A NUMERICAL METHOD FOR SOLVING EQUILIBRIUM PROBLEMS OF MASONRY-LIKE SOLIDS

Internal Report C92-14

M. Lucchesi
C. Padovani
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MASSIMILIANO LUCCHESI
Universita' di Chieti, Viale Pindaro 42, 65100 Pescara, Italy

CRISTINA PADOVANI and ANDREA PAGNI
Istituto CNUCE-CNR, Via S. Maria 36, 56100 Pisa, Italy

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ABSTRACT. This paper proposes a numerical method for the solution of equilibrium problems of solids which do not support tension. Some boundary-value problems are solved numerically and the solution obtained is compared to the exact one.

SOMMARIO. In questo lavoro viene proposto un metodo numerico per la soluzione di problemi di equilibrio di solidi non resistenti a trazione. Vengono successivamente risolti numericamente alcuni problemi di equilibrio e la soluzione ottenuta e' confrontata con quella esatta.

KEY WORDS: Masonry-like materials, Finite element method.

1. Introduction

The masonry-like materials considered in this paper are characterized by the constitutive hypothesis that the total strain $\mathbf{E}$ can be split into the sum of an elastic part $\mathbf{E}^e$ and an inelastic part $\mathbf{E}^a$: $\mathbf{E} = \mathbf{E}^a + \mathbf{E}^e$, with $\mathbf{E}^a$ positive semi-definite; and that the stress tensor $\mathbf{T}$ is negative semi-definite, depends linearly on $\mathbf{E}^e$ and is orthogonal to $\mathbf{E}^a$. Since $\mathbf{E}^a$ only depends on the current strain, masonry-like materials are non-linear elastic materials although $\mathbf{E}$ infinitesimal. The solution of the constitutive equation defined in this way exists and is unique [1] and was explicitly calculated for isotropic and transversely isotropic materials [2], [3]. The determination of the solution of the equilibrium problems for solids which do not support tension is, in general, very complex. Some very restrictive conditions have been found, which guarantee the existence of the solution [4], [5]; this one can only be calculated explicitly only in very simple cases [6], [7], [8], and consequently, it is necessary to use numerical techniques in the applications.

Numerical techniques using the finite element method have been proposed in some previous papers: in [9] the equilibrium problem for masonry-like solids was solved by minimizing the complementary energy under suitable constraints on the stress. In [10] an iterative procedure leading to the progressive reduction of tractions inside the studied structure, was analyzed. Finally in [11] the displacement field is determined with the secant matrix method.

The numerical method proposed in this paper uses the fact that a masonry-like material is a non-linear elastic material for which, at least in the isotropic case, it is possible to calculate explicitly the derivative of the stress with respect to the total strain. The knowledge of this derivative allows one to calculate the tangent matrix and determine the displacements, by solving the non-linear system obtained with the discretization into finite elements with the Newton Raphson method. For the sake of simplicity, the study is limited to plane problems.

The algorithm, implemented in the finite element code NOSA [12,13], is used to solve numerically some equilibrium problems; the numerical solution is compared with the exact one.

2. The constitutive equation of masonry-like materials

In this section, after briefly recalling the main properties of the constitutive equation of masonry-like materials, we calculate explicitly the derivative of the stress with respect to the strain. Let us state some notations. Let $\mathcal{V}$ be the three-dimensional linear space and Lin the linear space of all linear applications of $\mathcal{V}$ into $\mathcal{V}$, equipped with the inner product

$$\mathbf{A}\cdot\mathbf{B} = \mathrm{tr}\,(\mathbf{A}^T\mathbf{B})\ , \quad \mathbf{A}, \mathbf{B}\in \mathrm{Lin}\ ,$$

with $\mathbf{A}^T$ the transpose of $\mathbf{A}$. Let us indicate with Sym, Sym^+ and Sym^- the subsets of Lin constituted respectively, by symmetric, symmetric positive semi-definite and symmetric negative semi-definite tensors.

Let us assume that the tensor of infinitesimal strain $\mathbf{E}$ is the sum of an elastic part $\mathbf{E}^e$ and of an inelastic part $\mathbf{E}^a$:

$$\mathbf{E} = \mathbf{E}^e + \mathbf{E}^a\ , \quad \mathbf{E}^a\in \mathrm{Sym}^+\ , \tag{2.1}$$

and that the Cauchy stress tensor $\mathbf{T}$ depends linearly and isotropically on $\mathbf{E}^e$,

$$\mathbf{T} = 2\mu\mathbf{E}^e + \lambda\ \mathrm{tr}(\mathbf{E}^e)\mathbf{I}\ , \tag{2.2}$$

where μ and λ are the Lame' moduli of the material satisfying the inequalities

$$\mu > 0\ , \quad 2\mu + 3\lambda > 0\ . \tag{2.3}$$

Moreover, let us suppose that

$$\mathbf{T}\in \mathrm{Sym}^-\ , \quad \mathbf{T}\cdot\mathbf{E}^a = 0\ . \tag{2.4}$$

It is known that, given $\mathbf{E}\in \mathrm{Sym}$ and the elastic moduli μ and λ verifying (2.3), tensors $\mathbf{T}$ and $\mathbf{E}^a$ exist and are unique in satisfying (2.1), (2.2) and (2.4). Moreover, $\mathbf{T}$, $\mathbf{E}$ and $\mathbf{E}^a$ are coaxial by virtue of (2.1), (2.2) and (2.4), and this property allows one to calculate explicitly the solution of the constitutive equation.

Let us treat in great detail the case of plane strain, then afterwards we shall briefly describe the changes which need to be made for the plane stress. Let $\{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\}$ be an orthonormal basis of $\mathcal{V}$ constituted by eigenvectors of $\mathbf{E}$, such that $\mathbf{E}\mathbf{g}_3\cdot\ \mathbf{g}_3 = 0$. Let us put $\alpha = \lambda/\mu$[1] and indicate with e_1, e_2 and a_1, a_2, a_3 the eigenvalues of $\mathbf{E}$ and $\mathbf{E}^a$ respectively. Relations (2.1), (2.2) and (2.4) can be rewritten by using principal components of $\mathbf{E}$ and $\mathbf{E}^a$:

[1] Here we assume that $\lambda \geq 0$, consequently $\alpha \geq 0$.

$$(2.5)\qquad \begin{cases} (2\,(e_1 - a_1) + \alpha\,(e_1 + e_2 - a_1 - a_2 - a_3))\cdot a_1 = 0 \\ (2\,(e_2 - a_2) + \alpha\,(e_1 + e_2 - a_1 - a_2 - a_3))\cdot a_2 = 0 \\ (-\,2a_3 + \alpha\,(e_1 + e_2 - a_1 - a_2 - a_3))\cdot a_3 = 0 \\ a_1 \geq 0\ ,\quad a_2 \geq 0\ ,\quad a_3 \geq 0 \\ 2\,(e_1 - a_1) + \alpha\,(e_1 + e_2 - a_1 - a_2 - a_3) \leq 0 \\ 2\,(e_2 - a_2) + \alpha\,(e_1 + e_2 - a_1 - a_2 - a_3) \leq 0 \\ -\,2a_3 + \alpha\,(e_1 + e_2 - a_1 - a_2 - a_3) \leq 0\,. \end{cases}$$

