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A NUMERICAL METHOD FOR SOLVING EQUILIBRIUM PROBLEMS OF NO-TENSION SOLIDS SUBJECTED TO THERMAL LOADS

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Abstract

The paper starts out by recalling a constitutive equation of no-tension materials that accounts for thermal dilatation and the temperature dependence of the material constants. Subsequently, a numerical method is presented for solving, via the finite element method, equilibrium problems of no-tension solids subjected to thermal loads. Finally, three examples are solved and discussed: a spherical container subjected to two uniform radial pressures and a steady temperature distribution, a masonry arch subjected to a uniform temperature distribution and a converter used in the steel and iron industry.

KEY WORDS. No-tension materials, thermal loads, finite element method.

1. Introduction

In many applications it is necessary to model the behaviour of solids not withstanding tension in the presence of thermal dilatation. For example, molten metal production processes, in particular integrated steel manufacturing, require refractory linings able to withstand the thermo-mechanical actions produced by high-temperature fluids [1]. Analysis of these coverings is usually carried out by considering the refractory materials to be linear elastic, though they are actually non-resistant to traction. Results obtained by applying such a constitutive model are generally characterised by considerable tensile stresses and are thus quite unrealistic.

However, there are many other engineering problems in which thermal dilatation must be accounted for: consider, for example geological problems connected with the presence of a volcanic caldera, such as that of Pozzuoli [2], or the influence of thermal variations on stress fields in masonry bridges [3]. In many such cases the thermal variation is so high that the dependence of the material constants on temperature cannot be ignored.

In [4] the authors present a constitutive equation for no-tension materials in the presence of thermal expansion which accounts for the temperature-dependence of the material's constants. In particular, under the hypothesis that the strain minus the thermal dilatation is infinitesimal, explicit expressions for stress and inelastic strain are given, from which free energy, internal energy and entropy are then obtained, and both coupled and uncoupled equations of the thermo-mechanical equilibrium of a no-tension solid have been developed.

In this paper we recall the constitutive equation presented in [4] and, by limiting ourselves to thermo-mechanical uncoupling, we propose a numerical method for solution of the equilibrium problem of solids not supporting tension that are subjected to thermal loads.

In Section 2 the solution to the constitutive equation proposed in [4] is calculated for the three-dimensional case. It is assumed that the thermal dilatation is a temperature-dependent spherical tensor, and that the total strain minus the thermal dilatation is the sum of two components: an elastic part, on which the stress, negative semi-definite depends isotropically and linearly, and an inelastic part, positive semi-definite and orthogonal to the stress. We thereby obtain a non-linear elastic material conforming to a masonry-like material when no temperature change occurs. The solution to the equation for plane stress problems is presented in the Appendix.

By applying a procedure similar to that used in [5], it can be proved that the solution to the equilibrium problem is unique in terms of stress, and independent of the particular loading process chosen. This latter result is necessary in order to justify application of incremental numerical techniques that must usually be used because of the non-linearity of the constitutive equation.

At the end of Section 2 the derivative of the stress with respect to the strain is explicitly calculated for three-dimensional problems (the derivative for plane stress is reported in the Appendix). This is needed in order to calculate the tangent stiffness matrix used in applying the Newton-Raphson method for solution of the non-linear system obtained by discretising the structure in question into finite elements.

The constitutive equation and the numerical techniques for solving the non-linear boundary-value problem have been implemented in the finite element code NOSA [6].

In Section 3, the stress and displacement fields are explicitly calculated for a spherical container subjected to two uniform radial pressures and a steady temperature distribution. Finally, two problems are numerically solved by applying NOSA: a masonry arch subjected to temperature changes and a converter used in the steel and iron industry.

2. The numerical method

In this section, after recalling the constitutive equation of isotropic no-tension material in the presence of thermal expansion introduced in [4], we present a numerical procedure for solving the equilibrium problem via the finite element method. In the following, $\theta \in [\theta_1, \theta_2]$ will be the current absolute temperature, with $\theta_1 > 0$ and $\theta_0 \in [\theta_1, \theta_2]$ the reference temperature.

Let $\mathcal{V}$ be a three-dimensional linear space, and Lin the space of all linear transformations from $\mathcal{V}$ into $\mathcal{V}$, equipped with the inner product $\mathbf{A} \bullet \mathbf{B} = \mathrm{tr}\,(\mathbf{A}^T\mathbf{B})$, $\mathbf{A}, \mathbf{B} \in$ Lin, with $\mathbf{A}^T$ the transpose of $\mathbf{A}$. Let us indicate as Sym, Sym$^+$ and Sym$^-$, the subsets of Lin constituted by symmetric, symmetric positive semi-definite and symmetric negative semi-definite tensors, respectively.

Let us now assume that the thermal dilatation due to a temperature change $\theta - \theta_0$ is $\beta(\theta)\mathbf{I}$, where β is a sufficiently smooth function, called *thermal expansion*, such that $\beta(\theta_0) = 0$, and $\mathbf{I}$ is the second-order identity tensor. When $\theta - \theta_0$ is small, the expression of thermal dilatation can be written in the usual way, $\alpha(\theta - \theta_0)\mathbf{I}$, where $\alpha = \dfrac{d\beta(\theta_0)}{d\theta}$ is the *linear coefficient of thermal expansion*. Denoting $\mathbf{E}$ as the symmetric part of the displacement gradient, we assume that tensor $\mathbf{E} - \beta(\theta)\mathbf{I}$ is $O(\delta)$[1]. The strain $\mathbf{E} - \beta(\theta)\mathbf{I}$ is assumed to be the sum of a part $\mathbf{E}^e$ and a part $\mathbf{E}^a$ positive semi-definite

[1] Given a mapping B from a neighbourhood of 0 in $\mathbb{R}$ into a vector space with norm $\| \ \|$, we write $B(\delta) = O(\delta)$ if there exist $k > 0$ and $k' > 0$ such that $\| B(\delta)\| < k\,|\delta|$ whenever $|\delta| < k'$.

$$\mathbf{E} - \beta(\theta)\mathbf{I} = \mathbf{E}^e + \mathbf{E}^a \,, \tag{2.1}$$

$$\mathbf{E}^a \in \mathrm{Sym}^+ . \tag{2.2}$$

Moreover, we suppose that the stress $\mathbf{T}$, orthogonal to $\mathbf{E}^a$ and negative semi-definite, depends linearly and isotropically on $\mathbf{E}^e$,

$$\mathbf{T} \bullet \mathbf{E}^a = 0 \,, \tag{2.3}$$

$$\mathbf{T} \in \mathrm{Sym}^- \,, \tag{2.4}$$

$$\mathbf{T} = \frac{E(\theta)}{1+\nu(\theta)} \{\mathbf{E}^e + \frac{\nu(\theta)}{1-2\nu(\theta)} \operatorname{tr}(\mathbf{E}^e)\} \,, \tag{2.5}$$

where Young's modulus $E(\theta)$ and Poisson's ratio $\nu(\theta)$ depend on the temperature and satisfy the inequalities

$$E(\theta) > 0, \quad 0 \leq \nu(\theta) < 0.5 \,, \qquad \text{for each } \theta \in [\theta_1, \theta_2] \,. \tag{2.6}$$

Tensor $\mathbf{E}^a$ can be interpreted as fracture strain, because one can expect fractures in the regions where $\mathbf{E}^a$ is different from zero.

By a procedure similar to that used in [7], it is possible to prove that, given $(\mathbf{E}, \theta) \in \mathrm{Sym} \times [\theta_1, \theta_2]$ and the functions β, E and ν, the constitutive equation (2.1)-(2.5) has a unique solution $(\mathbf{T}, \mathbf{E}^a)$. Moreover, tensors $\mathbf{E}$, $\beta(\theta)\mathbf{I}$, $\mathbf{E}^a$, $\mathbf{E}^e$ and $\mathbf{T}$ are coaxial, and the constitutive equation (2.1)-(2.5) can be written with respect to the basis $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ of the eigenvectors of $\mathbf{E}$. Let $\{e_1, e_2, e_3\}$, $\{a_1, a_2, a_3\}$ and $\{t_1, t_2, t_3\}$ be the eigenvalues of $\mathbf{E}$, $\mathbf{E}^a$ and $\mathbf{T}$, respectively, with e_1, e_2 and e_3 ordered in such a way that $e_1 \leq e_2 \leq e_3$. The quantities $\{a_1, a_2, a_3\}$ and $\{t_1, t_2, t_3\}$ satisfying (2.1)-(2.5) can be calculated as functions of $e_1 - \beta(\theta)$, $e_2 - \beta(\theta)$, $e_3 - \beta(\theta)$, and their expression has already been presented in [4]. At this point, we recall the solution to the constitutive equation (2.1)-(2.5) for three-dimensional problems; to this end let us define [4] the following subsets of $\mathrm{Sym} \times [\theta_1, \theta_2]$

$$\mathcal{R}_1 = \{(\mathbf{E}, \theta) \mid 2(e_3 - \beta(\theta)) + \gamma(\theta)\,(\operatorname{tr}\mathbf{E} - 3\,\beta(\theta)) \leq 0\} \,, \tag{2.7}$$

$$\mathcal{R}_2 = \{(\mathbf{E}, \theta) \mid e_1 - \beta(\theta) \geq 0\} \,, \tag{2.8}$$

$$\mathcal{R}_3 = \{(\mathbf{E}, \theta) \mid e_1 - \beta(\theta) \leq 0 \,,\ \gamma(\theta)\,(e_1 - \beta(\theta)) + 2\,(1 + \gamma(\theta))\,(e_2 - \beta(\theta)) \geq 0\} \,, \tag{2.9}$$

$$\mathcal{R}_4 = \{(\mathbf{E}, \theta) \mid \gamma(\theta)\,(e_1 - \beta(\theta)) + 2\,(1 + \gamma(\theta))\,(e_2 - \beta(\theta)) \leq 0 \,,$$
$$2(e_3 - \beta(\theta)) + \gamma(\theta)\,(\operatorname{tr}\mathbf{E} - 3\,\beta(\theta)) \geq 0\} \,, \tag{2.10}$$

where we have put $\gamma(\theta) = \dfrac{2\nu(\theta)}{1-2\nu(\theta)}$. For later use, we observe that from the definition of $\mathcal{R}_3$ and $\mathcal{R}_4$, it clearly follows that in $\mathcal{R}_3$ and $\mathcal{R}_4$ we have $e_1 < e_2$ and $e_2 < e_3$, respectively.

