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On the Hong–Krahn–Szego inequality for the p-Laplace operator

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Abstract. Given an open set Ω, we consider the problem of providing sharp lower bounds for $\lambda_2(\Omega)$, i.e. its second Dirichlet eigenvalue of the p-Laplace operator. After presenting the nonlinear analogue of the *Hong–Krahn–Szego inequality*, asserting that the disjoint unions of two equal balls minimize λ_2 among open sets of given measure, we improve this spectral inequality by means of a quantitative stability estimate. The extremal cases $p = 1$ and $p = \infty$ are considered as well.

1. Introduction

In this paper, we are concerned with Dirichlet eigenvalues of the p-Laplace operator

$$-\Delta_p u := -\mathrm{div}(|\nabla u|^{p-2}\,\nabla u),$$

where $1 < p < \infty$. For every open set $\Omega \subset \mathbb{R}^N$ having finite measure, these are defined as the real numbers λ such that the boundary value problem

$$-\Delta_p u = \lambda\,|u|^{p-2}\,u, \quad \text{in } \Omega, \qquad u = 0, \quad \text{on } \partial\Omega$$

has non trivial (weak) solutions. In particular, we are mainly focused on the following *spectral optimization problem*

$$\min\{\lambda_2(\Omega)\ :\ |\Omega| = c\}, \tag{1.1}$$

where $c > 0$ is a given number, $\lambda_2(\cdot)$ is the second Dirichlet eigenvalue of the p-Laplacian and $|\cdot|$ stands for the N-dimensional Lebesgue measure. We will go back on the question of the well-posedness of this problem in a while, for the moment let us focus on the particular case $p = 2$. In this case we are facing the eigenvalue problem for the usual Laplace operator. As it is well known (see [20]), Dirichlet eigenvalues form a discrete nondecreasing sequence of positive real numbers $0 < \lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \lambda_3(\Omega) \leq \cdots$, going to ∞, where each eigenvalue

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is counted with its multiplicity. In particular, it is meaningful to speak of a second eigenvalue so that problem (1.1) is well-posed and we know that its solution is given by any disjoint union of two balls having measure $c/2$. Moreover, these are the only sets which minimize λ_2 under volume constraint.

Using the scaling properties both of the eigenvalues of $-\Delta$ and of the Lebesgue measure, we can reformulate the previous result in scaling invariant form as follows

$$|\Omega|^{2/N}\,\lambda_2(\Omega) \geq 2^{2/N}\,\omega_N^{2/N}\,\lambda_1(B), \tag{1.2}$$

with equality if and only if Ω is a disjoint union of two equal balls. Here and in what follows, B will always denote a N-dimensional ball of radius one and $\omega_N := |B|$. Observe that for $\Omega = B_1 \cup B_2$, with B_1 and B_2 disjoint balls having $|B_1| = |B_2|$, the first eigenvalue has multiplicity two, i.e. $\lambda_1(\Omega) = \lambda_2(\Omega)$ and these are equal to the first eigenvalue of one of the two balls.

This "isoperimetric" property of balls has been discovered (at least) three times: first by Edgar Krahn [24] in the '20s, but then the result has been probably neglected, since in 1955 George Pólya attributes this observation to Peter Szego (see the final remark of [29]). However, almost in the same years as Pólya's paper, there appeared the paper [21] by Imsik Hong, giving once again a proof of this result. It has to be noticed that Hong's paper appeared in 1954, just one year before Pólya's one. For this reason, in what follows we will refer to (1.2) as the *Hong–Krahn–Szego inequality* (HKS inequality for short).

We briefly recall that for successive Dirichlet eigenvalues of the Laplacian, much less is known. In general existence, regularity and characterization of optimal shapes for a problem like (1.1) are still open issues. As for existence, a general (positive) answer has been given only very recently, independently by Bucur [9] and Mazzoleni and Pratelli [26].

For the case of the p-Laplace operator, this is clearly a completely different story. The nonlinearity of the operator and the lack of an underlying Hilbertian structure complicate things a lot. For example, though there exists a variational procedure to produce an infinite sequence of eigenvalues of $-\Delta_p$ (the so called *eigenvalues of Ljusternik–Schnirelmann type*, see [16,19] for example), up to now it is not clear whether the resulting variational spectrum coincides with the whole spectrum of $-\Delta_p$ or there exist some other eigenvalues. Negative answers were given in [6,12] for slightly different nonlinear eigenvalue problems. Moreover, it is not even known whether or not the collection of the eigenvalues of $-\Delta_p$ forms a discrete set.

This said, while it is easy to define the first eigenvalue λ_1, in principle it becomes quite difficult even to start speaking of the second eigenvalue, the third one and so on, since discreteness of the spectrum is not guaranteed. However, as it is well known, it turns out that also in the case of $-\Delta_p$ one can speak of a second eingevalue λ_2. This means that there is a gap between λ_1 and λ_2, as for $p = 2$. Moreover, this second eigenvalue is a variational one, which has a mountain-pass characterization (see Sect. 2 for more details).

The main aim of the present paper is the study of the spectral optimization problem (1.1) for a general $1 < p < \infty$. As we will see, the Hong–Krahn–Szego inequality still holds in the case of the p-Laplace operator (Theorem 3.2). Namely, any disjoint union of two equal balls minimizes the second eigenvalue of $-\Delta_p$ among sets of given measure, that is

$$|\Omega|^{p/N}\,\lambda_2(\Omega)\geq 2^{p/N}\,\omega_N^{p/N}\,\lambda_1(B). \tag{1.3}$$

The proof runs very similarly to the case $p=2$ and it is based exactly on the same two ingredients, which still hold in the nonlinear setting:

- The Faber–Krahn inequality (see next section) for the first eigenvalue of $-\Delta_p$;
- The fact that for a connected open set the only eigenfunction of constant sign is the first one.

We will then turn our attention to the *stability issue*. Indeed, when dealing with shape optimization problems having unique solution (possibly up to some suitable group of rigid transformations, like rotations or translations, for example), a very interesting and natural question is to know whether this optimal shape is stable or not. For example, specializing this question to our problem (1.1), we are interested in addressing the following issue:

$$|\Omega_0|=c\; and\; \lambda_2(\Omega_0)\simeq\min\{\lambda_2(\Omega)\;:\;|\Omega|=c\}\overset{?}{\Longrightarrow}\Omega_0\text{“}near\text{''}\;to\,the\,optimal\,shape$$

In this paper, we give a positive answer to this question, by proving a *quantitative* version of (1.3). By "quantitative" we mean the following: actually, inequality (1.3) can be improved by adding a reminder term, which measures (in a suitable sense) the distance of the generic set Ω from the "manifold" of optimizers $\mathcal{O}$, i.e. the collection of all disjoint unions of two equal balls. Then the result we provide (Theorem 4.2) is an improvement of (1.3) of the type

$$|\Omega|^{p/N}\lambda_2(\Omega)-2^{p/N}\,\omega_N^{p/N}\,\lambda_1(B)\geq\Phi(d(\Omega,\mathcal{O})),$$

where $d(\cdot,\mathcal{O})$ is a suitable "distance" from $\mathcal{O}$ and $\Phi:[0,\infty)\to[0,\infty)$ is a continuous strictly increasing function, with $\Phi(0)=0$. More precisely, in Theorem 4.2 we prove a result like this with d equal to the L^1 distance of characteristic functions (a variant of the so-called *Fraenkel asymmetry*, see Sect. 4) and Φ given by a power function. This quantitative estimate guarantees that if the second eigenvalue λ_2 of a set Ω is almost equal to the λ_1 of a ball having measure $|\Omega|/2$, then Ω is almost the disjoint union of two equal balls, i.e. we have stability of optimal shapes for our spectral optimization problem. Our analysis will cover the whole range of p. Indeed, we will show that the same proof can be adapted to cover the cases $p=1$ and $p=\infty$ as well, where λ_2 becomes the *second Cheeger constant* and the *second eigenvalue of the* ∞*-Laplacian*, respectively (see Sect. 5 for the precise definitions). In the case of the first eigenvalue λ_1, we recall that quantitative results of this type have been derived in [18,27,30] (linear case, $p=2$) and [4,17] (general case, $1<p<\infty$).