It is easy to prove that $a_3 = 0$. In fact if $a_3 > 0$, from $(2.5)_7$ we must have $-\,2a_3 + \alpha\,(e_1 + e_2 - a_1 - a_2 - a_3) = 0$, and therefore

$$(2.6)\qquad a_3 = \frac{\alpha}{2+\alpha}\,(e_1 + e_2 - a_1 - a_2)\ > 0\,.$$

By substituting (2.6) into $(2.5)_5$ and $(2.5)_6$ we obtain

$$\begin{cases} 4(1+\alpha)(e_1 - a_1) + 2\alpha(e_2 - a_2) \leq 0 \\ 4(1+\alpha)(e_2 - a_2) + 2\alpha(e_1 - a_1) \leq 0 \end{cases}$$

and summing these inequalities we have

$$e_1 + e_2 - a_1 - a_2 \leq 0$$

which is not in agreement with (2.6). Finally we can conclude that $a_3 = 0$.
For plane strain the constitutive relations (2.1), (2.2) and (2.4) become[2]

$$(2.7)\qquad \begin{cases} (2\,(e_1 - a_1) + \alpha\,(e_1 + e_2 - a_1 - a_2\,))\cdot a_1 = 0 \\ (2\,(e_2 - a_2) + \alpha\,(e_1 + e_2 - a_1 - a_2))\cdot a_2 = 0 \\ a_1 \geq 0\ ,\quad a_2 \geq 0\ , \\ 2\,(e_1 - a_1) + \alpha\,(e_1 + e_2 - a_1 - a_2\,) \leq 0 \\ 2\,(e_2 - a_2) + \alpha\,(e_1 + e_2 - a_1 - a_2\,) \leq 0\,. \end{cases}$$

[2] From the equalities $e_3 = a_3 = 0$, we obtain $g_3\cdot Tg_3 = \alpha\,(t_1 + t_2\,)/2(1 + \alpha) \leq 0$.

Let us continue to indicate with $\mathbf{E}$ and $\mathbf{E}^a$ their restrictions to the linear subspace of $\mathcal{V}$ of dimension two, orthogonal to the vector $\mathbf{g}_3$. The calculation of a_1 and a_2 satisfying (2.7) requires the definition of the following subsets of Sym:

$$\mathcal{S}_1 = \{ \mathbf{E} \in \mathrm{Sym}\,;\ \alpha e_2 + (2+\alpha)e_1 \le 0\ ,\ \ \alpha e_1 + (2+\alpha)e_2 \le 0 \}\ ,$$

$$\mathcal{S}_2 = \{ \mathbf{E} \in \mathrm{Sym}\,;\ e_1 \ge 0\ ,\ \ e_2 > 0 \}\ ,$$

$$\mathcal{S}_3 = \{ \mathbf{E} \in \mathrm{Sym}\,;\ e_1 < 0\ ,\ \ \alpha e_1 + (2+\alpha)e_2 > 0 \}\ ,$$

where we suppose the eigenvalues e_1 and e_2. ordered in such a way that $e_1 \le e_2$.
The principal components of $\mathbf{E}^a$ can be calculated, as is known, from the relations

$$\text{(2.8)}\qquad \begin{array}{lll} \text{if } \mathbf{E} \in \mathcal{S}_1 \text{ then} & a_1 = 0\,, & a_2 = 0\,, \\ \text{if } \mathbf{E} \in \mathcal{S}_2 \text{ then} & a_1 = e_1\ , & a_2 = e_2\ , \\ \text{if } \mathbf{E} \in \mathcal{S}_3 \text{ then} & a_1 = 0\ , & a_2 = e_2 + \dfrac{\alpha}{2+\alpha}\, e_1\,. \end{array}$$

We observe that in $\mathcal{S}_3$, the eigenvalues e_1 and e_2 are distinct and different from zero, in particular in this region $\mathbf{E}$ is invertible.
In order to calculate the derivative of $\mathbf{T}$ with respect to $\mathbf{E}$, let us begin by observing that from the coaxiality of $\mathbf{E}$ and $\mathbf{E}^a$ and the fact that the eigenvalues of $\mathbf{E}^a$ only depend on the eigenvalues of $\mathbf{E}$, it easily follows that the non-linear function $\mathbf{E}^a = \hat{\mathbf{E}}^a(\mathbf{E})$ is isotropic. By virtue of a well known representation theorem, two scalar functions γ_0 and γ_1 exist of principal invariants of $\mathbf{E}$,

$$I_1(\mathbf{E}) = \mathrm{tr}\,\mathbf{E} = e_1 + e_2\ ,\qquad I_2(\mathbf{E}) = \det(\mathbf{E}) = e_1\, e_2\ ,$$

such that

$$\text{(2.9)}\qquad \mathbf{E}^a = \gamma_0\, \mathbf{I} + \gamma_1\, \mathbf{E}\,.$$

In view of (2.8), we have

$$\text{(2.10)}\qquad \begin{array}{l} \text{if } \mathbf{E} \in \mathcal{S}_1 \text{ then } \gamma_0 = \gamma_1 = 0, \\[2ex] \text{if } \mathbf{E} \in \mathcal{S}_2 \text{ then } \gamma_0 = 0\ ,\ \gamma_1 = 1\,; \end{array}$$

in $\mathcal{S}_3$ the pair γ_0, γ_1 is the unique solution of the linear system

$$\gamma_0 + \gamma_1\, e_1 = 0$$

$$\gamma_0 + \gamma_1 e_2 = e_2 + \frac{\alpha}{2+\alpha} e_1 ,$$

therefore

$$\text{(2.11)} \qquad \text{if } \mathbf{E} \in \mathcal{S}_3 \text{ then} \qquad \begin{aligned} \gamma_0 &= -\frac{e_1}{e_2 - e_1}\left(e_2 + \frac{\alpha}{2+\alpha} e_1 \right) \\ \gamma_1 &= \frac{1}{e_2 - e_1}\left(e_2 + \frac{\alpha}{2+\alpha} e_1 \right). \end{aligned}$$

The eigenvalues e_1 and e_2 are the roots of the characteristic polynomial

$$\lambda^2 - I_1(\mathbf{E})\,\lambda + I_2(\mathbf{E}) = 0 \ ;$$

then we can write

$$\text{(2.12)} \qquad e_1 = \frac{I_1 - \sqrt{I_1^2 - 4\,I_2}}{2} \ , \quad e_2 = \frac{I_1 + \sqrt{I_1^2 - 4\,I_2}}{2} .$$

From these relations we easily obtain the expressions of γ_0 and γ_1 as functions of I_1 and I_2, in the region $\mathcal{S}_3$

$$\text{(2.13)} \qquad \begin{aligned} \gamma_0 &= -\frac{1}{2(2+\alpha)\sqrt{I_1^2 - 4\,I_2}} \left(\sqrt{I_1^2 - 4\,I_2} + (1+\alpha)I_1\right)\left(I_1 - \sqrt{I_1^2 - 4\,I_2}\right) \ , \\ \gamma_1 &= \frac{1}{2+\alpha}\left(1 + \frac{(1+\alpha)I_1}{\sqrt{I_1^2 - 4\,I_2}}\right) . \end{aligned}$$

In view of (2.1), (2.2) and (2.8) we have

$$\text{(2.14)} \qquad \mathbf{T} = (\lambda\,(1-\gamma_1)\,I_1 - 2\,\gamma_0(\lambda + \mu))\,\mathbf{I} + 2\mu\,(1-\gamma_1)\,\mathbf{E} =$$

$$\beta_0\,\mathbf{I} + \beta_1\,\mathbf{E} \ ,$$

where, as we did for $\mathbf{E}$ and $\mathbf{E}^a$, we continue to indicate with $\mathbf{T}$ the restriction of the stress tensor to the subspace of $\mathcal{V}$ orthogonal to $\mathbf{g}_3$.
From (2.10) and (2.13) we obtain the expressions of β_0 and β_1 as functions of principal invariants of $\mathbf{E}$,

$$\text{(2.15)} \qquad \begin{aligned} &\text{if } \mathbf{E} \in \mathcal{S}_1 \text{ then} && \beta_0 = \lambda\,I_1 \ , && \beta_1 = 2\mu, \\ &\text{if } \mathbf{E} \in \mathcal{S}_2 \text{ then} && \beta_0 = 0 \ , && \beta_1 = 0 \\ &\text{if } \mathbf{E} \in \mathcal{S}_3 \text{ then} && \beta_0 = \varphi\,\frac{I_2}{\sqrt{I_1^2 - 4I_2}} \ , && \beta_1 = -\,\varphi\,\frac{I_1 - \sqrt{I_1^2 - 4I_2}}{2\sqrt{I_1^2 - 4I_2}} \ , \end{aligned}$$

where $\varphi = \dfrac{4\mu\,(1+\alpha)}{(2+\alpha)}$.

now we are able, with the help of (2.14) and (2.15), to calculate the derivative $D_E\mathbf{T}$ of $\mathbf{T}$ with respect to $\mathbf{E}$.