Given $\mathbf{E} - \beta(\theta)\mathbf{I}$, with spectral representation $\sum_{i=1}^{3}(e_i - \beta(\theta))\, \mathbf{q}_i \otimes \mathbf{q}_i$, by solving the constitutive equation (2.1) - (2.5), we establish that tensors $\mathbf{E}^a$ and $\mathbf{T}$ have the following expressions

if $(\mathbf{E}, \theta) \in \mathcal{R}_1$, then

$$\mathbf{E}^a = \mathbf{0},$$

$$\mathbf{T} = \frac{E(\theta)}{1+\nu(\theta)} \{\mathbf{E} - \beta(\theta)\,\mathbf{I} + \frac{\nu(\theta)}{1-2\nu(\theta)} \operatorname{tr}(\mathbf{E} - \beta(\theta)\,\mathbf{I})\,\mathbf{I}\} ; \tag{2.11}$$

if $(\mathbf{E}, \theta) \in \mathcal{R}_2$, then

$$\mathbf{E}^a = \mathbf{E} - \beta(\theta)\mathbf{I} ,$$

$$\mathbf{T} = \mathbf{0} ; \tag{2.12}$$

if $(\mathbf{E}, \theta) \in \mathcal{R}_3$, then

$$\mathbf{E}^a = [e_2 - \beta(\theta) + \nu(\theta)\,(e_1 - \beta(\theta))]\ \mathbf{q}_2 \otimes \mathbf{q}_2$$

$$+ [e_3 - \beta(\theta) + \nu(\theta)\,(e_1 - \beta(\theta))]\ \mathbf{q}_3 \otimes \mathbf{q}_3 ,$$

$$\mathbf{T} = E(\theta)\,(e_1 - \beta(\theta))\ \mathbf{q}_1 \otimes \mathbf{q}_1 ; \tag{2.13}$$

if $(\mathbf{E}, \theta) \in \mathcal{R}_4$, then

$$\mathbf{E}^a = \frac{1}{1-\nu(\theta)} [e_3 - \beta(\theta) + \nu(\theta)\,(e_1 + e_2 - e_3 - \beta(\theta))]\ \mathbf{q}_3 \otimes \mathbf{q}_3 ,$$

$$\mathbf{T} = \frac{E(\theta)}{1-\nu^2(\theta)} \{[e_1 - \beta(\theta) + \nu(\theta)\,(e_2 - \beta(\theta))]\ \mathbf{q}_1 \otimes \mathbf{q}_1 +$$

$$[e_2 - \beta(\theta) + \nu(\theta)\,(e_1 - \beta(\theta))]\ \mathbf{q}_2 \otimes \mathbf{q}_2 \} . \tag{2.14}$$

We shall denote by $\hat{\mathbf{T}}$, the function $\hat{\mathbf{T}}$: Sym × [θ_1, θ_2] → Sym which associates the stress $\mathbf{T} = \hat{\mathbf{T}}(\mathbf{E}, \theta)$ given in (2.11)-(2.14) to every ($\mathbf{E}$, θ); $\hat{\mathbf{T}}$ is an isotropic(2), continuous, non-linear, non-injective function, positively homogeneous of degree one and differentiable with respect to $\mathbf{E}$ in the internal part of every region $\mathcal{R}_i$.

The uncoupled equations governing the thermo-mechanical equilibrium of no-tension solids in the presence of thermal variations have been set forth in [4], under the hypotheses that $\mathbf{E}$, $\beta(\theta)$, $\beta'(\theta)$, $\dot{\mathbf{E}}$ and $\dot{\theta}$ are O(δ). In order to solve the equilibrium problems for masonry-like solids by using the finite element method, we must consider loading processes and associated incremental equilibrium problems.

As in the isothermal case [5], it is possible to prove that the numerical solution obtained by using an incremental procedure is independent of the particular loading process chosen: it depends solely on the final assigned load, provided that the loading process considered is admissible, in the sense specified as follows.

Let $\mathcal{B}$ be a body made of a no-tension material, $\mathcal{S}_u$ and $\mathcal{S}_f$ two subsets of the boundary $\partial\mathcal{B}$ of $\mathcal{B}$ such that their union covers $\partial\mathcal{B}$ and their interiors are disjoint. A loading process $l(t)$ = [$\mathbf{b}(x, t)$, $\theta(x, t)$, $\mathbf{s}(x, t)$], with $\mathbf{b}$ and θ defined on $\mathcal{B}\times[0, \bar{t}\,]$, and $\mathbf{s}$ defined on $\mathcal{S}_f \times[0,\bar{t}\,]$, is *admissible* if for each parameter $t \in [0, \bar{t}\,]$ the corresponding boundary-value problem has a solution, *i.e.* if there exists a triple [$\mathbf{u}(t)$, $\mathbf{E}(t)$, $\mathbf{T}(t)$] differentiable with respect to t, constituted by stress field $\mathbf{T}$, strain field $\mathbf{E}$ and displacement field $\mathbf{u}$ defined on the closure of $\mathcal{B}$ such that they satisfy the equations

$$\mathbf{E} = \frac{\nabla\mathbf{u} + \nabla\mathbf{u}^T}{2}, \tag{2.15}$$

$$\operatorname{div}\mathbf{T} + \mathbf{b} = \mathbf{0}, \tag{2.16}$$

$$\mathbf{T} = \hat{\mathbf{T}}(\mathbf{E}, \theta), \tag{2.17}$$

on $\mathcal{B}$, and the boundary conditions

$$\mathbf{T}\mathbf{n} = \mathbf{s} \text{ on } \mathcal{S}_f, \tag{2.18}$$

$$\mathbf{u} = \bar{\mathbf{u}} \text{ on } \mathcal{S}_u, \tag{2.19}$$

where $\mathbf{n}$ is the unit outward normal to $\mathcal{S}_f$ and $\bar{\mathbf{u}}$ an assigned displacement on $\mathcal{S}_u \times[0, \bar{t}\,]$.

We now turn to the numerical procedure used for finite element analysis of masonry solids. Let $\mathbf{w}$ be a vector field such that $\mathbf{w} = \mathbf{0}$ on $\mathcal{S}_u$, From (2.16) and (2.18) it follows that at every t, the following equilibrium equation must be verified

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2 $\hat{\mathbf{T}}$ is an isotropic function in the sense that $\hat{\mathbf{T}}(\mathbf{Q}\mathbf{E}\mathbf{Q}^T, \theta) = \mathbf{Q}\hat{\mathbf{T}}(\mathbf{E}, \theta)\mathbf{Q}^T$ for each orthogonal tensor $\mathbf{Q}$, for each $\mathbf{E}$ $\in$ Sym and $\theta \in [\theta_1, \theta_2]$.

$$\int_{\mathcal{B}} \mathbf{T} \bullet \nabla \mathbf{w} \, dV = \int_{\mathcal{S}_f} \mathbf{s} \cdot \mathbf{w} \, dA + \int_{\mathcal{B}} \mathbf{b} \cdot \mathbf{w} \, dV . \tag{2.20}$$

Since **T** depends non-linearly on **E**, we must also consider the following incremental equilibrium equation

$$\int_{\mathcal{B}} \{ \mathrm{D_E} \hat{\mathbf{T}}(\mathbf{E}, \theta)[\, \dot{\mathbf{E}} \,] \} \bullet \nabla \mathbf{w} \, dV =$$

$$= \int_{\mathcal{S}_f} \dot{\mathbf{s}} \cdot \mathbf{w} \, dA + \int_{\mathcal{B}} \dot{\mathbf{b}} \cdot \mathbf{w} \, dV - \int_{\mathcal{B}} \{ \mathrm{D}_\theta \hat{\mathbf{T}}(\mathbf{E}, \theta) \, \dot{\theta} \} \bullet \nabla \mathbf{w} \, dV , \tag{2.21}$$

where the dot denotes the derivative with respect to t.
The finite element method allows us to transform the incremental equation (2.21) into a non-linear system of algebraic equations which can be solved by means of a Newton-Raphson procedure. By using standard methods, the incremental equation (2.21) can be transformed into the evolution system

$$[K]\{ \dot{a} \} = \{ \dot{f} \}, \tag{2.22}$$

where $\{ \dot{a} \}$ is the vector of nodal velocities, matrix $[K]$ is obtained from the relation

$$\{c\} \cdot [K] \{ \dot{a} \} = \int_{\mathcal{B}} \{ \mathrm{D_E} \hat{\mathbf{T}}(\mathbf{E}, \theta)[\, \dot{\mathbf{E}} \,] \} \bullet \nabla \mathbf{w} \, dV , \tag{2.23}$$

with $\{c\}$ the vector of nodal values of field **w**, and finally,

$$\{c\} \cdot \{ \dot{f} \} = \int_{\mathcal{S}_f} \dot{\mathbf{s}} \cdot \mathbf{w} \, dA + \int_{\mathcal{B}} \dot{\mathbf{b}} \cdot \mathbf{w} \, dV - \int_{\mathcal{B}} \{ \mathrm{D}_\theta \hat{\mathbf{T}}(\mathbf{E}, \theta) \, \dot{\theta} \} \bullet \nabla \mathbf{w} \, dV . \tag{2.24}$$

We assume that equation (2.20) holds in correspondence of t, and that the body is therefore in equilibrium; subsequently, we assign a load increment $\{\Delta f\}$ defined by means of relation

$$\{c\} \cdot \{\Delta f\} = \int_{\mathcal{S}_f} (\mathbf{s}(t + \Delta t) - \mathbf{s}(t)) \cdot \mathbf{w} \, dA + \int_{\mathcal{B}} (\mathbf{b}(t + \Delta t) - \mathbf{b}(t)) \cdot \mathbf{w} \, dV$$

$$- \int_{\mathcal{B}} \{ \hat{\mathbf{T}}(\mathbf{E}, \theta_2) - \hat{\mathbf{T}}(\mathbf{E}, \theta_1) \} \bullet \nabla \mathbf{w} \, dV , \tag{2.25}$$

where $\theta_2 = \theta(t + \Delta t)$, $\theta_1 = \theta(t)$. It is easy to verify that the following equality

$$- \int_{\mathcal{B}} \{ \hat{\mathbf{T}}(\mathbf{E},\theta_2) - \hat{\mathbf{T}}(\mathbf{E},\theta_1)\} \cdot \nabla \mathbf{w} \, dV =$$

$$= \int_{\mathcal{B}} \{\mathbb{C}(\theta_2)[\Delta \mathbf{E}^a] + 3\chi(\theta_2)\Delta\beta \, \mathbf{I} - \Delta\mathbb{C}[\mathbf{E} - \mathbf{E}_1^a] + 3\Delta\chi \, \beta(\theta_1)\mathbf{I} \} \cdot \nabla \mathbf{w} \, dV \qquad (2.26)$$

holds, where we have

$$\mathbb{C}(\theta) = \frac{E(\theta)}{1+\nu(\theta)} \{\mathbb{I} + \frac{\nu(\theta)}{1-2\nu(\theta)} \mathbf{I} \otimes \mathbf{I}\},$$

$$\Delta\mathbb{C} = \mathbb{C}(\theta_2) - \mathbb{C}(\theta_1),$$

$$\Delta\beta = \beta(\theta_2) - \beta(\theta_1),$$

$$\Delta\chi = \chi(\theta_2) - \chi(\theta_1),$$

$$3\chi = \frac{E}{1-2\nu},$$

$$\Delta \mathbf{E}^a = \mathbf{E}_2^a - \mathbf{E}_1^a,$$

and $\mathbf{E}_1^a$ and $\mathbf{E}_2^a$ are the solutions to the constitutive equation corresponding to ($\mathbf{E}$, θ_1) and ($\mathbf{E}$, θ_2), respectively.

We then solve the linear system

$$[K(a)]\{\Delta a\} = \{\Delta f\} \qquad (2.27)$$

and follow the iterative procedure described in [7].

In order to determine the matrix $[K(a)]$ of system (2.27) while accounting for (2.23), the derivative of the stress with respect to the strain must be calculated in the four regions $\mathcal{R}_i$.