We point out that though problem (1.1) is a very natural one also for $-\Delta_p$, we have not been able to find in literature any paper recording a proof of (1.3). Only after the completion of this work, we found a related recent paper by Kennedy ([23]), dealing with problem (1.1), but for the second eigenvalue of $-\Delta_p$ with Robin boundary conditions. For this reason, we decided to write properly the complete proof of (1.3); on the contrary, the quantitative stability results of Theorem 4.2 and Theorem 5.2 in this paper are certainly new, though probably not sharp, except for the case $p = \infty$ (see the discussion in Sect. 6).

We conclude this introduction with the plan of the paper. In order to make the work as self-contained as possible, Sect. 2 recalls the basic facts about the first two eigenvalues of $-\Delta_p$ that we will need in the following; in Sect. 3 we prove the Hong–Krahn–Szego inequality for λ_2, while Sect. 4 provides a quantitative version of the latter, thus extending to the nonlinear case a result recently proven in [8]. In Sect. 5, for the sake of completeness, we consider the shape optimization problem (1.1) for the "extremal" cases, i.e. for $p = 1$ and $p = \infty$: in this case, the first two eigenvalues λ_1 and λ_2 become two purely geometrical objects and we study stability of optimal shapes for them. Finally, Sect. 6 concludes the paper with some examples, remarks and conjectures concerning the sharpness of the quantitative estimates derived in this work.

2. Tools: the first two eigenvalues of $-\Delta_p$

Given an open set $\Omega \subset \mathbb{R}^N$ having finite measure and $p \in (1, \infty)$, we define the L^p unitary sphere

$$\mathcal{B}_p(\Omega) = \{u \in L^p(\Omega) \, : \, \|u\|_{L^p(\Omega)} = 1\},$$

and we indicate with $W_0^{1,p}(\Omega)$ the usual Sobolev space, given by the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\| = \left(\int_\Omega |\nabla u(x)|^p \, dx \right)^{1/p}.$$

If for a certain λ we have that there exists a non trivial $u \in W_0^{1,p}(\Omega)$ satisfying

$$-\Delta_p u = \lambda |u|^{p-2} u, \quad \text{in } \Omega, \tag{2.1}$$

in a weak sense, i.e.

$$\int_\Omega \langle |\nabla u(x)|^{p-2} \nabla u(x), \nabla \varphi(x) \rangle \, dx = \lambda \int_\Omega |u(x)|^{p-2} u(x) \, \varphi(x) \, dx,$$

$$\text{for every } \varphi \in W_0^{1,p}(\Omega),$$

then we call λ a *Dirichlet eigenvalue* of $-\Delta_p$ in Ω: correspondingly u will be a *Dirichlet eigenfunction* of $-\Delta_p$. In particular, observe that for every such a pair (λ, u) there results

$$\int_\Omega |\nabla u(x)|^p \, dx = \lambda \int_\Omega |u(x)|^p \, dx.$$

Though we will not need it in the sequel, we recall that it is possible to show the existence of a diverging sequence of eigenvalues of $-\Delta_p$, see [16,19].

Remark 2.1 Observe that in general solutions of (2.1) are just in $C^{1,\alpha}$ (see [13]). In fact, the second derivatives cannot exist in a weak sense either, unless $1 < p \leq 2$ (see [1]). Then eigenfunctions in general are not classical solutions of the equation (2.1).

The first Dirichlet eigenvalue of the p-Laplacian of a set has the following variational definition

$$\lambda_1(\Omega) = \min_{u \in \mathcal{B}_p(\Omega) \cap W_0^{1,p}(\Omega)} \int_\Omega |\nabla u(x)|^p \, dx, \tag{2.2}$$

i.e. the quantity $1/\lambda_1(\Omega)$ is the sharp constant in the usual Poincaré inequality

$$\int_\Omega |u(x)|^p \, dx \leq C_\Omega \int_\Omega |\nabla u(x)|^p \, dx, \qquad u \in W_0^{1,p}(\Omega),$$

and this in particular implies that $\lambda_1(\Omega) > 0$.

Remark 2.2 It is easily seen by a standard compactness argument that the minimum in (2.2) is attained, then this $\lambda_1(\Omega)$ is indeed an eigenvalue of $-\Delta_p$, since (2.1) is precisely the Euler–Lagrange equation for (2.2). The fact that $\lambda_1(\Omega)$ is the minimal one follows observing that if λ is an eigenvalue with eigenfunction $v \in \mathcal{B}_p(\Omega) \cap W_0^{1,p}(\Omega)$, then $\int_\Omega |\nabla v(x)|^p \, dx = \lambda$ and thus

$$\lambda_1(\Omega) = \min_{u \in \mathcal{B}_p(\Omega) \cap W_0^{1,p}(\Omega)} \int_\Omega |\nabla u(x)|^p \, dx \leq \int_\Omega |\nabla v(x)|^p \, dx = \lambda.$$

The first important result that we need concerns the simplicity of λ_1 on a connected open set.

Theorem 2.3 *Let $\Omega \subset \mathbb{R}^N$ be an open connected set, having $|\Omega| < +\infty$. Then $\lambda_1(\Omega)$ is simple, i.e. the corresponding eigenfuctions form a 1-dimensional linear space.*

Proof. A very short and elegant proof of this fact can be found in [3]. Their proof is based on the strict convexity of $\int_\Omega |\nabla u|^p$ along curves of the form

$$\sigma_t = \left((1-t)\, u_0^p + t\, u_1^p\right)^{\frac{1}{p}}, \qquad t \in [0,1], \tag{2.3}$$

for every pair of strictly positive functions $u_0, u_1 \in W_0^{1,p}(\Omega)$. Actually, the result in [3] is stated for the case of Ω being a bounded set, but it can be easily seen that this hypothesis plays no role and the same proof still works for Ω having finite measure. □

Throughout the paper, we will use the following convention: when Ω is a disconnected open set, the set of its Dirichlet eigenvalues is made of the collection of the eigenvalues of its connected components. The eigenvalues are obtained by gathering and ordering increasingly the eigenvalues on the single pieces; correspondingly, each eigenfunction is solution of (2.1) on a certain connected component and vanishes on the others.

The following result plays a crucial role: it asserts that any eigenfunction having constant sign is the first one of some connected component of the open set Ω.

Theorem 2.4 *Let $\Omega \subset \mathbb{R}^N$ be an open set, having finite measure. Let $u \in W_0^{1,p}(\Omega)$ be a Dirichlet eigenfunction relative to some eigenvalue λ. If u has constant sign in Ω, then $\lambda = \lambda_1(\Omega_0)$ for some connected component Ω_0 of Ω, i.e. u is a first eigenfunction of Ω_0. In particular $\lambda = \lambda_1(\Omega)$ if Ω is connected.*

Proof. If Ω is connected, a straightforward proof of this fact has been recently given by the authors in [7], again based on the convexity of $\int_\Omega |\nabla u|^p$ along curves of the form (2.3).

On the other hand, if Ω is disconnected, then λ has to be a Dirichlet eigenvalue of a certain connected component Ω_0; correspondigly u is an eigenfunction of Ω_0, having constant sign. Then it suffices to apply the first part to conclude. □

We give now a precise definition of what we mean by the *second eigenvalue of* $-\Delta_p$. This definition keeps into account the multiplicity of the first eigenvalue. As we will see, this is necessary in order to properly deal with our spectral optimization problem (1.1).

