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Cahn–Hilliard equation with dynamic boundary conditions and mass constraint on the boundary

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A B S T R A C T

The well-known Cahn–Hilliard equation entails mass conservation if a suitable boundary condition is prescribed. In the case when the equation is also coupled with a dynamic boundary condition, including the Laplace–Beltrami operator on the boundary, the total mass on the inside of the domain and its trace on the boundary should be conserved. The new issue of this paper is the setting of a mass constraint on the boundary. The effect of this additional constraint is the appearance of a Lagrange multiplier; in fact, two Lagrange multipliers arise, one for the bulk, the other for the boundary. The well-posedness of the resulting Cahn–Hilliard system with dynamic boundary condition and mass constraint on the boundary is obtained. The theory of evolution equations governed by subdifferentials is exploited and a complete characterization of the solution is given.

Contents

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1. Introduction

The famous Cahn–Hilliard equation [7,14] offers a realistic description of the evolution phenomena related to solid–solid phase separation processes. In this paper, we are interested in the mathematical investigation of it and aim to analyze questions like existence and continuous dependence of solutions for a generalized Cahn–Hilliard equation with dynamic boundary conditions and mass constraints on the boundary. Actually, we can solve the mathematical problem and, in particular, characterize the constraint with the help of a Lagrange multiplier.

Let $0 < T < +\infty$ and let $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, be the bounded smooth domain occupied by the material. Also the boundary Γ of Ω is supposed to be smooth enough. We recall the isothermal Cahn–Hilliard equation in the following generalized form:

$$\frac{\partial u}{\partial t} - \Delta \mu = 0 \quad \text{in } Q := \Omega \times (0,T),$$
$$\mu = \tau \frac{\partial u}{\partial t} - \Delta u + \xi + \pi(u) - f, \quad \xi \in \beta(u) \quad \text{in } Q,$$

where the unknowns $u := u(x,t)$ and $\mu := \mu(x,t)$ stand for the order parameter and the chemical potential, respectively. Moreover, τ is a viscosity coefficient which can be greater or equal to 0 (we treat both cases); β stands for the subdifferential of the convex part $\widehat{\beta}$ and π stands for the derivative of the concave perturbation $\widehat{\pi}$ of a double well potential $W = \widehat{\beta} + \widehat{\pi}$, for example $W(r) = (r^2 - 1)^2/4$ with $\beta(r) = r^3$ and $\pi(r) = -r$ for all $r \in \mathbb{R}$. In general, β is assumed to be a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$. Recently, this equation was treated in some papers [11,12,16,17] when coupled with a dynamic boundary condition of the following form:

$$u_\Gamma = u_{|_\Gamma} \quad \text{on } \Sigma := \Gamma \times (0,T),$$
$$\partial_\nu u + \frac{\partial u_\Gamma}{\partial t} - \Delta_\Gamma u_\Gamma + \xi_\Gamma + \pi_\Gamma(u_\Gamma) = f_\Gamma, \quad \xi_\Gamma \in \beta_\Gamma(u_\Gamma) \quad \text{on } \Sigma,$$

where, $u_{|_\Gamma}$ denotes the trace of u and ∂_ν represents the outward normal derivative on Γ. Δ_Γ stands for the Laplace–Beltrami operator on Γ (see, e.g., [19, Chapter 3]), β_Γ and π_Γ have the same property as β and π, respectively.

About dynamic boundary conditions, let us point out that the mathematical research for the various problems was already running in the 1990's. For example, the Stefan problem with dynamic boundary conditions was treated in the series of Aiki [1–3]. Recent advances in the Cahn–Hilliard equation with dynamic boundary conditions can be found in [11,16–18,24] and the references therein.

As is well known, conservation of u is required. Therefore, under the homogeneous Neumann boundary condition

$$\partial_\nu \mu = 0 \quad \text{on } \Sigma,$$

we can realize that

$$\frac{1}{|\Omega|} \int_\Omega u(t) dx = m_0 := \frac{1}{|\Omega|} \int_\Omega u_0 dx \quad \text{for all } t \in [0,T],$$

for a given initial data u_0. The new issue of this paper is the setting of a mass constraint on the boundary. More precisely, we require that the solution u satisfies

$$k_* \le \int_\Gamma w_\Gamma u_\Gamma(t) d\Gamma \le k^* \quad \text{for all } t \in [0,T],$$

where k_* and k^* are fixed constants fulfilling $k_* \le k^*$ and w_Γ is given weight function on Γ. This kind of problem for the Allen–Cahn equation was treated in [10], by applying the abstract theory developed in [15]. In the case of the Cahn–Hilliard equation, the essential structure of the constraint has been studied in [21,22]. We can also find a similar treatment for the preservation of the constraint in [3,9].

A brief outline of the present paper along with a short description of the various items is as follows.

In Section 2, we present the main results, consisting in the well-posedness of the Cahn–Hilliard equation with dynamic boundary conditions and mass constraints on the boundary. We write the system as an evolution inclusion and characterize the solution with the help of the Lagrange multipliers. We also remark that actually there will be two Lagrange multipliers.

In Section 3, we prove the continuous dependence and of course this result entails the uniqueness property.

In Section 4, we prove the existence result. The proof is split in several steps. First, we construct an approximate solution by substituting the maximal monotone graphs with their Moreau–Yosida regularizations, in the case when $\tau > 0$. The solvability of the approximate problem is guaranteed by the abstract theory of doubly nonlinear evolution inclusions [13]. Moreover, arguing in a similar way as in [15], we show that the solution satisfies suitable regularity properties and obtain a strong characterization of the approximate problem by the Lagrange multiplier: in fact, we are able to prove uniform a priori estimates on all the components of the solution. And finally, from these estimates, we can pass to the limit and conclude the existence proof in the case $\tau > 0$. Next, we can proceed by considering the limiting problem as $\tau \to 0$ and derive the well-posedness result in the pure Cahn–Hilliard case as well.

2. Main results

In this section, we present our main result, which states the well-posedness of the Cahn–Hilliard equation with the dynamic boundary conditions and mass constraints on the boundary. We apply the treatment of the dynamic boundary conditions as in [8,10] and exploit the abstract theory of the evolution inclusion, essentially referring to [15,21].

2.1. Definition of the solution by the Lagrange multiplier

Let $0 < T < +\infty$ and $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, be the bounded domain with smooth boundary $\Gamma := \partial\Omega$. We use the notation:

$$H_0 := L^2(\Omega)_0 := \Big\{ z \in L^2(\Omega) \ : \ \int_\Omega z dx = 0 \Big\},$$

$$H_\Gamma := L^2(\Gamma), \quad V_0 := H^1(\Omega) \cap H_0, \quad V_\Gamma := H^1(\Gamma),$$

with usual norms $|\cdot|_{H_0}$, $|\cdot|_{H_\Gamma}$,

$$|z|_{V_0} := |\nabla z|_{L^2(\Omega)^d} \ \text{ for } \ z \in V_0, \qquad |z_\Gamma|_{V_\Gamma} := \left\{ \int_\Gamma \left(|z_\Gamma|^2 + |\nabla_\Gamma z_\Gamma|^2 \right) d\Gamma \right\}^{\frac{1}{2}} \ \text{ for } \ z_\Gamma \in V_\Gamma,$$

respectively. Here, ∇_Γ denotes the surface gradient on Γ (see, e.g., [19, Chapter 3]). Moreover, let V_0^* be the dual space of V_0 and $F : V_0 \to V_0^*$ denote the duality mapping defined by

$$\langle Fy, z\rangle_{V_0^*,V_0} := \int_\Omega \nabla y \cdot \nabla z dx \quad \text{for all } y, z \in V_0.$$

Then, the form $(\cdot,\cdot)_{V_0^*} : V_0^* \times V_0^* \to \mathbb{R}$,

$$(y^*, z^*)_{V_0^*} := \int_\Omega \nabla F^{-1} y^* \cdot \nabla F^{-1} z^* dx \quad \text{for all } y^*, z^* \in V_0^*,$$

yields the inner product in V_0^*. Here, F^{-1} is the inverse operator of F and its restriction to H_0 works as follows: if $z \in H_0$, $y = F^{-1}z$ uniquely solves the boundary value problem

$$\begin{cases} -\Delta y &= z \quad \text{a.e. in } \Omega, \\ \partial_\nu y &= 0 \quad \text{a.e. on } \Gamma, \\ \displaystyle\int_\Omega y dx &= 0, \end{cases}$$

and consequently lies in $H^2(\Omega)$, due to well-known elliptic regularity results. The reader can check that testing $-\Delta y = z$ by some $\tilde{z} \in V_0$ leads to

$$\int_\Omega \nabla y \cdot \nabla \tilde{z} dx = \int_\Omega z \tilde{z} dx \quad \text{for all } \tilde{z} \in V_0,$$

that is, $z = Fy$ as expected. Finally, by virtue of the Poincaré–Wirtinger inequality there exists a constant $C_0 > 0$ such that

$$|z|_{H_0}^2 \le C_0 |z|_{V_0}^2 \quad \text{for all } z \in V_0. \tag{2.1}$$

Then, we obtain $V_0 \hookrightarrow\hookrightarrow H_0 \hookrightarrow\hookrightarrow V_0^*$, where "$\hookrightarrow\hookrightarrow$" stands for the dense and compact embedding, namely (V_0, H_0, V_0^*) is a standard Hilbert triplet. The same considerations hold for H_Γ and V_Γ. Now, we set

$$\boldsymbol{H}_0 := H_0 \times H_\Gamma, \quad \boldsymbol{V}_0 := \big\{(u, u_\Gamma) \in V_0 \times V_\Gamma \ : \ u_{|_\Gamma} = u_\Gamma \text{ a.e. on } \Gamma\big\},$$

where $u_{|_\Gamma}$ denotes the trace of u. Observe that $\boldsymbol{H}_0$ and $\boldsymbol{V}_0$ are Hilbert spaces with the inner products

$$(\boldsymbol{u}, \boldsymbol{z})_{\boldsymbol{H}_0} := (u, z)_{H_0} + (u_\Gamma, z_\Gamma)_{H_\Gamma} \quad \text{for all } \boldsymbol{u} := (u, u_\Gamma),\ \boldsymbol{z} := (z, z_\Gamma) \in \boldsymbol{H}_0,$$
$$(\boldsymbol{u}, \boldsymbol{z})_{\boldsymbol{V}_0} := (u, z)_{V_0} + (u_\Gamma, z_\Gamma)_{V_\Gamma} \quad \text{for all } \boldsymbol{u} := (u, u_\Gamma),\ \boldsymbol{z} := (z, z_\Gamma) \in \boldsymbol{V}_0$$

and related norms. Then, we obtain $\boldsymbol{V}_0 \hookrightarrow\hookrightarrow \boldsymbol{H}_0 \hookrightarrow\hookrightarrow \boldsymbol{V}_0^*$ (see, e.g., [10, Appendix]). As a remark, let us restate that if $\boldsymbol{u} = (u, u_\Gamma) \in \boldsymbol{V}_0$ then u_Γ is exactly the trace of u on Γ, while, if $\boldsymbol{u} = (u, u_\Gamma)$ is just in $\boldsymbol{H}_0$, then $u \in L^2(\Omega)$ and $u_\Gamma \in H_\Gamma$ are independent.

The initial-value problem for the Cahn–Hilliard equation with dynamic boundary conditions can be set as the following system (2.2)–(2.7):

$$\frac{\partial u}{\partial t} - \Delta \mu = 0 \quad \text{in } Q, \tag{2.2}$$

$$\mu = \tau \frac{\partial u}{\partial t} - \Delta u + \xi + \pi(u) - f, \quad \xi \in \beta(u) \quad \text{in } Q, \tag{2.3}$$

$$\partial_\nu \mu = 0 \quad \text{on } \Sigma, \tag{2.4}$$

$$u_\Gamma = u_{|_\Gamma}, \quad \text{on } \Sigma, \tag{2.5}$$

$$\partial_\nu u + \frac{\partial u_\Gamma}{\partial t} - \Delta_\Gamma u_\Gamma + \xi_\Gamma + \pi_\Gamma(u_\Gamma) = f_\Gamma, \quad \xi_\Gamma \in \beta_\Gamma(u_\Gamma) \quad \text{on } \Sigma, \tag{2.6}$$

$$u(0) = u_0 \quad \text{in } \Omega, \quad u_\Gamma(0) = u_{0\Gamma} \quad \text{on } \Gamma, \tag{2.7}$$

where $\tau \geq 0$ is a viscosity coefficient. Testing (2.2) by the constant function 1 and using the boundary condition (2.4), we realize that $\partial u/\partial t$ has zero mean value in Ω. Then, a formal test of (2.2) and (2.3) by an arbitrary element $z \in V_0$ and a subsequent combination produce, with the help of the definition of F and the conditions in (2.4)–(2.6), the variational formulation

$$\begin{aligned}
&\int_\Omega F^{-1}\left(\frac{\partial u}{\partial t}(t)\right) z dx + \tau \int_\Omega \frac{\partial u}{\partial t}(t) z dx + \int_\Gamma \frac{\partial u_\Gamma}{\partial t}(t) z_\Gamma d\Gamma \\
&+ \int_\Omega \nabla u(t) \cdot \nabla z dx + \int_\Gamma \nabla_\Gamma u_\Gamma(t) \cdot \nabla_\Gamma z_\Gamma d\Gamma + \int_\Omega \xi(t) z dx + \int_\Gamma \xi_\Gamma(t) z_\Gamma d\Gamma \\
&+ \int_\Omega \pi\big(u(t)\big) z dx + \int_\Gamma \pi_\Gamma\big(u_\Gamma(t)\big) z_\Gamma d\Gamma = \int_\Omega f(t) z dx + \int_\Gamma f_\Gamma(t) z_\Gamma d\Gamma,
\end{aligned} \tag{2.8}$$

for a.a. $t \in (0, T)$, for all $z \in V_0$ with $z_\Gamma = z_{|_\Gamma}$. We are now interested to deal not directly with (2.8) but with a variational inequality replacing it, where the solution and the test function vary in a suitable convex set.