If $\mathbf{E} \in \mathcal{S}_1$

$$D_E\mathbf{T} = 2\mu\,\mathbb{1} + \lambda\,\mathbf{I} \otimes \mathbf{I}\,; \tag{2.16}$$

if $\mathbf{E} \in \mathcal{S}_2$

$$D_E\mathbf{T} = \mathbb{O}\,, \tag{2.17}$$

where $\mathbb{1}$ and $\mathbb{O}$ are respectively the fourth-order identity tensor and the fourth-order null tensor.

In the region $\mathcal{S}_3$, deriving (2.14) we have

$$D_E\mathbf{T}\,[\mathbf{H}] = (D_E\beta_0 \cdot \mathbf{H}\,)\,\mathbf{I} + (D_E\beta_1 \cdot \mathbf{H}\,)\,\mathbf{E} + \beta_1\,\mathbf{H} \quad , \quad \mathbf{H} \in \mathrm{Sym} \tag{2.18}$$

and therefore, taking into account the fact that β_0 and β_1 are functions of principal invariants of $\mathbf{E}$ and using the well known expression of the derivatives of the principal invariants with respect to $\mathbf{E}$,

$$D_E I_1(\mathbf{E})\,[\mathbf{H}] = \mathbf{I} \cdot \mathbf{H}\,, \quad D_E I_2(\mathbf{E})\,[\mathbf{H}] = I_2(\mathbf{E})\,\mathbf{E}^{-1} \cdot \mathbf{H}\,, \quad \mathbf{H} \in \mathrm{Sym}\,, \tag{2.19}$$

we obtain

$$D_E\mathbf{T} = \frac{\partial\beta_0}{\partial I_1}\,\mathbf{I} \otimes \mathbf{I} + I_2\,\frac{\partial\beta_0}{\partial I_2}\,\mathbf{I} \otimes \mathbf{E}^{-1} + \frac{\partial\beta_1}{\partial I_1}\,\mathbf{E} \otimes \mathbf{I} + I_2\,\frac{\partial\beta_1}{\partial I_2}\,\mathbf{E} \otimes \mathbf{E}^{-1} + \beta_1\,\mathbb{1}\,. \tag{2.20}$$

By virtue of the Hamilton-Cayley theorem

$$\mathbf{E}^{-1} = \frac{1}{I_2}\,(\,I_1\,\mathbf{I} - \mathbf{E})\,, \tag{2.21}$$

therefore

$$D_E\mathbf{T} = (\,\frac{\partial\beta_0}{\partial I_1} + I_1\,\frac{\partial\beta_0}{\partial I_2}\,)\,\mathbf{I} \otimes \mathbf{I} \;-\; \frac{\partial\beta_0}{\partial I_2}\,\mathbf{I} \otimes \mathbf{E} \,+ \tag{2.22}$$

$$(\,\frac{\partial\beta_1}{\partial I_1} + I_1\,\frac{\partial\beta_1}{\partial I_2}\,)\,\mathbf{E} \otimes \mathbf{I} - \frac{\partial\beta_1}{\partial I_2}\,\mathbf{E} \otimes \mathbf{E} + \beta_1\,\mathbb{1}$$

where, in view of $(2.15)_3$,

$$\frac{\partial\beta_0}{\partial I_1} = -\varphi\,\frac{I_1\,I_2}{(I_1^2 - 4I_2)^{3/2}}\,,$$

$$\frac{\partial\beta_0}{\partial I_2} = \varphi\,\frac{I_1^2 - 2I_2}{(I_1^2 - 4I_2)^{3/2}}\,,$$

(2.23)

$$\frac{\partial\beta_1}{\partial I_1} = 2\,\varphi\,\frac{I_2}{(I_1^2 - 4I_2)^{3/2}}\,,$$

$$\frac{\partial\beta_1}{\partial I_2} = -\varphi\,\frac{I_1}{(I_1^2 - 4I_2)^{3/2}}\,.$$

Taking into account the fact that $-\frac{\partial\beta_0}{\partial I_2} = \frac{\partial\beta_1}{\partial I_1} + I_1\,\frac{\partial\beta_1}{\partial I_2}$, from (2.23) and (2.22) we finally obtain the derivative of $\mathbf{T}$ with respect to $\mathbf{E}$ in the region $\mathcal{S}_3$:

(2.24)
$$D_E\mathbf{T} = \frac{\varphi I_1(I_1^2 - 3I_2)}{(I_1^2 - 4I_2)^{3/2}}\,\mathbf{I}\otimes\mathbf{I} - \frac{\varphi(I_1^2 - 2I_2)}{(I_1^2 - 4I_2)^{3/2}}\,(\mathbf{I}\otimes\mathbf{E} + \mathbf{E}\otimes\mathbf{I}) +$$

$$\frac{\varphi\,I_1}{(I_1^2 - 4I_2)^{3/2}}\,\mathbf{E}\otimes\mathbf{E} - \varphi\,\frac{I_1 - \sqrt{I_1^2 - 4I_2}}{2\sqrt{I_1^2 - 4I_2}}\,\mathbb{1}\,.$$

We observe that $D_E\mathbf{T}$ is a symmetric fourth-order tensor; this result is in agreement with the fact that the material is hyperelastic, and the potential

$$\psi(\mathbf{E}) = \begin{cases} \frac{\mu}{2}\left((2+\alpha)I_1^2 - 4\,I_2\right)\,, & \mathbf{E}\in\mathcal{S}_1,\\ 0\,, & \mathbf{E}\in\mathcal{S}_2,\\ \frac{\varphi}{4}\left(I_1^2 - 2\,I_2 - \sqrt{I_1^2 - 4\,I_2}\,\right)\,, & \mathbf{E}\in\mathcal{S}_3, \end{cases}$$

calculated with the help of (2.14) and (2.15), is a function of class C^2 in the internal part of every region $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$.

Now we briefly present the calculation of the derivative of $\mathbf{T}$ with respect to $\mathbf{E}$ in the case of plane stress. Let $\{g_1, g_2, g_3\}$, be the basis of $\mathcal{V}$ with respect to which the stress components T_{13}, T_{23} and T_{33} vanish; let us again indicate with e_1, e_2, e_3 and a_1, a_2, a_3 the eigenvalues of $\mathbf{E}$ and $\mathbf{E}^a$ respectively.

