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# Hydrodynamic Performances of Aeronautical Propeller for Drones

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**Abstract.** Currently, several countries already invest on the application of drones for different scopes. These purposes are mainly civilian and military and some of these include the presence of water as search and rescue, inspection of pipelines, getting a bird's eyes view over oil spills or structural integrity monitoring of bridges. For this reason, an interesting feature for drones is their ability to maneuver in fluids with different density as air and water. This research topic, shows strong scientific potentiality but it also represents a great challenge from scientific and technical view point: i) aeronautical propeller for drones in water must generate reverse thrust or braking force through an off-design rotation in off-design condition (different fluid's density); ii) the propeller rotational velocity in water is generally one order of magnitude lower than the one for air application requiring the motors to be properly desing to meet such a wide operational range. In this paper, an experimental campaign aimed at the performance quantification of a three-bladed propeller for drone propulsion in water has been carried out in a towing tank. In particular, the generated forces are acquired through a load cell for different advance ratios  $J$  provided by varying both rotational regime and free stream velocity. The results show expected losses in the performance for the off-design rotation in terms of both thrust and efficiency. At low rotational speeds, higher values of efficiency are presented for small advance ratio. The maximum efficiency increase for higher RPS and it's slightly influenced by small variation of propeller disk angle.

## 1. Introduction

Drones represent one of the most promising technology for the future of many fields in terms of research, development and innovation. Currently, several countries already invest on the application of drones for different scopes in order to enhance the efficacy of human intervention like in an military or civilian use [1]. Thus, drones are the ultimate solution to replicate the human presence in several areas and its adaptability in different condition may represent a key feature in order to have reliable device for many purposes [2]. Many studies face the difficulties of a drone that can change its configuration in function of the mission [3, 4]. Some of these purposes include the presence of water like inspection of pipelines, getting a bird's eyes view over oil spills or structural integrity monitoring of bridges. For these reasons, the most useful feature in which the industrial market is interested is the ability to maneuver in fluids with different density like air and water. An amphibious drone shows a strong scientific potentiality before



the industrial one as an underdeveloped topic that can contribute in terms of innovation. The crucial components of drones are for sure the propellers. This research topic face hard challenges for two main reason: i) aeronautical propeller for drones are designed for one rotational direction but in water they must generate reverse thrust or braking force through an off-design rotation added to the off-design condition due to the different fluid's density; ii) the propeller rotational speed in water is about 1-10 RPS. The rotational speed of the propellers for drones in air can reach values one order of magnitude greater thus the motors must be *had-oc* for the purpose. The fields of propeller aerodynamics and hydrodynamics is one of the most complex scientific topic in terms of design and simulation. Many numerical methods were developed to investigate this topic starting from the 1960s based on the lifting line theory [5]. With the advancement of computer performance the Computational Fluids Dynamics (CFD) methods became more accurately and with the modern numerical model like Blade element theory (BET), Blade element momentum theory (BEMT) [6], Free Vortex Method (FVM) [7]) it becomes possible to model very complex fluid flow problems and solve the RANS equations. The numerical simulation allow to reduce drastically the time and cost of a propeller design but need some input parameter from experiments to produce accurate results. It's therefore crucial in order to develop new numerical method and improve their accuracy to carry on experimental campaigns that provide empirical data. The separation between the aerodynamic and the hydrodynamic propeller is no longer clear-cut since the firsts prototypes of amphibious drones are already developed and industrially available. In order to evaluate the performance of this new technology is therefore important and interesting analyze the drone propeller performances in an off-design condition like in the water. To analyze these performances can contribute in future in order to accelerate the design of innovative propeller for an amphibious missions and their numerical simulations. In this paper an experimental campaign aimed at quantifying the performance of an aeronautical propeller for drone in water is carried out. A three-blade aeronautical propeller for drones is experimentally characterized in water at the Institute of Marine Engineering in Rome, CNR-INM. In particular, the forces generated are acquired through a load cell for different advance ratios provided by varying both rotational regime and free stream velocity. The results shows expected losses in the performance for the off-design rotation in terms of thrust and efficiency. The low rotational speed evaluated present their best efficiency for small advance ratio and the maximum increase with higher RPS and small variation of propeller disk angle appear to slightly affect the performances.

## 2. Test Cases and Experimental Setup

The experimental campaign has been carried out at the Institute of Marine Engineering (CNR-INM) in the "Umberto Pugliese" towing tank. The facility is 470m long, with 13.5m width and 6.5m depth. The carriage used to move along the canal has an electric drive system with 4 pairs of drive wheels. Each pair of wheels is coupled with one electrical motor of 92 kW while the carriage advancement speeds reach 15.0 m/s with an accuracy of 1 mm/s. The propulsion motor is a CC shunt wound-IP 23 with 5 kW power and the velocity field is from 60 to 3000 revolution per minute.