Concerning the data, we assume that

(A1) β, β_Γ, maximal monotone graphs in $\mathbb{R} \times \mathbb{R}$, are the subdifferentials

$$\beta = \partial \widehat{\beta}, \quad \beta_\Gamma = \partial \widehat{\beta}_\Gamma$$

of some continuous and convex functions

$$\widehat{\beta},\ \widehat{\beta}_\Gamma : \mathbb{R} \to [0, +\infty) \ \text{ such that } \ \widehat{\beta}(0) = \widehat{\beta}_\Gamma(0) = 0;$$

(A2) π, $\pi_\Gamma : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous functions with Lipschitz constants L and L_Γ, respectively;

(A3) $\boldsymbol{f} := (f, f_\Gamma) \in L^2(0, T; L^2(\Omega)) \times L^2(0, T; H_\Gamma)$ and $\boldsymbol{u}_0 := (u_0, u_{0\Gamma}) \in H^1(\Omega) \times V_\Gamma$, where $u_{0\Gamma} := u_{0|_\Gamma}$.

In particular, by (A1) we are asking that $D(\beta) = D(\beta_\Gamma) = \mathbb{R}$, $0 \in \beta(0)$ and $0 \in \beta_\Gamma(0)$.

In this paper, we are interested in the setting of the constraint

$$k_* \leq \int_\Gamma w_\Gamma u_{|_\Gamma}(t) d\Gamma \leq k^* \quad \text{for all } t \in [0, T], \tag{2.9}$$

for the solution to the related variational inequality (cf. (2.8)). Here, k_* and k^* are real constants with $k_* \leq k^*$, and $\boldsymbol{w} := (0, w_\Gamma) \in \boldsymbol{H}_0$ is fixed. We require that the weight function w_Γ satisfies

(A4) $w_\Gamma \in H_\Gamma$, $w_\Gamma \geq 0$ a.e. on Γ and $\sigma_0 := \int_\Gamma w_\Gamma d\Gamma > 0$.

The last inequality can be seen as a nondegeneracy condition on the weight element $\boldsymbol{w}$.

Hence, let us term (P) the initial-value problem related to the variational inequality and to the constraint in (2.9). Now, we define precisely the notion of solution to the problem (P) by means of a Lagrange multiplier. In order to set H_0 as the pivot space, put $m_0 := (1/|\Omega|) \int_\Omega u_0 dx$ and let $v(x,t) := u(x,t) - m_0$ be the new unknown function and define analogously $v_0 := u_0 - m_0$ in Ω, $v_{0\Gamma} := u_{0\Gamma} - m_0$ on Γ, $h_* := k_* - m_0\sigma_0$ and $h^* := k^* - m_0\sigma_0$, respectively.

Definition 2.1. The quadruplet $(\boldsymbol{v}, \boldsymbol{\xi}, \omega, \lambda)$ is called the solution of (P) if

$$\begin{aligned} \boldsymbol{v} = (v, v_\Gamma) \quad \text{with} \quad & v \in H^1(0,T;H_0) \cap C\big([0,T];V_0\big) \cap L^2\big(0,T;H^2(\Omega)\big), \\ & v_\Gamma \in H^1(0,T;H_\Gamma) \cap C\big([0,T];V_\Gamma\big) \cap L^2\big(0,T;H^2(\Gamma)\big), \\ & \boldsymbol{\xi} = (\xi, \xi_\Gamma) \in L^2(0,T;\boldsymbol{H}_0), \quad \omega, \lambda \in L^2(0,T), \end{aligned}$$

and v, v_Γ, ξ, ξ_Γ, ω, λ satisfy

$$F^{-1}\left(\frac{\partial v}{\partial t}\right) + \tau \frac{\partial v}{\partial t} - \Delta v + \xi + \pi(v + m_0) = f + \omega \quad \text{a.e. in } Q, \tag{2.10}$$

$$\xi \in \beta(v + m_0) \quad \text{a.e. in } Q, \tag{2.11}$$

$$v_\Gamma = v_{|\Gamma}, \quad \partial_\nu v + \frac{\partial v_\Gamma}{\partial t} - \Delta_\Gamma v_\Gamma + \xi_\Gamma + \pi_\Gamma(v_\Gamma + m_0) + \lambda w_\Gamma = f_\Gamma \quad \text{a.e. on } \Sigma, \tag{2.12}$$

$$\xi_\Gamma \in \beta_\Gamma(v_\Gamma + m_0) \quad \text{a.e. on } \Sigma, \tag{2.13}$$

$$v(0) = v_0 \quad \text{a.e. in } \Omega, \quad v_\Gamma(0) = v_{0\Gamma} \quad \text{a.e. on } \Gamma, \tag{2.14}$$

$$h_* \le \int_\Gamma w_\Gamma v_\Gamma(t) d\Gamma \le h^* \quad \text{for a.a. } t \in (0,T), \tag{2.15}$$

$$\lambda(t) \int_\Gamma w_\Gamma \big(v_\Gamma(t) - z_\Gamma\big) d\Gamma \ge 0 \quad \text{for a.a. } t \in (0,T)$$

$$\text{and for all } \boldsymbol{z} = (z, z_\Gamma) \in \boldsymbol{V}_0 \text{ such that } \; h_* \le \int_\Gamma w_\Gamma z_\Gamma \, d\Gamma \le h^*. \tag{2.16}$$

In the case $\tau = 0$, the regularity of v should be modified into

$$v \in H^1(0,T;V_0^*) \cap L^\infty(0,T;V_0) \cap L^2\big(0,T;H^2(\Omega)\big).$$

2.2. Remark for the Lagrange multipliers

By comparing (2.3) with (2.10)–(2.11), we realize that

$$\mu = -F^{-1}\left(\frac{\partial v}{\partial t}\right) + \omega \quad \text{a.e. in } Q,$$

so that ω turns out to be the mean value of the chemical potential μ

$$\omega(t) = \frac{1}{|\Omega|} \int_\Omega \mu(t) dx.$$

On the other hand, λ has the role of a Lagrange multiplier related to the constraint in (2.15) on the boundary. Then, the two Lagrange multipliers ω and λ have different meaning; in particular, λ is obtained by solving the problem and it explicitly appears in the variational formulation, while ω does not show up in the variational inequality and it can be only identified a posteriori. Indeed, if we test (2.10) by a function $z \in V_0$, then ω disappears and we obtain (cf. also (2.8))

$$\begin{aligned} &\int_\Omega F^{-1}\left(\frac{\partial v}{\partial t}(t)\right) z dx + \tau \int_\Omega \frac{\partial v}{\partial t}(t) z dx + \int_\Gamma \frac{\partial v_\Gamma}{\partial t}(t) z_\Gamma d\Gamma \\ &\quad + \int_\Omega \nabla v(t) \cdot \nabla z dx + \int_\Gamma \nabla_\Gamma v_\Gamma(t) \cdot \nabla_\Gamma z_\Gamma d\Gamma + \int_\Omega \xi(t) z dx + \int_\Gamma \xi_\Gamma(t) z_\Gamma d\Gamma \\ &\quad + \int_\Omega \pi\big(v(t)+m_0\big) z dx + \int_\Gamma \pi_\Gamma\big(v_\Gamma(t)+m_0\big) z_\Gamma d\Gamma + \int_\Gamma \lambda(t) w_\Gamma z_\Gamma d\Gamma \\ &= \int_\Omega f(t) z dx + \int_\Gamma f_\Gamma(t) z_\Gamma d\Gamma, \end{aligned} \tag{2.17}$$

for all $z \in V_0$ satisfying $z_{|_\Gamma} = z_\Gamma$, because $(\omega(t), z)_{H_0} = 0$. On the contrary, if we simply integrate (2.10) and set

$$q := \xi + \pi(v+m_0) - f \quad \text{a.e. in } Q, \qquad q_\Gamma := \xi_\Gamma + \pi_\Gamma(v_\Gamma + m_0) - f_\Gamma \quad \text{a.e. on } \Sigma, \tag{2.18}$$

with the help of (2.12) we obtain

$$\omega(t) = \frac{1}{|\Omega|}\left\{ \int_\Omega q(t) dx + \int_\Gamma \left(\frac{\partial v_\Gamma}{\partial t}(t) + q_\Gamma(t) + \lambda(t) w_\Gamma \right) d\Gamma \right\} \quad \text{for all } t \in [0,T]. \tag{2.19}$$

In the last part of this section, we show how to recover (2.10) and (2.12) from the variational equality (2.17). Define the projection $P_0 : L^2(\Omega) \to H_0$ by

$$P_0 z := z - \frac{1}{|\Omega|} \int_\Omega z dx \quad \text{for all } z \in L^2(\Omega).$$

Take $z \in H^1_0(\Omega)$ (so that $z_{|_\Gamma} = 0$ a.e. on Γ) and use $P_0 z$ as test function in (2.17). We note that $(P_0 z)_{|_\Gamma} = -(1/|\Omega|) \int_\Omega z dx$ and infer

$$\begin{aligned} &\int_\Omega F^{-1}\left(\frac{\partial v}{\partial t}(t)\right) z dx + \tau \int_\Omega \frac{\partial v}{\partial t}(t) z dx + \int_\Gamma \frac{\partial v_\Gamma}{\partial t}(t) d\Gamma \left(-\frac{1}{|\Omega|} \int_\Omega z d\tilde{x} \right) \\ &\quad + \int_\Omega \nabla v(t) \cdot \nabla z dx + \int_\Omega \Big(\xi(t) + \pi\big(v(t)+m_0\big) - f(t) \Big) \left(z - \frac{1}{|\Omega|} \int_\Omega z d\tilde{x} \right) dx \\ &\quad + \int_\Gamma \Big(\xi_\Gamma(t) + \pi_\Gamma\big(v_\Gamma(t)+m_0\big) - f_\Gamma(t) \Big) d\Gamma \left(-\frac{1}{|\Omega|} \int_\Omega z d\tilde{x} \right) = 0. \end{aligned}$$

Then, recalling the notation (2.18) we easily obtain the equation in the interior, i.e.,

$$F^{-1}\left(\frac{\partial v}{\partial t}\right)+\tau\frac{\partial v}{\partial t}-\Delta v+P_0 q-\frac{1}{|\Omega|}\int_\Gamma\left(\frac{\partial v_\Gamma}{\partial t}+q_\Gamma\right)d\Gamma=0 \quad \text{a.e. in } Q$$

and, in view of (2.19), we find out that

$$F^{-1}\left(\frac{\partial v}{\partial t}\right)+\tau\frac{\partial v}{\partial t}-\Delta v+q=\omega \quad \text{a.e. in } Q.$$

Next, we take a general $\boldsymbol{z}:=(z,z_\Gamma)\in \boldsymbol{V}_0$ and note that (2.17) reduces to

$$\begin{aligned}&\int_\Omega F^{-1}\left(\frac{\partial v}{\partial t}(t)\right)zdx+\tau\int_\Omega\frac{\partial v}{\partial t}(t)zdx+\int_\Gamma\frac{\partial v_\Gamma}{\partial t}(t)z_\Gamma d\Gamma-\int_\Omega\Delta v(t)zdx+\int_\Gamma\partial_\nu v(t)z_\Gamma d\Gamma\\&+\int_\Gamma\nabla_\Gamma v_\Gamma(t)\cdot\nabla_\Gamma z_\Gamma d\Gamma+\int_\Omega q(t)zdx+\int_\Gamma q_\Gamma(t)z_\Gamma d\Gamma+\int_\Gamma\lambda(t)w_\Gamma z_\Gamma d\Gamma=0,\end{aligned}$$

which means that

$$\int_\Omega\omega(t)zdx+\int_\Gamma\left(\partial_\nu v(t)+\frac{\partial v_\Gamma}{\partial t}(t)-\Delta_\Gamma v_\Gamma(t)+q_\Gamma(t)+\lambda(t)w_\Gamma\right)z_\Gamma d\Gamma=0.$$

By virtue of the fact that $\int_\Omega\omega(t)zdx=\omega(t)\int_\Omega zdx=0$, we finally have (cf. (2.12))

$$\partial_\nu v+\frac{\partial v_\Gamma}{\partial t}-\Delta_\Gamma v_\Gamma+q_\Gamma+\lambda w_\Gamma=0 \quad \text{a.e. on } \Sigma.$$

2.3. Well-posedness

The first result states the continuous dependence on the data. The uniqueness of the component $\boldsymbol{v}$ of the solution is also guaranteed by this theorem.