In view of (2.1) and (2.2) we have

$$(2.25) \qquad e_3 - a_3 = \frac{\alpha}{2+\alpha}(a_1 + a_2 - e_1 - e_2).;$$

moreover, a_3 , being arbitrary by virtue of $(2.4)_2$, is assumed to be equal to zero. The principal components of $\mathbf{E}^a$ are calculated by means of the relations

$$(2.26) \qquad \begin{array}{lll} \text{if } \mathbf{E} \in \tilde{\mathcal{S}}_1 \text{ then} & a_1 = 0\,, & a_2 = 0\,, \\ \text{if } \mathbf{E} \in \tilde{\mathcal{S}}_2 \text{ then} & a_1 = e_1\,, & a_2 = e_2\,, \\ \text{if } \mathbf{E} \in \tilde{\mathcal{S}}_3 \text{ then} & a_1 = 0\,, & a_2 = e_2 + \dfrac{\alpha}{2(1+\alpha)}\, e_1\,, \end{array}$$

where

$$\tilde{\mathcal{S}}_1 = \{\, \mathbf{E} \in \mathrm{Sym}\,;\ \alpha e_2 + 2(1+\alpha)e_1 \le 0\ ,\ \ \alpha e_1 + 2(1+\alpha)e_2 \le 0 \,\}\,,$$

$$\tilde{\mathcal{S}}_2 = \{\, \mathbf{E} \in \mathrm{Sym}\,;\ e_1 \ge 0\ ,\ \ e_2 > 0 \,\}\,,$$

$$\tilde{\mathcal{S}}_3 = \{\, \mathbf{E} \in \mathrm{Sym}\,;\ e_1 < 0\ ,\ \ \alpha e_1 + 2(1+\alpha)e_2 > 0 \}\ ,$$

with $e_1 \le e_2$.
From (2.2) and (2.25) we obtain the expression of the stress

$$\mathbf{T} = 2\mu(\mathbf{E} - \mathbf{E}^a) + \frac{2\lambda}{2+\alpha}\,\mathrm{tr}\,(\mathbf{E} - \mathbf{E}^a)\,\mathbf{I}\ ,$$

its derivative with respect to $\mathbf{E}$ is calculated with a similar procedure as the one for the plane strain state,

$$(2.27) \qquad D_E\mathbf{T} = 2\mu\,\mathbb{1} + \frac{2\lambda}{2+\alpha}\,\mathbf{I}\otimes\mathbf{I}\ ,\ \text{if } \mathbf{E} \in \tilde{\mathcal{S}}_1\ ,$$

$$(2.28) \qquad D_E\mathbf{T} = \mathbb{O}\ ,\ \text{if } \mathbf{E} \in \tilde{\mathcal{S}}_2,$$

$$(2.29) \qquad D_E\mathbf{T} = \frac{\varphi_1 I_1 (I_1^2 - 3I_2)}{(I_1^2 - 4I_2)^{3/2}}\,\mathbf{I}\otimes\mathbf{I} - \frac{\varphi_1 (I_1^2 - 2I_2)}{(I_1^2 - 4I_2)^{3/2}}\,(\mathbf{I}\otimes\mathbf{E} + \mathbf{E}\otimes\mathbf{I}) +$$

$$\frac{\varphi_1\, I_1}{(I_1^2 - 4I_2)^{3/2}}\,\mathbf{E}\otimes\mathbf{E} - \varphi_1\,\frac{I_1 - \sqrt{I_1^2 - 4I_2}}{2\sqrt{I_1^2 - 4I_2}}\,\mathbb{1}\ ,\ \text{if } \mathbf{E} \in \tilde{\mathcal{S}}_3\ ,$$

Where we put $\varphi_1 = \dfrac{\mu\,(2 + 3\,\alpha)}{1 + \alpha}$.

3. Description of the algorithm

In this section we describe the algorithm which has been used in NOSA, a code using isoparametric finite elements for the calculation of the numerical solutions presented in Section 4.
Let us consider the following quantities relating to the i-th iteration:

$\mathbf{u}^{(i)}$	vector of nodal displacements,
$D(\mathbf{u}^{(i)})$	matrix of the engineering components of $D_E\mathbf{T}$ (see Appendix),
$K_T(\mathbf{u}^{(i)})$	tangent stiffness matrix,
$\mathbf{f}^{(i)}$	nodal equivalent of given loads $i = 0$, nodal equivalent of residual loads if $i \geq 1$,
$\mathbf{e}_G^{(i)}$	vector of the engineering components of the total strain,
$\mathbf{a}_G^{(i)}$	vector of the engineering components of inelastic strain ,
$\mathbf{t}_G^{(i)}$	vector of the engineering components of stress,

where the subscript G indicates the Gauss point in which these quantities are calculated.
At the first iteration $\mathbf{u}^{(0)}$ is null and $D(\mathbf{u}^{(0)})$ coincides with the matrix of elastic moduli.
Let us suppose we have calculated the displacement $\mathbf{u}^{(i)}$, the tangent stiffness matrix $K_T(\mathbf{u}^{(i)})$ and the nodal equivalent loads $\mathbf{f}^{(i)}$, corresponding to the i - th iteration; we solve the linear system

$$(3.1) \qquad K_T(\mathbf{u}^{(i)})\, \Delta\mathbf{u}^{(i)} = \mathbf{f}^{(i)}\,,$$

in order to determine the displacement $\mathbf{u}^{(i+1)} = \mathbf{u}^{(i)} + \Delta\mathbf{u}^{(i)}$ relative to the $(i + 1)$ - th iteration.
Then for every Gauss point of every element we calculate the total strain $\mathbf{e}_G^{(i+1)}$ relating to the displacement $\mathbf{u}^{(i+1)}$, we calculate its eigenvalues which are needed to calculate the inelastic strain $\mathbf{a}_G^{(i+1)}$, by using relations (2.8)-(2.11); then we calculate the stress $\mathbf{t}_G^{(i+1)}$ using the constitutive relation (2.2). We observe that the stress is negative semi-definite because it is calculated by directly solving the constitutive equation (2.1), (2.2), (2.4).
Moreover, using relations (A.1), matrix $D(\mathbf{u}^{(i+1)})$ is calculated which if necessary may be used in the next iteration.
Finally, we calculate the vector of residual loads $\mathbf{f}^{(i+1)}$ and we perform the convergence control

$$(3.2) \qquad \frac{|\mathbf{f}^{(i+1)}|}{|\mathbf{f}^{(0)}|} \leq \xi_c\,;$$

if the convergence has not been reached, we repeat all operations beginning with the solution of the system (3.1).

4. Examples

In this Section we solve numerically some equilibrium problems and we compare the results obtained with the corresponding exact solutions. In all these examples, for the discretization we use isoparametric elements with eight nodes and nine Gauss points. In the following ν is the Poisson ratio and E is the Young modulus.

Example 1. *Half circular ring subjected to non uniform radial loads.*
In the polar reference system $\{o, \rho, \theta\}$ of Figure 1, let us consider the half circular ring $\Omega = \{(\rho, \theta)\ ;\ \rho \in (a, b), \theta \in (0, \pi)\}$.
On $\partial\Omega_1 = \{(\rho, \theta) \mid \rho = b, \theta \in [0, \pi]\}$ and $\partial\Omega_2 = \{(\rho, \theta) \mid \rho = a, \theta \in [0, \pi]\}$ the pressures

$$\hat{p}_e(\theta) = p_e - \chi \frac{\sin\theta}{b}\ , \quad \hat{p}_i(\theta) = p_i - \chi \frac{\sin\theta}{a}\ ,$$

are given respectively, wherep_e and p_i are constants and χ is a parameter such that $0 \le \chi \le ap_i$; on $\partial\Omega_3 = \{(\rho, \theta) \mid \rho \in [a, b]\ , \theta = 0\}$ and $\partial\Omega_4 = \{(\rho, \theta) \mid \rho \in [a, b]\ , \theta = \pi\}$ the circumferential displacement

$$(4.1) \qquad v(\rho, 0) = \chi \frac{1+\nu}{E} (\nu + (1-\nu) \ln\rho)\ , \quad v(\rho, \pi) = -\chi \frac{1+\nu}{E} (\nu + (1-\nu) \ln\rho)\ ,$$

is given and the shear stress $\tau_{\rho\theta}$ is null [14].