For each $\mathbf{E} \in$ Sym with eigenvalues e_1, e_2, e_3 and eigenvectors $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$, let us consider the orthonormal basis of Sym

$$\mathbf{O}_1 = \mathbf{q}_1 \otimes \mathbf{q}_1, \quad \mathbf{O}_2 = \mathbf{q}_2 \otimes \mathbf{q}_2, \quad \mathbf{O}_3 = \mathbf{q}_3 \otimes \mathbf{q}_3,$$

$$\mathbf{O}_4 = \frac{1}{\sqrt{2}} (\mathbf{q}_1 \otimes \mathbf{q}_2 + \mathbf{q}_2 \otimes \mathbf{q}_1), \; \mathbf{O}_5 = \frac{1}{\sqrt{2}} (\mathbf{q}_1 \otimes \mathbf{q}_3 + \mathbf{q}_3 \otimes \mathbf{q}_1), \qquad (2.28)$$

$$\mathbf{O}_6 = \frac{1}{\sqrt{2}} (\mathbf{q}_2 \otimes \mathbf{q}_3 + \mathbf{q}_3 \otimes \mathbf{q}_2).$$

The fourth-order tensor $D_E\hat{\mathbf{T}}(\mathbf{E},\theta)$ in the four regions $\mathcal{R}_i$ can be calculated by accounting for (2.11)-(2.14) and recalling that [5]

$$D_E e_1 = \mathbf{O}_1 ,$$

$$D_E e_2 = \mathbf{O}_2 ,$$

$$D_E e_3 = \mathbf{O}_3 ;$$

$$D_E \mathbf{O}_1 = \frac{1}{e_1 - e_2}\, \mathbf{O}_4 \otimes \mathbf{O}_4 + \frac{1}{e_1 - e_3}\, \mathbf{O}_5 \otimes \mathbf{O}_5 ,$$

$$D_E \mathbf{O}_2 = \frac{1}{e_2 - e_1}\, \mathbf{O}_4 \otimes \mathbf{O}_4 + \frac{1}{e_2 - e_3}\, \mathbf{O}_6 \otimes \mathbf{O}_6 ,$$

$$D_E \mathbf{O}_3 = \frac{1}{e_3 - e_1}\, \mathbf{O}_5 \otimes \mathbf{O}_5 + \frac{1}{e_3 - e_2}\, \mathbf{O}_6 \otimes \mathbf{O}_6 ,$$

where the operator $\mathbf{A} \otimes \mathbf{B}$, with $\mathbf{A}, \mathbf{B} \in$ Lin is the fourth-order tensor defined by $\mathbf{A} \otimes \mathbf{B}[\mathbf{H}] = (\mathbf{B} \bullet \mathbf{H})\, \mathbf{A}$, for every $\mathbf{H} \in$ Lin.
For the derivative of $\hat{\mathbf{T}}$ with respect to $\mathbf{E}$ we have

if $(\mathbf{E}, \theta) \in \mathcal{R}_1$, then

$$D_E \hat{\mathbf{T}}(\mathbf{E},\theta) = \frac{E(\theta)}{1+\nu(\theta)} \left\{ \mathbb{I} + \frac{\nu(\theta)}{1-2\nu(\theta)}\, \mathbf{I} \otimes \mathbf{I} \right\} , \tag{2.29}$$

if $(\mathbf{E}, \theta) \in \mathcal{R}_2$, then

$$D_E \hat{\mathbf{T}}(\mathbf{E},\theta) = \mathbb{O} , \tag{2.30}$$

if $(\mathbf{E}, \theta) \in \mathcal{R}_3$, then

$$D_E \hat{\mathbf{T}}(\mathbf{E},\theta) = E(\theta) \Big\{ \mathbf{O}_1 \otimes \mathbf{O}_1 - \frac{e_1 - \beta(\theta)}{e_2 - e_1}\, \mathbf{O}_4 \otimes \mathbf{O}_4 - \frac{e_1 - \beta(\theta)}{e_3 - e_1}\, \mathbf{O}_5 \otimes \mathbf{O}_5 \Big\} , \tag{2.31}$$

if $(\mathbf{E}, \theta) \in \mathcal{R}_4$, then

$$D_E\hat{\mathbf{T}}(\mathbf{E},\theta) = \frac{E(\theta)}{2(1+\nu(\theta))}\{(\mathbf{O}_1 - \mathbf{O}_2)\otimes(\mathbf{O}_1 - \mathbf{O}_2)$$

$$+\frac{1+\nu(\theta)}{1-\nu(\theta)}\{(\mathbf{O}_1 + \mathbf{O}_2)\otimes(\mathbf{O}_1 + \mathbf{O}_2) + 2\,\mathbf{O}_4\otimes\mathbf{O}_4$$

$$-2\,\frac{e_1+\nu(\theta)e_2-\beta(\theta)(1+\nu(\theta))}{(1-\nu(\theta))(e_3-e_1)}\,\mathbf{O}_5\otimes\mathbf{O}_5$$

$$-2\,\frac{e_2+\nu(\theta)e_1-\beta(\theta)(1+\nu(\theta))}{(1-\nu(\theta))(e_3-e_2)}\,\mathbf{O}_6\otimes\mathbf{O}_6\}\,, \tag{2.32}$$

where $\mathbb{I}$ and $\mathbb{O}$ are the fourth-order identity tensor and the fourth-order null tensor, respectively. Since in $\mathcal{R}_3$ and $\mathcal{R}_4$ we have $e_1 < e_2$ and $e_2 < e_3$, respectively, the derivatives given in (2.31) and (2.32) are well-defined.

3. Numerical examples

Three different examples are dealt with in this section. In the first, the explicit solution of an equilibrium problem is determined, thus highlighting the difference between the thermo-mechanical behaviour of a no-tension material and a linear elastic one having the same elastic constants and thermal expansion. Moreover, the solution is a test case for validating the numerical method proposed and its implementation in the finite element program, NOSA.

Subsequently, we analyse a masonry arch subjected to its own weight and a temperature distribution representing the mean seasonal thermal variation, and finally, we consider a converter used in the steel and iron industry, subjected to its own weight and a highly non-uniform temperature distribution. The two structures, made of no-tension materials having constitutive equation (2.1)-(2.5), have been discretised into finite elements and analysed with the NOSA code. For the arch, the stress fields and corresponding lines of thrust have been determined both for when it is subjected to its own weight alone and when subjected to this weight as well as a temperature distribution. For the converter, on the other hand, in addition to determining the stress field, we also characterise any fracturing that would occur. In Section 2 it was stated that if the inelastic part $\mathbf{E}^a$ of strain is non-null in any region of the structure, then we can expect fractures to be present in that region. Nevertheless, a simple analysis of the components of $\mathbf{E}^a$ does not generally yield any information about the direction of eventual fractures. To this end, we point out that if for any $\mathbf{v}$, $\mathbf{v}\cdot\mathbf{E}^a\mathbf{v} > 0$, $\mathbf{v}$ is not necessarily a fracture direction; in other words, $\mathbf{v}$ is not necessarily an eigenvector of $\mathbf{T}$ corresponding to the zero eigenvalue. Nonetheless, there must surely exist at least one eigenvector $\mathbf{q}$ of $\mathbf{E}^a$ (in view of the coaxiality of $\mathbf{E}^a$ and $\mathbf{T}$, $\mathbf{q}$ is also an eigenvector of $\mathbf{T}$) such that $\mathbf{q}\cdot\mathbf{E}^a\mathbf{q} > 0$ and then, by virtue of the orthogonality of $\mathbf{E}^a$ and $\mathbf{T}$, $\mathbf{q}\cdot\mathbf{T}\mathbf{q} = 0$. By such reasoning, we deduce that if $\mathcal{F}$ is a fracture surface, then every vector orthogonal to $\mathcal{F}$ is an eigenvector of $\mathbf{T}$ corresponding to the eigenvalue 0. This criterion has been used in the last example in order to reveal the regions were fractures are present and their corresponding directions.

3.1 Spherical container subjected to two uniform radial pressures and a steady temperature distribution.

A spherical container with inner radius a and outer radius b is subjected to two uniform radial pressures p_1 and p_2 acting on the internal and external boundary, respectively, and to a steady temperature distribution θ varying with the radius r

$$\theta(r) = \frac{ab(\vartheta_1 - \vartheta_2)}{b-a}\,\frac{1}{r} + \frac{b\vartheta_2 - a\vartheta_1}{b-a} + \theta_0 \,. \tag{3.1}$$

In (3.1) ϑ_1 and ϑ_2, with $\vartheta_1 > \vartheta_2$ are defined by $\vartheta_1 = \theta_1 - \theta_0$ and $\vartheta_2 = \theta_2 - \theta_0$, with $\theta_1 = \theta(a)$ and $\theta_2 = \theta(b)$, respectively. For the thermal expansion β, we assume that

$$\beta(\theta) = \alpha(\theta - \theta_0)\,, \tag{3.2}$$

with constant α; Poisson's ratio is taken equal to zero and finally, Young's modulus is a decreasing function of temperature

$$E(\theta) = \omega\,\frac{\dfrac{ab(\vartheta_1 - \vartheta_2)}{b-a}}{\theta - \dfrac{b\vartheta_2 - a\vartheta_1}{b-a} - \theta_0}\,, \tag{3.3}$$

with $\omega = \dfrac{E_1}{a}$ a positive constant. In view of (3.1) we have

$$\hat{E}(r) = E(\theta(r)) = \omega\, r\,, \tag{3.4}$$

in particular $\hat{E}(a) = E_1$, $\hat{E}(b) = \dfrac{b}{a}E_1 > E_1$.

For the moment, let us suppose that the spherical container is made of a linear elastic material. We denote by $\sigma_r^{(l)}$ and $\sigma_t^{(l)}$, the radial and tangent stress, respectively. By using a procedure similar to that described in [8] we get

$$\sigma_r^{(l)}(r) = \omega\left(\mu_1 C_1 r^{\mu_1} + \mu_2 C_2 r^{\mu_2} - \alpha\frac{ab(\vartheta_1 - \vartheta_2)}{b-a}\right), \tag{3.5}$$

$$\sigma_t^{(l)}(r) = \omega\left(C_1 r^{\mu_1} + C_2 r^{\mu_2} - \alpha\frac{ab(\vartheta_1 - \vartheta_2)}{b-a}\right), \tag{3.6}$$

where $\mu_1 = -(1+\sqrt{3})$, $\mu_2 = -1+\sqrt{3}$ and (C_1, C_2) is the unique solution of the linear system

$$\begin{pmatrix} \mu_1 a^{\mu_1 - 1} & \mu_2 a^{\mu_2 - 1} \\ \mu_1 b^{\mu_1 - 1} & \mu_2 b^{\mu_2 - 1} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} -\dfrac{p_1}{\omega\, a} + \dfrac{\alpha\, b(\vartheta_1 - \vartheta_2)}{b-a} \\ -\dfrac{p_2}{\omega\, b} + \dfrac{\alpha\, a(\vartheta_1 - \vartheta_2)}{b-a} \end{pmatrix}. \tag{3.7}$$

The radial displacement is

$$u^{(l)}(r) = C_1 r^{\mu_1} + C_2 r^{\mu_2} + \alpha\frac{b\vartheta_2 - a\vartheta_1}{b-a} r\ . \tag{3.8}$$

It can be proved that for suitable values of parameters p_1, p_2, ϑ_1 and ϑ_2, the stress field with components (3.5) and (3.6) is purely compressive, while, on the contrary, there exist values of p_1, p_2, ϑ_1 and ϑ_2 such that the radial stress is still negative, whereas the tangent stress is negative for $a \le r \le \bar{r}$ with $\bar{r}$ belonging to (a, b), and then becomes positive. Therefore, the stress field in (3.5)-(3.6) is not the solution to the equilibrium problem of a spherical container made of a material with constitutive equation (2.1)-(2.5). In this case, it is possible to prove that the spherical container is compressed for $a \le r \le r_0$, and the region $r_0 \le r \le b$ is characterised by a null tangent stress and is therefore cracked. The transition radius r_0 is the only solution belonging to the interval [a, b] to the non-linear equation

$$C_1(r)\, r^{\mu_1} + C_2(r)\, r^{\mu_2} - \alpha\, \frac{ab(\vartheta_1 - \vartheta_2)}{b-a} = 0\,, \tag{3.9}$$

and $(C_1(r),\ C_2(r))$ is given by

$$\begin{pmatrix} C_1(r) \\ C_2(r) \end{pmatrix} = \begin{pmatrix} \mu_1 a^{\mu_1 - 1} & \mu_2 a^{\mu_2 - 1} \\ \mu_1 r^{\mu_1 - 1} & \mu_2 r^{\mu_2 - 1} \end{pmatrix}^{-1} \begin{pmatrix} -\dfrac{p_1}{\omega\, a} + \dfrac{\alpha\, b(\vartheta_1 - \vartheta_2)}{b-a} \\ -\dfrac{p_2 b^2}{\omega\, r^3} + \dfrac{\alpha\, ab(\vartheta_1 - \vartheta_2)}{(b-a)r} \end{pmatrix}. \tag{3.10}$$