Theorem 2.1. *Let $\tau\ge 0$. Assume* (A1)–(A4)*. For $i=1,2$, let $(\boldsymbol{v}^{(i)},\boldsymbol{\xi}^{(i)},\omega^{(i)},\lambda^{(i)})$, with $\boldsymbol{v}^{(i)}=(v^{(i)},v_\Gamma^{(i)})$ and $\boldsymbol{\xi}^{(i)}=(\xi^{(i)},\xi_\Gamma^{(i)})$ be a solution to* (P) *corresponding to the data $\boldsymbol{f}^{(i)}=(f^{(i)},f_\Gamma^{(i)})$ and $\boldsymbol{v}_0^{(i)}=(v_0^{(i)},v_{0\Gamma}^{(i)})$. Then, there exists a positive constant $C>0$, depending on L, L_Γ and T, such that*

$$\begin{aligned}&\left|v^{(1)}(t)-v^{(2)}(t)\right|_{V_0^*}^2+\tau\left|v^{(1)}(t)-v^{(2)}(t)\right|_{H_0}^2+\left|v_\Gamma^{(1)}(t)-v_\Gamma^{(2)}(t)\right|_{H_\Gamma}^2\\&+\int_0^t\left|v^{(1)}(s)-v^{(2)}(s)\right|_{V_0}^2ds+2\int_0^t\left|\nabla_\Gamma v_\Gamma^{(1)}(s)-\nabla_\Gamma v_\Gamma^{(2)}(s)\right|_{H_\Gamma^d}^2ds\\&\le C\left\{\left|v_0^{(1)}-v_0^{(2)}\right|_{V_0^*}^2+\tau\left|v_0^{(1)}-v_0^{(2)}\right|_{H_0}^2+\left|v_{0\Gamma}^{(1)}-v_{0\Gamma}^{(2)}\right|_{H_\Gamma}^2+\int_0^T\left|f^{(1)}(s)-f^{(2)}(s)\right|_{L^2(\Omega)}^2ds\right.\\&\left.+\int_0^T\left|f_\Gamma^{(1)}(s)-f_\Gamma^{(2)}(s)\right|_{H_\Gamma}^2ds\right\}\quad \textit{for all } t\in[0,T]. \qquad (2.20)\end{aligned}$$

The second result deals with the existence of the solution. To the aim, we further assume that

(A5) there exist positive constants c_0, $\varrho > 0$ such that

$$|s| \le c_0\bigl(1 + \widehat{\beta}(r)\bigr) \quad \text{for all } r \in \mathbb{R} \text{ and } s \in \beta(r), \tag{2.21}$$

$$|s| \le c_0\bigl(1 + \widehat{\beta}_\Gamma(r)\bigr) \quad \text{for all } r \in \mathbb{R} \text{ and } s \in \beta_\Gamma(r), \tag{2.22}$$

$$\bigl|\beta^\circ(r)\bigr| \le \varrho\bigl|\beta_\Gamma^\circ(r)\bigr| + c_0 \quad \text{for all } r \in \mathbb{R}; \tag{2.23}$$

(A6) for the initial data $\boldsymbol{v}_0 = (v_0, v_{\Gamma 0}) \in \boldsymbol{V}_0$ the compatibility conditions

$$h_* \le \int_\Gamma w_\Gamma v_{0\Gamma} d\Gamma \le h^*, \quad \widehat{\beta}(v_0 + m_0) \in L^1(\Omega), \quad \widehat{\beta}_\Gamma(v_{0\Gamma} + m_0) \in L^1(\Gamma) \tag{2.24}$$

must hold.

The minimal section β° of β is specified by $\beta^\circ(r) := \{r^* \in \beta(r) : |r^*| = \min_{s\in\beta(r)} |s|\}$ and the same definition applies to β_Γ°. The reader can compare these assumptions with the analogous ones in [10, (2.17)–(2.21)].

We have to distinguish between the cases $\tau > 0$ and $\tau = 0$. To this aim, we introduce the additional regularity assumption for f:

(A7) $f \in H^1(0,T;L^2(\Omega))$ or $f \in L^2(0,T;H^1(\Omega))$.

Theorem 2.2. *Let $\tau > 0$. Then, under the assumptions* (A1)–(A6)*, there exists a unique solution of* (P)*. Moreover, if $\tau = 0$ and* (A7) *holds, then the problem* (P) *has a unique solution as well.*

2.4. Abstract formulation

In this subsection, an abstract formulation of the problem is given. We can write the problem as an evolution inclusion governed by a subdifferential operator, with essentially the same approach as in [10,21,22].

The point of emphasis is that our mass constraint (2.15) reads

$$h_* \le \bigl(\boldsymbol{w}, \boldsymbol{v}(t)\bigr)_{\boldsymbol{H}_0} \le h^* \quad \text{for all } t \in [0,T],$$

with $\boldsymbol{w} := (0, w_\Gamma) \in \boldsymbol{H}_0$. Then, by introducing the convex constraint set

$$\boldsymbol{K} := \bigl\{\boldsymbol{z} \in \boldsymbol{V}_0 \ : \ h_* \le (\boldsymbol{w}, \boldsymbol{z})_{\boldsymbol{H}_0} \le h^*\bigr\},$$

let $I_{\boldsymbol{K}} : \boldsymbol{H}_0 \to [0, +\infty]$ denote the indicator function of $\boldsymbol{K}$. Now, define the proper, lower semicontinuous and convex functional $\varphi : \boldsymbol{H}_0 \to [0, +\infty]$ by

$$\varphi(\boldsymbol{z}) := \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |\nabla z|^2 dx + \int_\Omega \widehat{\beta}(z + m_0)dx + \dfrac{1}{2}\int_\Gamma |\nabla_\Gamma z_\Gamma|^2 d\Gamma + \int_\Gamma \widehat{\beta}_\Gamma(z_\Gamma + m_0)d\Gamma \\ \qquad\qquad \text{if } \boldsymbol{z} \in \boldsymbol{V}_0, \widehat{\beta}(z + m_0) \in L^1(\Omega) \text{ and } \widehat{\beta}_\Gamma(z_\Gamma + m_0) \in L^1(\Gamma), \\ +\infty \quad \text{otherwise.} \end{cases}$$

Then, the problem (P) can be stated as the Cauchy problem for an evolution inclusion with a perturbation, namely

$$\boldsymbol{A}_\tau \boldsymbol{v}'(t) + \partial(\varphi + I_{\boldsymbol{K}})(\boldsymbol{v}(t)) \ni P\Big(\boldsymbol{f}(t) - \boldsymbol{\Pi}_0(\boldsymbol{v}(t))\Big) \quad \text{in } \boldsymbol{H}_0, \ \text{ for a.a. } t \in (0,T), \tag{2.25}$$

$$\boldsymbol{v}(0) = \boldsymbol{v}_0 \quad \text{in } \boldsymbol{H}_0, \tag{2.26}$$

where $\boldsymbol{A}_\tau \boldsymbol{z} := (F^{-1}z + \tau z, z_\Gamma)$ for $\tau \geq 0$, $P\boldsymbol{z} := (P_0 z, z_\Gamma - (1/|\Omega|)\int_\Omega z dx)$ and $\boldsymbol{\Pi}_0(\boldsymbol{z}) := (\pi(z + m_0), \pi_\Gamma(z_\Gamma + m_0))$ for all $\boldsymbol{z} \in \boldsymbol{H}_0$.

Hence, let us recall the paper [13] and express our expectation that (2.25)–(2.26) can be solved by the abstract theory of doubly nonlinear evolution inclusions. All this will be discussed in Section 4. On the other hand, Theorem 2.2 allows a characterization in terms of regularity of the solution and presence of the Lagrange multipliers.

We aim to point out that analogous remarks were emphasized in [10] for an Allen–Cahn equation with dynamic boundary conditions and mass constraints; the reader can compare the two problems. In connection with [10], we also quote the abstract approach carried out in [15], which however does not comply here with the structure of (2.25)–(2.26).

3. Continuous dependence

In this section, we prove Theorem 2.1.

Proof of Theorem 2.1. For $i = 1, 2$ let $(\boldsymbol{v}^{(i)}, \boldsymbol{\xi}^{(i)}, \omega^{(i)}, \lambda^{(i)})$ be a solution of (P) corresponding to the data $(f^{(i)}, f_\Gamma^{(i)}, v_0^{(i)}, v_{0\Gamma}^{(i)})$. We consider the difference between (2.10) written for $v^{(1)}(s)$ of $\boldsymbol{v}^{(1)}(s) = (v^{(1)}(s), v_\Gamma^{(1)}(s))$ and (2.10) written for $v^{(2)}(s)$ of $\boldsymbol{v}^{(2)}(s) = (v^{(2)}(s), v_\Gamma^{(2)}(s))$ at the time $s \in (0,T)$. Then, we take the inner product with $v^{(1)}(s) - v^{(2)}(s)$ in $L^2(\Omega)$. Using the monotonicity of β and the fact $\int_\Omega (v^{(1)}(s) - v^{(2)}(s))dx = 0$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{ds}\big|v^{(1)}(s) - v^{(2)}(s)\big|_{V_0^*}^2 + \frac{\tau}{2}\frac{d}{ds}\big|v^{(1)}(s) - v^{(2)}(s)\big|_{H_0}^2 \\
&\quad + \big|v^{(1)}(s) - v^{(2)}(s)\big|_{V_0}^2 - \big(\partial_\nu v^{(1)}(s) - \partial_\nu v^{(2)}(s), v_\Gamma^{(1)}(s) - v_\Gamma^{(2)}(s)\big)_{H_\Gamma} \\
&\leq \big(f^{(1)}(s) - f^{(2)}(s), v^{(1)}(s) - v^{(2)}(s)\big)_{L^2(\Omega)} \\
&\quad - \Big(\pi\big(v^{(1)}(s) + m_0\big) - \pi\big(v^{(2)}(s) + m_0\big), v^{(1)}(s) - v^{(2)}(s)\Big)_{L^2(\Omega)},
\end{aligned} \tag{3.27}$$

for a.a. $s \in (0,T)$. Moreover, we take the difference between (2.12) written for $v_\Gamma^{(1)}(s)$ of and (2.12) written for $v_\Gamma^{(2)}(s)$ of at the time $t = s$, and take the inner product with $v_\Gamma^{(1)}(s) - v_\Gamma^{(2)}(s)$ in H_Γ; hence, we can replace the term

$$-\big(\partial_\nu v^{(1)}(s) - \partial_\nu v^{(2)}(s), v_\Gamma^{(1)}(s) - v_\Gamma^{(2)}(s)\big)_{H_\Gamma}$$

with the corresponding quantity in (3.27). Then, by exploiting the monotonicity of β_Γ and the Lipschitz continuities of π and π_Γ, we obtain

$$\begin{aligned}
&\frac{d}{ds}\Big\{\big|v^{(1)}(s) - v^{(2)}(s)\big|_{V_0^*}^2 + \tau\big|v^{(1)}(s) - v^{(2)}(s)\big|_{H_0}^2 + \big|v_\Gamma^{(1)}(s) - v_\Gamma^{(2)}(s)\big|_{H_\Gamma}^2\Big\} \\
&\quad + 2\big|v^{(1)}(s) - v^{(2)}(s)\big|_{V_0}^2 + 2\big|\nabla_\Gamma v_\Gamma^{(1)}(s) - \nabla_\Gamma v_\Gamma^{(2)}(s)\big|_{H_\Gamma^d}^2 \\
&\leq \big|f^{(1)}(s) - f^{(2)}(s)\big|_{L^2(\Omega)}^2 + (1 + 2L)\big|v^{(1)}(s) - v^{(2)}(s)\big|_{H_0}^2 + \big|f_\Gamma^{(1)}(s) - f_\Gamma^{(2)}(s)\big|_{H_\Gamma}^2
\end{aligned}$$

$$+ (1 + 2L_\Gamma)\big|v_\Gamma^{(1)}(s) - v_\Gamma^{(2)}(s)\big|_{H_\Gamma}^2,$$

for a.a. $s \in (0,T)$. If $\tau > 0$, by applying directly the Gronwall lemma, it is straightforward to find a constant $C > 0$, depending only on L, L_Γ and T, such that the continuous dependence holds. If $\tau = 0$, a known compactness inequality (see, e.g., [23, Thm. 16.4, p. 102]) states that for each $\delta > 0$ there exists a positive constant C_δ such that

$$|z|_{H_0} \le \delta |z|_{V_0} + C_\delta |z|_{V_0^*} \quad \text{for all } z \in V_0.$$

Therefore, taking $\delta^2 < 1/(2 + 4L)$ we have

$$\begin{aligned}
&(1 + 2L)\big|v^{(1)}(s) - v^{(2)}(s)\big|_{H_0}^2 \\
&\le (1 + 2L)\left\{2\delta^2\big|v^{(1)}(s) - v^{(2)}(s)\big|_{V_0}^2 + 2C_\delta^2\big|v^{(1)}(s) - v^{(2)}(s)\big|_{V_0^*}^2\right\} \\
&\le \big|v^{(1)}(s) - v^{(2)}(s)\big|_{V_0}^2 + \tilde{C}\big|v^{(1)}(s) - v^{(2)}(s)\big|_{V_0^*}^2,
\end{aligned}$$

for a.a. $s \in (0,T)$ and some constant $\tilde{C} > 0$ depending only on L. At this point, we can analogously apply the Gronwall lemma and find a constant $C > 0$, with the same dependencies as above, such that (2.20) holds. Thus, Theorem 2.1 is completely proved. □