Definition. Let Ω be an open set having finite measure. Then its *second eigenvalue* is given by

$$\lambda_2(\Omega) = \begin{cases} \min\{\lambda > \lambda_1(\Omega) \, : \, \lambda \text{ is an eigenvalue } \} & \text{if } \lambda_1(\Omega) \text{ is simple} \\ \\ \lambda_1(\Omega) & \text{otherwise.} \end{cases} \tag{2.4}$$

When λ_1 is simple, some words about the consistency of this definition are in order. Indeed, using Theorem 2.4 it can be proven that if Ω is connected, then λ_1 is isolated in the spectrum, the latter being a closed set ([25, Theorem 3]): this shows that the minimum in (2.4) is well-defined. On the other hand, if Ω consists of infinitely many connected components, one only has to check that the collection of the first eigenvalues on the single components cannot accumulate at any value $\lambda \geq \lambda_1(\Omega)$. This follows by combining the assumption $|\Omega| < \infty$ and the Faber–Krahn inequality (see below), which in particular implies that if $|E| \to 0$, then $\lambda_1(E) \to \infty$. This shows again that the minimum in (2.4) is meaningful.

Remark 2.5 By definition, the nodal domains of an eigenfunction u are the connected components of the sets $\{x \, : \, u(x) > 0\}$ and $\{x \, : \, u(x) < 0\}$. If Ω is connected, we recall that every eigenfunction corresponding to λ_2 has exactly two nodal domains (see [10]), in which case by Theorem 2.4 we can infer

$$\lambda_2(\Omega) = \min\{\lambda > \lambda_1(\Omega) : \lambda \text{ admits a sign-changing eigenfunction}\}.$$

For the sake of completeness, we recall that one can give a variational characterization also for λ_2: in order to introduce it, we need some further notations. Given a pair of functions $u, v \in \mathcal{B}_p(\Omega) \cap W_0^{1,p}(\Omega)$, let us denote by $\Gamma_\Omega(u, v)$ the set of continuous (in the $W^{1,p}$ topology) paths in $\mathcal{B}_p(\Omega) \cap W_0^{1,p}(\Omega)$ connecting u to v, i.e.

$$\Gamma_\Omega(u, v) = \left\{\gamma : [0, 1] \to \mathcal{B}_p(\Omega) \cap W_0^{1,p}(\Omega) \, : \, \gamma \text{ is continuous and } \gamma(0) = u, \ \gamma(1) = v\right\}.$$

Theorem 2.6 *Let $\Omega \subset \mathbb{R}^N$ be an open set having finite measure, not necessarily connected. Let $u_1 \in \mathcal{B}_p(\Omega) \cap W_0^{1,p}(\Omega)$ be one of its first eigenfunctions. Then $\lambda_2(\Omega)$ has the following mountain pass characterization*

$$\lambda_2(\Omega) = \inf_{\gamma \in \Gamma_\Omega(u_1,-u_1)} \max_{u \in \gamma([0,1])} \int_\Omega |\nabla u(x)|^p \, dx. \tag{2.5}$$

If $\lambda_1(\Omega)$ is not simple, this characterization is independent of the particular u_1 we choose.

Proof. If Ω is connected, this has been proven in the paper [11], to which we refer for the proof. Here, we just show how (2.5) can be extended to the case of general open sets: anyway, since we will not need this result in the sequel, the uninterested reader may skip the proof at a first reading.

Let us take Ω not connected, then the following alternative holds: either $\lambda_1(\Omega)$ is simple or not. *Case λ_1 simple:* in this case, a first eigenfunction $u_1 \in \mathcal{B}_p(\Omega) \cap W_0^{1,p}(\Omega)$ is unique and let us consider a second eigenfunction u_2, still normalized by $\|u_2\|_{L^p(\Omega)} = 1$.

If u_2 is sign-changing, then it is supported on some connected component Ω_0 of Ω such that $\lambda_i(\Omega) = \lambda_i(\Omega_0)$, $i = 1, 2$: in particular the mountain pass characterization of $\lambda_2(\Omega_0)$ holds, with the maximum performed on the restricted class of curves $\Gamma_{\Omega_0}(u_1, -u_1) \subset \Gamma_\Omega(u_1, -u_1)$. Thus setting

$$\lambda := \inf_{\gamma \in \Gamma_\Omega(u_1,-u_1)} \max_{u \in \gamma([0,1])} \int_\Omega |\nabla u(x)|^p \, dx$$

on the one hand we have $\lambda \le \lambda_2(\Omega_0) = \lambda_2(\Omega)$, while on the other hand we get $\lambda_1(\Omega) < \lambda$, since λ gives a Dirichlet eigenvalue of $-\Delta_p$ in any case (see [11], Sect. 2). Summarizing, we obtain $\lambda_1(\Omega) < \lambda \le \lambda_2(\Omega)$ which gives the thesis in this case, thanks to (2.4).

On the contrary, if $\lambda_1(\Omega)$ is simple but u_2 has constant sign, then we have $\lambda_1(\Omega) = \lambda_1(\Omega_0)$ and $\lambda_2(\Omega) = \lambda_1(\Omega_1)$, with Ω_0 and Ω_1 distinct connected components. We construct a special element of $\Gamma_\Omega(u_1, -u_1)$, a continuous path γ defined as follows

$$\gamma(t) = \varphi_t, \ \text{with } \varphi_t(x) = \frac{\cos(\pi\, t)u_1(x) + t(1-t)u_2(x)}{(|\cos(\pi\, t)|^p + t^p(1-t)^p)^{1/p}}, \ x \in \Omega,$$

for all $t \in [0, 1]$. It is easy to see that γ has the following properties

$$\gamma(t) \in \mathcal{B}_p(\Omega) \cap W_0^{1,p}(\Omega), \quad \text{for every } t \in [0,1], \quad \text{and} \quad \gamma(0) = u_1, \ \gamma(1) = -u_1,$$

i.e. the curve γ is admissible for the variational problem (2.5). Hence we get

$$\begin{aligned}\lambda &\le \max_{t\in[0,1]} \int_\Omega |\nabla \varphi_t(x)|^p \, dx = \max_{t\in[0,1]} \frac{|\cos(\pi\, t)|^p \, \lambda_1(\Omega_0) + t^p(1-t)^p \lambda_1(\Omega_1)}{|\cos(\pi\, t)|^p + t^p(1-t)^p} \\ &\le \lambda_1(\Omega_1) = \lambda_2(\Omega),\end{aligned}$$

where we used $\lambda_1(\Omega_0) < \lambda_1(\Omega_1)$. Thus we get $\lambda_1(\Omega) < \lambda \le \lambda_2(\Omega)$ and we can conclude as before.

Case λ_1 *multiple:* if Ω is not connected and its corresponding first eigenvalue is not simple, we just take two linearly independent first eigenfunctions $u_1, u_2 \in \mathcal{B}_p(\Omega) \cap W_0^{1,p}(\Omega)$, which are thus supported on different connected components of Ω. Repeating the construction of the curve γ above, we obtain

$$\lambda_1(\Omega) \le \lambda = \inf_{\gamma\in\Gamma_\Omega(u,-u)} \max_{u\in\gamma([0,1])} \int_\Omega |\nabla u(x)|^p \, dx \le \lambda_1(\Omega),$$

which shows that $\lambda = \lambda_1(\Omega) = \lambda_2(\Omega)$. Observe that if we exchange the role of u_1 and u_2, we still arrive at the same conclusion, thus proving that in this case formula (2.5) is independent of the choice of the particular first eigenfunction. □

Remark 2.7 It is useful to recall at this point that usually the variational eigenvalues $\{\lambda_k\}_{k\ge1}$ of $-\Delta_p$ are defined through a minimax problem on $\mathcal{B}_p \cap W_0^{1,p}$ for the integral $\int_\Omega |\nabla u|^p$, involving the concept of *Krasnosel'skii genus*. The previous result gives in particular that for $k = 2$ this characterization coincides with the mountain-pass one given by (2.5).

Finally, since our aim is that of considering a particular class of shape optimization problems involving the spectrum of $-\Delta_p$, we conclude this introduction by recalling some further properties of λ_1 and λ_2 that we will need in the sequel. In particular, they are monotone decreasing with respect to set inclusion, while as for their scaling properties we have

$$\lambda_i(t\,\Omega) = t^{-p}\lambda_i(\Omega), \quad t > 0,\ i = 1, 2,$$

which in particular implies that the shape functional $\Omega \mapsto |\Omega|^{p/N}\,\lambda_i(\Omega)$ is scaling invariant. Thus the two problems

$$\min\{\lambda_i(\Omega)\ :\ |\Omega| = c\} \text{ and } \min |\Omega|^{p/N}\,\lambda_i(\Omega), \qquad i = 1, 2,$$

are equivalent, in the sense that they both provide the same optimal shapes, up to a scaling. For $i = 1$, the solution to the previous problem is given by any ball: this is the celebrated *Faber–Krahn inequality*. The classical proof combines the Schwarz symmetrization with the so called Pólya-Szegő principle (see [20, Chap. 3], for example).

