alpha\beta\delta\gamma}\nabla^{\alpha}\left(\frac{\pi^{\beta}}{\rho_0}\right)\nabla^{\gamma}\left(\frac{\pi^{\delta}}{\rho_0}\right) + \frac{\kappa J}{\rho_0 T}\nabla^{\eta}\nabla_{\eta}T. \end{aligned} \tag{32}$$

In general, equations as (32) are extremely complicated to treat because of the very non-linear way in which the unknown functions, i.e. the fields $\vec{\zeta}(\vec{a}, t)$, $\vec{\pi}(\vec{a}, t)$and $s(\vec{a}, t)$, participate in the various operators. For instance, one should underline how the gradients of $\vec{\zeta}(\vec{a}, t)$ appear in A^i_{α} through the inverse Jacobians (see (I. 17)), or in the determinants J, not to mention the operators ∇^{η}, especially when they are squared, so that, for instance, one has:

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$\nabla^{\beta}\nabla_{\beta}T = (J^{-1})^{i}_{\beta}\frac{\partial}{\partial a^{i}}\left[(J^{-1})^{j\beta}\frac{\partial}{\partial a^{j}}T\right]$in the entropy evolution equation. The intriguing thing is that this high complexity of the LF equations is due to the necessity of taking into account of how the fluid parcel deforms during its motion, which is actually the meaning of J, and of J^{-1}.

The viscous fluid is an example of dissipative system that, if it is *isolated* (i.e., without energetic supply from the environment), will relax to an *asymptotically stable field configuration*. Correspondingly, the total entropy of the isolated system (24) will grow up to a certain maximum value allowable S_{max}, as a *Lyapunov functional* is expected to do (Courbage & Prigogine, 1983).

The algebrization of the viscous fluid dynamics in LF will follow the same route proposed in Fish (2005) and references therein (see, in particular, Morrison (1984)), or in Materassi & Tassi (2012 b): the inspiration comes from the fact that, in the vicinity of the equilibrium, it is possible to represent the convergence of the system to the asymptotically stable point as a *metric system* (Morrison, 2009).

A metric system is a dynamical system of variables ψ that has a smooth function $S(\psi)$ on its phase space that generates the dynamics via a *symmetric*, *negative semidefinite* operator G, so that one has:

$$\dot{\psi}_i = -G_i^j(\psi)\frac{\partial S}{\partial \psi_j}. \tag{33}$$

It is possible, then, to define a symmetric Leibniz bracket (Guha, 2007) for two any observables f and g by stating:

$$(f,g) = -\frac{\partial f}{\partial \psi_i}G_i^j(\psi)\frac{\partial g}{\partial \psi_j}, \tag{34}$$

so that *the dynamics of the metric system* is prescribed as:

$$\dot{f} = (f,S). \tag{35}$$

It is possible to see that the asymptotically stable configuration will be a stationary point for $S(\psi)$, $\frac{\partial S}{\partial \psi_j} \Rightarrow \dot{\psi}_i = 0$, and that it will be a maximum, since from (34) and (35) and the negative seminidefiniteness of G, the function $S(\psi)$ turns out to be monotonically growing with time:

$$\dot{S} = (S,S) = -\frac{\partial S}{\partial \psi_i}G_i^j(\psi)\frac{\partial S}{\partial \psi_j} \geq 0. \tag{36}$$

The smooth function $S(\psi)$ then appears to be *a Lyapunov function for the metric system*.

Since the metric system appears to be a good tool to describe the convergence to an asymptotically stable equilibrium, in order to let the fluid be described by a bracket algebra of observables that allows for relaxation, next to the Poisson bracket (21), one introduces a *metric bracket* (f, g), i.e.:

$$(f,g)_{\mathrm{LF}} = (g,f)_{\mathrm{LF}}, \quad (f,f)_{\mathrm{LF}} \leq 0 \quad \forall \quad f,g \in O_{\mathrm{fluid}}^{\mathrm{LF}}. \tag{37}$$

If the choice of the metric bracket is clever enough, i.e. provided the requirement

$$(H,f)_{\mathrm{LF}} = 0, \quad \forall \quad f \in O_{\mathrm{fluid}}^{\mathrm{LF}} \tag{38}$$

is satisfied, then the dissipative part of dynamics may be entrusted to this new algebraic object, with the choice of the total entropy $S[s]$ as the *metric generating function*, since we already know it is a Lyapunov functional of relaxation:

$$\dot{f}_{\mathrm{diss}} = \lambda (f,S)_{\mathrm{LF}}, \quad \forall \quad f \in O_{\mathrm{fluid}}^{\mathrm{LF}} \tag{39}$$

(this λ is a suitable constant that only makes a definite physical sense at the equilibrium).