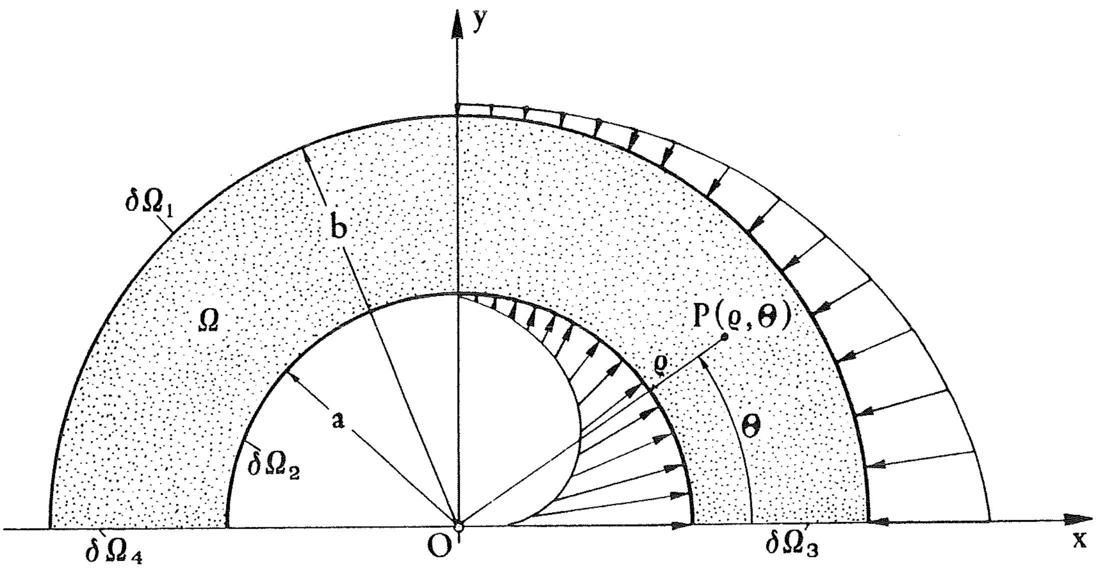

Figure 1. Half circular ring subjected to non-uniform radial loads.

If the condition $\frac{a}{b} \leq \frac{p_e}{p_i} \leq \frac{a^2 + b^2}{2\,b^2}$ is satisfied, then the negative semi-definite stress field, in equilibrium with the loads $\hat{p}_e$ and $\hat{p}_i$, has components

$$\sigma_\rho(\rho,\theta) = \begin{cases} -\frac{a}{\rho}\,p_i + \chi\,\frac{\sin\theta}{\rho}\,, & \rho \in [a, \rho_0] \\ -\,p_i\,\frac{a\,\rho_0}{2\rho^2} - \frac{a}{2\rho_0}\,p_i + \chi\,\frac{\sin\theta}{\rho}\,, & \rho \in [\rho_0, b] \end{cases}$$

(4.2)
$$\sigma_\theta(\rho,\theta) = \begin{cases} 0\,, & \rho \in [a, \rho_0] \\ p_i\,\frac{a\,\rho_0}{2\rho^2} - \frac{a}{2\rho_0}\,p_i\,, & \rho \in [\rho_0, b] \end{cases}$$

$$\tau_{\rho\theta}(\rho,\theta) = 0\,,$$

where $\rho_0 = \frac{b}{a}\,\frac{bp_e - \sqrt{(bp_e)^2 - (ap_i)^2}}{p_i}$ is the transition radius from the region in which $\mathbf{E}^a \neq \mathbf{0}$ to the one in which $\mathbf{E}^a = \mathbf{0}$.

For a plane strain state, the radial displacement, the circumferential displacement and the circumferential component of the inelastic strain are univocally determined and have the expressions

$$u(\rho,\theta) = \frac{1+\nu}{E}\,\{\, ap_i\,(1-\nu)\ln\left(\frac{\rho_0}{\rho}\right) + \nu ap_i + (1-\nu)\,\chi\,\sin\theta\,\ln\rho\; - $$

$$\frac{1-2\nu}{2}\,\chi\,(\sin\theta + (\theta - \frac{\pi}{2})\cos\theta\,)\}\,, \qquad \rho \in [a, \rho_0]\,,$$

(4.3)

$$u(\rho,\theta) = \frac{1+\nu}{E}\,\{\,\frac{ap_i}{2}\,(\frac{\rho_0}{\rho} - (1-2\nu)\,\frac{\rho}{\rho_0}) + (1-\nu)\,\chi\,\sin\theta\,\ln\rho\; - $$

$$\frac{1-2\nu}{2}\,\chi\,(\sin\theta + (\theta - \frac{\pi}{2})\cos\theta\,)\}\,, \qquad \rho \in [\rho_0, b]\,;$$

(4.4) $$v(\rho,\theta) = -\,\chi\,\frac{1+\nu}{E}\,\{-\cos\theta\,(\nu + (1-\nu)\ln\rho) + \frac{1-2\nu}{2}\,\sin\theta\,(\frac{\pi}{2} - \theta)\}\,, \quad \rho \in [a, b]\,;$$

(4.5) $$\varepsilon^a_\theta(\rho) = \begin{cases} \frac{1-\nu^2}{E}\,\frac{a}{\rho}\,p_i\,\ln(\rho_0/\rho)\,, & \rho \in [a, \rho_0] \\ 0\,, & \rho \in [\rho_0, b]. \end{cases}$$

If we put $\chi = 0$, the half circular ring Ω is subjected to two uniform radial pressures p_e and p_i [8]. In this case, the circumferential displacement is null, the inelastic strain still has the expression (4.5) and the radial displacement is obtained from (4.3), by putting $\chi = 0$,

$$(4.6) \qquad u(\rho) = \begin{cases} \dfrac{1+\nu}{E} a\, p_i \left[\nu + (1-\nu) \ln(\rho_0/\rho) \right], & \rho \in [a, \rho_0] \\[2ex] \dfrac{1+\nu}{2E} a\, p_i \left[\dfrac{\rho_0}{\rho} - (1-2\nu)\dfrac{\rho}{\rho_0} \right], & \rho \in [\rho_0, b]\,. \end{cases}$$

For the numerical calculus of the solutions relating to cases $\chi > 0$ and $\chi = 0$, the following values of constants have been used:

$a = 1000$ cm
$b = 2000$ cm
$p_i = 10$ Kg/cm^2
$p_e = 5.5$ Kg/cm^2
$\chi = 10000$ Kg/cm ($\chi = 0$ Kg/cm)
$\nu = 0.1$
$E = 50000$ Kg/cm^2

With these values the transition radius ρ_0 is approximately 1283 cm.
In the finite element analysis, for symmetry reasons, only a quarter of the circular ring was studied, and was discretized into one hundred elements; the tolerance ξ_c is equal to 10^{-7}, the convergence was reached in twelve iterations and the norm of residual forces is equal to $0.3 10^{-9} \, |f^{(0)}|$.
The following figures show the behaviour of the components of stress, of inelastic strain and of displacement; the continuous line represents the exact solution, the bold dots the numerical solution.
Figure 2 shows the behaviour of the radial stress for $\theta = 1.016°$; Figures 3 and 4 show the behaviour of circumferential stress and circumferential inelastic strain, which do not depend on θ.
In Figure 5 the radial displacement versus radius is plotted for $\theta = 88.98°$.
Figures 6 and 7 show the radial displacement (continuous line) and the circumferential displacement (dotted line) as functions of the radius, for the values $\theta = 0°, 45°, 90°$, and as functions of θ for $\rho = a$ and $\rho = b$.