The propeller under investigation is a KDE propeller, model no. KDE-CF155-TP 15.5" x 5.3, Triple-Edition Series. The described propeller is made of Carbon-Fiber supplied for military, commercial and industrial applications. The assembled propeller reach a diameter of 393,32 mm and the loads are measured with the six axis dynamometer Kempf & Remmers H29/6. The measurable range of torque is from -15 to +15 Nm and the thrust can varies from -400 to +400 N. The time series of the loads are acquired with a sample frequency of 1000 Hz through the software Dewesoft X3. The propeller is designed for a right rotational direction so the sign convention is determined as it's described in table 1.

The four quadrant are acquired for a rotational speed of 3.87 RPS and the carriage velocities

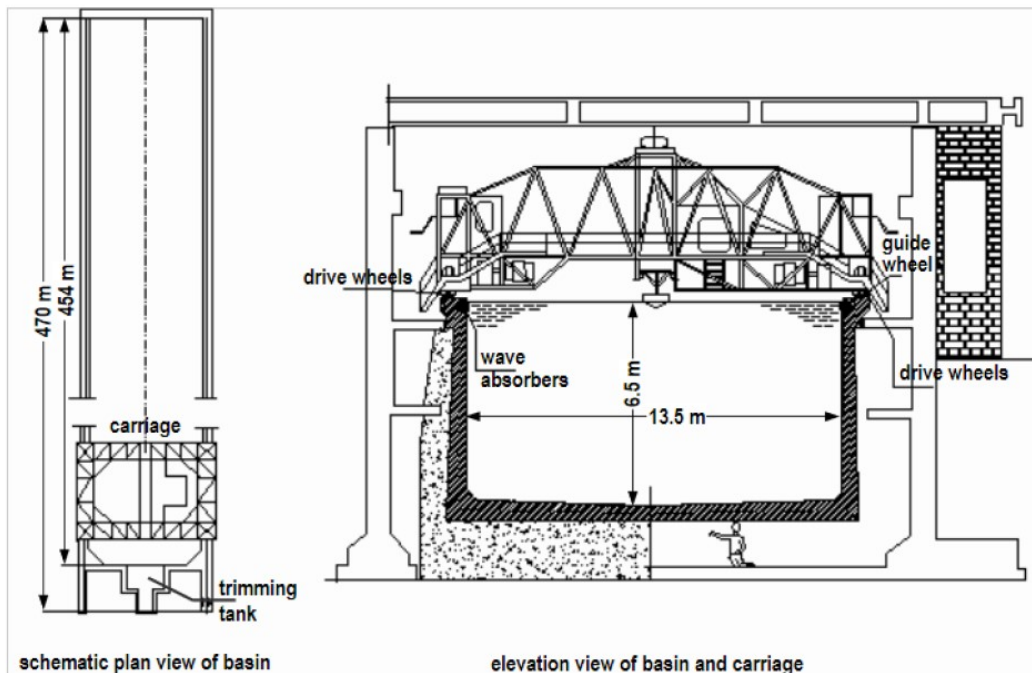


Figure 1. The townig tank "Umberto Pugliese" at CNR-INM.

	Rotation	Advance Ratio	Thrust	Torque
Quadrant 1	R	+ (carriage forward)	+	+
Quadrant 2	R	- (carriage backward)	+	+
Quadrant 3	L	- (carriage backward)	-	-
Quadrant 4	L	+ (carriage forward)	-	-

Table 1. Sign convention for the experimental campaign at the Institute of Marine Engineering (CNR-INM) in the townig tank "Umberto Pugliese"

are identified in order to generate specific advance ratios from about -0.7 to 0.7 with 0.1 step. For the rotational speed of 5.27 and 5.86 RPS the quadrants under investigation are the first and third, relative to positive advance ratio with right rotation and negative advance ratio with left rotation. The test matrix is prepared in order to investigate the quadrants in different condition by varying rotational direction/speed and carriage velocity. The four conditions that identify the four quadrants are represented in fig. 2

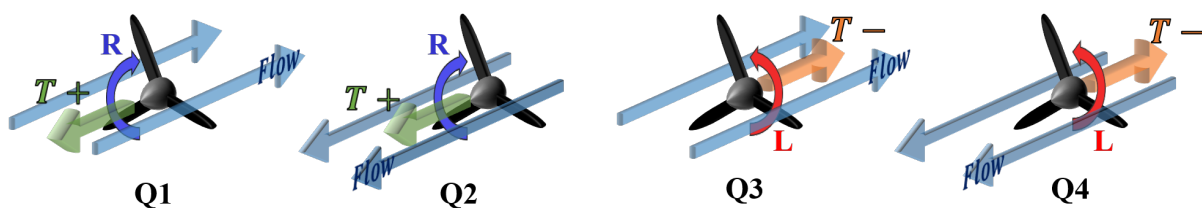


Figure 2. Representation of the four conditions that identify the four quadrants.