By setting

$$p_0 = \frac{b^2}{r_0^2} p_2\ , \tag{3.11}$$

through the same procedure used in [9], we arrive at the stress field and radial displacement for the container made of a no-tension material with constitutive equation (2.1)-(2.5)

$$\sigma_r^{(m)}(r) = \begin{cases} \mu_1 C_1(r_0)\, r^{\mu_1} + \mu_2 C_2(r_0)\, r^{\mu_2} - \alpha \dfrac{ab(\vartheta_1 - \vartheta_2)}{b-a}\,, & r \in [a,\, r_0]\,, \\[2ex] -\, p_2\, \dfrac{b^2}{r^2}\,, & r \in [r_0\,, b]\,; \end{cases} \tag{3.12}$$

$$\sigma_t^{(m)}(r) = \begin{cases} C_1(r_0)\, r^{\mu_1} + C_2(r_0)\, r^{\mu_2} - \alpha \dfrac{ab(\vartheta_1 - \vartheta_2)}{b-a}\,, & r \in [a,\, r_0]\,, \\[2ex] 0\,, & r \in [r_0\,, b]\,; \end{cases} \tag{3.13}$$

$$u^{(m)}(r) = \begin{cases} C_1(r_0)\, r^{\mu_1} + C_2(r_0)\, r^{\mu_2} + \alpha \dfrac{b\vartheta_2 - a\vartheta_1}{b-a}\, r\,, & r \in [a,\, r_0]\,, \\[2ex] u_0 + \dfrac{p_2\, b^2}{2\omega\, r^2} + \alpha \left(\dfrac{ab(\vartheta_1 - \vartheta_2)}{b-a} \ln r + \dfrac{b\vartheta_2 - a\vartheta_1}{b - a}\, r \right), & r \in [r_0\,, b]\,, \end{cases} \tag{3.14}$$

where the constant u_0 is determined by imposing the continuity of the radial displacement at r_0,

$$u_0 = C_1(r_0)\, r_0^{\mu_1} + C_2(r_0)\, r_0^{\mu_2} + \alpha \frac{b\vartheta_2 - a\vartheta_1}{b-a}\, r_0$$

$$- \frac{p_2\, b^2}{2\omega\, r_0^2} - \alpha \left(\frac{ab(\vartheta_1 - \vartheta_2)}{b-a} \ln r_0 + \frac{b\vartheta_2 - a\vartheta_1}{b - a}\, r_0 \right). \tag{3.15}$$

The radial inelastic strain is equal to zero, and the tangent one is

$$\varepsilon_t^{(m)}(r) = \begin{cases} 0\,, & r \in [a,\, r_0]\,, \\[2ex] \dfrac{u^{(m)}(r)}{r} - \alpha \left(\dfrac{ab(\vartheta_1 - \vartheta_2)}{(b-a)r} + \dfrac{b\vartheta_2 - a\vartheta_1}{b - \mathrm{a}} \right), & r \in [r_0\,, b]\,. \end{cases} \tag{3.16}$$

Figures 3.1, 3.2 and 3.3 show the behaviour of the radial and tangent stresses and the radial displacement for a spherical container made of a linear elastic material (grey line) and a no-tension elastic material (black line). The graphs have been obtained from expressions (3.5), (3.12), (3.6), (3.13) and (3.8), (3.14), using the following parameter values

$$a = 1 \text{ m}, \quad b = 2 \text{ m},$$
$$\nu = 0, \quad E_1 = 2.5\ 10^9 \text{ Pa},$$
$$\alpha = 1.\ 10^{-5}\ (^\circ\text{C})^{-1},$$
$$p_1 = 1.\ 10^6 \text{ Pa}, \ p_2 = 0.4\ 10^6 \text{ Pa},$$
$$\theta_0 = 30.\ ^\circ\text{C}, \quad \vartheta_1 = 400.\ ^\circ\text{C}, \quad \vartheta_2 = -10.\ ^\circ\text{C}.$$

With the aim of comparing the explicit solution with the numerical one calculated by means of the finite element program NOSA, we have considered a half spherical container discretised with 800 axisymmetric elements with 8 nodes and 9 Gauss points. Figures 3.4, 3.5, 3.6 and 3.7 show the values of the stress components, radial displacement and tangent inelastic strain calculated by using expressions (3.12), (3.13), (3.14) and (3.16) (continuous line) and those furnished by NOSA code (markers).

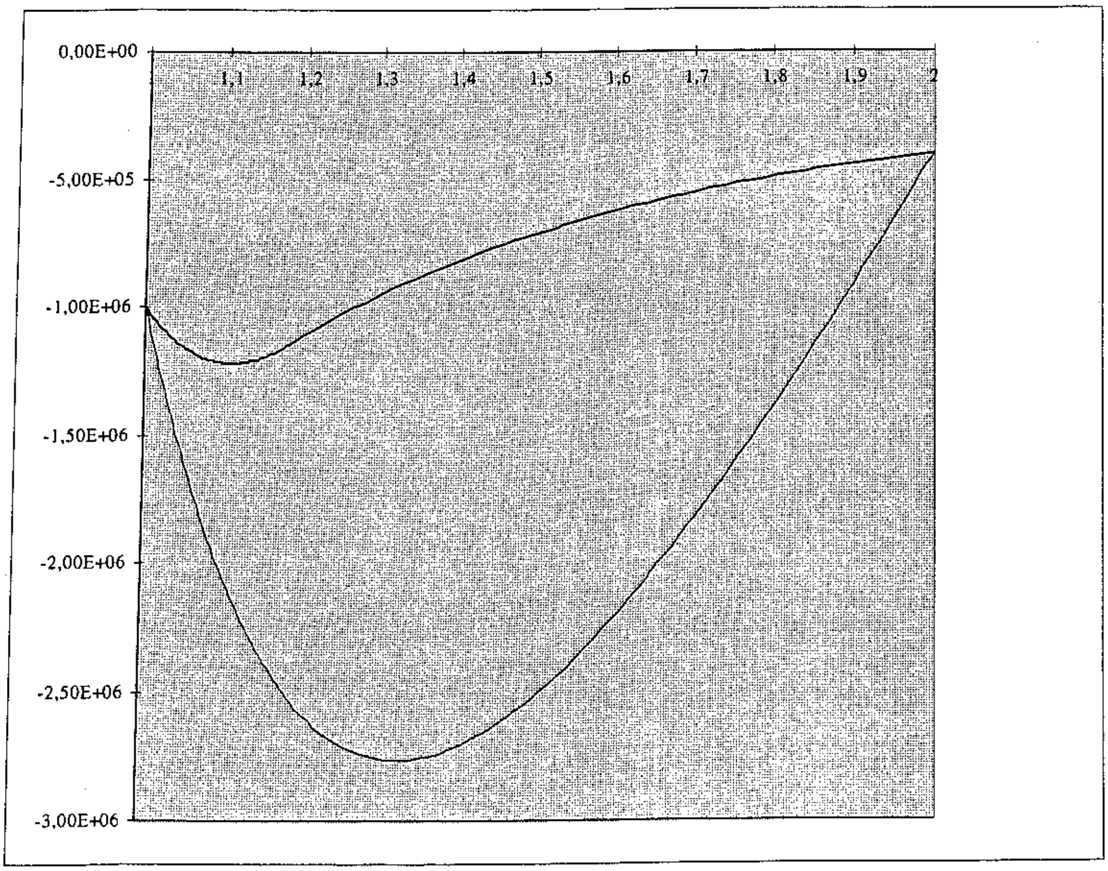

Figure 3.1. $\sigma_r^{(l)}$ (Pa) and $\sigma_r^{(m)}$ (Pa) vs. r (m).

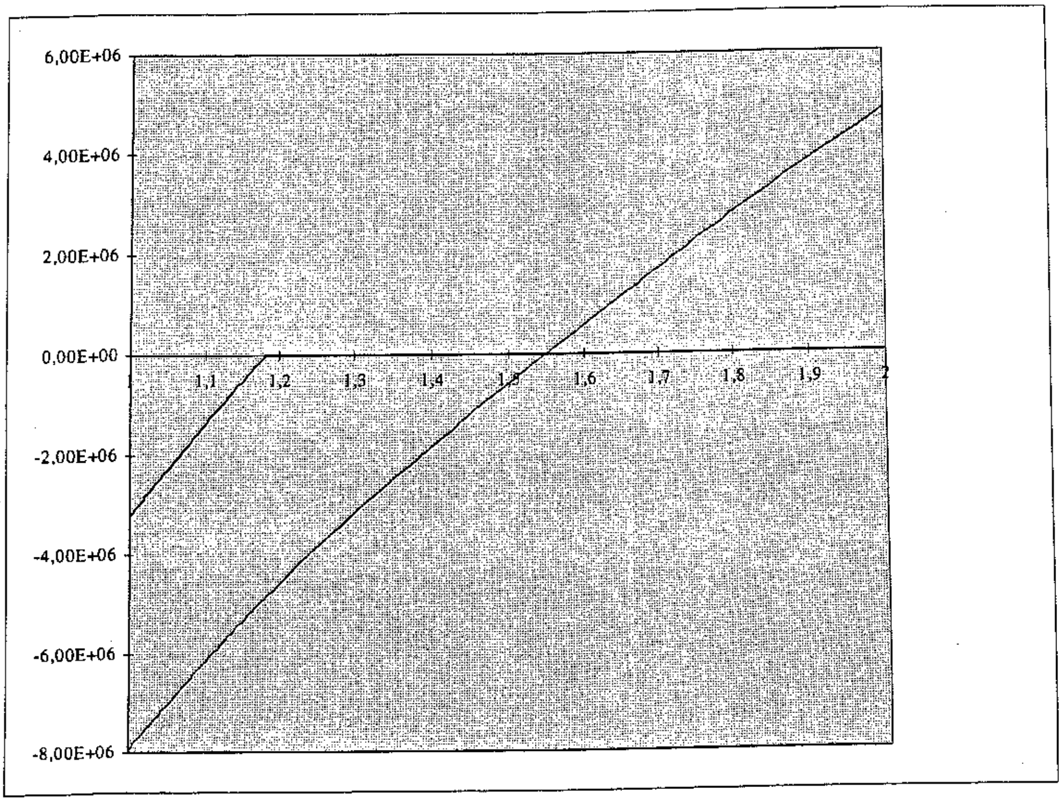

Figure 3.2. $\sigma_t^{(l)}$ (Pa) and $\sigma_t^{(m)}$ (Pa) vs. r (m).