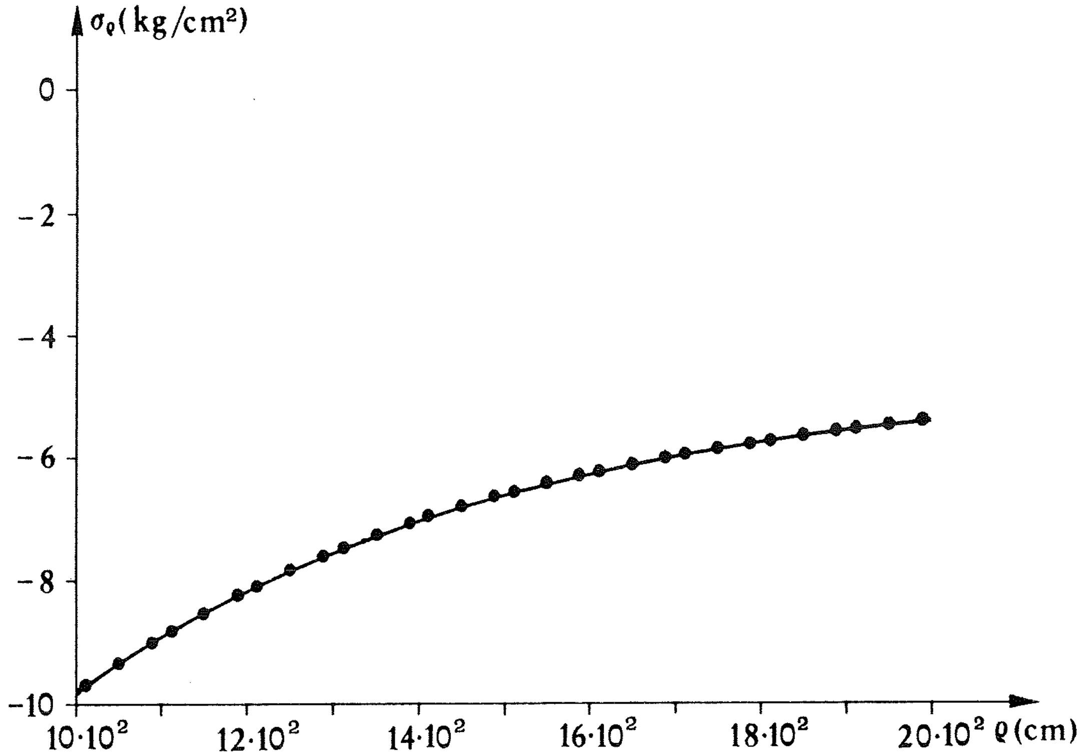

Figure 2. Radial stress vs. ρ, for θ = 1.016°.

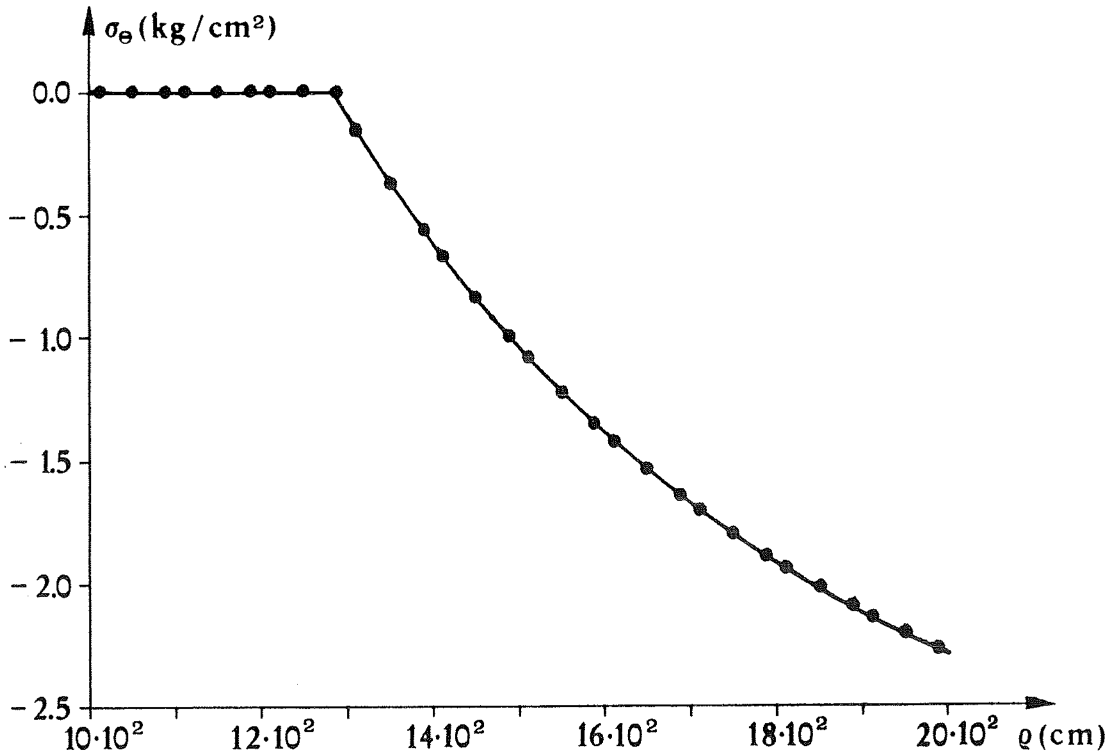

Figure 3. Circumferential stress vs. ρ, for θ = 1.016°.

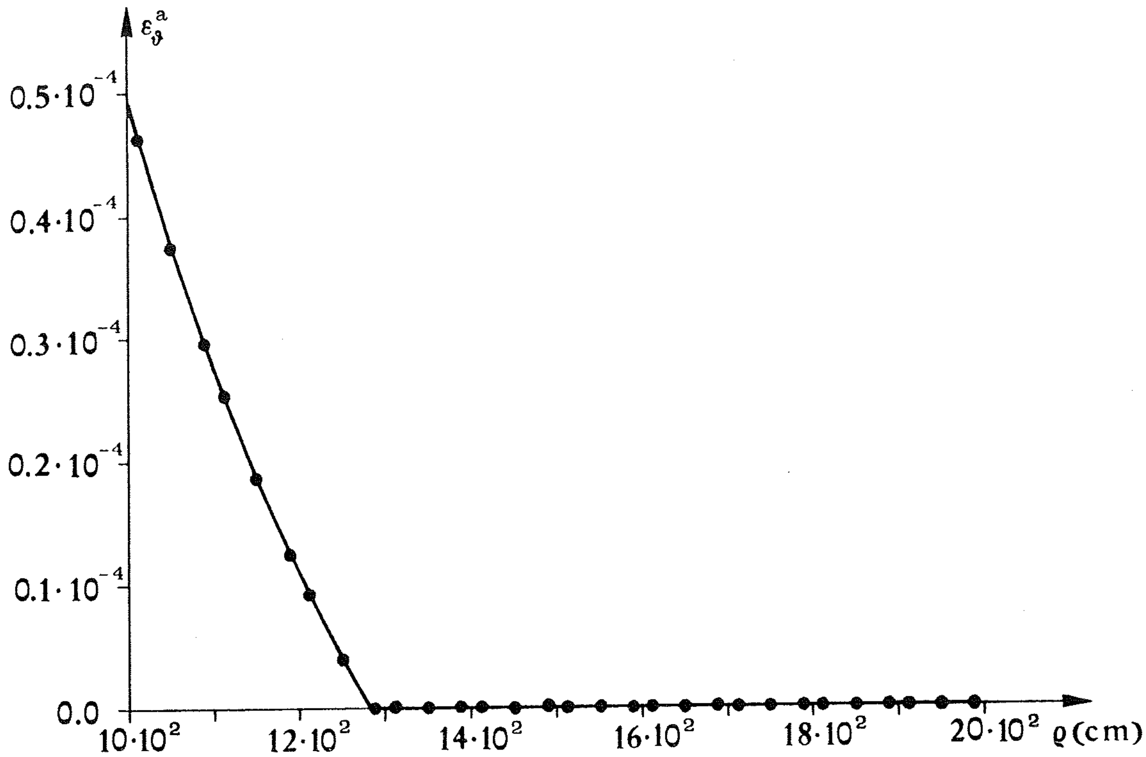

Figure 4. Circumferential inelastic strain vs. ρ.