Faber–Krahn Inequality: *Let* $1 < p < \infty$. *For every open set* $\Omega \subset \mathbb{R}^N$ *having finite measure, we have*

$$|\Omega|^{p/N}\lambda_1(\Omega) \ge \omega_N^{p/N}\lambda_1(B), \tag{2.6}$$

where B *is the* N*-dimensional ball of radius 1 and* $\omega_N := |B|$. *Moreover, equality sign in* (2.6) *holds if and only if* Ω *is a ball.*

In other words, for every c>0 the unique solutions of the following spectral optimization problem

$$\min\{\lambda_1(\Omega)\ :\ |\Omega| = c\},$$

are given by balls having measure c.

3. The Hong–Krahn–Szego inequality

In this section, we are going to prove that the disjoint unions of equal balls are the only sets minimizing λ_2 under volume constraint, i.e. we will prove the Hong–Krahn–Szego inequality for the p-Laplacian. The key step in the proof is the following technical result: this is an adaptation of a similar result for the linear case $p = 2$ (see [8, Lemma 3.1], for example).

Lemma 3.1 *Let $\Omega \subset \mathbb{R}^N$ be an open set with $|\Omega| < \infty$. Then there exists Ω_+, Ω_- disjoint subsets of Ω such that*

$$\lambda_2(\Omega) = \max\{\lambda_1(\Omega_+), \lambda_1(\Omega_-)\}. \tag{3.1}$$

Proof. Let us take $u_1, u_2 \in \mathcal{B}_p(\Omega) \cap W_0^{1,p}(\Omega)$ a first and second eigenfunction, respectively: notice that if $\lambda_1(\Omega)$ is not simple, we mean that u_1 and u_2 are two linearly independents eigenfunctions corresponding to $\lambda_1(\Omega)$. We can distinguish two alternatives:

(i) u_2 is sign-changing;
(ii) u_2 has constant sign in Ω.

Let us start with (i): in this case, by Remark 2.5 u_2 has exactly two nodal domains

$$\Omega_+ = \{x \in \Omega : u_2(x) > 0\} \text{ and } \Omega_- = \{x \in \Omega : u_2(x) < 0\},$$

which by definition are connected sets. The restriction of u_2 to Ω_+ is an eigenfunction of constant sign for Ω_+, then Theorem 2.4 implies that u_2 must be a first eigenfunction for it. Replacing Ω_+ with Ω_-, the previous observation leads to

$$\lambda_2(\Omega) = \lambda_1(\Omega_-) = \lambda_1(\Omega_+).$$

which implies in particular (3.1) in this case. In case (ii), let us set

$$\Omega_+ = \{x \in \Omega : |u_1(x)| > 0\} \text{ and } \Omega_- = \{x \in \Omega : |u_2(x)| > 0\}.$$

Using Theorem 2.4, we have that Ω_+ and Ω_- have to be two distinct connected components of Ω: in addition u_1, u_2 are eigenfunctions (with constant sign) of Ω_+ and Ω_-, respectively. Then

$$\lambda_1(\Omega_-) = \int_{\Omega_-} |\nabla u_2(x)|^p\, dx = \int_{\Omega} |\nabla u_2(x)|^p = \lambda_2(\Omega).$$

Clearly, we also have $\lambda_1(\Omega_+) = \lambda_1(\Omega) \le \lambda_2(\Omega)$, which finally gives (3.1) also in this case. □

We are now ready for the main result of this section.

Theorem 3.2 (HKS inequality for the p-Laplacian) *For every $\Omega \subset \mathbb{R}^N$ open set having finite measure, we have*

$$|\Omega|^{p/N}\,\lambda_2(\Omega) \geq 2^{p/N}\,\omega_N^{p/N}\,\lambda_1(B), \tag{3.2}$$

where B is the N-dimensional ball of radius 1 and $\omega_N := |B|$. Moreover, equality sign in (3.2) holds if and only if Ω is the disjoint union of two equal balls.

In other words, for every $c > 0$ the unique solutions of the following spectral optimization problem

$$\min\{\lambda_2(\Omega)\ :\ |\Omega| = c\},$$

are given by disjoint unions of two balls, both having measure $c/2$.

Proof. With the notation of Lemma 3.1, an application of the Faber–Krahn inequality yields

$$\lambda_2(\Omega) = \max\{\lambda_1(\Omega_+), \lambda_1(\Omega_-)\} \geq \max\{\lambda_1(B_+), \lambda_1(B_-)\}, \tag{3.3}$$

where B_+, B_- are balls such that $|B_+| = |\Omega_+|$ and $|B_-| = |\Omega_-|$. Thanks to the scaling properties of λ_1, we have

$$\lambda_1(B_+) = \left(\frac{\omega_N}{|\Omega_+|}\right)^{p/N} \lambda_1(B) \text{ and } \lambda_1(B_-) = \left(\frac{\omega_N}{|\Omega_-|}\right)^{p/N} \lambda_1(B),$$

so that from (3.3) we obtain

$$\lambda_2(\Omega) \geq \omega_N^{p/N}\lambda_1(B)\ \max\{|\Omega_+|^{-p/N}, |\Omega_-|^{-p/N}\}.$$

Finally, observe that since $|\Omega_+| + |\Omega_-| \leq |\Omega|$, we get

$$\max\{|\Omega_+|^{-p/N}, |\Omega_-|^{-p/N}\} \geq \left(\frac{|\Omega|}{2}\right)^{-p/N}, \tag{3.4}$$

which concludes the proof of the inequality.

As for the equality cases, we start observing that we just used two inequalities, namely (3.3) and (3.4). On the one hand, equality in (3.3) holds if and only if at least one among the two subsets is a ball, say $\Omega_+ = B_+$, with $\lambda_1(B_+) \geq \lambda_1(\Omega_-)$; on the other hand, if equality holds in (3.4) then we must have $|\Omega_+| = |\Omega_-| = |\Omega|/2$. Since Ω_+ and Ω_- have the same measure and the one with the greatest λ_1 is a ball, we can conclude that both have to be a ball, thanks to the equality cases in the Faber–Krahn inequality. □

4. The stability issue

We now come to the question of stability for optimal shapes of λ_2 under measure constraint. In particular, we will enforce the lower bound on $|\Omega|^{2/N}\,\lambda_2(\Omega)$ provided by the Hong–Krahn–Szego inequality, by adding a remainder terms in the right-hand side of (3.2). At this aim, we need to introduce some further tools. Given an open set $\Omega \subset \mathbb{R}^N$ having $|\Omega| < \infty$, its *Fraenkel asymmetry* is defined by

$$\mathcal{A}(\Omega) = \inf \left\{ \frac{\|1_\Omega - 1_B\|_{L^1}}{|\Omega|} \ :\ B \text{ is a ball such that } |B| = |\Omega| \right\}.$$

This is a scaling invariant quantity such that $0 \leq \mathcal{A}(\Omega) < 2$, with $\mathcal{A}(\Omega) = 0$ if and only if Ω coincides with a ball, up to a set of measure zero. Then we recall the following quantitative improvement of the Faber–Krahn inequality, proven in [4] (case $N = 2$) and [17] (general case). For every $\Omega \subset \mathbb{R}^N$ open set with $|\Omega| < \infty$, we have

$$|\Omega|^{p/N}\, \lambda_1(\Omega) \geq \omega_N^{p/N}\, \lambda_1(B) \left[1 + \gamma_{N,p}\, \mathcal{A}(\Omega)^{\kappa_1}\right], \tag{4.1}$$

where $\gamma_{N,p}$ is a constant depending only on N and p and the exponent $\kappa_1 = \kappa_1(N, p)$ is given by

$$\kappa_1(N, p) = \begin{cases} 3, & \text{if } N = 2, \\ 2 + p, & \text{if } N \geq 3. \end{cases}$$