4. Existence

This section is devoted to the proof of Theorem 2.2. We make use of Yosida approximations for the maximal monotone operators β, β_Γ and of well-known results of this theory (see, [4,5,20]). For each $\varepsilon \in (0,1]$, we define $\beta_\varepsilon, \beta_{\Gamma,\varepsilon} : \mathbb{R} \to \mathbb{R}$, along with the associated resolvent operators $J_\varepsilon, J_{\Gamma,\varepsilon} : \mathbb{R} \to \mathbb{R}$ by

$$\beta_\varepsilon(r) := \frac{1}{\varepsilon}\big(r - J_\varepsilon(r)\big) := \frac{1}{\varepsilon}\big(r - (I + \varepsilon\beta)^{-1}(r)\big),$$

$$\beta_{\Gamma,\varepsilon}(r) := \frac{1}{\varepsilon\varrho}\big(r - J_{\Gamma,\varepsilon}(r)\big) := \frac{1}{\varepsilon\varrho}\big(r - (I + \varepsilon\varrho\beta_\Gamma)^{-1}(r)\big) \quad \text{for all } r \in \mathbb{R},$$

where $\varrho > 0$ is the same constant as in (2.23). Note that the two definitions are not symmetric since in the second it is $\varepsilon\varrho$ and not directly ε to be used as approximation parameter. Now, we easily have $\beta_\varepsilon(0) = \beta_{\Gamma,\varepsilon}(0) = 0$. Moreover, the related Moreau–Yosida regularizations $\widehat{\beta}_\varepsilon$, $\widehat{\beta}_{\Gamma,\varepsilon}$ of $\widehat{\beta}, \widehat{\beta}_\Gamma : \mathbb{R} \to \mathbb{R}$ fulfill

$$\widehat{\beta}_\varepsilon(r) := \inf_{s \in \mathbb{R}}\left\{\frac{1}{2\varepsilon}|r - s|^2 + \widehat{\beta}(s)\right\} = \frac{1}{2\varepsilon}\big|r - J_\varepsilon(r)\big|^2 + \widehat{\beta}(J_\varepsilon r) = \int_0^r \beta_\varepsilon(s)\,ds,$$

$$\widehat{\beta}_{\Gamma,\varepsilon}(r) := \inf_{s \in \mathbb{R}}\left\{\frac{1}{2\varepsilon\varrho}|r - s|^2 + \widehat{\beta}_\Gamma(s)\right\} = \int_0^r \beta_{\Gamma,\varepsilon}(s)\,ds \quad \text{for all } r \in \mathbb{R}.$$

It is well known that β_ε is Lipschitz continuous with Lipschitz constant $1/\varepsilon$ and $\beta_{\Gamma,\varepsilon}$ is also Lipschitz continuous with constant $1/(\varepsilon\varrho)$. In addition, we have the standard properties

$$\big|\beta_\varepsilon(r)\big| \le \big|\beta^\circ(r)\big|, \quad \big|\beta_{\Gamma,\varepsilon}(r)\big| \le \big|\beta_\Gamma^\circ(r)\big| \quad \text{for all } r \in \mathbb{R},$$

$$0 \le \widehat{\beta}_\varepsilon(r) \le \widehat{\beta}(r), \quad 0 \le \widehat{\beta}_{\Gamma,\varepsilon}(r) \le \widehat{\beta}_\Gamma(r) \quad \text{for all } r \in \mathbb{R}.$$

Here, we note that from the assumptions (2.21), (2.22) and the above properties we also obtain

$$\left|\beta_\varepsilon(r)\right| \le c_0\bigl(1+\widehat{\beta}_\varepsilon(r)\bigr), \tag{4.28}$$

$$\left|\beta_{\Gamma,\varepsilon}(r)\right| \le c_0\bigl(1+\widehat{\beta}_{\Gamma,\varepsilon}(r)\bigr) \quad \text{for all } r \in \mathbb{R}, \tag{4.29}$$

with the same constant c_0. Moreover, thanks to (2.23) and [8, Lemma 4.4], the inequality

$$\left|\beta_\varepsilon(r)\right| \le \varrho\left|\beta_{\Gamma,\varepsilon}(r)\right| + c_0 \quad \text{for all } r \in \mathbb{R}, \tag{4.30}$$

holds for β_ε and $\beta_{\Gamma,\varepsilon}$.

4.1. Approximation of the problem

In this subsection, we consider the approximation of problem (P) in the case when $\tau > 0$. The limiting case as $\tau \to 0$ will be discussed later. We introduce the following Cauchy problem: for each $\varepsilon \in (0,1]$ find $\boldsymbol{v}_\varepsilon$ satisfying

$$\boldsymbol{A}_\tau \boldsymbol{v}_\varepsilon'(t) + \partial(\varphi_\varepsilon + I_{\boldsymbol{K}})\bigl(\boldsymbol{v}_\varepsilon(t)\bigr) \ni P\Bigl(\boldsymbol{f}(t) - \boldsymbol{\Pi}_0\bigl(\boldsymbol{v}_\varepsilon(t)\bigr)\Bigr) \quad \text{in } \boldsymbol{H}_0, \text{ for a.a. } t \in (0,T), \tag{4.31}$$

$$\boldsymbol{v}_\varepsilon(0) = \boldsymbol{v}_0 \quad \text{in } \boldsymbol{H}_0, \tag{4.32}$$

with $\boldsymbol{v}_0 = (v_0, v_{0\Gamma}) \in \boldsymbol{K}$ satisfying the compatibility conditions (2.24). Here, $\varphi_\varepsilon : \boldsymbol{H}_0 \to [0,+\infty]$ is defined by

$$\varphi_\varepsilon(\boldsymbol{z}) := \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |\nabla z|^2 dx + \int_\Omega \widehat{\beta}_\varepsilon(z+m_0)dx \\ \quad + \dfrac{1}{2}\displaystyle\int_\Gamma |\nabla_\Gamma z_\Gamma|^2 d\Gamma + \int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}(z_\Gamma + m_0)d\Gamma + \frac{\varepsilon}{2}\int_\Gamma |z_\Gamma|^2 d\Gamma & \text{if } \boldsymbol{z} \in \boldsymbol{V}_0, \\ +\infty \quad \text{if } \boldsymbol{z} \in \boldsymbol{H}_0 \setminus \boldsymbol{V}_0, \end{cases}$$

moreover, it is understood that $\boldsymbol{A}_\tau \boldsymbol{z} := (F^{-1}z + \tau z, z_\Gamma)$, $P\boldsymbol{z} := (P_0 z, z_\Gamma - (1/|\Omega|)\int_\Omega z dx)$ and $\boldsymbol{\Pi}_0(\boldsymbol{z}) := (\pi(z+m_0), \pi_\Gamma(z_\Gamma + m_0))$ for all $\boldsymbol{z} = (z, z_\Gamma) \in \boldsymbol{H}_0$.

As a remark, thanks to the Poincaré–Wirtinger inequality for functions with 0 mean value, there is no need to introduce an approximating term like $(\varepsilon/2)\int_\Omega |z|^2 dx$ in the expression of φ_ε above. Denote by $\partial_* \varphi_\varepsilon$ the subdifferential of $\varphi_\varepsilon : \boldsymbol{V}_0 \to [0,+\infty]$ from $\boldsymbol{V}_0$ to $\boldsymbol{V}_0^*$. From [10, Lemma 3.1], we obtain the characterization of $\partial_*\varphi_\varepsilon$ by

$$\bigl\langle \partial_*\varphi_\varepsilon(\boldsymbol{z}), \bar{\boldsymbol{z}} \bigr\rangle_{\boldsymbol{V}_0^*, \boldsymbol{V}_0} = \bigl(\nabla z, \nabla \bar{z}\bigr)_{L^2(\Omega)^d} + \bigl(\beta_\varepsilon(z+m_0), \bar{z}\bigr)_{L^2(\Omega)} + \bigl(\nabla_\Gamma z_\Gamma, \nabla_\Gamma \bar{z}_\Gamma\bigr)_{H_\Gamma^d} + \bigl(\beta_{\Gamma,\varepsilon}(z_\Gamma+m_0), \bar{z}_\Gamma\bigr)_{H_\Gamma} + \varepsilon(z_\Gamma, \bar{z}_\Gamma)_{H_\Gamma} \quad \text{for all } \boldsymbol{z} = (z, z_\Gamma),\ \bar{\boldsymbol{z}} = (\bar{z}, \bar{z}_\Gamma) \in \boldsymbol{V}_0. \tag{4.33}$$

Moreover, there exists a positive constant C_ε depending on $\varepsilon > 0$ such that

$$\bigl|\partial_*\varphi_\varepsilon(\boldsymbol{z})\bigr|_{\boldsymbol{V}_0^*} \le C_\varepsilon\bigl(1 + \varphi_\varepsilon(\boldsymbol{z})\bigr) \quad \text{for all } \boldsymbol{z} \in \boldsymbol{V}_0. \tag{4.34}$$

Now, we recall the fact that the closure $\overline{\boldsymbol{K}}$ of $\boldsymbol{K}$ in $\boldsymbol{H}_0$ is characterized by

$$\overline{\boldsymbol{K}} = \bigl\{\boldsymbol{z} \in \boldsymbol{H}_0 \ : \ h_* \le (\boldsymbol{w}, \boldsymbol{z})_{\boldsymbol{H}_0} \le h^*\bigr\},$$

which is closed convex subset of $\boldsymbol{H}_0$. Moreover, there exists a function $z_c \in C^1(\overline{\Omega})$ such that

$$\int_\Omega z_c dx = 0, \quad z_{c|_\Gamma} = \frac{1}{\sigma_0},$$

whence $\boldsymbol{z}_c := (z_c, 1/\sigma_0) \in \boldsymbol{V}_0$. Then, we can deduce the following result.

Proposition 4.1. *Let $\tau > 0$. For each $\varepsilon \in (0,1]$, there exist a unique*

$$\boldsymbol{v}_\varepsilon \in H^1(0,T;\boldsymbol{H}_0) \cap L^\infty(0,T;\boldsymbol{V}_0)$$

and a pair of functions $\boldsymbol{v}^*_\varepsilon \in L^2(0,T;\boldsymbol{H}_0)$ and $\lambda_\varepsilon \in L^2(0,T)$ such that

$$\boldsymbol{u}_\varepsilon(t) \in \overline{\boldsymbol{K}} \quad \text{for all } t \in [0,T],$$

and

$$\boldsymbol{A}_\tau \boldsymbol{v}'_\varepsilon(t) + \boldsymbol{v}^*_\varepsilon(t) + \lambda_\varepsilon(t)\boldsymbol{w} = P\Big(\boldsymbol{f}(t) - \boldsymbol{\Pi}_0\big(\boldsymbol{v}_\varepsilon(t)\big)\Big) \quad \text{in } \boldsymbol{H}_0, \text{ for a.a. } t \in (0,T), \tag{4.35}$$

$$\boldsymbol{v}^*_\varepsilon(t) := \big(v^*_\varepsilon(t), v^*_{\Gamma,\varepsilon}(t)\big) = \partial\varphi_\varepsilon\big(\boldsymbol{v}_\varepsilon(t)\big) \quad \text{in } \boldsymbol{H}_0, \text{ for a.a. } t \in (0,T), \tag{4.36}$$

$$\lambda_\varepsilon(t)\boldsymbol{w} := \lambda_\varepsilon(t)(0, w_\Gamma) \in \partial I_{\overline{\boldsymbol{K}}}\big(\boldsymbol{v}_\varepsilon(t)\big) \quad \text{in } \boldsymbol{H}_0, \text{ for a.a. } t \in (0,T), \tag{4.37}$$

$$\boldsymbol{v}_\varepsilon(0) = \boldsymbol{v}_0 \quad \text{in } \boldsymbol{H}_0. \tag{4.38}$$

Proof. We sketch the basic steps of the proof.

1. We claim that for a given $\bar{\boldsymbol{v}} \in C([0,T];\boldsymbol{H}_0)$ there exists a unique

$$\boldsymbol{v} \in H^1(0,T;\boldsymbol{H}_0) \cap L^\infty(0,T;\boldsymbol{V}_0) \subset C\big([0,T];\boldsymbol{H}_0\big)$$

such that

$$\boldsymbol{A}_\tau \boldsymbol{v}'(t) + \partial(\varphi_\varepsilon + I_{\boldsymbol{K}})\big(\boldsymbol{v}(t)\big) \ni P\Big(\boldsymbol{f}(t) - \boldsymbol{\Pi}_0\big(\bar{\boldsymbol{v}}(t)\big)\Big) \quad \text{in } \boldsymbol{H}_0, \text{ for a.a. } t \in (0,T),$$

$$\boldsymbol{v}(0) = \boldsymbol{v}_0 \quad \text{in } \boldsymbol{H}_0.$$

Indeed, it suffices to apply the abstract theory of doubly nonlinear evolution inclusions (see, e.g., [13, Thm. 2.1]). We point out that, thanks to $\tau > 0$, the operator $\boldsymbol{A}_\tau$ is coercive in $\boldsymbol{H}_0$. Then, we construct the map

$$\Psi : \bar{\boldsymbol{u}} \mapsto \boldsymbol{u},$$

from $C([0,T];\boldsymbol{H}_0)$ into itself.