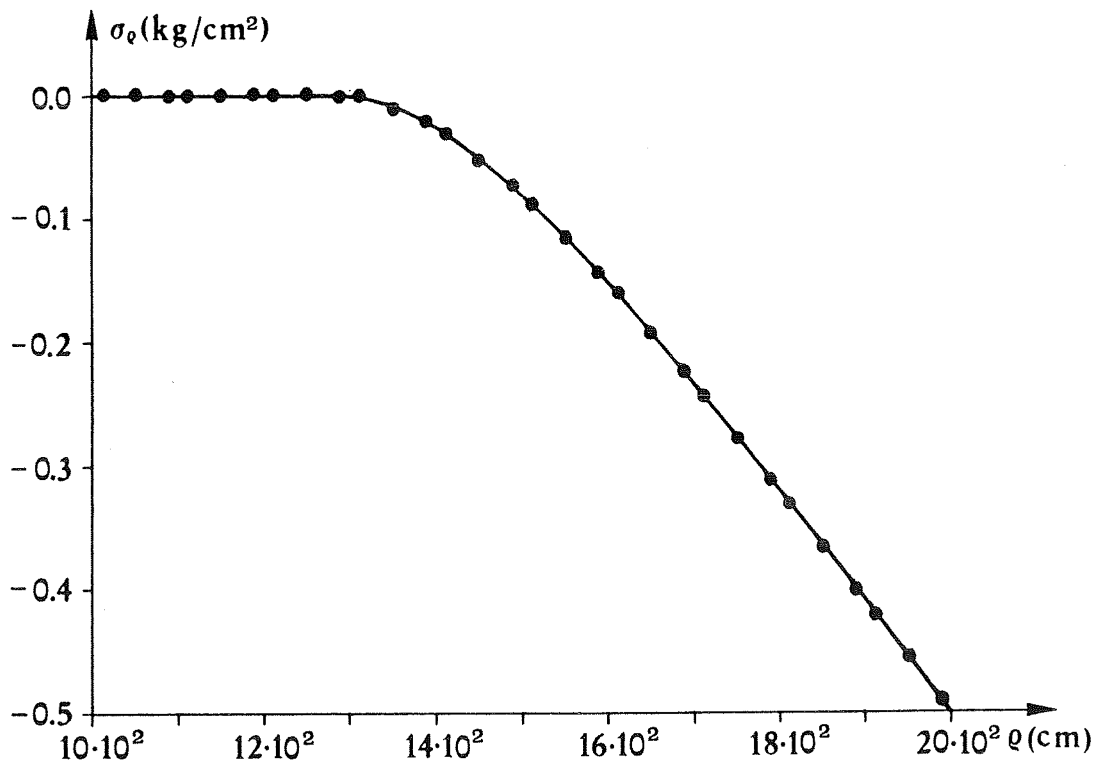

Figure 5. Radial stress vs. ρ, for θ = 88.98°.

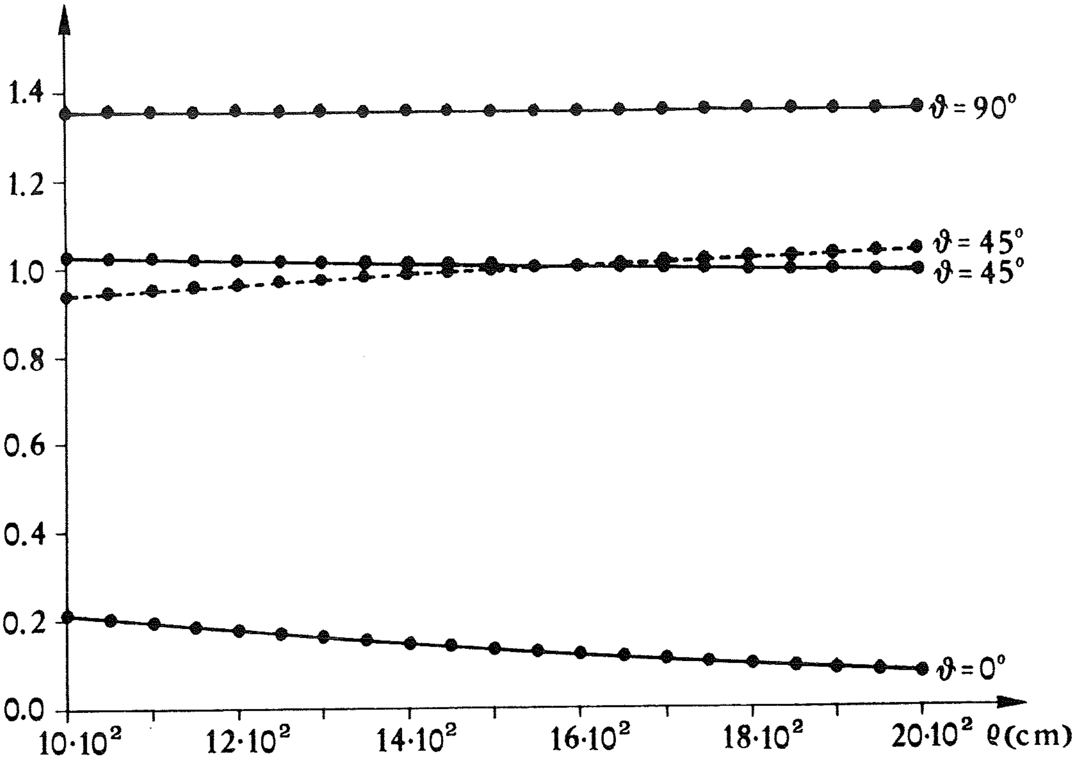

Figure 6. Radial and circumferential displacements vs. ρ for θ = 0°, 45°, 90°.

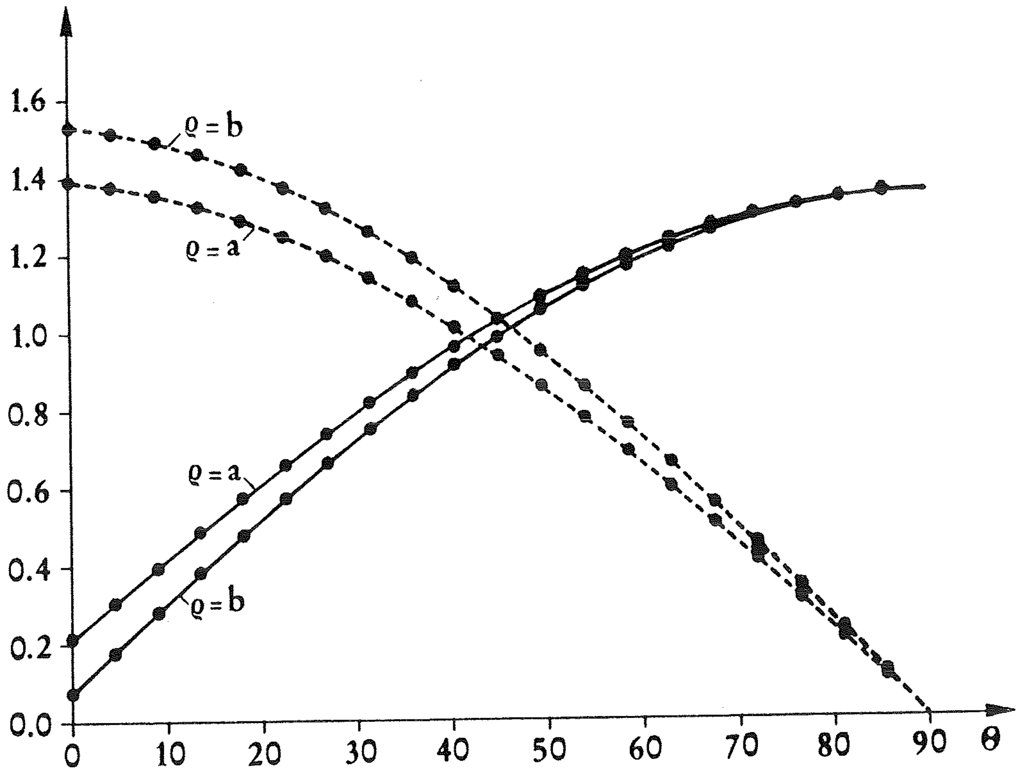

Figure 7. Radial and circumferential displacements vs. θ for ρ = a and ρ = b.