Remark 4.1 One may ask wheter the exponent κ_1 in (4.1) is sharp or not. By introducing the *deficit*

$$FK(\Omega) := \frac{|\Omega|^{p/N}\, \lambda_1(\Omega)}{\omega_N^{p/N}\, \lambda_1(B)} - 1,$$

one would like to prove the existence of suitable deformations $\{\Omega_\varepsilon\}_{\varepsilon>0}$ of a ball B, such that

$$\lim_{\varepsilon\to 0} FK(\Omega_\varepsilon) = 0 \text{ and } \lim_{\varepsilon\to 0} \frac{\mathcal{A}(\Omega_\varepsilon)^{\kappa_1}}{FK(\Omega_\varepsilon)} = \ell \neq \{0, +\infty\}.$$

i.e. the asymmetry to the power κ_1 and the deficit have the same decay rate to zero. At least in the case $p = 2$, the answer should be *no*, since the conjectured sharp exponent is two (see [5, p. 56]), while $\kappa_1(N, 2) \geq 3$. At present, a proof of this fact still lacks.

In the case of the Hong–Krahn–Szego inequality, the relevant notion of asymmetry is the *Fraenkel* 2*−asymmetry*, introduced in [8]

$$\mathcal{A}_2(\Omega) = \inf \left\{ \frac{\|1_\Omega - 1_{B_1\cup B_2}\|_{L^1}}{|\Omega|} \ :\ B_1, B_2 \text{ balls such that } |B_1 \cap B_2| = 0,\ |B_i| = \frac{|\Omega|}{2},\ i = 1, 2 \right\}.$$

The main result of this section is the following quantitative version of Theorem 3.2.

Theorem 4.2 *Let $\Omega \subset \mathbb{R}^N$ be an open set, with $|\Omega| < \infty$ and $p \in (1, \infty)$. Then*

$$|\Omega|^{p/N}\lambda_2(\Omega) \geq 2^{p/N}\, \omega_N^{p/N}\, \lambda_1(B) \left[1 + C_{N,p}\, \mathcal{A}_2(\Omega)^{\kappa_2}\right], \tag{4.2}$$

with $C_{N,p} > 0$ constant depending on N and p only and $\kappa_2 = \kappa_2(N, p)$ given by

$$\kappa_2(N, p) = \kappa_1(N, p) \cdot \frac{N + 1}{2}.$$

Proof. Thanks to Lemma 3.1, we have existence of two disjoint sets $\Omega_+, \Omega_- \subset \Omega$ such that (3.1) holds. We then set

$$\delta_+ = |\Omega_+| - \frac{|\Omega|}{2},\ \delta_- = |\Omega_-| - \frac{|\Omega|}{2},$$

and we observe that it must be $\delta_+ + \delta_- \le 0$, since $|\Omega_+| + |\Omega_-| \le |\Omega|$. To simplify a bit formulas, let us introduce the *deficit functional*

$$HKS(\Omega) := \frac{|\Omega|^{p/N}\lambda_2(\Omega)}{2^{p/N}\,\omega_N^{p/N}\,\lambda_1(B)} - 1.$$

In order to prove (4.2), we just need to show that

$$HKS(\Omega) \ge C_{N,p} \max\left\{\mathcal{A}(\Omega_+)^{\kappa_1} + \left|\frac{\delta_+}{|\Omega|}\right|,\ \mathcal{A}(\Omega_-)^{\kappa_1} + \left|\frac{\delta_-}{|\Omega|}\right|\right\}, \tag{4.3}$$

then a simple application of Lemma 4.3 below will conclude the proof. To obtain (4.3), it will be useful to distinguish between the case $p \le N$ and the case $p > N$. For both of them, we will in turn treat separately the case where both δ_+ and δ_- are non positive and the case where they have opposite sign. Finally, since the quantities appearing in the right-hand side of (4.3) are all universally bounded, it is not restrictive to prove (4.3) under the additional assumption

$$HKS(\Omega) \le \frac{1}{4}. \tag{4.4}$$

Indeed, it is straightforward to see that if $HKS(\Omega) > 1/4$ then (4.3) trivially holds with constant

$$C_{N,p} = \frac{1}{2}\,\frac{1}{2^{\kappa_1+1}+1} > 0.$$

Case A: $p \le N$. In this case the proof runs very similarly to the linear case $p = 2$ treated in [8]. We start applying the quantitative Faber–Krahn inequality (4.1) to Ω_+. If we indicate with B the ball of unit radius, recalling (3.1) and using the definition of δ_+, we find

$$\gamma_{N,p}\,\mathcal{A}(\Omega_+)^{\kappa_1} \le \frac{|\Omega_+|^{p/N}\,\lambda_1(\Omega_+)}{\omega_N^{p/N}\,\lambda_1(B)} - 1 \le \frac{(|\Omega|+2\,\delta_+)^{p/N}\lambda_2(\Omega)}{2^{p/N}\omega_N^{p/N}\,\lambda_1(B)} - 1$$

Since $p \le N$, the power function $t \mapsto (|\Omega| + t)^{p/N}$ is concave, thus we have

$$(|\Omega| + 2\delta_+)^{p/N} \le |\Omega|^{p/N} + \frac{2p}{N}\,|\Omega|^{p/N}\,\frac{\delta_+}{|\Omega|}.$$

Using this information in the previous inequality, we get

$$\gamma_{N,p}\,\mathcal{A}(\Omega_+)^{\kappa_1} \le HKS(\Omega) + \frac{2p}{N}\,\frac{\delta_+}{|\Omega|}\,\frac{|\Omega|^{p/N}\,\lambda_2(\Omega)}{2^{p/N}\,\omega_N^{p/N}\,\lambda_1(B)},$$

that we can rewrite as follows

$$\gamma_{N,p}\,\mathcal{A}(\Omega_+)^{\kappa_1} \le HKS(\Omega) + \frac{2p}{N}\frac{\delta_+}{|\Omega|}\,(HKS(\Omega)+1)\,. \tag{4.5}$$

Replacing Ω_+ with Ω_-, one obtains a similar estimate for Ω_-.

Case A.1: δ_+ and δ_- are both non-positive. In this case, it is enough to observe that $HKS(\Omega) \ge 0$ while $\delta_+ \le 0$, thus from (4.5) we get

$$\gamma_{N,p}\,\mathcal{A}(\Omega_+)^{\kappa_1} + \frac{2p}{N}\frac{|\delta_+|}{|\Omega|} \le HKS(\Omega).$$

The same computations with Ω_- in place of Ω_+ yield (4.3).

Case A.2: δ_+ and δ_- have opposite sign. Let us assume for example that $\delta_+ \ge 0$ and $\delta_- \le 0$: the main difference with the previous case is that now the larger piece Ω_+ could be so large that the information provided by (3.1) is useless. However, estimate (4.5) still holds true for both Ω_+ and Ω_-. Using this and the fact that $\delta_+ + \delta_- \le 0$, we can thus infer

$$HKS(\Omega) \ge -\frac{2p}{N}\frac{\delta_-}{|\Omega|} \ge \frac{2p}{N}\frac{\delta_+}{|\Omega|},$$

i.e. the deficit is controlling the error term $|\delta_+|/|\Omega|$. To finish, we still have to control the asymmetry of the larger piece Ω_+ in terms of the deficit: it is now sufficient to introduce the previous information into (4.5), thus getting

$$\gamma_{N,p}\,\mathcal{A}(\Omega_+)^{\kappa_1} \le HKS(\Omega)(2 + HKS(\Omega)).$$

Since we are assuming $HKS(\Omega) \le 1/4$, the previous implies that $HKS(\Omega)$ controls $\mathcal{A}(\Omega_+)^{\kappa_1}$, modulo a constant depending only on N and p. These estimates on Ω_+, together with the validity of (4.5) for Ω_- and with the fact that $\delta_- \le 0$, ensure that (4.3) holds also in this case.