2. For given $\bar{\boldsymbol{u}}^{(i)} \in C([0,T];\boldsymbol{H}_0)$, put $\boldsymbol{u}^{(i)} := \Psi\bar{\boldsymbol{u}}^{(i)}$ for $i = 1,2$. Then, using the monotonicity of $\partial(\varphi_\varepsilon + I_{\boldsymbol{K}})$ and the special form of $\boldsymbol{A}_\tau$, it is not difficult to deduce the estimate

$$\big|\boldsymbol{u}^{(1)}(t) - \boldsymbol{u}^{(2)}(t)\big|^2_{\boldsymbol{H}_0} \le C_\tau \int_0^t \big|\bar{\boldsymbol{u}}^{(1)}(s) - \bar{\boldsymbol{u}}^{(2)}(s)\big|^2_{\boldsymbol{H}_0} ds \quad \text{for all } t \in [0,T], \tag{4.39}$$

where $C_\tau > 0$ is a constant depending on L, L_Γ and τ. Owing to (4.39), we can prove that there exists a suitable $k \in \mathbb{N}$ such that Ψ^k is a contraction mapping in $C([0,T];\boldsymbol{H}_0)$. Hence, being $\tau > 0$, there exists a unique fixed point for Ψ which yields the unique solution $\boldsymbol{v}_\varepsilon$ of the problem (4.31)–(4.32).

3. The third step is essentially the same as in the abstract theory developed in [15]. Put

$$\boldsymbol{y}_\varepsilon(t) := -\boldsymbol{A}_\tau \boldsymbol{v}'_\varepsilon(t) + P\Big(\boldsymbol{f}(t) - \boldsymbol{\Pi}_0\big(\boldsymbol{v}_\varepsilon(t)\big)\Big) \quad \text{in } \boldsymbol{H}_0, \text{ for a.a. } t \in (0,T),$$

and observe that $\boldsymbol{y}_\varepsilon \in L^2(0,T;\boldsymbol{H}_0)$. In general, for each $\boldsymbol{z} \in \boldsymbol{V}_0$ we have that

$$\partial(\varphi_\varepsilon + I_{\boldsymbol{K}})(\boldsymbol{z}) \subset \partial_*(\varphi_\varepsilon + I_{\boldsymbol{K}})(\boldsymbol{z}) = \partial_*\varphi_\varepsilon(\boldsymbol{z}) + \partial_* I_{\boldsymbol{K}}(\boldsymbol{z}).$$

Thus, there exists $\boldsymbol{v}^{**}_\varepsilon(t) \in \partial_* I_{\boldsymbol{K}}(\boldsymbol{v}_\varepsilon(t))$ such that

$$\boldsymbol{y}_\varepsilon(t) = \partial_*\varphi_\varepsilon\big(\boldsymbol{v}_\varepsilon(t)\big) + \boldsymbol{v}^{**}_\varepsilon(t) \quad \text{in } \boldsymbol{V}_0^*, \text{ for a.a. } t \in (0,T).$$

Moreover, taking advantage of [15, Prop. 2] and using $\boldsymbol{z}_c = (z_c, 1/\sigma_0) \in \boldsymbol{V}_0$, we set

$$\lambda_\varepsilon(t) := (\boldsymbol{y}_\varepsilon(t), \boldsymbol{z}_c)_{\boldsymbol{H}_0} - \Big\langle \partial_*\varphi_\varepsilon\big(\boldsymbol{v}_\varepsilon(t)\big), \boldsymbol{z}_c \Big\rangle_{\boldsymbol{V}_0^*, \boldsymbol{V}_0} \quad \text{for a.a. } t \in (0,T) \tag{4.40}$$

and obtain

$$\boldsymbol{v}^{**}_\varepsilon(t) = \lambda_\varepsilon(t)\boldsymbol{w} \in \partial I_{\overline{\boldsymbol{K}}}(\boldsymbol{v}_\varepsilon(t)) \quad \text{in } \boldsymbol{H}_0, \text{ for a.a. } t \in (0,T),$$

where $\boldsymbol{w} = (0, w_\Gamma)$ (cf. (A4)). Note that $\lambda_\varepsilon \in L^2(0,T)$ thanks to (4.40) and (4.33). As a consequence, both $\boldsymbol{v}^{**}_\varepsilon$ and $\boldsymbol{v}^*_\varepsilon := \partial_*\varphi(\boldsymbol{v}_\varepsilon)$ are in $L^2(0,T;\boldsymbol{H}_0)$ and (4.35)–(4.37) follow with the right regularity. □

Let $\tau > 0$. Using Proposition 4.1 with the characterization (4.33) of $\partial_*\varphi_\varepsilon$ we obtain the following weak formulation:

$$\begin{aligned}
&\int_\Omega F^{-1}\left(\frac{\partial v_\varepsilon}{\partial t}(t)\right) z\,dx + \tau \int_\Omega \frac{\partial v_\varepsilon}{\partial t}(t) z\,dx + \int_\Gamma \frac{\partial v_{\Gamma,\varepsilon}}{\partial t}(t) z_\Gamma\,d\Gamma + \int_\Omega \nabla v_\varepsilon(t)\cdot\nabla z\,dx \\
&\quad + \int_\Gamma \nabla_\Gamma v_{\Gamma,\varepsilon}(t)\cdot\nabla_\Gamma z_\Gamma\,d\Gamma + \int_\Omega q_\varepsilon(t) z\,dx + \int_\Gamma q_{\Gamma,\varepsilon}(t) z_\Gamma\,d\Gamma + \int_\Gamma \lambda_\varepsilon(t) w_\Gamma z_\Gamma\,d\Gamma \\
&= 0 \quad \text{for all } \boldsymbol{z} := (z, z_\Gamma) \in \boldsymbol{V}_0,
\end{aligned} \tag{4.41}$$

where

$$\begin{aligned}
q_\varepsilon &:= \beta_\varepsilon(v_\varepsilon + m_0) + \pi(v_\varepsilon + m_0) - f \in L^2(0,T;L^2(\Omega)),\\
q_{\Gamma,\varepsilon} &:= \varepsilon v_{\Gamma,\varepsilon} + \beta_{\Gamma,\varepsilon}(v_{\Gamma,\varepsilon} + m_0) + \pi_\Gamma(v_{\Gamma,\varepsilon} + m_0) - f_\Gamma \in L^2(0,T;H_\Gamma).
\end{aligned}$$

We also introduce the auxiliary quantity

$$\omega_\varepsilon(t) := \frac{1}{|\Omega|}\int_\Omega q_\varepsilon(t)\,dx + \frac{1}{|\Omega|}\int_\Gamma \left(\frac{\partial v_{\Gamma,\varepsilon}}{\partial t}(t) + q_{\Gamma,\varepsilon}(t) + \lambda_\varepsilon(t) w_\Gamma\right) d\Gamma \tag{4.42}$$

for a.a. $t \in (0,T)$. By noting that $\partial v_{\Gamma,\varepsilon}/\partial t$ and $\lambda_\varepsilon w_\Gamma$ lie in $L^2(0,T;H_\Gamma)$, it turns out that $\omega_\varepsilon \in L^2(0,T)$. Moreover, according to [10, Prop. 3.2], for each $\varepsilon \in (0,1]$ we can infer that $v_\varepsilon \in L^2(0,T;H^2(\Omega))$ and $v_{\Gamma,\varepsilon} \in L^2(0,T;H^2(\Gamma))$. By virtue of this regularity, our approximate problem can be written as

$$F^{-1}\left(\frac{\partial v_\varepsilon}{\partial t}\right)+\tau\frac{\partial v_\varepsilon}{\partial t}-\Delta v_\varepsilon+q_\varepsilon=\omega_\varepsilon \quad \text{a.e. in } Q, \tag{4.43}$$

$$v_{\Gamma,\varepsilon}=v_{\varepsilon|_\Gamma}, \quad \partial_\nu v_\varepsilon+\frac{\partial v_{\Gamma,\varepsilon}}{\partial t}-\Delta_\Gamma v_{\Gamma,\varepsilon}+q_{\Gamma,\varepsilon}+\lambda_\varepsilon w_\Gamma=0 \quad \text{a.e. on } \Sigma, \tag{4.44}$$

$$v_\varepsilon(0)=v_0 \quad \text{a.e. in } \Omega, \quad v_{\Gamma,\varepsilon}(0)=v_{0\Gamma} \quad \text{a.e. on } \Gamma, \tag{4.45}$$

$$h_*\le h_\varepsilon(t):=\int_\Gamma w_\Gamma v_{\Gamma,\varepsilon}(t)d\Gamma\le h^* \quad \text{for all } t\in[0,T], \tag{4.46}$$

$$\lambda_\varepsilon(t)\in\partial I_{[h_*,h^*]}\big(h_\varepsilon(t)\big) \quad \text{for a.a. } t\in(0,T). \tag{4.47}$$

Due to the regularity of the solution, $\boldsymbol{v}_\varepsilon(t)$ is in $\overline{\boldsymbol{K}}$ for all $t\in[0,T]$. Another remark is that the last condition (4.47) is equivalent to (see, e.g., [10, Remark 3.2])

$$\lambda_\varepsilon(t)\boldsymbol{w}\in\partial I_{\overline{\boldsymbol{K}}}\big(\boldsymbol{v}_\varepsilon(t)\big) \quad \text{in } \boldsymbol{H}_0, \text{ for a.a. } t\in(0,T). \tag{4.48}$$

4.2. A priori estimates

Let $\tau>0$. In this subsection, we obtain the uniform estimates independent of $\varepsilon>0$. Moreover, our second objective will be to study the limiting behavior as $\tau\to 0$. Therefore, under the additional regularity assumption (A7) for f we also obtain some uniform estimates independent of $\varepsilon>0$ and $\tau>0$.

Lemma 4.1. *There exists a positive constant M_1, independent of $\varepsilon\in(0,1]$, such that*

$$|v_\varepsilon|_{H^1(0,T;V_0^*)}+\tau^{1/2}|v_\varepsilon|_{H^1(0,T;H_0)}+|v_\varepsilon|_{L^\infty(0,T;V_0)}+\sup_{t\in(0,T)}\int_\Omega\widehat{\beta}_\varepsilon\big(v_\varepsilon(t)+m_0\big)dx$$
$$+|v_{\Gamma,\varepsilon}|_{H^1(0,T;H_\Gamma)}+|v_{\Gamma,\varepsilon}|_{L^\infty(0,T;V_\Gamma)}+\sup_{t\in(0,T)}\int_\Gamma\widehat{\beta}_{\Gamma,\varepsilon}\big(v_{\Gamma,\varepsilon}(t)+m_0\big)d\Gamma\le M_1.$$

Moreover, if (A7) *is assumed, then $M_1>0$ is obtained independent of $\varepsilon\in(0,1]$ and $\tau>0$.*

Proof. We test (4.43) by $v_\varepsilon'=\partial v_\varepsilon/\partial t\in L^2(0,T;L^2(\Omega))$. Moreover, we add $v_{\Gamma,\varepsilon}$ to both sides of (4.44) and use it as the boundary condition, obtaining

$$\int_0^t\big|v_\varepsilon'(s)\big|_{V_0^*}^2ds+\tau\int_0^t\big|v_\varepsilon'(s)\big|_{H_0}^2ds+\frac12\big|v_\varepsilon(t)\big|_{V_0}^2+\int_\Omega\widehat{\beta}_\varepsilon\big(v_\varepsilon(t)+m_0\big)dx$$
$$+\int_0^t\big|v_{\Gamma,\varepsilon}'(s)\big|_{H_\Gamma}^2ds+\frac12\big|v_{\Gamma,\varepsilon}(t)\big|_{V_\Gamma}^2+\int_\Gamma\widehat{\beta}_{\Gamma,\varepsilon}\big(v_{\Gamma,\varepsilon}(t)+m_0\big)d\Gamma+\frac{\varepsilon}{2}\big|v_{\Gamma,\varepsilon}(t)\big|_{H_\Gamma}^2$$
$$+\int_0^t\lambda_\varepsilon(s)\left\{\int_\Gamma w_\Gamma v_{\Gamma,\varepsilon}'(s)d\Gamma\right\}ds$$
$$\le\frac12|v_0|_{V_0}^2+\int_\Omega\widehat{\beta}_\varepsilon(v_0+m_0)dx+\frac12|v_{0\Gamma}|_{V_\Gamma}^2+\int_\Gamma\widehat{\beta}_{\Gamma,\varepsilon}(v_{0\Gamma}+m_0)d\Gamma+\frac{\varepsilon}{2}|v_{0\Gamma}|_{H_\Gamma}^2$$

$$+\int_0^t \Big(f(s)-\pi\big(v_\varepsilon(s)+m_0\big), v'_\varepsilon(s)\Big)_{L^2(\Omega)} ds$$

$$+\int_0^t \Big(f_\Gamma(s)+v_{\Gamma,\varepsilon}(s)-\pi_\Gamma\big(v_{\Gamma,\varepsilon}(s)+m_0\big), v'_{\Gamma,\varepsilon}(s)\Big)_{H_\Gamma} ds \tag{4.49}$$

for all $t\in[0,T]$. We note that (cf. (2.24))

$$\int_\Omega \widehat{\beta}_\varepsilon(v_0+m_0)dx \le \int_\Omega \widehat{\beta}(v_0+m_0)dx < +\infty, \tag{4.50}$$

$$\int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}(v_{0\Gamma}+m_0)d\Gamma \le \int_\Gamma \widehat{\beta}_\Gamma(v_{0\Gamma}+m_0)d\Gamma < +\infty. \tag{4.51}$$