The experimental campaign is summarized in table 2.

	RPS	Rotation	Carriage Vel [m/s]	Advance Ratio	Propeller Disk Angle[°]
1	Variable	R	0	0	0
2	Variable	L	0	0	0
3	3.87	R	Variable (+)	-0.7<J<0.7	0
4	3.87	R	Variable (-)	-0.7<J<0.7	0
5	3.87	L	Variable (+)	-0.7<J<0.7	0
6	3.87	L	Variable (-)	-0.7<J<0.7	0
7	5.27	R	Variable (+)	-0.7<J<0.7	0
8	5.27	L	Variable (-)	-0.7<J<0.7	0
9	5.86	R	Variable (+)	-0.7<J<0.7	0
10	5.86	L	Variable (-)	-0.7<J<0.7	0
11	3.87	R	Variable (+)	-0.7<J<0.7	2.5
12	3.87	L	Variable (+)	-0.7<J<0.7	10
13	5.86	R	Variable (+)	-0.7<J<0.7	2.5
14	5.86	L	Variable (+)	-0.7<J<0.7	10

**Table 2.** Test matrix of the experimental campaign at CNR-INM

In order to test at the same conditions, a fluid dynamics analogy is generated producing the same Reynolds number of the air conditions. In table 3 the fluid dynamics analogy is presented with a comparison of the rotational regime of the propeller in air and water for the same Reynolds number.

RPS <sub>AIR</sub>	RPS <sub>WATER</sub>	Re
20	1.41	35636
26.7	1.87	47515
30	2.11	53454
43	3.02	76618
55	3.87	97999
66.3	4.66	118193
75	5.27	133636
83.3	5.86	148484

**Table 3.** Test matrix of the fluid dynamic analogy

Following the scientific conventional method the Reynolds number is calculated for the propeller chord length  $c$  at 70% radius. From the definitions of Reynolds number :

$$Re = \frac{VL}{\nu} \quad (1)$$

Where  $L$  for the case of naval propeller is generally the chord length at the 70% of the propeller radius [ $m$ ],  $V$  is the free stream velocity [ $m/s$ ] and  $\nu$  is the cinematic viscosity [ $m^2/s$ ]. It's important to consider that the free stream velocity  $V$  of the chord depends on the longitudinal velocity and also on the rotational velocity of the propeller. The contribution of the rotational velocity is calculated from the periperhal rotational velocity of the propeller as:

$$v_p = 2\pi \cdot f \cdot r_{70\%} \quad (2)$$

Where  $f$  is the rotational frequency and  $r_{70\%}$  is the 70% of propeller radius. Known the

advancing velocity and the periperhal rotational velocity the actual speed is:

$$v_{eff} = \sqrt{v^2 + v_p^2} \quad (3)$$

Where  $v$  is the advancing velocity and  $v_p$  is the periperhal velocity.

### 3. Data Analysis

Several studies developed several theories for analyze propeller performance. The most common theories are Momentum theory and Blade element theory [8]. Both of these theoretical approach have been improved in time. These analysis starting from replacing the propeller with a disc that produces uniform changes in stream velocity of the flow that pass through it [9]. The sophistication continued by starting to consider the number of blade and their thickness till the Goldstein lifting-line model, where the propeller is replaced by a series of horse shoes vortices [10]. At the actual state of the art several studies provide many analytical models that can be simulated with numerical models in order to produce a prediction with small approximation and higher levels of accuracy. All the theories mentioned above allow to calculate the main magnitudes and coefficients that describe the propeller performance. The most common way to quantify the efficiency of propeller is to calculate the dimensionless coefficients of thrust ( $K_T$ ), torque ( $K_Q$ ) and power ( $K_P$ ).

$$K_T = \frac{T}{\rho n^2 D^4} \quad K_Q = \frac{Q}{\rho n^2 D^5} \quad K_P = \frac{P}{\rho n^3 D^5} \quad (4)$$

Where  $D$  is the propeller diameter [ $m$ ],  $T$  is the thrust [ $N$ ],  $Q$  is the torque [ $Nm$ ],  $n$  is the rotation velocity [RPS] and  $\rho$  is the fluid density [ $Kg/m^3$ ]. These coefficients are generally investigated along the advance ratio ( $J$ ) calculated as it follows:

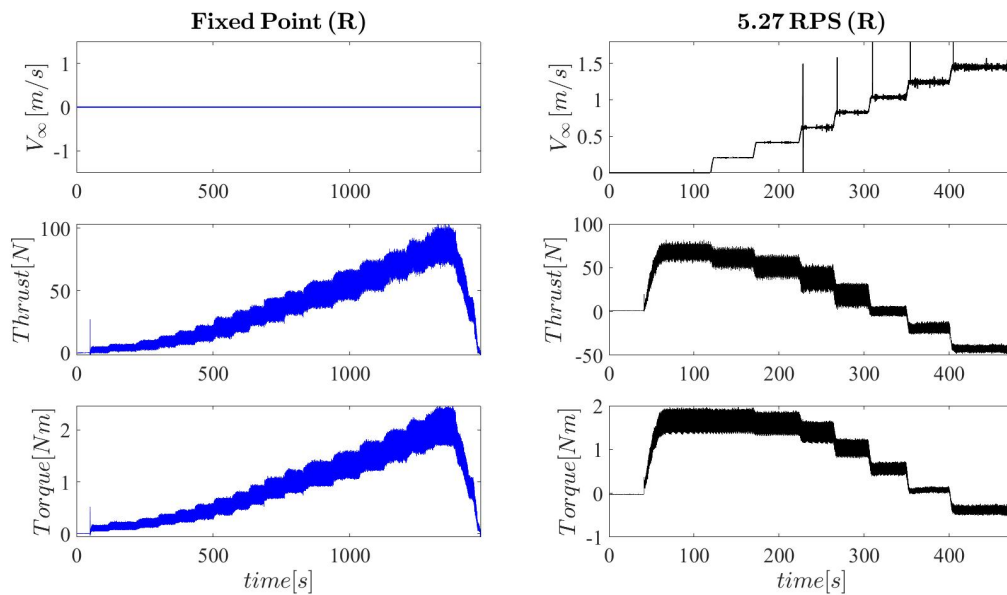
$$J = \frac{2\pi V_\infty}{\omega D} = \frac{V_\infty}{nD} \quad (5)$$

Where  $\omega$  is the rotational speed [ $rad/s$ ] and  $V_\infty$  is the free stream velocity [ $m/s$ ]. The ultimate useful coefficient for the propellers is the efficiency calculated as:

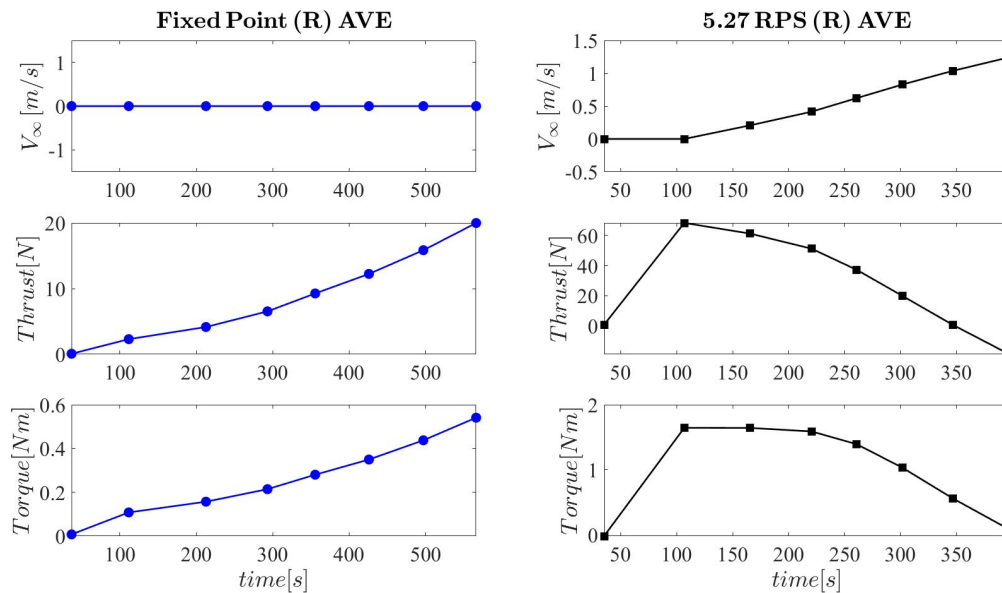
$$\eta = \frac{K_T J}{K_P} = \frac{K_T J}{2\pi K_Q} = \frac{TV_\infty}{P} \quad (6)$$

The main differences of the design condition for the propeller under investigation are due to the rotational direction, rotational velocity and fluid density. The aeronautical propeller are generally designed for certain flight condition and one rotational direction (right direction for the investigated one). In this experimental campaign the left rotational direction is investigated producing the first off-design condition. The second off-design condition is due to the rotational speed that in water is generally about 1-10 RPS while in air the rotational speed can overcome the 100 RPS. The main difference from the in-design condition is ascribed to the extreme different fluid density. It is known that the water has a density about  $1000 \text{ Kg/m}^3$  that is about 1000 times the air density. This condition imposes extreme off-design dynamics that can generate forces and coefficients totally discord from the ones of the in-design condition. The acquisition of the main forces was carried out at the "Umberto Pugliese" towing tank through the software DEWESOFT X3 and the time series are processed. Two example of different cases are reported in figure 3.