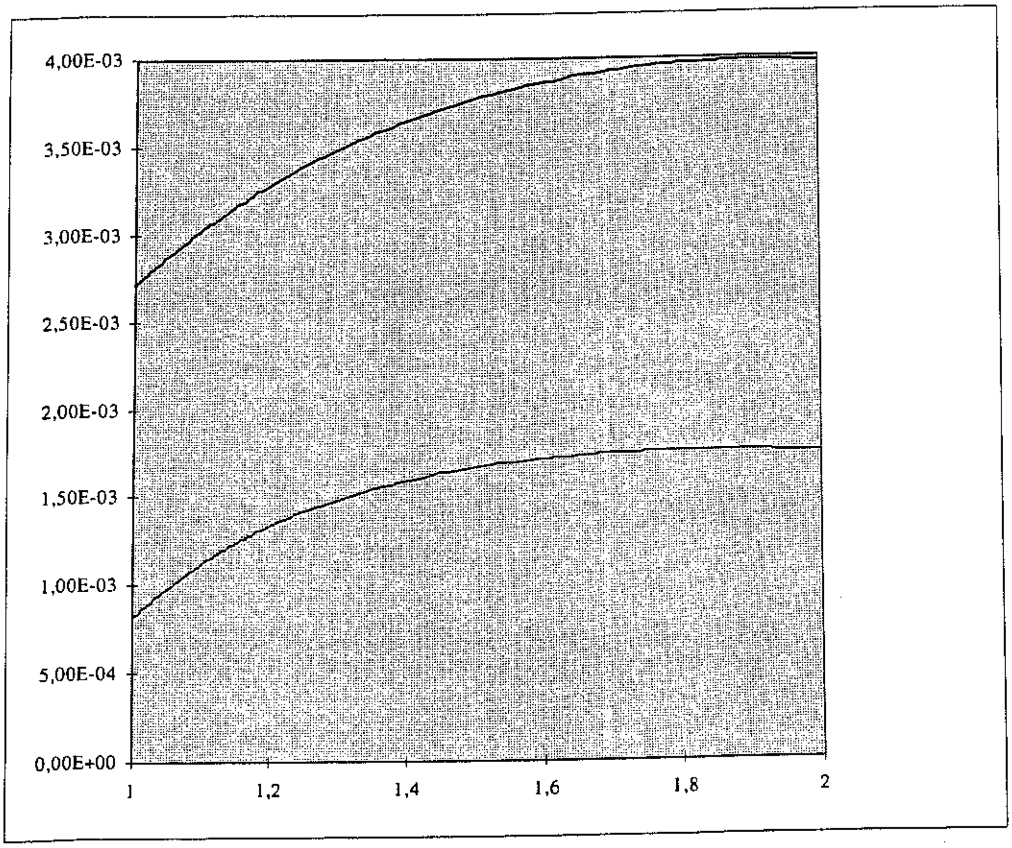

Figure 3.3. $u^{(l)}$ (m) and $u^{(m)}$ (m) vs. r (m).

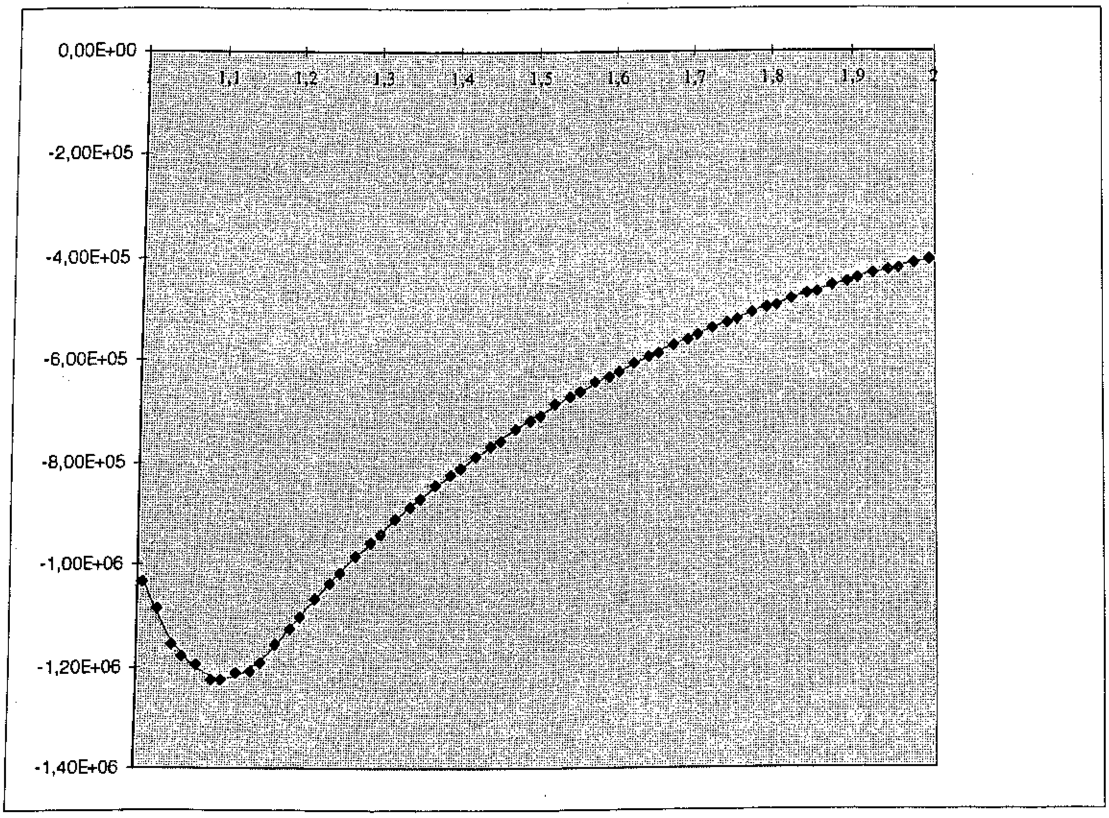

Figure 3.4. Radial stress (Pa) vs. *r* (m), explicit solution and numerical solution.

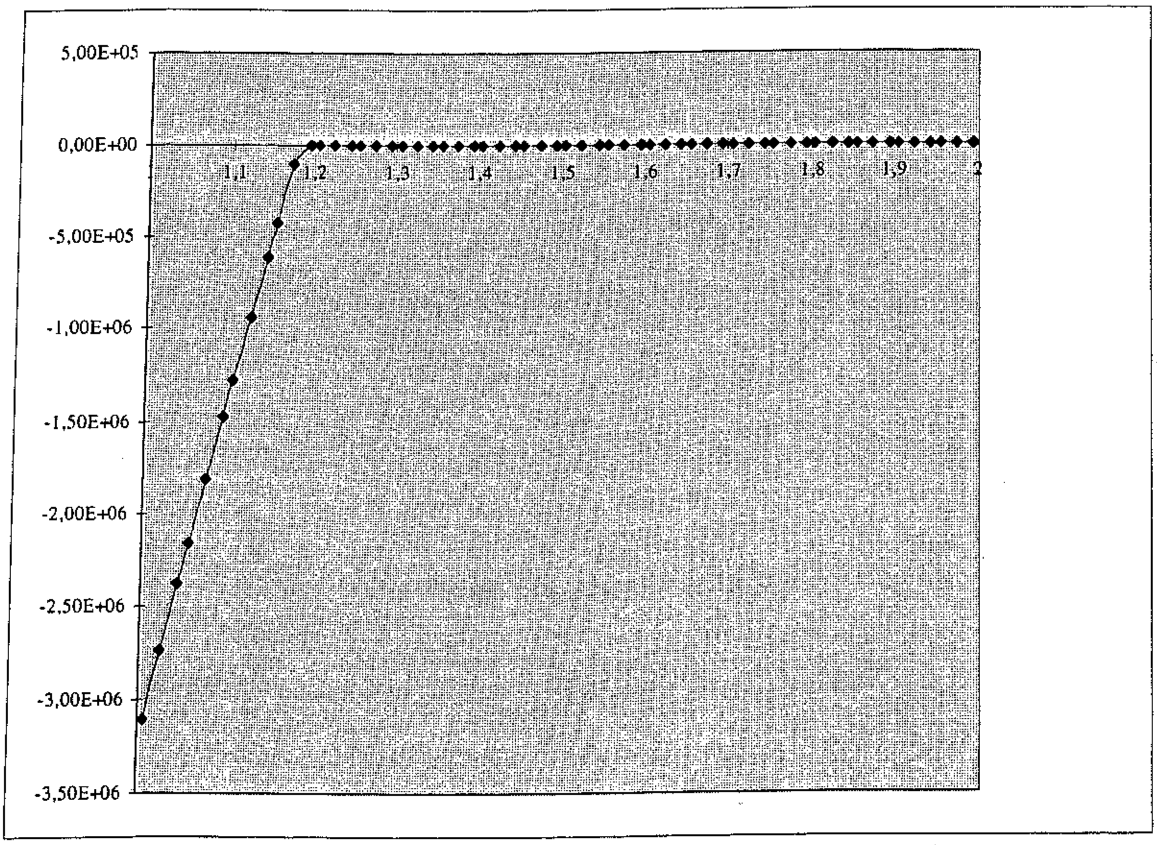

Figure 3.5. Tangent stress (Pa) vs. *r* (m), explicit solution and numerical solution.

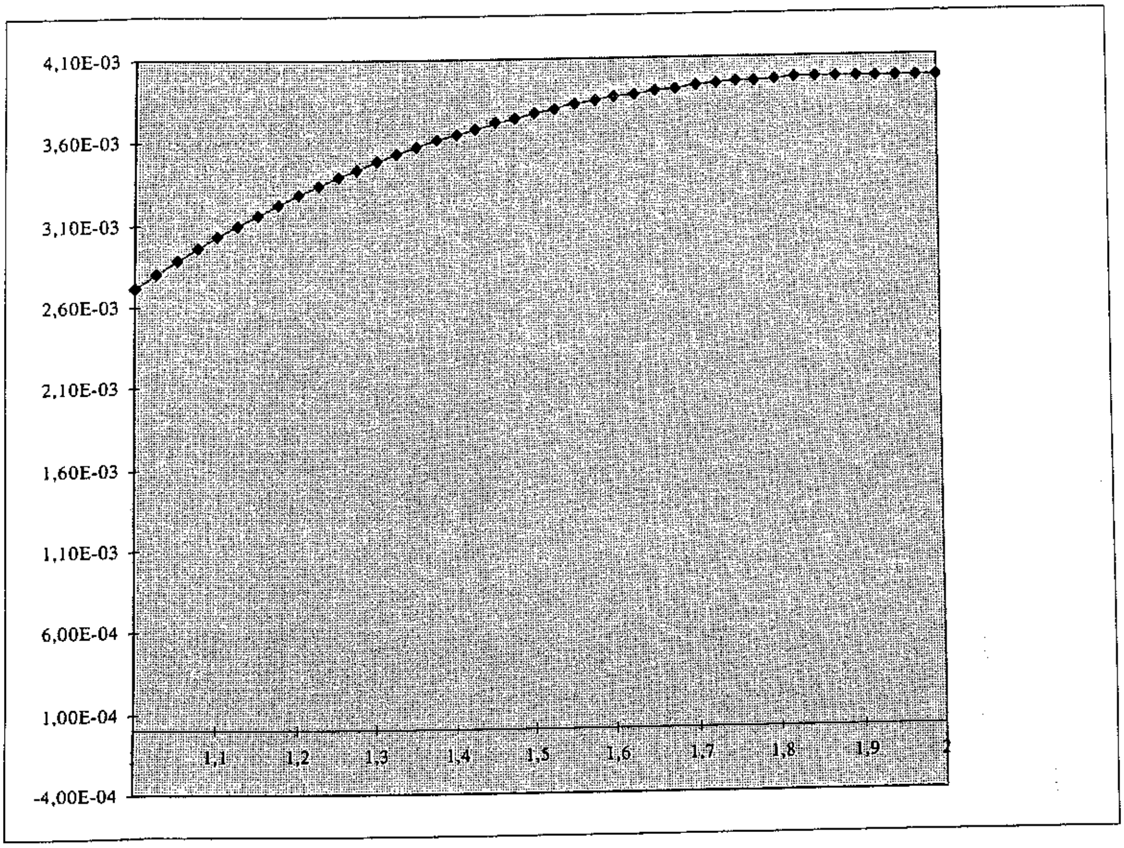

Figure 3.6. Radial displacement (m) vs. *r* (m), explicit solution and numerical solution.