Case B: $p > N$. Let us start once again with Ω_+. Using (3.1) and the quantitative Faber–Krahn (4.1) as before, we get

$$HKS(\Omega) \ge \frac{|\Omega|^{p/N}\,\lambda_1(\Omega_+)}{2^{p/N}\,\omega_N^{p/N}\,\lambda_1(B)} - 1 \ge \left[\left(\frac{|\Omega|}{2\,|\Omega_+|}\right)^{p/N}(1 + \gamma_{N,p}\,\mathcal{A}(\Omega_+)^{\kappa_1}) - 1\right].$$

Then using the definition of δ_+ and the convexity of the function $t \mapsto (1+t)^{p/N}$ (since $p > N$), we have

$$\left(\frac{|\Omega|}{2\,|\Omega_+|}\right)^{p/N} = \left(1 - \frac{\delta_+}{|\Omega_+|}\right)^{p/N} \ge 1 - \frac{p}{N}\frac{\delta_+}{|\Omega_+|}.$$

Inserted in the previous estimate, this yields

$$HKS(\Omega) \ge \left[\gamma_{N,p}\left(1 - \frac{p}{N}\frac{\delta_+}{|\Omega_+|}\right)\mathcal{A}(\Omega_+)^{\kappa_1} - \frac{p}{N}\frac{\delta_+}{|\Omega_+|}\right]. \tag{4.6}$$

In the same way, using Ω_- in place of Ω_+, we obtain a similar estimate for Ω_-.

Case B.1: δ_+ and δ_- are both non positive. In this case, in (4.6) we can drop the terms

$$-\frac{p}{N}\,\frac{\delta_+}{|\Omega_+|}\,\gamma_{N,p}\,\mathcal{A}(\Omega_+)^{\kappa_1}\ \text{and}\ -\frac{p}{N}\,\frac{\delta_-}{|\Omega_-|}\,\gamma_{N,p}\,\mathcal{A}(\Omega_-)^{\kappa_1},$$

since these are positive, thus we arrive once again at (4.3), keeping into account that

$$-\frac{\delta_+}{|\Omega_+|}\geq-\frac{\delta_+}{|\Omega|}\ \text{and}\ -\frac{\delta_-}{|\Omega_-|}\geq-\frac{\delta_-}{|\Omega|}.$$

Case B.2: δ_+ and δ_- have opposite sign. Let us suppose as before that $\delta_+ \geq 0$ and $\delta_- \leq 0$. Now the main problem is the term in front of the asymmetry $\mathcal{A}(\Omega_+)$ in (4.6), which could be negative. Since $\delta_+ + \delta_- \leq 0$, applying (4.6) to Ω_- we obtain

$$\frac{\delta_+}{|\Omega|}\leq-\frac{\delta_-}{|\Omega|}\leq\frac{N}{p}\,HKS(\Omega). \tag{4.7}$$

We then observe that if

$$\delta_+\leq\frac{N}{p}\,\frac{|\Omega|}{4}, \tag{4.8}$$

we have

$$\left(1-\frac{p}{N}\,\frac{\delta_+}{|\Omega_+|}\right)\geq1-\frac{1}{4}\,\frac{|\Omega|}{|\Omega_+|}\geq\frac{1}{2},$$

thanks to the fact that $|\Omega| \leq 2\,|\Omega_+|$, which easily follows from the assumption that $\delta_+ \geq 0$. From (4.6) we can now infer

$$HKS(\Omega)\geq\frac{\gamma_{N,p}}{2}\,\mathcal{A}(\Omega_+)^{\kappa_1}-\frac{p}{N}\,\frac{\delta_+}{|\Omega_+|},$$

then (4.3) follows as before, using (4.7) and the fact that

$$-\frac{\delta_+}{|\Omega_+|}\geq-2\,\frac{\delta_+}{|\Omega|}.$$

This would prove the thesis under the additional hypothesis (4.8): however, if this is not satisfied, then (4.7) would imply $HKS(\Omega) > 1/4$, which is in contrast with our assumption (4.4). □

The following technical Lemma of geometrical content completes the proof of Theorem 4.2. This is the same as [8, Lemma 3.3] and we omit the proof.

Lemma 4.3 *Let $\Omega \subset \mathbb{R}^N$ be an open set, with finite measure. For every $\Omega_+, \Omega_- \subset \Omega$ such that $|\Omega_+ \cap \Omega_-| = 0$, we have*

$$\mathcal{A}_2(\Omega)\leq C_N\left(\mathcal{A}(\Omega_+)+\left|\frac{1}{2}-\frac{|\Omega_+|}{|\Omega|}\right|+\mathcal{A}(\Omega_-)+\left|\frac{1}{2}-\frac{|\Omega_-|}{|\Omega|}\right|\right)^{2/(N+1)} \tag{4.9}$$

for a suitable dimensional constant $C_N > 0$.

5. Extremal cases: p=1 and p=∞

To complete the analysis of our spectral optimization problem in the nonlinear setting, it is natural to give a brief look at what happens for (1.1), when p tends to the extrema of its possible range, i.e. $p = 1$ and $p = \infty$. It is known that in these cases, some shape functionals of geometric flavour appear, in place of the eigenvalues of an elliptic operator.

To enter more in this question, we need some definitions: for $\Omega \subset \mathbb{R}^N$ open set with $|\Omega| < \infty$, $\mathcal{C}_1(\Omega)$ and $\mathcal{C}_2(\Omega)$ stand for the *first two Cheeger constants*, which are defined respectively by

$$\mathcal{C}_1(\Omega) = \inf_{E \subset \Omega} \frac{P(E)}{|E|} \quad \text{and}$$

$$\mathcal{C}_2(\Omega) = \inf \left\{ t \, : \, \begin{array}{l} \text{there exist } E_1, E_2 \subset \Omega \\ \text{such that } E_1 \cap E_2 = \emptyset \end{array} \text{ and } \max_{i=1,2} \frac{P(E_i)}{|E_i|} \leq t \right\},$$

where $P(E)$ equals the distributional perimeter of a set E if this is a finite perimeter sets and is $+\infty$ otherwise. Also, if $|E| = 0$ we use the convention $P(E)/|E| = +\infty$.

We denote by $\Lambda_1(\Omega)$ the inverse of the radius r_1 of the largest ball included in Ω, while $\Lambda_2(\Omega)$ will denote the inverse of the largest positive number r_2 such that there exist two disjoint balls of radius r_2 contained in Ω. It is remarkable to notice that Λ_1 and Λ_2 are indeed two eigenvalues, precisely they coincide with the *first two eigenvalues of the* ∞*Laplacian* (see [22]).

Our interest in these quantities is motivated by the following Theorem, collecting various results about the asymptotic behaviour of λ_1 and λ_2.

Limiting behaviour of eigenvalues. For every set $\Omega \subset \mathbb{R}^N$, there holds

$$\lim_{p \to 1^+} \lambda_i(\Omega) = \mathcal{C}_i(\Omega), \quad i=1,2 \quad \text{and} \quad \lim_{p \to \infty} \lambda_i(\Omega)^{1/p} = \Lambda_i(\Omega), \quad i=1,2. \tag{5.1}$$

Proof. The first fact is proven in [15] and [28], respectively. For the second, one can consult [22] and the references therein. □

Remark 5.1 At this point, one could be tempted to use the previous results for λ_1, in order to improve inequality (4.1). For example, using the subadditivity of the function $t \mapsto (1+t)^{1/p}$, it is not difficult to see that

$$\lim_{p \to \infty} FK(\Omega)^{1/p} \geq \frac{|\Omega|^{1/N} \Lambda_1(\Omega)}{\omega_N^{1/N}} - 1 \geq \frac{1}{2N} \mathcal{A}(\Omega),$$

where in the last inequality we used the (sharp) quantitative stability estimate[1] for Λ_1 (see [22], Eq. (2.6)). Then one could bravely guess that for p "very large", inequality (4.1) has to hold with the exponent $\kappa_1(N, p)$ replaced by p, which is strictly small if $N \geq 3$. This would prove that (4.1) is not sharp, at least for $N \geq 3$ and p going to ∞. Needless to say, this argument (and the related one for $p \to 1$) is only a heuristic one, since these limits are not uniform with respect to the sets Ω.