In order to process the acquired database the averaged values of the magnitude of interest are extracted from the time series. The net thrust and torque are identified for each segment of



**Figure 3.** Two cases of the experimental campaign. On the left column the acquisition of thrust and torque with stationary carriage (hovering condition for the propeller) at different RPS of the propeller (right rotation). In the right column the acquisition of thrust and torque with advancing carriage for a rotational regime of 5.27 RPS (right rotation)



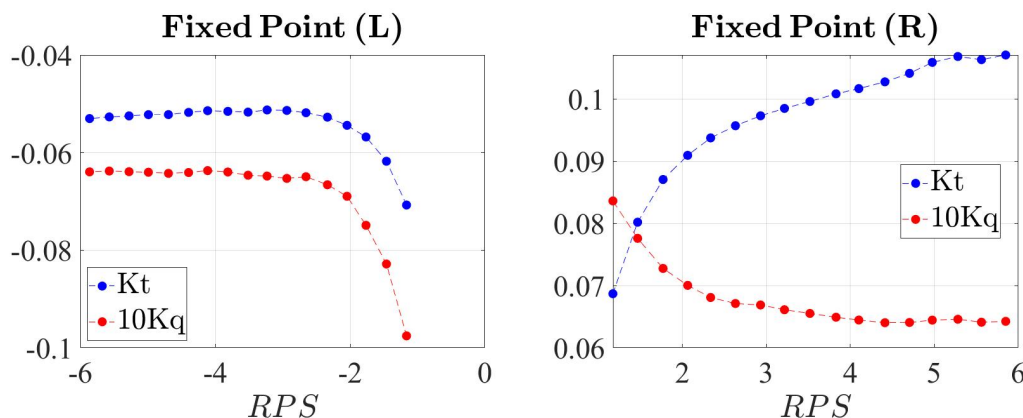
**Figure 4.** Averaged values of the two cases of the experimental campaign shown in figure 3. On the left column the acquisition of thrust and torque with stationary carriage (hovering condition for the propeller) at different RPS of the propeller (right rotation). In the right column the acquisition of thrust and torque with advancing carriage for a rotational regime of 5.27 RPS (right rotation)

the time series where the carriage velocity or the rotational regime is approximately constant. The averaged values of the magnitude in figure 3 are shown in figure 4

Starting from this pre-qualification of the signals, the mail coefficients and indexes can be calculated. Thus, the quantification of the performances of the aeronautical propeller under investigation in water can be done.

#### 4. Results

The propeller performances, as introduced in Sec.3 are indicated in terms of thrust coefficient  $K_T$ , torque coefficient  $K_Q$  and propeller efficiency  $\eta$ . All these parameter are expressed in function of the advance ratio  $J$ . The results in figure 5 describe the coefficients in function of the RPS acquired in hovering condition (stationary carriage). The plots state that for the hovering condition the propeller increase its thrust coefficient increasing the RPS for the design rotational direction (right). The torque coefficient in right rotation decrease as the increasing RPS till a near constant value for regimes over 3 RPS. Differently from the typical trends of the coefficient for the design rotation, the left rotation in hovering condition show an unusual trend. The thrust coefficient for the left rotation in hovering increase in a transitory state with increasing RPS till a constant value over 3 RPS. The trend is the same for the torque coefficient increasing at low regimes and stabilizing subsequently. This brings to the conclusion that in the inverse rotation in hovering condition the aeronautical propeller reaches a maximum thrust earlier with respect to design rotation increasing the rotational regime. This is probably due to the propeller design, that is not optimized for both rotational direction.



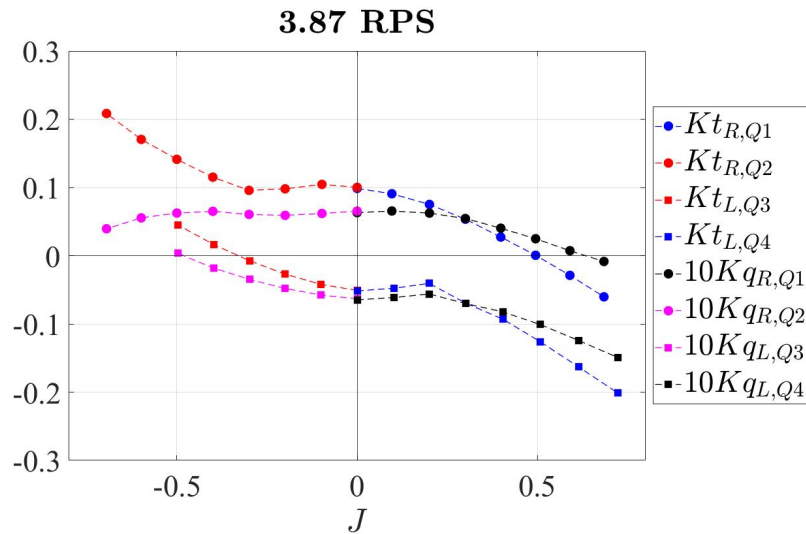
**Figure 5.** Thrust and torque coefficients for the hovering condition with left (left side) and right (right side) rotation.