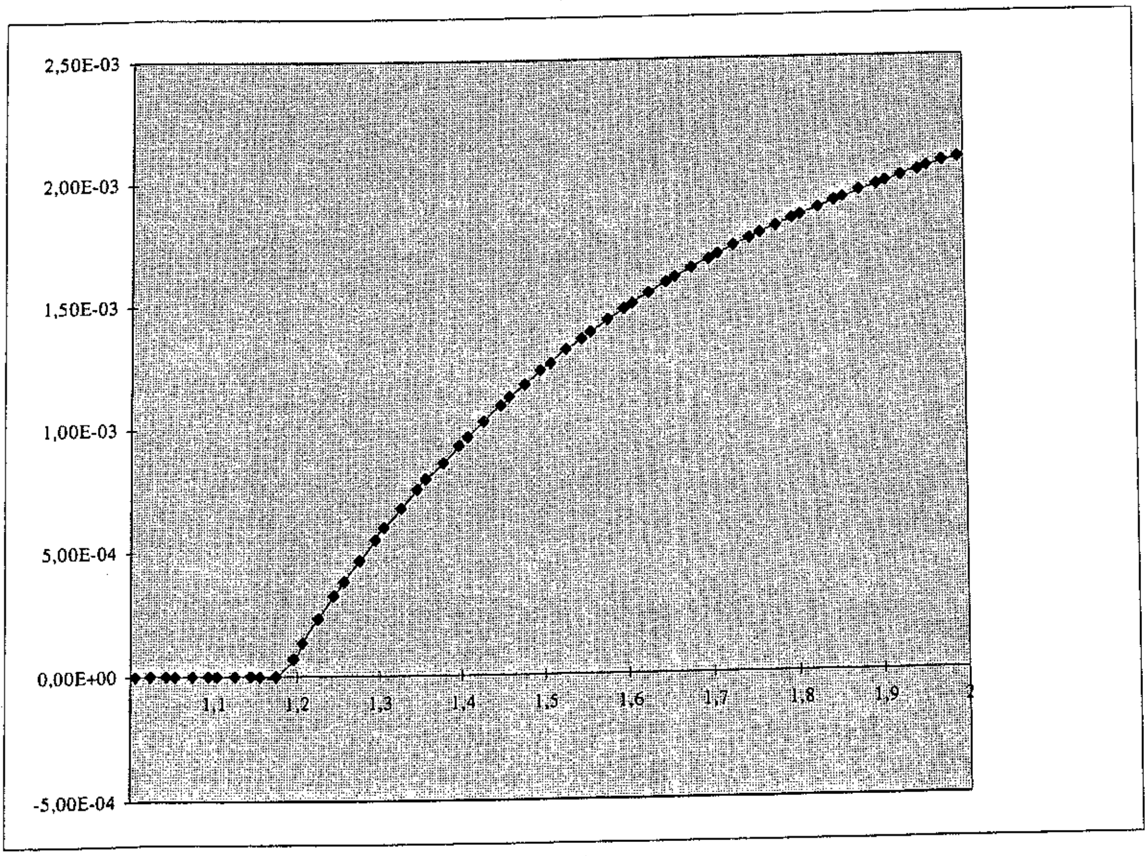

Figure 3.7. Inelastic tangent strain vs. *r* (m), explicit solution and numerical solution.

3.2 Masonry arch subjected to its own weight and a uniform temperature distribution.

A circular masonry arch having mean radius 110 cm and thickness 20 cm has been discretized into 4800 isoparametric plane stress elements. The arch, whose springings are fixed, is subjected to its own weight. The reference temperature is 30° C, and the arch subsequently reaches the temperature of -10° C. The elastic constants, assumed to be independent of temperature, are E = 50000 kg/cm^2, $\nu = 0.1$, while the thermal expansion is $\beta(\theta) = -40\ \alpha$, with $\alpha = 1.\ 10^{-5}$ (° C)$^{-1}$. For the first increment the weight alone has been assigned, and the temperature variation is divided into the four subsequent increments. Figures 3.8 and 3.9 show the lines of thrust for the arch subjected to its weight alone, as well as under the action of both weight and a temperature variation of - 40° C. Figures 3.10 and 3.11 present plots of the normal force and bending moment per unit length vs. the anomaly, for the first (black line) and fifth (grey line) increment.

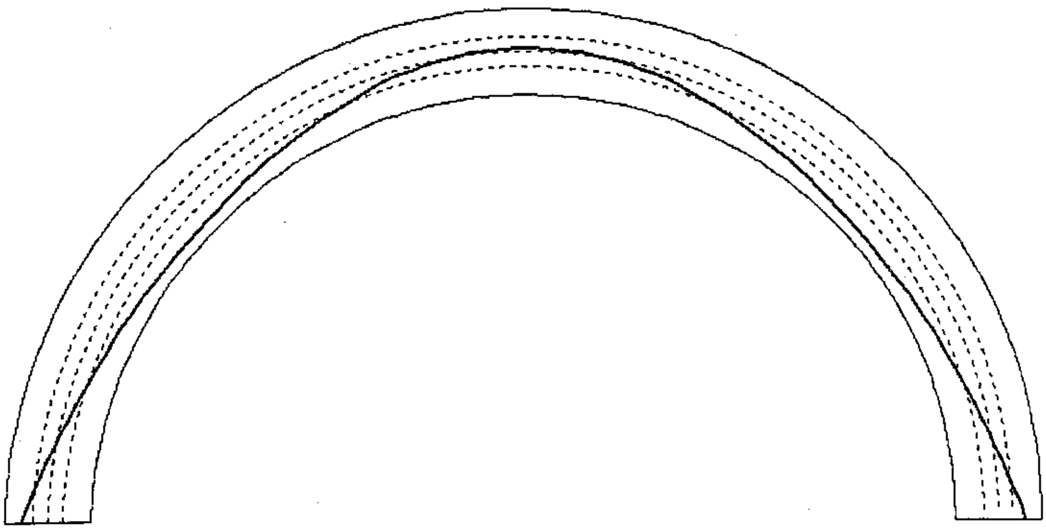

Figure 3.8. Line of thrust of the arch subjected to its own weight.

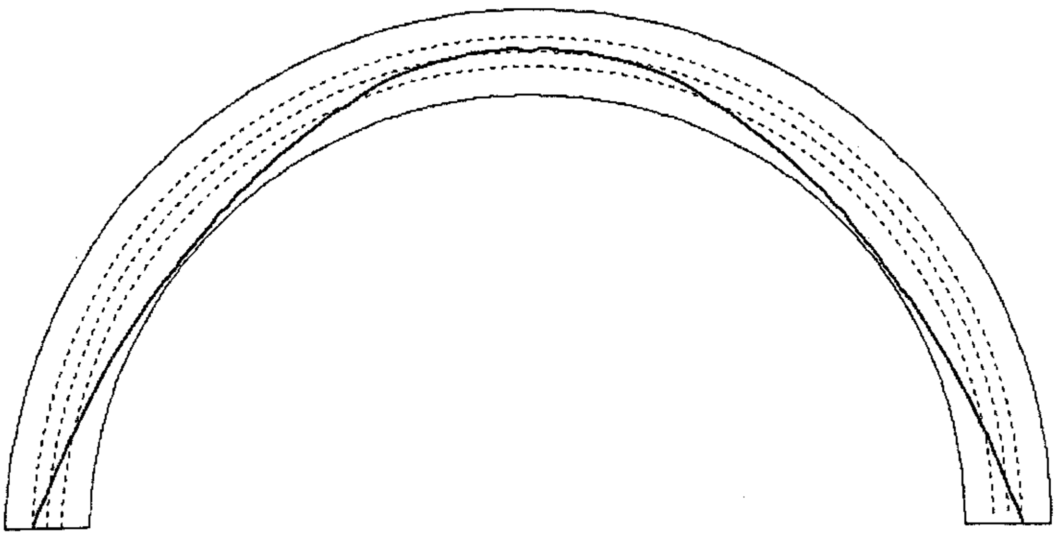

Figure 3.9. Line of thrust of the arch subjected to its own weight and temperature variation of - 40° C.

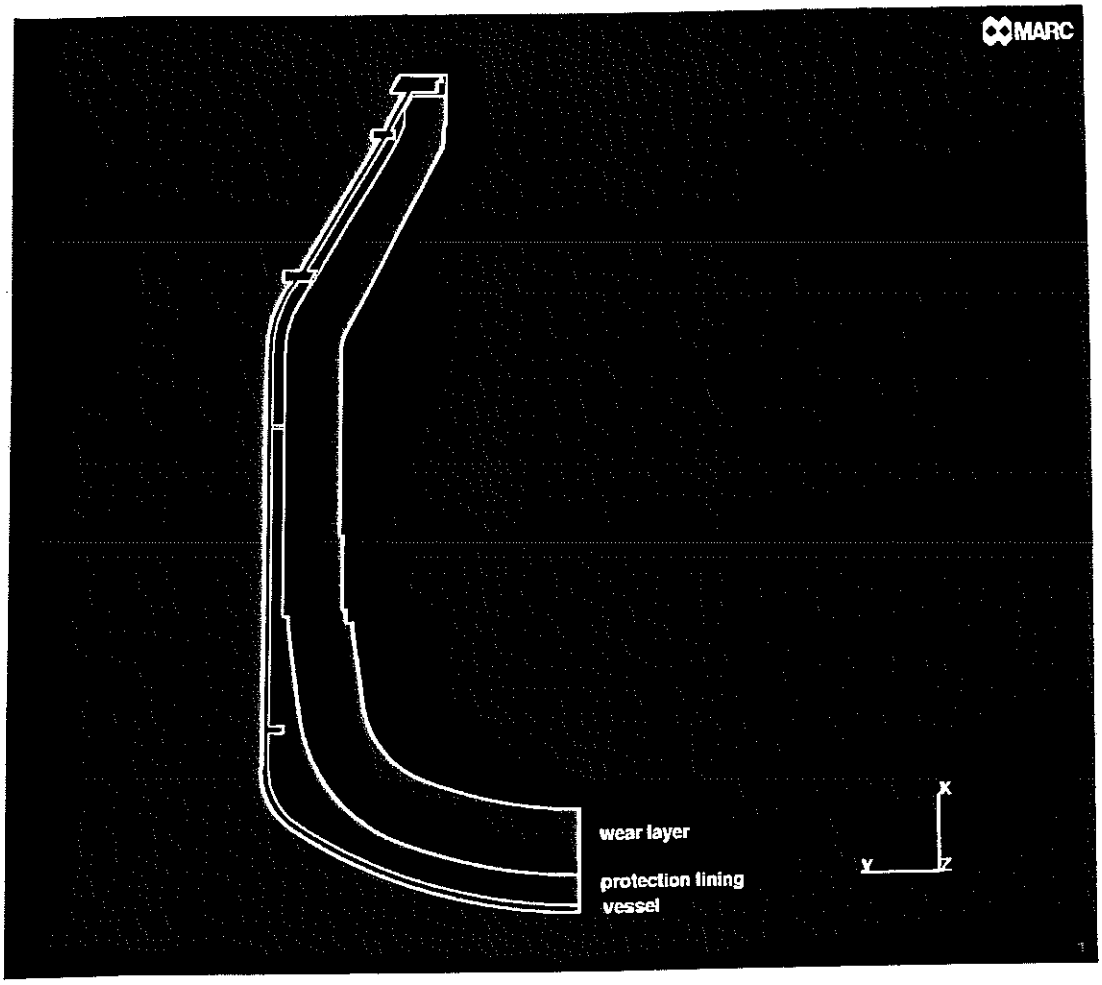

Figure 3.13. Wear layer, protective lining and steel vessel in the converter.