Also by the chain rule differentiation lemma (see, e.g., [4, Lemma 4.4, p. 158] or [5, Lemme 3.3, p. 73]) and in view of (4.46)–(4.47), the last term on the left hand side is exactly

$$\int_0^t \lambda_\varepsilon(s)h'_\varepsilon(s)ds = I_{[h_*,h^*]}\big(h_\varepsilon(t)\big) - I_{[h_*,h^*]}(h_0) \equiv 0 \quad \text{for all } t\in[0,T], \tag{4.52}$$

where $h_0 := (w_\Gamma, v_{0\Gamma})_{H_\Gamma}$. We easily see that there exists a positive constant $\tilde{M}_1$, depending on L, L_Γ, $|\pi(m_0)|$, $|\pi_\Gamma(m_0)|$, $|\Omega|$ and $|\Gamma|$ (but independent of $\varepsilon\in(0,1]$ and $\tau>0$), such that

$$\begin{aligned}
&\int_0^t \Big(f(s)-\pi\big(v_\varepsilon(s)+m_0\big), v'_\varepsilon(s)\Big)_{L^2(\Omega)} ds \\
&\le \frac{\tau}{2}\int_0^t \big|v'_\varepsilon(s)\big|^2_{H_0} ds + \frac{1}{\tau}\int_0^t \left(\big|f(s)\big|^2_{L^2(\Omega)} + \Big|\pi\big(v_\varepsilon(s)+m_0\big)\Big|^2_{L^2(\Omega)}\right) ds \\
&\le \frac{\tau}{2}\int_0^t \big|v'_\varepsilon(s)\big|^2_{H_0} ds + \frac{\tilde{M}_1}{\tau}\int_0^t \Big(1+\big|f(s)\big|^2_{L^2(\Omega)} + \big|v_\varepsilon(s)\big|^2_{V_0}\Big) ds
\end{aligned} \tag{4.53}$$

and

$$\begin{aligned}
&\int_0^t \Big(f_\Gamma(s)+v_{\Gamma,\varepsilon}(s)-\pi_\Gamma\big(v_{\Gamma,\varepsilon}(s)+m_0\big), v'_{\Gamma,\varepsilon}(s)\Big)_{H_\Gamma} ds \\
&\le \frac{1}{2}\int_0^t \big|v'_{\Gamma,\varepsilon}(s)\big|^2_{H_\Gamma} ds + \tilde{M}_1\int_0^t \Big(1+\big|f_\Gamma(s)\big|^2_{H_\Gamma} + \big|v_{\Gamma,\varepsilon}(s)\big|^2_{H_\Gamma}\Big) ds
\end{aligned} \tag{4.54}$$

for all $t\in[0,T]$. Now, we collect the information in (4.50)–(4.54) and then apply the Gronwall lemma to the inequality resulting from (4.49). Hence, we prove the lemma in this case and we see from (4.53) that the constant M_1 depends on $\tau>0$.

On the contrary, if (A7) is assumed, the key estimate (4.53) is modified. Thanks to the Young inequality, we see that

$$\int_0^t \Big(-\pi\big(v_\varepsilon(s)+m_0\big), v_\varepsilon'(s)\Big)_{L^2(\Omega)} ds \le \delta \int_0^t \big|v_\varepsilon'(s)\big|_{V_0^*}^2 ds + \frac{\tilde{M}_1}{\delta} \int_0^t \Big(1+\big|v_\varepsilon(s)\big|_{V_0}^2\Big) ds, \tag{4.55}$$

for all $\delta > 0$. If we assume $f \in H^1(0,T;L^2(\Omega))$, then we can integrate by parts and use the Young inequality and (2.1), as follows:

$$\begin{aligned}
&\int_0^t \big(f(s), v_\varepsilon'(s)\big)_{L^2(\Omega)} ds \\
&= -\int_0^t \big(f'(s), v_\varepsilon(s)\big)_{L^2(\Omega)} ds + \big(f(t), v_\varepsilon(t)\big)_{L^2(\Omega)} - \big(f(0), v_0\big)_{L^2(\Omega)} \\
&\le \frac{1}{2}\int_0^t \big|f'(s)\big|_{L^2(\Omega)}^2 ds + \frac{C_0}{2}\int_0^t \big|v_\varepsilon(s)\big|_{V_0}^2 ds + \frac{1}{4}\big|v_\varepsilon(t)\big|_{V_0}^2 + \frac{1}{4}|v_0|_{H_0}^2 + (C_0+1)|f|_{C([0,T];L^2(\Omega))}^2,
\end{aligned}$$

for all $t \in [0,T]$. Thus, taking $\delta < 1$ we can apply the Gronwall lemma to obtain the estimate with a certain positive constant M_1 independent of $\tau > 0$. On the other hand, if we assume $f \in L^2(0,T;H^1(\Omega))$, then we have

$$\int_0^t \big(f(s), v_\varepsilon'(s)\big)_{L^2(\Omega)} ds \le \frac{\delta}{2}\int_0^t \big|v_\varepsilon'(s)\big|_{V_0^*}^2 ds + \frac{1}{2\delta}\int_0^t \big|f(s)\big|_{H^1(\Omega)}^2 ds \quad \text{for all } t \in [0,T].$$

Thus, by taking $\delta < 2/3$, the Gronwall inequality works again to the conclusion. □

Thanks to the growth conditions (2.21)–(2.22) (see also (4.28)–(4.29)), we obtain the following estimate.

Lemma 4.2. *There exists a positive constant M_2, independent of $\varepsilon \in (0,1]$, such that*

$$|\lambda_\varepsilon|_{L^2(0,T)} \le M_2.$$

Proof. From the expression of λ_ε, given by (4.40), we infer that

$$\begin{aligned}
\lambda_\varepsilon(t) = &-\int_\Omega \left\{ F^{-1}\left(\frac{\partial v_\varepsilon}{\partial t}(t)\right) + \tau \frac{\partial v_\varepsilon}{\partial t}(t) + q_\varepsilon(t) \right\} z_c dx - \int_\Omega \nabla v_\varepsilon(t)\cdot\nabla z_c dx \\
&- \frac{1}{\sigma_0}\int_\Gamma \left\{ \frac{\partial v_{\Gamma,\varepsilon}}{\partial t}(t) + q_{\Gamma,\varepsilon}(t) \right\} d\Gamma,
\end{aligned}$$

for a.a. $t \in (0,T)$. Therefore,

$$\begin{aligned}
|\lambda_\varepsilon|_{L^2(0,T)}^2 \le{}& 6|z_c|_{H_0}^2 \int_0^T \left\{ \Big|F^{-1}\big(v_\varepsilon'(t)\big)\Big|_{H_0}^2 + \tau^2 \big|v_\varepsilon'(t)\big|_{H_0}^2 \right\} dt + 6|z_c|_{C(\bar{\Omega})}^2 \int_0^T \big|q_\varepsilon(t)\big|_{L^1(\Omega)}^2 dt \\
&+ 6|z_c|_{V_0}^2 \int_0^T \big|v_\varepsilon(t)\big|_{V_0}^2 dt + \frac{6}{\sigma_0^2}|\Gamma| \int_0^T \big|v_{\Gamma,\varepsilon}'(t)\big|_{H_\Gamma}^2 dt + \frac{6}{\sigma_0^2}\int_0^T \big|q_{\Gamma,\varepsilon}(t)\big|_{L^1(\Gamma)}^2 dt.
\end{aligned}$$

By virtue of (4.28)–(4.29), there exists a positive constant $\tilde{M}_2 > 0$ depending only on c_0, L, L_Γ, $|\pi(m_0)|$ and $|\pi_\Gamma(m_0)|$ such that

$$\begin{aligned}
&\bigl|q_\varepsilon(t)\bigr|_{L^1(\Omega)} \\
&\le \int_\Omega c_0\Bigl(1+\widehat{\beta}_\varepsilon\bigl(v_\varepsilon(t)+m_0\bigr)\Bigr)dx + \int_\Omega \Bigl\{L\bigl|v_\varepsilon(t)\bigr| + \bigl|\pi(m_0)\bigr|\Bigr\}dx + \int_\Omega \bigl|f(t)\bigr|dx \\
&\le \tilde{M}_2\left\{1+\int_\Omega \widehat{\beta}_\varepsilon\bigl(v_\varepsilon(t)+m_0\bigr)dx + \bigl|v_\varepsilon(t)\bigr|_{L^1(\Omega)} + \bigl|f(t)\bigr|_{L^1(\Omega)}\right\}
\end{aligned}$$

and

$$\begin{aligned}
\bigl|q_{\Gamma,\varepsilon}(t)\bigr|_{L^1(\Gamma)} &\le \int_\Gamma \varepsilon\bigl|v_{\Gamma,\varepsilon}(t)\bigr|d\Gamma + \int_\Gamma c_0\Bigl(1+\widehat{\beta}_{\Gamma,\varepsilon}\bigl(v_{\Gamma,\varepsilon}(t)+m_0\bigr)\Bigr)d\Gamma \\
&\quad + \int_\Gamma \Bigl\{L\bigl|v_{\Gamma,\varepsilon}(t)\bigr| + \bigl|\pi_\Gamma(m_0)\bigr|\Bigr\}d\Gamma + \int_\Gamma \bigl|f_\Gamma(t)\bigr|d\Gamma \\
&\le \tilde{M}_2\left\{1+\int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}\bigl(v_{\Gamma,\varepsilon}(t)+m_0\bigr)d\Gamma + \bigl|v_{\Gamma,\varepsilon}(t)\bigr|_{L^1(\Gamma)} + \bigl|f_\Gamma(t)\bigr|_{L^1(\Gamma)}\right\}
\end{aligned}$$

for a.a. $t \in (0,T)$. Therefore, using Lemma 4.1 and taking into account that

$$\bigl|F^{-1}\bigl(v'_\varepsilon(t)\bigr)\bigr|^2_{H_0} \le C_0\bigl|F^{-1}\bigl(v'_\varepsilon(t)\bigr)\bigr|^2_{V_0} = C_0\bigl|v'_\varepsilon(t)\bigr|^2_{V_0^*},$$

we can find a positive constant M_2, independent of $\varepsilon \in (0,1]$, to prove the assertion. □

Lemma 4.3. *There exists a positive constant M_3, independent of $\varepsilon \in (0,1]$, such that*

$$|\omega_\varepsilon|_{L^2(0,T)} \le M_3.$$

Proof. From the expression of ω_ε, given by (4.42), we have

$$\begin{aligned}
|\omega_\varepsilon|^2_{L^2(0,T)} &\le \frac{4}{|\Omega|^2}\int_0^T \bigl|q_\varepsilon(t)\bigr|^2_{L^1(\Omega)}dt + \frac{4}{|\Omega|^2}\int_0^T \bigl|v'_{\Gamma,\varepsilon}(t)\bigr|^2_{L^1(\Gamma)}dt + \frac{4}{|\Omega|^2}\int_0^T \bigl|q_{\Gamma,\varepsilon}(t)\bigr|^2_{L^1(\Gamma)}dt \\
&\quad + \frac{4}{|\Omega|^2}|w_\Gamma|^2_{L^1(\Gamma)}\int_0^T \bigl|\lambda_\varepsilon(t)\bigr|^2 dt.
\end{aligned}$$

Thus, Lemmas 4.1 and 4.2 ensure the existence of a positive constant M_3, independent of $\varepsilon \in (0,1]$, which yields a bound for $|\omega_\varepsilon|_{L^2(0,T)}$. □

Lemma 4.4. *There exist two positive constants M_4 and M_5, independent of $\varepsilon \in (0,1]$, such that*

$$\begin{aligned}
\bigl|\beta_\varepsilon(v_\varepsilon+m_0)\bigr|_{L^2(0,T;L^2(\Omega))} + \bigl|\beta_\varepsilon(v_{\Gamma,\varepsilon}+m_0)\bigr|_{L^2(0,T;H_\Gamma)} &\le M_4, \\
|v_\varepsilon|_{L^2(0,T;H^{3/2}(\Omega))} + |\partial_\nu v_\varepsilon|_{L^2(0,T;H_\Gamma)} &\le M_5.
\end{aligned}$$