The results of the four quadrant acquisition data processing is shown in figure 6. The figure shows the propeller performance for the regime of 3.87 RPS in 4 configuration :

- (i) Right rotation, carriage moving onward (Quadrant 1)
- (ii) Right rotation, carriage moving backward (Quadrant 2)
- (iii) Left rotation, carriage moving backward (Quadrant 3)
- (iv) Left rotation, carriage moving onward (Quadrant 4)

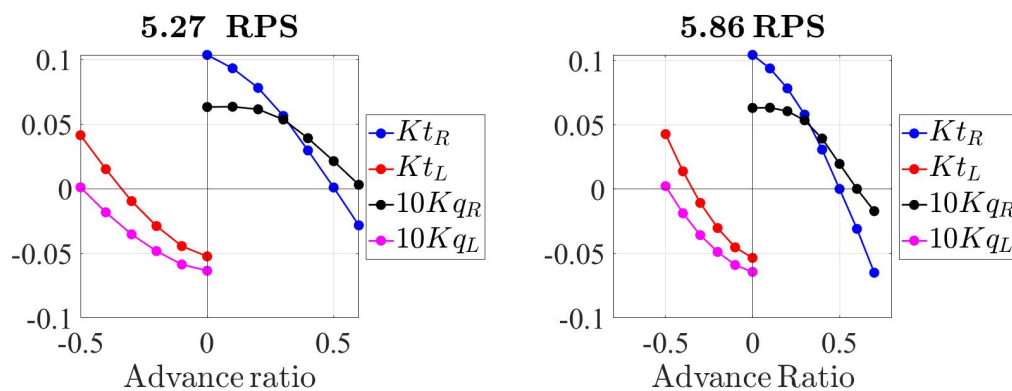
The behaviour of torque and thrust coefficients for the same condition but opposite rotational direction shows symmetrical trends on the x axis. For example, the thrust coefficients for the quadrant 1 and 3 describe specular trends on the plot by varying the advance ratio. The main noticeable difference is that the results for the off-design rotation (left rotation) seem to be





**Figure 6.** Thrust and torque coefficients of 4 quadrants at rotational regime of 3.87 RPS.

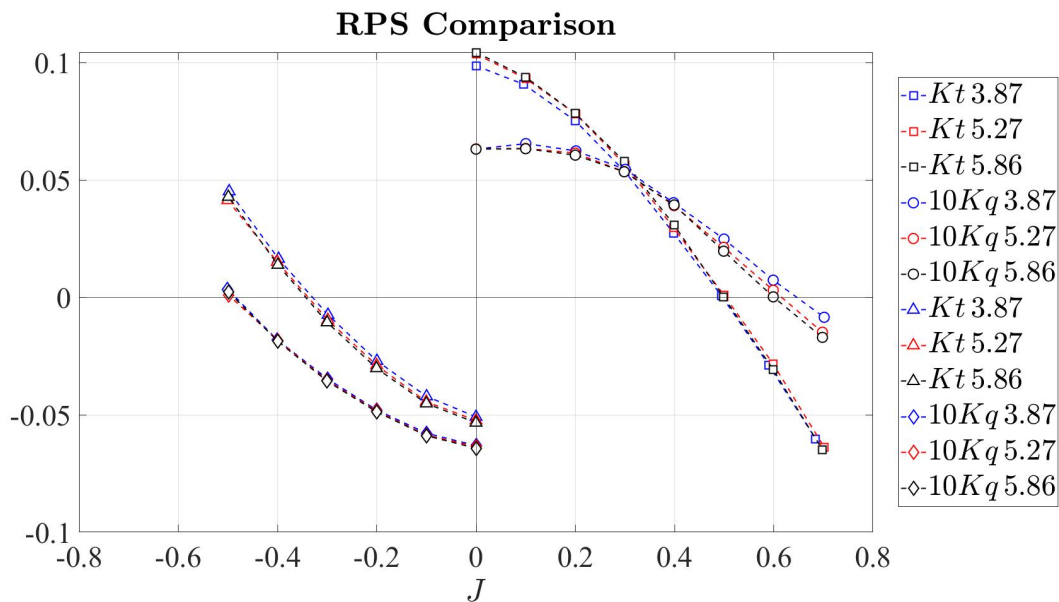
shifted towards the x axis. Indeed, the off-design rotation reaches the zero thrust point earlier than the right rotation. Thus, a loss over 50% is shown compared to the value of the in-design rotation (right rotation). The torque coefficients show specular trends also in the value of the non dimensional index. Thrust and torque coefficients are investigated for others rotational regime for the quadrants 1 and 3. The results are shown in figure 7.