The full dynamics may be obtained by putting together the *non-dissipative symplectic part* (I. 22) and the *dissipative metric part* (39), so to give rise to a properly algebrized dissipative system relaxing to some asymptotic equilibrium, namely a *metriplectic system* (Morrison, 1984; Guha, 2007):

$$\dot{f} = \{f,H\}_{\mathrm{LF}} + \lambda (f,S)_{\mathrm{LF}}, \quad \forall \quad f \in O_{\mathrm{fluid}}^{\mathrm{LF}}. \tag{40}$$

The Poisson bracket $\{.,.\}_{\mathrm{LF}}$ is the one defined in (21), while the metric component $(.,.)_{\mathrm{LF}}$ has been obtained in Materassi (2014) by reasoning on the application of the inverse map $\vec{\zeta} \mapsto \vec{a}$ (and of its implications in terms of functional derivatives) to the corresponding quantity presented in Morrison (1984) for the same system in the EF. The bracket reads:

$$\begin{aligned}(f,g)_{\mathrm{LF}} = \frac{1}{\lambda} \int_{D_0} J d^3 a \Bigg\{ &\kappa T^2 \nabla^{\eta} \left(\frac{1}{\rho_0 T} \frac{\delta f}{\delta s} \right) \nabla_{\eta} \left(\frac{1}{\rho_0 T} \frac{\delta g}{\delta s} \right) + \\ &+ T \Lambda_{\alpha\beta\gamma\delta} \left[\nabla^{\alpha} \left(\frac{\delta f}{\delta \pi_{\beta}} \right) - \frac{1}{\rho_0 T} \nabla^{\alpha} \left(\frac{\pi^{\beta}}{\rho_0} \right) \frac{\delta f}{\delta s} \right] \left[\nabla^{\gamma} \left(\frac{\delta g}{\delta \pi_{\delta}} \right) - \frac{1}{\rho_0 T} \nabla^{\gamma} \left(\frac{\pi^{\delta}}{\rho_0} \right) \frac{\delta g}{\delta s} \right] \Bigg\}.\end{aligned} \tag{41}$$

It is possible to check that the application of the prescription (40) to the dynamical variables $\vec{\zeta}$, $\vec{\pi}$and s reproduces the equations of motion for the dissipative fluids (32), as represented in the LF via parcel variables.

The former Poisson algebra of the observables of a fluid in the LF, the set $O_{\text{fluid}}^{\text{LF}}$, is now structured with both the brackets $\{.,.\}_{\text{LF}}$ and $(.,.)_{\text{LF}}$, so that a *metriplectic structure* may be defined by stating

$$\langle\langle f,g\rangle\rangle_{\text{LF}} = \{f,g\}_{\text{LF}} + (f,g)_{\text{LF}} \quad \forall \quad f,g \in O_{\text{fluid}}^{\text{LF}}. \tag{42}$$

In order for the metriplectic bracket $\langle\langle .,.\rangle\rangle_{\text{LF}}$ to generate a Leibniz dynamics (that corresponds to the dynamics of viscous fluids in the LF), the definition of the free energy

$$F[\vec{\zeta},\vec{\pi},s] = H[\vec{\zeta},\vec{\pi},s] + \lambda S[s] \tag{43}$$

is necessary. It is possible to see that, when the prescription

$$\dot{f} = \langle\langle f,F\rangle\rangle_{\text{LF}} \tag{44}$$

is applied to the dynamical variables ($\vec{\zeta}$, $\vec{\pi}$, s), the dynamics of the viscous fluid is re-obtained, as described by the equations of motion (32).

6. Conclusion

In this Chapter we have described the dynamics of viscous fluids in the formalism descending from the Lagrangian viewpoint, according to which the motion of the fluid parcels is followed along their evolution, just like it is done in discrete dynamics with the motion of point particles.

The "something different" that parcels have with respect to point particles is their internal structure, which is a "gas" (a fluid, truly) of microscopic particles, all in all described by their mass density, roughly J^{-1}, and their mass-specific entropy density s. The existence of this very rich internal structure has important consequences influencing the dynamics of the parcels moving through the space.

The particle ensemble may influence the parcel's motion both in a "conservative" and in a "dissipative" way.

On the one hand, particles may undergo "adiabatic interactions", hitting each other conserving their kinetic energy, and propagating, to the nearby particles, pressure: this is the effect of those motions of particles exerting forces, but not exchanging heat, i.e. not degrading the "ordered" energy of the macroscopic scales at which the parcel variables $\vec{\zeta}$ and $\vec{\pi}$ are defined. If the language of textbook Thermodynamics were to be used, we should state this is the kind of micro-macro interactions via which a parcel makes *mechanical work* on the nearby parcels, but does not give or receive heat.

These adiabatic interactions give rise to the pressure terms in (17) and (20), that still remain equations of motion of a dissipation-free system.

An ensemble of particles is however able to exchange *heat* too, i.e. transform the energy of macroscopic scales to micro-scale energy. This happens due to friction forces, that are representing in an effective way a huge amount of microscopic collisions in which particles literally drain kinetic energy from $\vec{\zeta}$ and $\vec{\pi}$ of nearby parcels. Or, the micro-motions and collisions among particles, from one parcel to the next one, give rise to *thermal conduction*, with a proper heat flux in which "nothing happens" at macroscopic scales, and nevertheless there is a transfer of internal energy throughout the fluid.

Two approaches are presented here to treat the dissipative phenomena of a viscous, thermally conducting fluid: writing the partial derivative equations of motion (32) in LF; or constructing the metriplectic algebra (42), through which the viscous, thermally conducting fluid is regarded as an isolated system with dissipation that has a way to relax to asymptotically stable equilibria.

In both approaches we have tried to stress the role of the *entropy functional* (24), that is fully enhanced in the metriplectic algebra (Materassi, 2014).

Let us conclude briefly by stressing again the limits of the LF as described here, i.e. via parcel variables. The fundamental requirements of parcel formulation in § 2, on which all the LF is based, render it possible to describe the evolution of fluid matter only in "rather regular" motions: no sub-fluid scale mixing, no evaporation, no droplet formation, no noisy micro-scale turbulence. Just the deformation of a body that can be regarded as an elastic rubber space-filling cloud, much more a shape shifting smooth Barbapapa rather than the water of a fountain dropping and fragmenting in pointy wavelets.

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