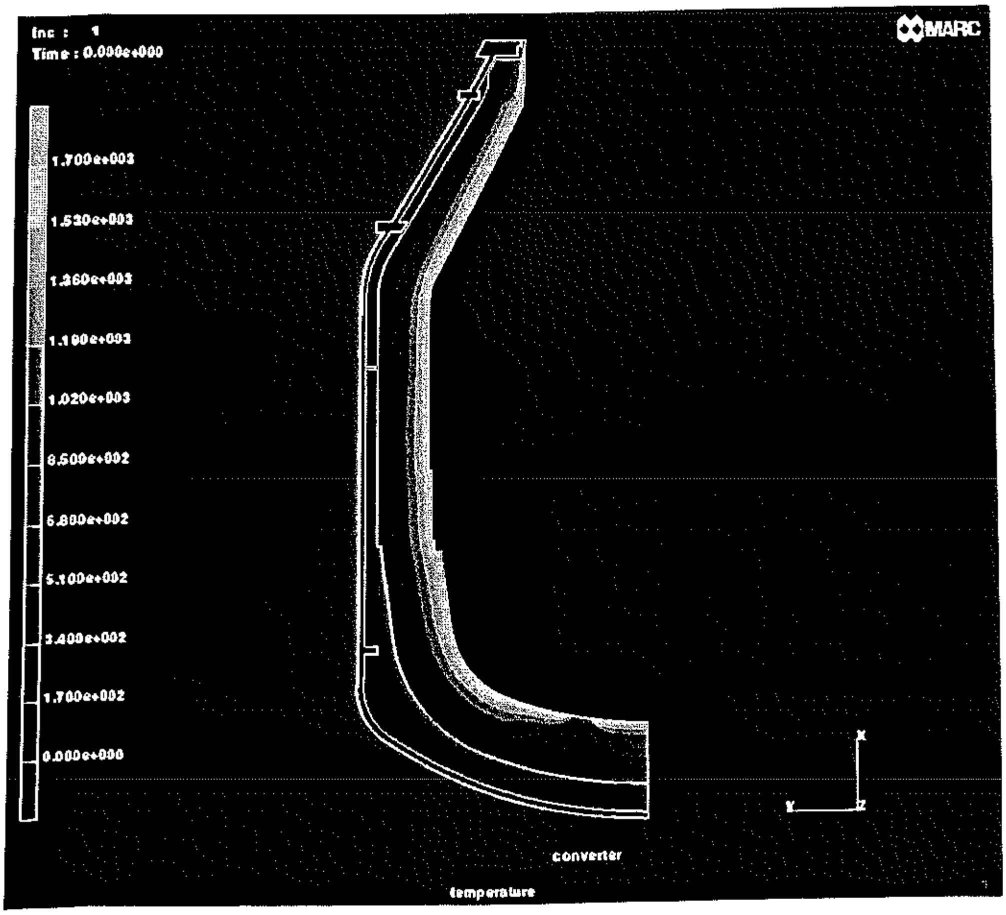

Figure 3.14. Temperature distribution within the converter.

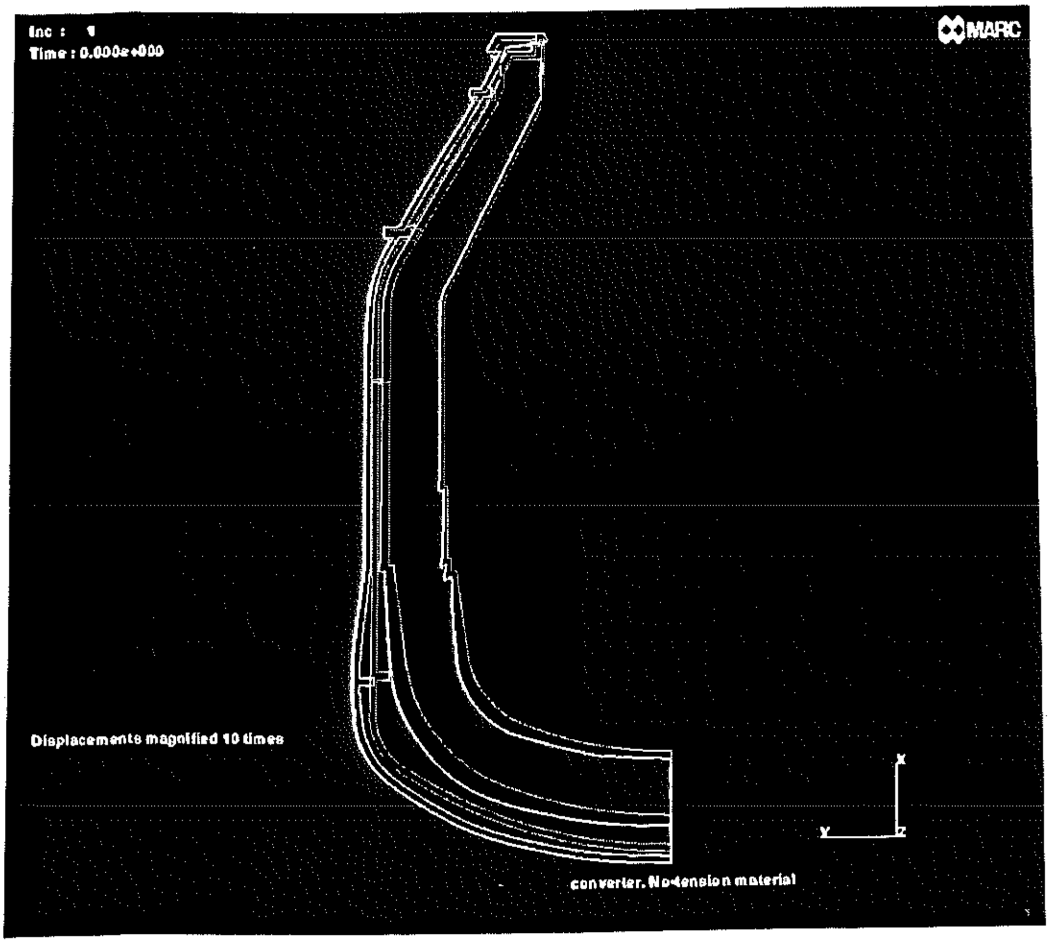

Figure 3.15. Displacements magnified 10 times, non-linear analysis.

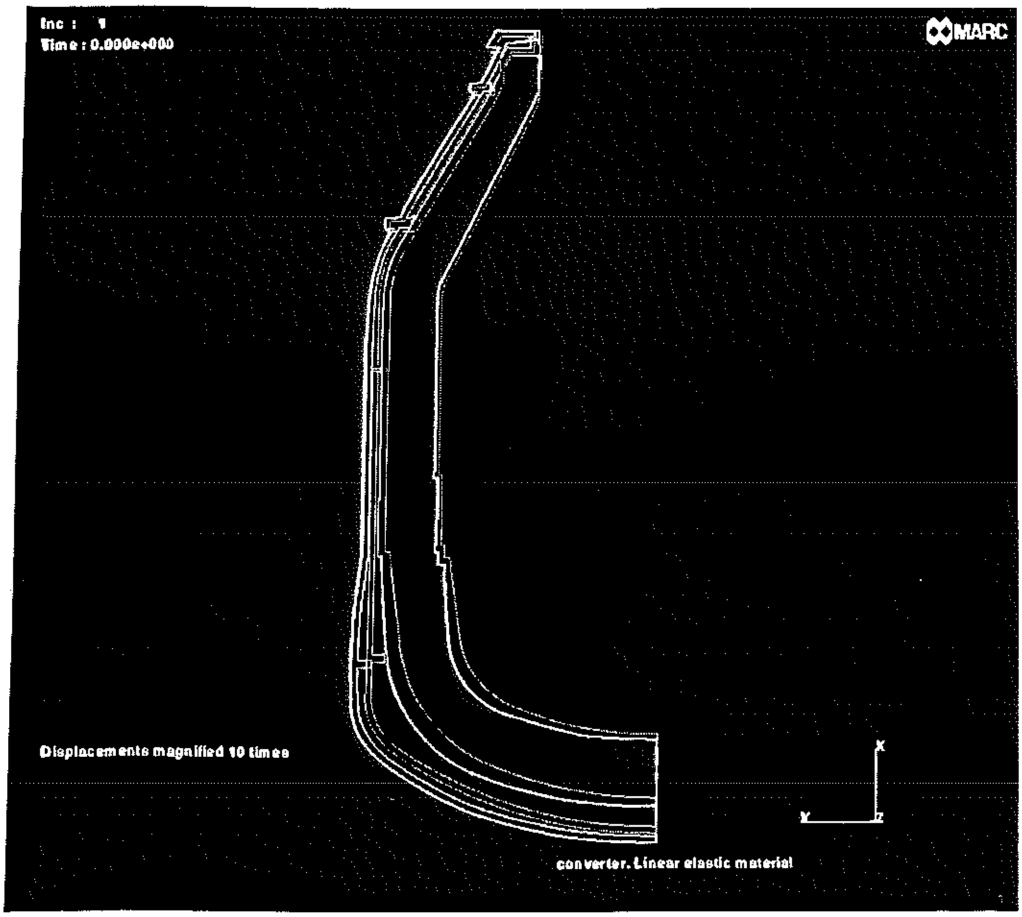

Figure 3.16. Displacements magnified 10 times, linear analysis.