**Figure 7.** Thrust and torque coefficients of quadrants 1 and 3 for rotational regime of 5.27 and 5.86 RPS.

As for the previous case the thrust coefficient shows specular trends in the quadrant under investigation. Also for these rotational regime a loss over 50% is noticeable for the off-design rotation. The torque coefficient describes similar trends in the two quadrants, reaching the zero torque earlier in quadrant 3. An important aspect of the three rotational regimes to highlight is that the coefficients decay very rapidly with small increase of the advance ratio. Thus, the propeller performance at low rotational regime can be acceptable only for small advance ratios. A comparison of the thrust and torque coefficients by varying the rotational regime is shown in figure 8.

The coefficients are almost overlaid, describing similar trajectories. The main deduction that can be done is that the coefficients are slightly dependent from the rotational speed. The same

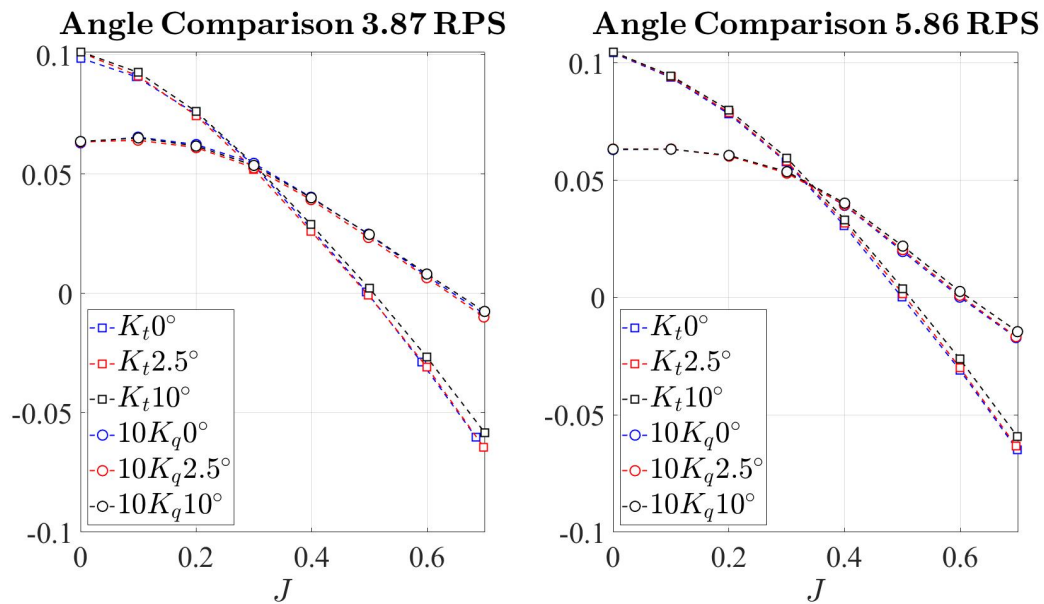


**Figure 8.** Thrust and torque coefficients of quadrants 1 and 3 for rotational regime of 3.87, 5.27 and 5.86 RPS in function of the advance ratio  $J$ .

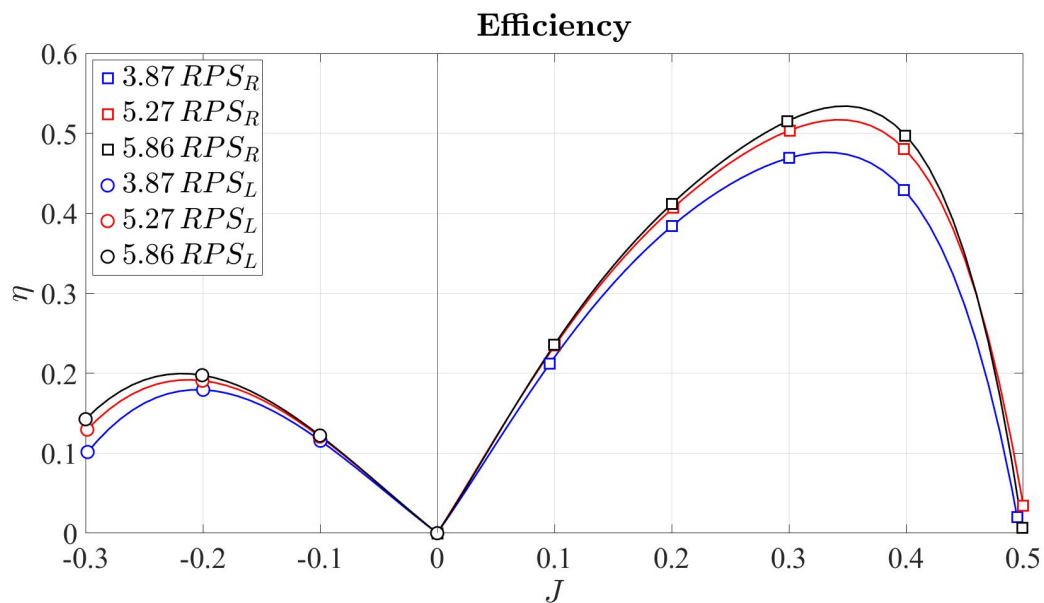
hypothesis can be done for the inverse rotational direction, in fact in quadrant 3 the trends are similar. The experimental campaign is carried out to investigate an orthogonal flow on the propeller in water. It's interesting to investigate an additional off-design condition due to different flow angle of incidence. The aeronautical propeller, especially the ones for drones, are designed for tangential flow on the propeller disk. In order to analyze this condition the propeller disk angle is varied to  $2.5^\circ$  and  $10^\circ$ . The results in terms of thrust and torque coefficients in function of the advance ratio are shown in figure 9 for different rotational regimes.