Proof. Testing (4.43) by $\beta_\varepsilon(v_\varepsilon+m_0) \in L^2(0,T;H^1(\Omega))$ and using (4.44). Then, integrating it over $\Omega \times (0,t)$ with respect to (x,s), we infer that

$$\begin{aligned}
&\int_0^t\int_\Omega \beta_\varepsilon'\bigl(v_\varepsilon(s)+m_0\bigr)\bigl|\nabla v_\varepsilon(s)\bigr|^2 dxds + \int_0^t \bigl|\beta_\varepsilon\bigl(v_\varepsilon(s)+m_0\bigr)\bigr|^2_{L^2(\Omega)} ds \\
&\quad + \int_0^t\int_\Gamma \beta_\varepsilon'\bigl(v_{\Gamma,\varepsilon}(s)+m_0\bigr)\bigl|\nabla_\Gamma v_{\Gamma,\varepsilon}(s)\bigr|^2 d\Gamma ds \\
&\quad + \int_0^t\int_\Gamma \beta_{\Gamma,\varepsilon}\bigl(v_{\Gamma,\varepsilon}(s)+m_0\bigr)\beta_\varepsilon\bigl(v_{\Gamma,\varepsilon}(s)+m_0\bigr) d\Gamma ds \\
&\le \int_0^t \Bigl(f(s) - F^{-1}\bigl(v_\varepsilon'(s)\bigr) - \tau v_\varepsilon'(s) - \pi\bigl(v_\varepsilon(s)+m_0\bigr) + \omega_\varepsilon(s), \beta_\varepsilon\bigl(v_\varepsilon(s)+m_0\bigr)\Bigr)_{L^2(\Omega)} ds \\
&\quad + \int_0^t \Bigl(f_\Gamma(s) - v_{\Gamma,\varepsilon}'(s) - \pi_\Gamma\bigl(v_{\Gamma,\varepsilon}(s)+m_0\bigr) - \lambda_\varepsilon(s) w_\Gamma, \beta_\varepsilon\bigl(v_{\Gamma,\varepsilon}(s)+m_0\bigr)\Bigr)_{H_\Gamma} ds \\
&\quad - \varepsilon \int_0^t \Bigl(v_{\Gamma,\varepsilon}(s), \beta_\varepsilon\bigl(v_{\Gamma,\varepsilon}(s)+m_0\bigr)\Bigr)_{H_\Gamma} ds \quad \text{for all } t \in [0,T],
\end{aligned}$$

where we should take care that $(\beta_\varepsilon(v_\varepsilon+m_0))_{|\Gamma} = \beta_\varepsilon(v_{\Gamma,\varepsilon}+m_0) \in L^2(0,T;H^1(\Gamma))$. Here, we use the assumption (4.30) to deduce that

$$\begin{aligned}
&\int_0^t\int_\Gamma \beta_{\Gamma,\varepsilon}\bigl(v_{\Gamma,\varepsilon}(s)+m_0\bigr)\beta_\varepsilon\bigl(v_{\Gamma,\varepsilon}(s)+m_0\bigr) d\Gamma ds \\
&= \int_0^t\int_\Gamma \bigl|\beta_{\Gamma,\varepsilon}\bigl(v_{\Gamma,\varepsilon}(s)+m_0\bigr)\bigr|\bigl|\beta_\varepsilon\bigl(v_{\Gamma,\varepsilon}(s)+m_0\bigr)\bigr| d\Gamma ds \\
&\ge \frac{1}{\varrho}\int_0^t\int_\Gamma \bigl|\beta_\varepsilon\bigl(v_{\Gamma,\varepsilon}(s)+m_0\bigr)\bigr|^2 d\Gamma ds - \frac{c_0}{\varrho}\int_0^t\int_\Gamma \bigl|\beta_\varepsilon\bigl(v_{\Gamma,\varepsilon}(s)+m_0\bigr)\bigr| d\Gamma ds \\
&\ge \frac{1}{2\varrho}\int_0^t \bigl|\beta_\varepsilon\bigl(v_{\Gamma,\varepsilon}(s)+m_0\bigr)\bigr|^2_{H_\Gamma} ds - \frac{c_0^2}{2\varrho} T|\Gamma| \quad \text{for all } t \in [0,T],
\end{aligned}$$

because $\beta_\varepsilon(r)$ and $\beta_{\Gamma,\varepsilon}(r)$ have the same sign for all $r \in \mathbb{R}$. We also note that

$$\int_0^t\int_\Omega \beta_\varepsilon'\bigl(v_\varepsilon(s)+m_0\bigr)\bigl|\nabla v_\varepsilon(s)\bigr|^2 dxds \ge 0,$$

$$\int_0^t\int_\Gamma \beta_\varepsilon'\bigl(v_{\Gamma,\varepsilon}(s)+m_0\bigr)\bigl|\nabla_\Gamma v_{\Gamma,\varepsilon}(s)\bigr|^2 d\Gamma ds \ge 0$$

for all $t \in [0,T]$. Moreover, using the Young inequality and the fact $\varepsilon \le 1$ we have

$$- \varepsilon \int_0^t \Big(v_{\Gamma,\varepsilon}(s), \beta_\varepsilon\big(v_{\Gamma,\varepsilon}(s) + m_0\big)\Big)_{H_\Gamma} ds$$

$$\le \frac{\delta}{2} \int_0^t \Big| \beta_\varepsilon\big(v_{\Gamma,\varepsilon}(s) + m_0\big)\Big|_{H_\Gamma}^2 ds + \frac{1}{2\delta} \int_0^t \big|v_{\Gamma,\varepsilon}(s)\big|_{H_\Gamma}^2 d\Gamma$$

for all $t \in [0,T]$ and $\delta > 0$. Now, there exists a positive constant $\tilde{M}_4$, which depends only on C_0, L, L_Γ, $|\pi(m_0)|$, $|\pi_\Gamma(m_0)|$, $|\Omega|$, $|\Gamma|$ and T, such that

$$\int_0^t \Big(f(s) - F^{-1}\big(v_\varepsilon'(s)\big) - \tau v_\varepsilon'(s) - \pi\big(v_\varepsilon(s) + m_0\big) + \omega_\varepsilon(s), \beta_\varepsilon\big(v_\varepsilon(s) + m_0\big)\Big)_{L^2(\Omega)} ds$$

$$\le \frac{1}{2} \int_0^t \Big|\beta_\varepsilon\big(v_\varepsilon(s) + m_0\big)\Big|_{L^2(\Omega)}^2 ds$$

$$+ \tilde{M}_4 \left(1 + |f|^2_{L^2(0,T;L^2(\Omega))} + |v_\varepsilon'|^2_{L^2(0,T;V_0^*)} + \tau^2 |v_\varepsilon'|^2_{L^2(0,T;H_0)} + |v_\varepsilon|^2_{L^2(0,T;H_0)} + |\omega_\varepsilon|^2_{L^2(0,T)}\right),$$

and

$$\int_0^t \Big(f_\Gamma(s) - v_{\Gamma,\varepsilon}'(s) - \pi_\Gamma\big(v_{\Gamma,\varepsilon}(s) + m_0\big) - \lambda_\varepsilon(s) w_\Gamma, \beta_\varepsilon\big(v_{\Gamma,\varepsilon}(s) + m_0\big)\Big)_{H_\Gamma} ds$$

$$\le \frac{\delta}{2} \int_0^t \Big|\beta_\varepsilon\big(v_{\Gamma,\varepsilon}(s) + m_0\big)\Big|_{H_\Gamma}^2 ds$$

$$+ \frac{\tilde{M}_4}{2\delta} \left(1 + |f_\Gamma|^2_{L^2(0,T;H_\Gamma)} + |v_{\Gamma,\varepsilon}'|^2_{L^2(0,T;H_\Gamma)} + |v_{\Gamma,\varepsilon}|^2_{L^2(0,T;H_\Gamma)} + |\lambda_\varepsilon|^2_{L^2(0,T)} |w_\Gamma|^2_{H_\Gamma}\right),$$

for all $t \in [0,T]$ and $\delta > 0$, with the help of the Young inequality. Thus, choosing $\delta < 1/(2\varrho)$ and recalling Lemmas 4.1–4.3 we deduce that there exists a positive constant M_4, independent of $\varepsilon \in (0,1]$, such that

$$\big|\beta_\varepsilon(v_\varepsilon + m_0)\big|_{L^2(0,T;L^2(\Omega))} + \big|\beta_\varepsilon(v_{\Gamma,\varepsilon} + m_0)\big|_{L^2(0,T;H_\Gamma)} \le M_4.$$

Next, we can compare the terms in (4.43) and conclude that

$$|\Delta v_\varepsilon|_{L^2(0,T;L^2(\Omega))} \text{ is bounded independently of } \varepsilon,$$

whence, taking Lemma 4.1 into account and applying the theory of the elliptic regularity (see, e.g., [6, Thm. 3.2, p. 1.79]), we have that

$$|v_\varepsilon|_{L^2(0,T;H^{3/2}(\Omega))} \le \tilde{M}_5,$$

and, owing to the trace theory (see, e.g., [6, Thm. 2.25, p. 1.62]), that

$$|\partial_\nu v_\varepsilon|_{L^2(0,T;H_\Gamma)} \le \tilde{M}_5,$$

for some constant $\tilde{M}_5$ independent of $\varepsilon \in (0,1]$. $\square$

Lemma 4.5. *There exist positive constants M_6, M_7 and M_8, independent of $\varepsilon \in (0,1]$, such that*

$$\big|\beta_{\Gamma,\varepsilon}(v_{\Gamma,\varepsilon}+m_0)\big|_{L^2(0,T;H_\Gamma)} \le M_6, \quad |v_{\Gamma,\varepsilon}|_{L^2(0,T;H^2(\Gamma))} \le M_7, \quad |v_\varepsilon|_{L^2(0,T;H^2(\Omega))} \le M_8.$$

Proof. We test (4.44) by $\beta_{\Gamma,\varepsilon}(v_{\Gamma,\varepsilon}+m_0) \in L^2(0,T;V_\Gamma)$ and integrate on the boundary, deducing that

$$\begin{aligned}
&\int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}\big(v_{\Gamma,\varepsilon}(t)+m_0\big)d\Gamma + \int_0^t\int_\Gamma \beta'_{\Gamma,\varepsilon}\big(v_{\Gamma,\varepsilon}(s)+m_0\big)\big|\nabla_\Gamma v_{\Gamma,\varepsilon}(s)\big|^2 d\Gamma ds \\
&\quad + \int_0^t \Big|\beta_{\Gamma,\varepsilon}\big(v_{\Gamma,\varepsilon}(s)+m_0\big)\Big|_{H_\Gamma}^2 ds \\
&\le \int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}\big(v_{0\Gamma}+m_0\big)d\Gamma - \int_0^t \Big(\varepsilon\, v_{\Gamma,\varepsilon}(s) + \partial_\nu v_{\Gamma,\varepsilon}(s), \beta_{\Gamma,\varepsilon}\big(v_{\Gamma,\varepsilon}(s)+m_0\big)\Big)_{H_\Gamma} ds \\
&\quad + \int_0^t \Big(f_\Gamma(s) - \pi_\Gamma\big(v_{\Gamma,\varepsilon}(s)+m_0\big) - \lambda_\varepsilon(s)w_\Gamma, \beta_{\Gamma,\varepsilon}\big(v_{\Gamma,\varepsilon}(s)+m_0\big)\Big)_{H_\Gamma} ds, \qquad (4.56)
\end{aligned}$$

for all $t \in [0,T]$. We note that

$$\int_0^t\int_\Gamma \beta'_{\Gamma,\varepsilon}\big(v_{\Gamma,\varepsilon}(s)+m_0\big)\big|\nabla_\Gamma v_{\Gamma,\varepsilon}(s)\big|^2 d\Gamma ds \ge 0,$$

due to the properties of $\beta_{\Gamma,\varepsilon}$, and

$$\int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}(v_{0\Gamma}+m_0)d\Gamma \le \int_\Gamma \widehat{\beta}_\Gamma(v_{0\Gamma}+m_0)d\Gamma < +\infty,$$

by virtue of (2.24). By applying the Young inequality in the last two terms of (4.56), we see that there exists a positive constant $\tilde{M}_6$ independent of $\varepsilon \in (0,1]$ such that

$$\big|\beta_{\Gamma,\varepsilon}(v_{\Gamma,\varepsilon}+m_0)\big|_{L^2(0,T;H_\Gamma)} \le \tilde{M}_6.$$

Hence, by comparison in (4.44) we also infer that

$$|\Delta_\Gamma v_{\Gamma,\varepsilon}|_{L^2(0,T;H_\Gamma)} \le \tilde{M}_7$$

and consequently (see, e.g., [19, Section 4.2])

$$\begin{aligned}
|v_{\Gamma,\varepsilon}|_{L^2(0,T;H^2(\Gamma))} &\le \Big(|v_{\Gamma,\varepsilon}|^2_{L^2(0,T;V_\Gamma)} + |\Delta_\Gamma v_{\Gamma,\varepsilon}|^2_{L^2(0,T;H_\Gamma)}\Big)^{\frac{1}{2}} \\
&\le \big(M_1^2 T + \tilde{M}_7^2\big)^{\frac{1}{2}} =: M_7.
\end{aligned}$$

Then, in view of Lemma 4.4, using the theory of the elliptic regularity (see, e.g., [6, Thm. 3.2, p. 1.79]) along with the boundedness of $|v_{\Gamma,\varepsilon}|_{L^2(0,T;H^{3/2}(\Gamma))}$, it turns out that

$$|v_\varepsilon|_{L^2(0,T;H^2(\Omega))} \le M_8$$

for some positive constant M_8 independent of $\varepsilon \in (0,1]$. □

Remark 4.1. All constants M_k, for k from 1 to 8, are obtained independently of $\tau > 0$ provided that (A7) is assumed. Actually, under the additional assumption (A7) the positive constant M_1 in Lemma 4.1 is independent of $\tau > 0$.