[1] The relation between the Fraenkel asymmetry $\alpha(\Omega)$ as defined in [22] and our definition is given by $\mathcal{A}(\Omega) = 2\,\alpha(\Omega)$. This explains the discrepancy between our constant $1/(2N)$ and the constant $1/N$ that can be found in [22], Eq. (2.6).

The analogues of problem (1.1) in these extremal cases are the following

$$\min\{\mathcal{C}_2(\Omega)\ :\ |\Omega|=c\}\ \text{and}\ \min\{\Lambda_2(\Omega)\ :\ |\Omega|=c\}.$$

Once again, they both have (unique) solution given by any disjoint union of two equal balls: for the first one, the reader can see [28, Proposition 3.14], while the second is easily derived thanks to the geometrical meaning of Λ_2. In scaling invariant form, these rewrite as

$$|\Omega|^{1/N}\,\mathcal{C}_2(\Omega)\geq 2^{1/N}\,N\omega_N^{1/N}\ \text{and}\ |\Omega|^{1/N}\,\Lambda_2(\Omega)\geq 2^{1/N}\,\omega_N^{1/N},$$

and they both can be improved in a quantitative form, as it is proved in the following.

Theorem 5.2 *Let Ω be an open subset of $\mathbb{R}^N$ having finite measure. Then*

$$|\Omega|^{1/N}\,\mathcal{C}_2(\Omega)\geq 2^{1/N}\,N\,\omega_N^{1/N}\,\Big[1+h_N\,\mathcal{A}_2(\Omega)^{N+1}\Big], \tag{5.2}$$

where the constant $h_N>0$ only depens on the dimension N. Moreover, for Λ_2 we have

$$|\Omega|^{1/N}\,\Lambda_2(\Omega)\geq 2^{1/N}\,\omega_N^{1/N}\,\left[1+\frac{1}{2N}\,\mathcal{A}_2(\Omega)\right]. \tag{5.3}$$

Proof. To prove (5.2), we start defining

$$T_\Omega=\left\{t>0\ :\ \text{there exist}\ \Omega_1,\Omega_2\subset\Omega\ \text{disjoint and s.t.}\ \max_{i=1,2}\frac{P(\Omega_i)}{|\Omega_i|}\leq t\right\}.$$

It is not difficult to see that if Ω is open, then $T_\Omega\neq\emptyset$, since Ω contains at least two disjoint small balls, which are in particular two sets with positive measure and finite perimeter. Then let us pick up a $t\in T_\Omega$. Correspondingly, there exist $\Omega^t_+,\Omega^t_-\subset\Omega$ disjoint and such that

$$t\geq\max\left\{\frac{P(\Omega^t_+)}{|\Omega^t_+|},\frac{P(\Omega^t_-)}{|\Omega^t_-|}\right\}\geq\max\{\mathcal{C}_1(\Omega^t_+),\mathcal{C}_1(\Omega^t_-)\}, \tag{5.4}$$

where we used the straightforward estimate $\mathcal{C}_1(E)\leq P(E)/|E|$, which is valid for every finite perimeter set E. Now, we introduce the following quantity

$$D_\Omega(t):=\frac{|\Omega|^{1/N}\,\max\{\mathcal{C}_1(\Omega^t_+),\mathcal{C}_1(\Omega^t_-)\}}{2^{1/N}\,N\,\omega_N^{1/N}}-1,$$

and we proceed exactly as in Case A of the proof of Theorem 4.2. We only need to replace $HKS(\Omega)$ by $D_\Omega(t)$ and the quantitative Faber–Krahn inequality by the following (sharp) quantitative Cheeger inequality (see [14]),

$$|\Omega|^{1/N}\mathcal{C}_1(\Omega)\geq N\,\omega_N^{1/N}\,\Big[1+\gamma_N\,\mathcal{A}(\Omega)^2\Big], \tag{5.5}$$

where $\gamma_N > 0$ is a constant depending only on the dimension N. In this way, one arrives at the estimate

$$D_\Omega(t) \geq h_N\, \mathcal{A}_2^{N+1}(\Omega), \quad \text{for every } t \in T_\Omega,$$

that is

$$\frac{|\Omega|^{1/N}\, t}{2^{1/N}\, N\, \omega_N^{1/N}} - 1 \geq h_N\, \mathcal{A}_2^{N+1}(\Omega), \quad \text{for every } t \in T_\Omega,$$

thanks to (5.4). Taking the infimum on T_Ω on both sides and using the definition of second Cheeger constant, we eventually prove the thesis.

In order to prove (5.3), let us take a pair of optimal disjoint balls $B(x_0, r)$, $B(x_1, r) \subset \Omega$, whose common radius r is given by

$$\Lambda_2(\Omega) = r^{-1},$$

and set for simplicity $\mathcal{O}_1 := B(x_0, r) \cup B(x_1, r)$, then obviously we have

$$|\Omega \setminus \mathcal{O}_1| = |\Omega| - 2\, \omega_N\, r^N.$$

Up to a rigid movement, we can assume that $x_0 = (M, 0, \ldots, 0)$ and $x_1 = (-M, 0, \ldots, 0)$, for some $M \geq r$, then for every $t \geq 1$ we define the new centers $x_0(t) := (M + (t-1)\, r, 0, \ldots, 0)$ and $x_1(t) := ((1-t)\, r - M, 0, \ldots, 0)$: observe that $x_i(1) = x_i$, $i = 0, 1$. Finally, we set

$$\mathcal{O}_t := B(x_0(t), t\, r) \cup B(x_1(t), t\, r), \quad t \geq 1,$$

i.e. for every $t \geq 1$ this is a disjoint union of two balls of radius $t\, r$ and moreover $\mathcal{O}_t \subset \mathcal{O}_s$ if $t < s$.The latter fact implies that the function $t \mapsto |\Omega \cap \mathcal{O}_t|$ is increasing, thus $t \mapsto |\Omega \setminus \mathcal{O}_t|$ is decreasing. We exploit this fact by taking $t_0 > 1$ such that $|\mathcal{O}_{t_0}| = |\Omega|$: then we have

$$|\Omega| - 2\, \omega_N\, r^N = |\Omega \setminus \mathcal{O}_1| \geq |\Omega \setminus \mathcal{O}_{t_0}| \geq \frac{1}{2}\, \mathcal{A}_2(\Omega)\, |\Omega|,$$

where in the last inequality we used that $\mathcal{O}_{t_0}$ is admissible for the problem defining $\mathcal{A}_2(\Omega)$. From the previous, we easily obtain

$$\frac{|\Omega|}{r^N} \geq \frac{2\, \omega_N}{(1 - 1/2\, \mathcal{A}_2(\Omega))},$$

which finally gives (5.3), just by raising both members to the power $1/N$, using the elementary inequality $(1-t)^{-1/N} \geq 1 + 1/N\, t$ for $t < 1$ and recalling that $r = \Lambda_2(\Omega)^{-1}$. □

6. Sharpness of the estimates: examples and open problems

In the estimates of Theorems 4.2 and 5.2, we have shown that for every set the relevant notion of *deficit* dominates a certain power κ of the asymmetry $\mathcal{A}_2$. If in addition to this, one could prove that for some sets converging to the optimal shape (i.e. a disjoint union of two equal balls), the deficit and $\mathcal{A}_2^\kappa$ have the same decay rate to 0, then these estimates would turn out to be sharp. We devote the last section to this interesting issue.