The evidence of this results is that there is a small influence of thrust and torque with small variation of the propeller disk angle. Thus, the propeller performance dependency on small variation of the flow angle of incidence is minimal. This phenomenon is probably due to the propeller design that, as previously discussed, optimize the performances for tangential flows. In this way, orthogonal flows or flows with small tangential components do not heavily influence the performances of the drone propeller under investigation. The figure 10 shows the propeller efficiency for the cases under investigation. The efficiency is generally defined as the ratio between the output and the input energy. For propeller aerodynamics/hydrodynamics it can be traduced as the ratio between thrust generated and the input power supplied by the motor. Generally propellers, especially the aeronautical propeller ones, are designed in order to reach the maximum efficiency for certain values of advance ratio/rotational regime. These value are usually the cruise condition of the aircraft. It is clear that subsequently the efficiency decreases in transitional condition like take off or landing. Thus, the expected qualitative behaviour is not different from the one in air even with the fluid density increased about 1000 times.

The results show a typical efficiency trend. The maximum efficiency for the quadrant 1 is located at an advance ratio between 0.3 and 0.4. For the quadrant 3 the maximum efficiency is slightly over the advance ratio equal to 0.2. It is noticeable that the maximum efficiency increase with the rotational regime, probably due to the increasing of the Reynolds number. In addition it is interesting to note that the maximum efficiency slightly shifts towards higher advance ratio with increasing RPM. The main results is that for the "reverse condition" of quadrant 3 the efficiency decreases drastically. The losses are about the 60%. These losses in



**Figure 9.** Thrust and torque coefficients of quadrants 1 for rotational regime of 3.87 (on the left) and 5.27 (on the right) in function of the advance ratio  $J$  for angle of disk propeller at  $0^\circ$ ,  $2.5^\circ$  and  $10^\circ$ .



**Figure 10.** Propeller efficiency of quadrants 1 and 3 for rotational regime of 3.87, 5.27 and 5.86 in function of the advance ratio  $J$ .

the performances are clearly due to the extreme off-design conditions applied on the propeller, it could be interesting to understand if these performances are sufficient to do their job to manoeuvre under water.

## 5. Summary, conclusions and future perspectives

The operative condition of drones nowadays could include the presence of water. Thus, an interesting feature for drones is their ability to maneuver in fluids with different density as air and water. An experimental campaign aimed at the performance quantification of a three-bladed propeller for drone propulsion in water has been carried out at the Institute of Marine Engineering (CNR-INM) in the "Umberto Pugliese" towing tank. The two main challenges from the scientific and technical viewpoints are: i) aeronautical propeller for drones in water must generate reverse thrust or braking force through an off-design rotation in off-design condition (different fluid's density); ii) the propeller rotational velocity in water is generally of one order of magnitude lower than the one for air application since the motors must be properly designed for such a wide operational range. The forces generated are acquired through a load cell for different advance ratios  $J$  provided by varying both rotational regime and free stream velocity. The results show over 50% loss in thrust coefficients between in-off design rotation in water. The thrust and torque coefficients describe a very rapid decaying trends with small increase of advance ratio. Thus, the propeller performance at low rotational regime can be acceptable only for small advance ratios. A comparison between three rotational regimes shows that the dimensionless coefficients are slightly dependent on the RPS. A small variation of  $2.5^\circ$  and  $10^\circ$  of the propeller disk angle demonstrates that orthogonal flows or flows with small tangential components have small influence on thrust and torque for the drone propeller under investigation. The efficiency shows a typical trend. The maximum efficiency increases with the rotational regime, probably due to the increasing of the Reynolds number. In addition maximum efficiency slightly shifts to higher advance ratio with increasing RPM. For the off-design rotation the efficiency decreases about the 60%. These losses in the performances are clearly due to the extreme off-design condition applied on the propeller, but can be interesting to evaluate if these performances are sufficient to do their job to manoeuvre under water. Future perspectives plan to carry on an experimental campaign aimed to the acquisition of the main forces in air. Compare the air and water performances could give important information about the propeller interaction with fluid at extreme different density. In addition, Particle Image Velocimetry (PIV) acquisition and spectral analysis can be interesting methodologies of analysis to apply on this research topic.

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