4.3. Passage to the limit as $\varepsilon \to 0$

In this subsection, we keep $\tau > 0$ fixed and conclude the existence proof by passage to the limit of the approximate solutions as $\varepsilon \to 0$. Indeed, owing to the uniform estimates stated in Lemmas from 4.1 to 4.5, there exist a subsequence of ε (not relabeled) and some limit functions v, v_Γ, ξ, ξ_Γ, ω, λ such that

$$v_\varepsilon \to v \quad \text{weakly star in } H^1(0,T;H_0) \cap L^\infty(0,T;V_0) \cap L^2\big(0,T;H^2(\Omega)\big), \tag{4.57}$$

$$v_{\Gamma,\varepsilon} \to v_\Gamma \quad \text{weakly star in } H^1(0,T;H_\Gamma) \cap L^\infty(0,T;V_\Gamma) \cap L^2\big(0,T;H^2(\Gamma)\big), \tag{4.58}$$

$$\beta_\varepsilon(v_\varepsilon + m_0) \to \xi \quad \text{weakly in } L^2\big(0,T;L^2(\Omega)\big), \tag{4.59}$$

$$\beta_{\Gamma,\varepsilon}(v_{\Gamma,\varepsilon} + m_0) \to \xi_\Gamma \quad \text{weakly in } L^2(0,T;H_\Gamma), \tag{4.60}$$

$$\omega_\varepsilon \to \omega \quad \text{weakly in } L^2(0,T), \tag{4.61}$$

$$\lambda_\varepsilon \to \lambda \quad \text{weakly in } L^2(0,T), \tag{4.62}$$

as $\varepsilon \to 0$. From (4.57) and (4.58), due to strong compactness results (see, e.g., [25, Sect. 8, Cor. 4]) we have that

$$v_\varepsilon \to v \quad \text{strongly in } C\big([0,T];H_0\big) \cap L^2(0,T;V_0), \tag{4.63}$$

$$v_{\Gamma,\varepsilon} \to v_\Gamma \quad \text{strongly in } C\big([0,T];H_\Gamma\big) \cap L^2(0,T;V_\Gamma), \tag{4.64}$$

as $\varepsilon \to 0$. Moreover, on account of (4.46) and (4.58) it is a standard matter to deduce that

$$h_\varepsilon \to h \quad \text{weakly in } H^1(0,T) \text{ and strongly in } C\big([0,T]\big), \tag{4.65}$$

where

$$h_* \le h(t) := \int_\Gamma w_\Gamma v_\Gamma(t) d\Gamma \le h^* \quad \text{for all } t \in [0,T].$$

We point out that (4.44), (4.57) and (4.58) imply that $v_\Gamma = v_{|_\Gamma}$ a.e. on Σ, while (4.45), (4.63), (4.64) entail

$$v(0) = v_0 \quad \text{a.e. in } \Omega, \quad v_\Gamma(0) = v_{0\Gamma} \quad \text{a.e. on } \Gamma.$$

Now, (4.62) and (4.65) and the maximal monotonicity of $\partial I_{[h_*,h^*]}$ allow us to conclude that

$$\lambda \in \partial I_{[h_*,h^*]}(h) \quad \text{a.e. in } (0,T),$$

that is equivalent to (2.16). Moreover, (4.63)–(4.64) and the Lipschitz continuity of π, π_Γ imply that

$$\pi(v_\varepsilon + m_0) \to \pi(v + m_0) \quad \text{strongly in } C\big([0,T];L^2(\Omega)\big),$$

$$\pi_\Gamma(v_{\Gamma,\varepsilon} + m_0) \to \pi_\Gamma(v_\Gamma + m_0) \quad \text{strongly in } C\big([0,T];H_\Gamma\big),$$

as $\varepsilon \to 0$. At this point, we can pass to the limit in (4.43) and (4.44) obtaining (2.10) and (2.12). Moreover, by applying [4, Prop. 2.2, p. 38] and using (4.59)–(4.60) with (4.63)–(4.64), we obtain

$$\xi \in \beta(v + m_0) \quad \text{a.e. in } Q, \qquad \xi_\Gamma \in \beta_\Gamma(v_\Gamma + m_0) \quad \text{a.e. on } \Sigma.$$

Thus, it turns out that the pair $\boldsymbol{v} = (v, v_\Gamma)$ yields, along with $\boldsymbol{\xi} = (\xi, \xi_\Gamma)$, ω and λ, a solution of the limit problem, which can be stated exactly as in (2.10)–(2.16). Also, we note the regularities $v \in C([0,T]; V_0)$ and $u_\Gamma \in C([0,T]; V_\Gamma)$ for the solution as a consequence of (4.57)–(4.58).

4.4. Passage to the limit as $\tau \to 0$

In this subsection, we discuss the limiting problem as $\tau \to 0$. We need to assume the additional regularity (A7) for f. For each $\tau > 0$, let now $\boldsymbol{v}_\tau := (v_\tau, v_{\Gamma,\tau})$ be the solution to (2.10)–(2.16) with related ω_τ, λ_τ and

$$h_\tau(t) := \int_\Gamma w_\Gamma v_{\Gamma,\tau}(t) d\Gamma \quad \text{for all } t \in [0,T].$$

On account of Lemma 4.1 with Remark 4.1, we use the uniform estimates in Lemmas 4.1–4.5 to perform the limit procedure as $\tau \to 0$.

As in the previous passage to the limit as $\varepsilon \to 0$, also in this case a subsequence of τ (not relabeled) and some limit functions v, v_Γ, ξ, ξ_Γ, ω, λ can be found in order that the same convergences as in (4.58)–(4.62) and

$$v_\tau \to v \quad \text{weakly star in } H^1(0,T; V_0^*) \cap L^\infty(0,T; V_0) \cap L^2\bigl(0,T; H^2(\Omega)\bigr) \tag{4.66}$$

hold as $\tau \to 0$. We can still deduce the same strong convergences as in (4.63)–(4.65) and the passage to the limit can be carried out in a similar way. Of course, here we have to point out that (cf. the estimate in Lemma 4.1)

$$\tau v_\tau' \to 0 \quad \text{strongly in } L^2(0,T; H_0)$$

as $\tau \to 0$, which is important when we pass to the limit in Eq. (2.10), obtaining

$$F^{-1}\left(\frac{\partial v}{\partial t}\right) - \Delta v + \xi + \pi(v + m_0) = f + \omega \quad \text{a.e. in } Q, \tag{4.67}$$

to be coupled with (2.11)–(2.16).

Remark 4.2. On the side of the proof, one can make the remark that the solution component $\boldsymbol{v} = (v, v_\Gamma)$ of the problem solves the abstract formulation (see Subsections 2.4 and 4.1)

$$\boldsymbol{v} \in H^1(0,T; \boldsymbol{V}_0^*) \cap L^\infty(0,T; \boldsymbol{V}_0),$$
$$\boldsymbol{v} \in H^1(0,T; \boldsymbol{H}_0) \quad \text{if } \tau > 0,$$
$$\boldsymbol{v}^* := (-\Delta v + \xi, \partial_\nu v - \Delta_\Gamma v_\Gamma + \xi_\Gamma) \in L^2(0,T; \boldsymbol{H}_0),$$
$$\lambda \in L^2(0,T),$$
$$\boldsymbol{A}_\tau \boldsymbol{v}'(t) + \boldsymbol{v}^*(t) + \lambda(t)\boldsymbol{w} = P\Bigl(\boldsymbol{f}(t) - \boldsymbol{\Pi}_0\bigl(\boldsymbol{v}(t)\bigr)\Bigr) \quad \text{in } \boldsymbol{H}_0, \text{ for a.a. } t \in (0,T),$$
$$\boldsymbol{v}^*(t) \in \partial\varphi\bigl(\boldsymbol{v}(t)\bigr) \quad \text{in } \boldsymbol{H}_0, \text{ for a.a. } t \in (0,T),$$
$$\lambda(t)\boldsymbol{w} \in \partial I_{\overline{\boldsymbol{K}}}\bigl(\boldsymbol{v}(t)\bigr) \quad \text{in } \boldsymbol{H}_0, \text{ for a.a. } t \in (0,T),$$
$$\boldsymbol{v}(0) = \boldsymbol{v}_0 \quad \text{in } \boldsymbol{H}_0.$$

Moreover, let us point out that

$$\boldsymbol{v}^*(t) + \lambda(t)\boldsymbol{w} \in \partial(\varphi + I_{\boldsymbol{K}})\big(\boldsymbol{v}(t)\big) \quad \text{in } \boldsymbol{H}_0, \text{ for a.a. } t \in (0,T).$$

Therefore, it is clear that $\boldsymbol{v}$ is the solution of the Cauchy problem expressed by (2.25)–(2.26). We note that although the solution $\boldsymbol{v}$ of this problem is uniquely determined, the auxiliary quantities $\boldsymbol{v}^*$ and λ are not unique in general (cf. [10, Remark 3.3], [15, Remark 2]).

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References

[1] T. Aiki, Two-phase Stefan problems with dynamic boundary conditions, Adv. Math. Sci. Appl. 2 (1993) 253–270.
[2] T. Aiki, Multi-dimensional Stefan problems with dynamic boundary conditions, Appl. Anal. 56 (1995) 71–94.
[3] T. Aiki, Periodic stability of solutions to some degenerate parabolic equations with dynamic boundary conditions, J. Math. Soc. Japan 48 (1996) 37–59.
[4] V. Barbu, Nonlinear Differential Equations of Monotone Types in Banach Spaces, Springer, London, 2010.
[5] H. Brézis, Opérateurs maximaux monotones et semi-groupes de contractions dans les especes de Hilbert, North-Holland, Amsterdam, 1973.
[6] F. Brezzi, G. Gilardi, Partial differential equations, in: H. Kardestuncer, D.H. Norrie (Eds.), Finite Element Handbook, McGraw-Hill Book Co., New York, 1987, Part 1.
[7] J.W. Cahn, J.E. Hilliard, Free energy of a nonuniform system I. Interfacial free energy, J. Chem. Phys. 2 (1958) 258–267.
[8] L. Calatroni, P. Colli, Global solution to the Allen–Cahn equation with singular potentials and dynamic boundary conditions, Nonlinear Anal. 79 (2013) 12–27.
[9] L. Cherfils, S. Gatti, A. Miranville, A variational approach to a Cahn–Hilliard model in a domain with nonpermeable walls, J. Math. Sci. (N. Y.) 189 (2013) 604–636.
[10] P. Colli, T. Fukao, Allen–Cahn equation with dynamic boundary conditions and mass constraints, Math. Methods Appl. Sci. (2015), http://dx.doi.org/10.1002/mma.3329, see also preprint arXiv:1405.0116 [math.AP], 2014, pp. 1–23.
[11] P. Colli, G. Gilardi, J. Sprekels, On the Cahn–Hilliard equation with dynamic boundary conditions and a dominating boundary potential, J. Math. Anal. Appl. 419 (2014) 972–994.
[12] P. Colli, G. Gilardi, J. Sprekels, A boundary control problem for the viscous Cahn–Hilliard equation with dynamic boundary conditions, Appl. Math. Optim. (2015), see also preprint arXiv:1407.3916 [math.AP], 2014, pp. 1–27.
[13] P. Colli, A. Visintin, On a class of doubly nonlinear evolution equations, Comm. Partial Differential Equations 15 (1990) 737–756.
[14] C.M. Elliott, S. Zheng, On the Cahn–Hilliard equation, Arch. Ration. Mech. Anal. 96 (1986) 339–357.
[15] T. Fukao, N. Kenmochi, Abstract theory of variational inequalities and Lagrange multipliers, in: Discrete and Continuous Dynamical Systems, Supplement 2013, 2013, pp. 237–246.
[16] G. Gilardi, A. Miranville, G. Schimperna, On the Cahn–Hilliard equation with irregular potentials and dynamic boundary conditions, Commun. Pure Appl. Anal. 8 (2009) 881–912.
[17] G. Gilardi, A. Miranville, G. Schimperna, Long-time behavior of the Cahn–Hilliard equation with irregular potentials and dynamic boundary conditions, Chin. Ann. Math. Ser. B 31 (2010) 679–712.
[18] G.R. Goldstein, A. Miranville, A Cahn–Hilliard–Gurtin model with dynamic boundary conditions, Discrete Contin. Dyn. Syst. Ser. S 6 (2013) 387–400.
[19] A. Grigor'yan, Heat Kernel and Analysis on Manifolds, American Mathematical Society, International Press, Boston, 2009.
[20] N. Kenmochi, Monotonicity and compactness methods for nonlinear variational inequalities, in: M. Chipot (Ed.), Handbook of Differential Equations: Stationary Partial Differential Equations, vol. 4, North-Holland, Amsterdam, 2007, pp. 203–298.
[21] N. Kenmochi, M. Niezgódka, Viscosity approach to modelling non-isothermal diffusive phase separation, Jpn. J. Ind. Appl. Math. 13 (1996) 135–169.
[22] M. Kubo, The Cahn–Hilliard equation with time-dependent constraint, Nonlinear Anal. 75 (2012) 5672–5685.
[23] J.-L. Lions, E. Magenes, Non-Homogeneous Boundary Value Problems and Applications, vol. I, Springer, Berlin, 1972.
[24] R. Racke, S. Zheng, The Cahn–Hilliard equation with dynamic boundary conditions, Adv. Differential Equations 8 (2003) 83–110.
[25] J. Simon, Compact sets in the spaces $L^p(0,T;B)$, Ann. Mat. Pura Appl. (4) 146 (1987) 65–96.