6.1. Quantitative Hong–Krahn–Szego inequality

Here, the question of sharpness is quite a delicate issue. First of all, observe that in contrast with the case of the Faber–Krahn inequality, the exponent of the asymmetry κ_2 blows up with N. For this reason, one could automatically guess that κ_2 is not the sharp exponent. However, it has to be noticed that this dependence on N is directly inherited from the geometrical estimate (4.9), *which is indeed sharp*. Let us fix a small parameter $\varepsilon > 0$ and consider the following set

$$\Omega^\varepsilon = \{(x_1, x') \in \mathbb{R}^N \,:\, (x_1 + 1 - \varepsilon)^2 + |x'|^2 < 1\} \\ \cup \{(x_1, x') \in \mathbb{R}^N \,:\, (x_1 - 1 + \varepsilon)^2 + |x'|^2 < 1\},$$

which is just the union of two balls of radius one, with an overlapping part whose area is of order $\varepsilon^{(N+1)/2}$. We set

$$\Omega^\varepsilon_+ = \{(x_1, x') \in \Omega^\varepsilon \,:\, x_1 \geq 0\} \text{ and } \Omega^\varepsilon_- = \{(x_1, x') \in \Omega^\varepsilon \,:\, x_1 \leq 0\},$$

and it is not difficult to see that $\mathcal{A}(\Omega^\varepsilon_+) = O(\varepsilon^{(N+1)/2})$, while on the contrary $\mathcal{A}_2(\Omega^\varepsilon) = O(\varepsilon)$ which means

$$\mathcal{A}_2(\Omega^\varepsilon)^{(N+1)/2} \simeq \mathcal{A}(\Omega^\varepsilon).$$

i.e. both sides in (4.9) are asymptotically equivalent, as the area of the overlapping region goes to 0 (see [8, Example 3.4], for more details on this example). And in fact one can use these sets Ω^ε to show that the *sharp exponent in* (4.2) *has to blow-up with the dimension*. Also observe that in the proof of Theorem 4.2, the precise value of κ_1 plays no role, so the same proof actually gives (4.2) with

$$\kappa_2 = (\text{sharp exponent for (4.1)}) \times \frac{N+1}{2}.$$

Though we strongly suspect this κ_2 not to provide the right decay rate, currently we are not able to solve this issue, which seems to be quite a changelling one even for $p = 2$.

6.2. Second Cheeger constant

Also in this case, the exponent $N + 1$ in (5.2) seems not to be sharp in the decay rate of the deficit. In order to shed some light on this fact, we estimate the deficit for $\mathcal{C}_2$ of the same set Ω^ε as before. First of all, thanks to the symmetries of Ω^ε, it is not difficult to see that $\mathcal{C}_2(\Omega^\varepsilon) = \mathcal{C}_1(\Omega^\varepsilon_+) = \mathcal{C}_1(\Omega^\varepsilon_-)$. Then we have

$$\begin{aligned} h_N\, \mathcal{A}_2(\Omega^\varepsilon)^{N+1} &\leq \frac{|\Omega^\varepsilon|^{1/N}\, \mathcal{C}_2(\Omega^\varepsilon)}{2^{1/N}\, N\, \omega_N^{1/N}} - 1 = \frac{|\Omega^\varepsilon|^{1/N}\, \mathcal{C}_1(\Omega^\varepsilon_+)}{2^{1/N}\, N\, \omega_N^{1/N}} - 1 \\ &\leq \frac{|\Omega^\varepsilon_+|^{1/N-1}\, P(\Omega^\varepsilon_+)}{N\, \omega_N^{1/N}} - 1, \end{aligned}$$

so that the deficit of this inequality is controlled from above by the isoperimetric deficit of one of the two cut balls. We then estimate the right-hand side in the previous expression: observe that setting $\vartheta = \arccos(1-\varepsilon)$, we have for instance

$$P(\Omega_+^\varepsilon) = N\omega_N + \omega_{N-1}\left[(\sin\vartheta)^{N-1} - (N-1)\int_0^{\sin\vartheta} \frac{t^{N-2}}{\sqrt{1-t^2}}\,d\varrho\right],$$

and

$$|\Omega_+^\varepsilon| = \omega_N - \omega_{N-1}\int_{\cos\vartheta}^{1} (1-t^2)^{\frac{N-1}{2}}\,dt,$$

then

$$P(\Omega_+^\varepsilon) \simeq N\,\omega_N - \frac{N-1}{N+1}\,\frac{\omega_{N-1}}{2}\,\vartheta^{N+1} \text{ and}$$
$$|\Omega_+^\varepsilon|^{1/N-1} \simeq \omega_N^{\frac{1-N}{N}}\left(1 + \frac{N-1}{N(N+1)}\,\frac{\omega_{N-1}}{\omega_N}\,\vartheta^{N+1}\right),$$

from which we can infer

$$|\Omega_+^\varepsilon|^{1/N-1}\,P(\Omega_+^\varepsilon) - N\,\omega^{1/N} \simeq \frac{N-1}{N+1}\,\omega_{N-1}\,\omega_N^{\frac{1-N}{N}}\,\vartheta^{N+1} \simeq c_N\,\varepsilon^{\frac{N+1}{2}}.$$

In the end, we get

$$C_1\,\mathcal{A}_2(\Omega^\varepsilon)^{N+1} \le |\Omega^\varepsilon|^{1/N}\,\mathcal{C}_2(\Omega^\varepsilon) - 2^{1/N}\,N\,\omega_N^{1/N} \le C_2\,\mathcal{A}_2(\Omega^\varepsilon)^{\frac{N+1}{2}}, \quad (6.1)$$

where we used that $\mathcal{A}_2(\Omega^\varepsilon) \simeq \varepsilon$. Notice that this estimate implies in particular that, also in this case, *the sharp exponent is dimension-dependent and it blows up as N goes to* ∞.

We point out that the previous computations give the correct decay rate to 0 of the quantity $\mathcal{C}_2(\Omega^\varepsilon)-\mathcal{C}_2(B)$, which is $O(\varepsilon^{(N+1)/2}) = O(\mathcal{A}_2(\Omega_\varepsilon)^{(N+1)/2})$. Indeed, from the right-hand side of (6.1) we can promptly infer that

$$\mathcal{C}_2(\Omega^\varepsilon) = \mathcal{C}_1(\Omega_+^\varepsilon) \le N + c\,\varepsilon^{\frac{N+1}{2}} = \mathcal{C}_1(B) + c\,\varepsilon^{\frac{N+1}{2}}.$$

Now assume that $\mathcal{C}_1(\Omega_+^\varepsilon) \le \mathcal{C}_1(B) + \omega(\varepsilon)$ for some modulus of continuity ω such that $\omega(\varepsilon) = o(\varepsilon^{(N+1)/2})$ as ε goes to zero, in this case we would obtain

$$0 \le |\Omega^\varepsilon|^{1/N}\,\mathcal{C}_1(\Omega_+^\varepsilon) - 2^{1/N}\,N\,\omega_N^{1/N} \le -K\,\varepsilon^{\frac{N+1}{2}},$$

for some constant $K > 0$ independent of ε. This gives a contradiction, thus proving that

$$\mathcal{C}_2(\Omega^\varepsilon) - \mathcal{C}_2(B) \simeq \varepsilon^{\frac{N+1}{2}}.$$

6.3. Second eigenvalue of $-\Delta_\infty$

On the contrary, it is not difficult to see that the quantitative estimate (5.3) is sharp. By still taking the set Ω^ε as before, we observe that

$$\Lambda_2(\Omega^\varepsilon) = \Lambda_1(\Omega^\varepsilon_+) = \frac{2}{2-\varepsilon} \simeq 1 + \frac{\varepsilon}{2} \quad \text{and}$$
$$|\Omega_\varepsilon|^{1/N} \simeq \omega_N^{1/N} \left(1 - \frac{\omega_{N-1}}{\omega_N} \frac{2^{\frac{N+1}{2}}}{N\,(N+1)}\, \varepsilon^{\frac{N+1}{2}}\right),$$

while $\mathcal{A}_2(\Omega_\varepsilon) = O(\varepsilon)$ as already observed. Then

$$|\Omega|^{1/N}\, \Lambda_2(\Omega) - \omega_N^{1/N} \simeq \mathcal{A}_2(\Omega),$$

proving the sharpness of (5.3). We remark that in this case the sharp exponent does not depend on the dimension, in contrast with the cases $p \in [1, \infty)$.

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