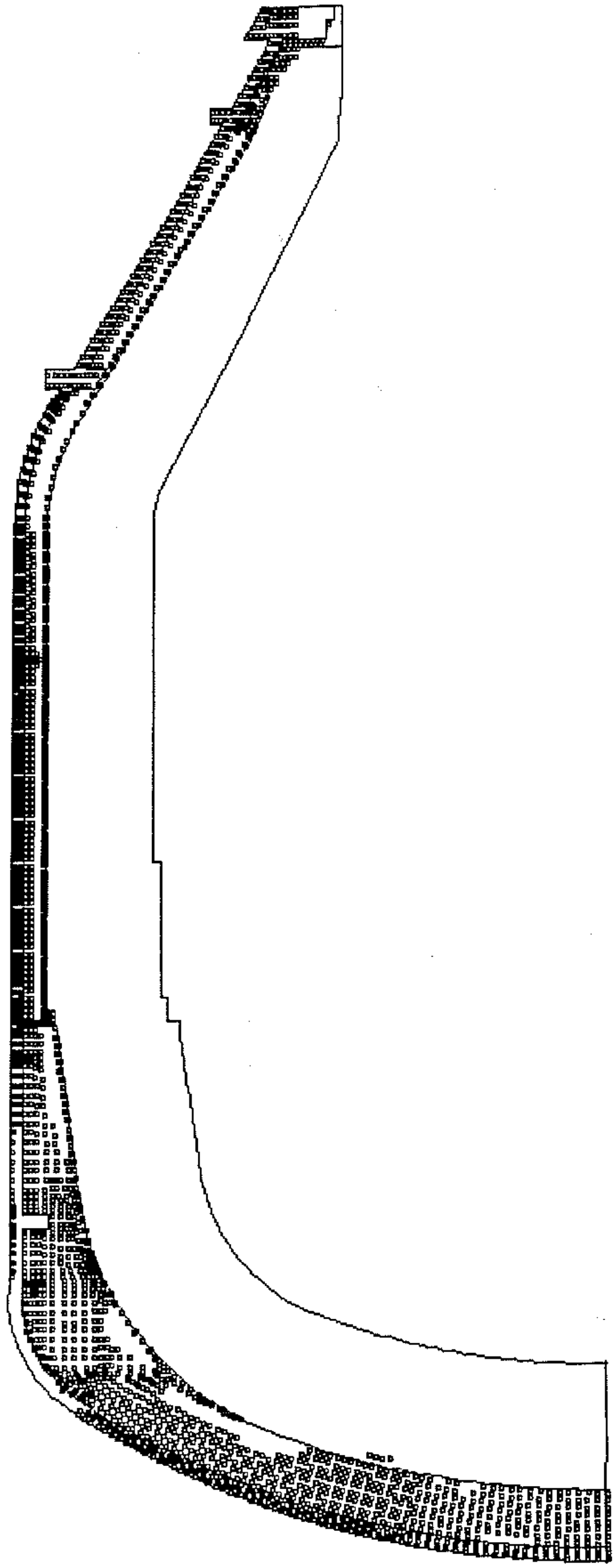

Figure 3.17. Fractures belonging to the meridian planes.

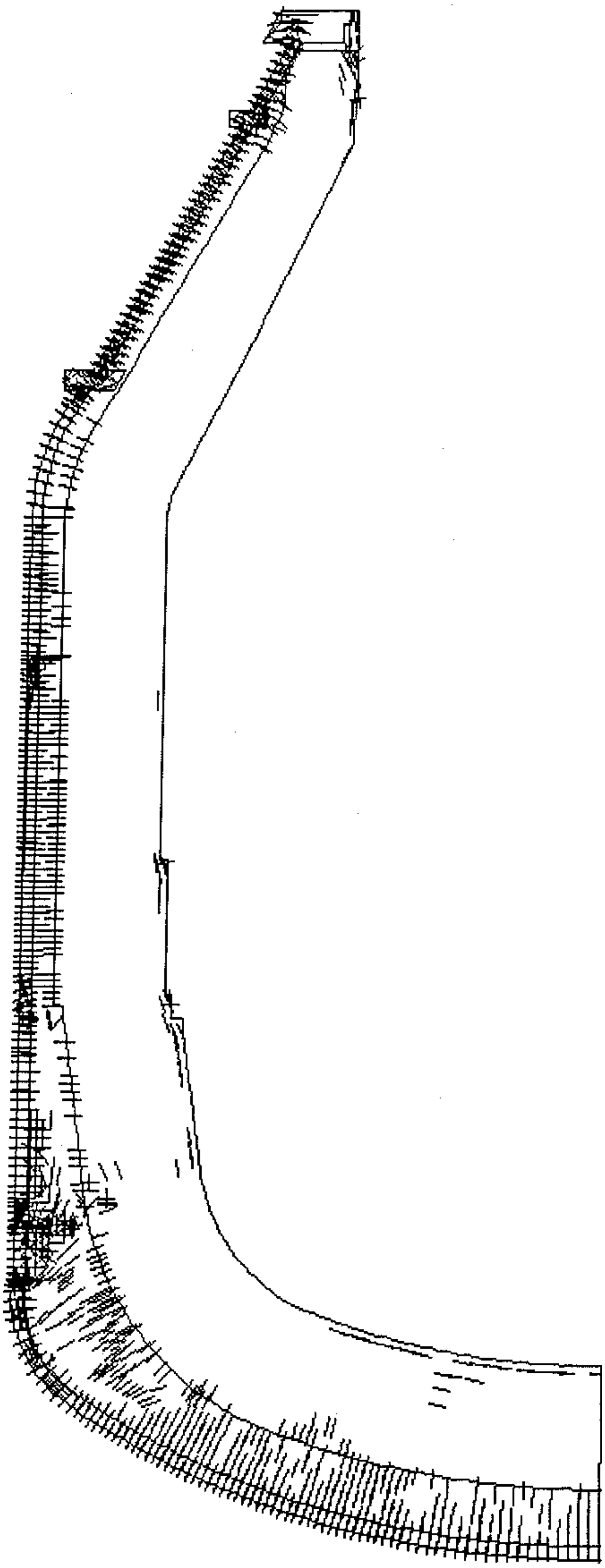

Figure 3.18. Fractures belonging to planes orthogonal to the meridians.

With regard to the stress field, it can be seen that the regions that result to be subjected to tensile stresses in the linear analysis are characterised by zero stress in the non-linear one. Moreover, although the stress fields in the other regions are essentially equal in the two cases, the maximum compressive stresses in the non-linear case are slightly lower. For the sake of comparison, Figures 3.15 and 3.16 show the deformed configuration superimposed upon the initial one for the two analyses. In the case of non-linear behaviour, a substantially greater displacement of the lower part is observed. Analysis aimed at revealing the fracture field was performed according to the criterion described above. Disregarding the filling zones, it can be observed that while fractures are absent along surfaces orthogonal to the direction of the thickness, they are concentrated in the protective layer, along the meridian planes (Figure 3.17) and along planes orthogonal to the meridians (Figure 3.18), particularly in the cylindrical wall zone and around the porous cooling tubes in the lower zone.

4. Conclusions

The constitutive equation and the numerical procedure proposed in this paper can be used in many applications. In particular, the equation of no-tension materials in the presence of thermal expansion is both realistic enough to describe the actual constitutive response of masonry and refractory materials, and simple enough to be employed in many engineering problems.

The third example provided shows that the proposed numerical method permits determination, not only of the stress field, but more importantly, the distribution of cracking within the refractory and its critical zones. The finite element code developed for thermo-mechanical analyses can be a useful tool for industries involved in the production of molten metals and refractories, as well as those concerned with the design of furnace metalwork.

Acknowledgements

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Appendix

An analyse of the plane stress is provided here, though for the sake of brevity, plane strain has been omitted.

Let us suppose $t_3 = \mathbf{q}_3 \cdot \mathbf{T}\mathbf{q}_3 = 0$. From (2.5) we obtain $e_3 - \beta(\theta) - a_3 = \frac{\nu(\theta)}{1-\nu(\theta)}(a_1 + a_2 - e_1 - e_2 + 2\,\beta(\theta))$; moreover, since by virtue of (2.3), a_3 is arbitrary, it can be assumed to be equal to zero. In order to calculate the values of a_1, a_2, t_1 and t_2 we define the following subsets of $\mathrm{Sym} \times [\theta_1, \theta_2]$

$$Q_1 = \{(\mathbf{E}, \theta) \mid \gamma(\theta)(e_1 - \beta(\theta)) + 2(1+\gamma(\theta))\,(e_2 - \beta(\theta)) \leq 0\}\,, \tag{A.1}$$

$$Q_2 = \{(\mathbf{E}, \theta) \mid \gamma(\theta)(e_1 - \beta(\theta)) + 2(1+\gamma(\theta))\,(e_2 - \beta(\theta)) > 0,\ e_1 - \beta(\theta) \leq 0\}\,, \tag{A.2}$$

$$Q_3 = \{(\mathbf{E}, \theta) \mid e_1 - \beta(\theta) > 0\} . \tag{A.3}$$

In this case $\mathbf{E}^a$ and $\mathbf{T}$ have the following expressions

if $(\mathbf{E}, \theta) \in Q_1$, then

$$\begin{aligned} &\mathbf{E}^a = \mathbf{0} , \\ &\mathbf{T} = \frac{E(\theta)}{1+\nu(\theta)} \{\mathbf{E} - \beta(\theta)\,\mathbf{I} + \frac{\nu(\theta)}{1-\nu(\theta)} \operatorname{tr}(\mathbf{E} - \beta(\theta)\,\mathbf{I})\,\mathbf{I}\} ; \end{aligned} \tag{A.4}$$

if $(\mathbf{E}, \theta) \in Q_2$, then

$$\begin{aligned} &\mathbf{E}^a = [\nu(\theta)\, e_1 + e_2 - (1 + \nu(\theta))\, \beta(\theta)]\ \mathbf{q}_2 \otimes \mathbf{q}_2 , \\ &\mathbf{T} = E(\theta)\,(e_1 - \beta(\theta))\ \mathbf{q}_1 \otimes \mathbf{q}_1 ; \end{aligned} \tag{A.5}$$

if $(\mathbf{E}, \theta) \in Q_3$, then

$$\begin{aligned} &\mathbf{E}^a = (e_1 - \beta(\theta))\ \mathbf{q}_1 \otimes \mathbf{q}_1 + (e_2 - \beta(\theta))\ \mathbf{q}_2 \otimes \mathbf{q}_2 , \\ &\mathbf{T} = \mathbf{0} . \end{aligned} \tag{A.6}$$

Recalling that $e_1 \le e_2$. are the eigenvalues of $\mathbf{E}$, and $\mathbf{q}_1$, $\mathbf{q}_2$ are the corresponding eigenvectors, we define the tensors

$$\mathbf{O}_1 = \mathbf{q}_1 \otimes \mathbf{q}_1 , \qquad \mathbf{O}_2 = \mathbf{q}_2 \otimes \mathbf{q}_2 , \qquad \mathbf{O}_3 = \frac{1}{\sqrt{2}} (\mathbf{q}_1 \otimes \mathbf{q}_2 + \mathbf{q}_2 \otimes \mathbf{q}_1) , \tag{A.7}$$

and we recall that [5]

$$\mathrm{D}_{\mathrm{E}} e_1 = \mathbf{O}_1 , \ \mathrm{D}_{\mathrm{E}} e_2 = \mathbf{O}_2 ,$$

$$\mathrm{D}_{\mathrm{E}} \mathbf{O}_1 = \frac{1}{e_1 - e_2}\ \mathbf{O}_3 \otimes \mathbf{O}_3 , \ \mathrm{D}_{\mathrm{E}} \mathbf{O}_2 = \frac{1}{e_2 - e_1}\ \mathbf{O}_3 \otimes \mathbf{O}_3 .$$

From (A.4)-(A.6) we obtain the derivative of $\hat{\mathbf{T}}$ with respect to $\mathbf{E}$

if $(\mathbf{E}, \theta) \in Q_1$, then

$$D_E\hat{\mathbf{T}}(\mathbf{E},\theta) = \frac{E(\theta)}{1+\nu(\theta)}\ \{\mathbb{I} + \frac{\nu(\theta)}{1-\nu(\theta)}\ \mathbf{I}\otimes\mathbf{I}\}\ , \tag{A.8}$$

if $(\mathbf{E}, \theta) \in Q_2$, then

$$D_E\hat{\mathbf{T}}(\mathbf{E},\theta) = (\theta)\ \{\ \mathbf{O}_1\otimes\mathbf{O}_1 - \frac{e_1-\beta(\theta)}{e_2-e_1}\ \mathbf{O}_3\otimes\mathbf{O}_3\ \}\ , \tag{A.9}$$

if $(\mathbf{E}, \theta) \in Q_3$, then

$$D_E\hat{\mathbf{T}}(\mathbf{E},\theta) = \mathbb{O}\ . \tag{A.10}$$

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