If the half circular ring is subjected to uniform pressures p_e and p_i ($\chi = 0$), the circumferential stress and the circumferential inelastic strain are the same as case $\chi > 0$, the radial stress, which now does not depend on θ, has a qualitative behaviour similar to the one in Figure 2; the radial displacement is plotted in Figure 8.
In this case, for $\xi_c = 10^{-7}$, the convergence was reached in three iterations and the norm of residual forces is $0.6\ 10^{-8}\,|f^{(0)}|$.

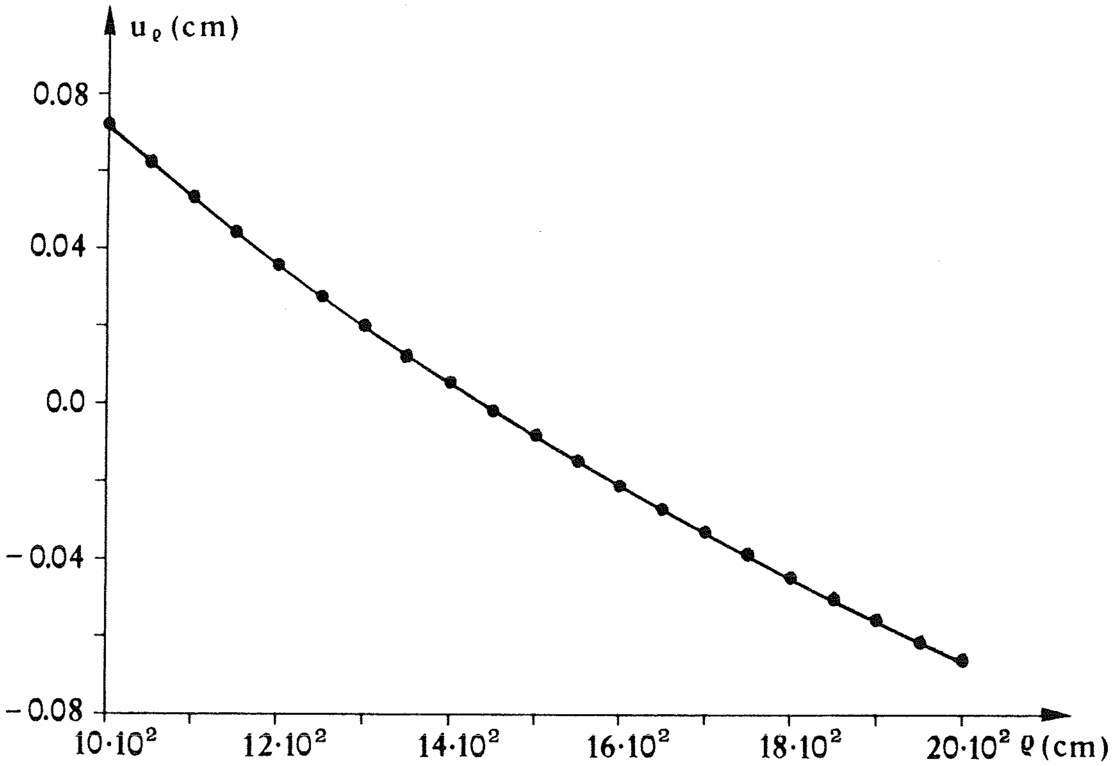

Figure 8 Radial displacement vs. ρ in the case of uniform radial pressures.

The two following examples concern a rectangular block, under the hypothesis of plane strain [6]. If the block with density γ is subjected to a vertical load p(x), distributed on the top, the stress tensor has components

(4.7) $\sigma_x(x, y) = 0\,, \quad \sigma_y(x, y) = -\,\gamma\,(h - y) - p(x)\,, \quad \tau_{xy}(x, y) = 0\,, \quad \sigma_z(x, y) = \nu\,\sigma_y(x, y)\,.$

It is known [6] that the vertical displacement is unique, whereas the inelastic strain ε_x^a and the horizontal displacement are not unique.

Example 2. *Supported block subjected to its own weight and to a distributed load p(x).*
A rectangular block with base 2b and height h, simply supported on a rigid plane, is subjected to its own weight $-\,\gamma\,\mathbf{j}$ and to the parabolic load $p(x) = \frac{p_0}{b^2}\,(b^2 - x^2)$, symmetric with respect to the y axis, with $p_0 > 0$ (Fig. 9).

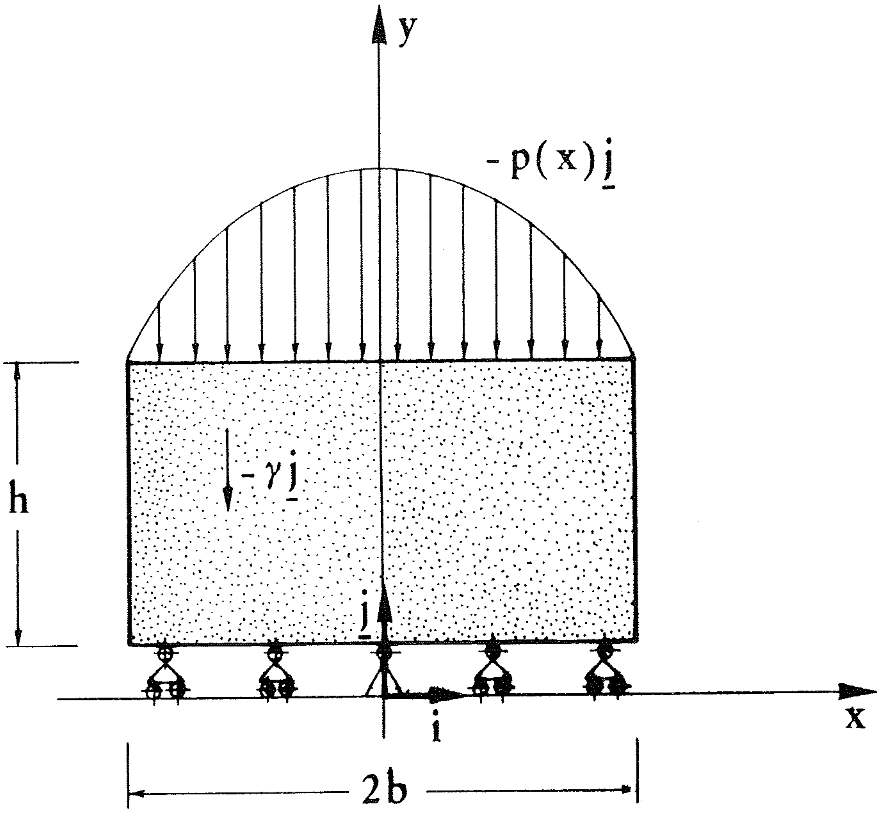

Figure 9. Supported block subjected to its own weight and to parabolic load p(x).

The vertical displacement corresponding to stress field (4.7) is

(4.8) $$v(x, y) = -\frac{1-\nu^2}{E}\left(p(x)\,y + \gamma\left(hy - \frac{y^2}{2}\right)\right) \quad , \quad x\in[-b, b], \quad y\in[0, h]\ .$$

For symmetry reasons we studied only half structure, which was discretized into two hundred elements; we use the following constants:

$b = 50$ cm
$h = 100$ cm
$\gamma = 0.002$ Kg/cm^3
$p_0 = 1$ Kg/cm^2
$\nu = 0.1$
$E = 50000$ Kg/cm^2

If $\xi_c = 10^{-2}$, the convergence is reached in nine iterations and the norm of residual forces is $0.5\ 10^{-2}\,|f^{(0)}|$.
Figures 10 and 11 show respectively the behaviour of σ_y as a function of x for y = 0.56 cm and the behaviour of σ_y as a function of y for x = 0.56 cm. In Figure 12 we see the vertical displacement v on the top of the block.

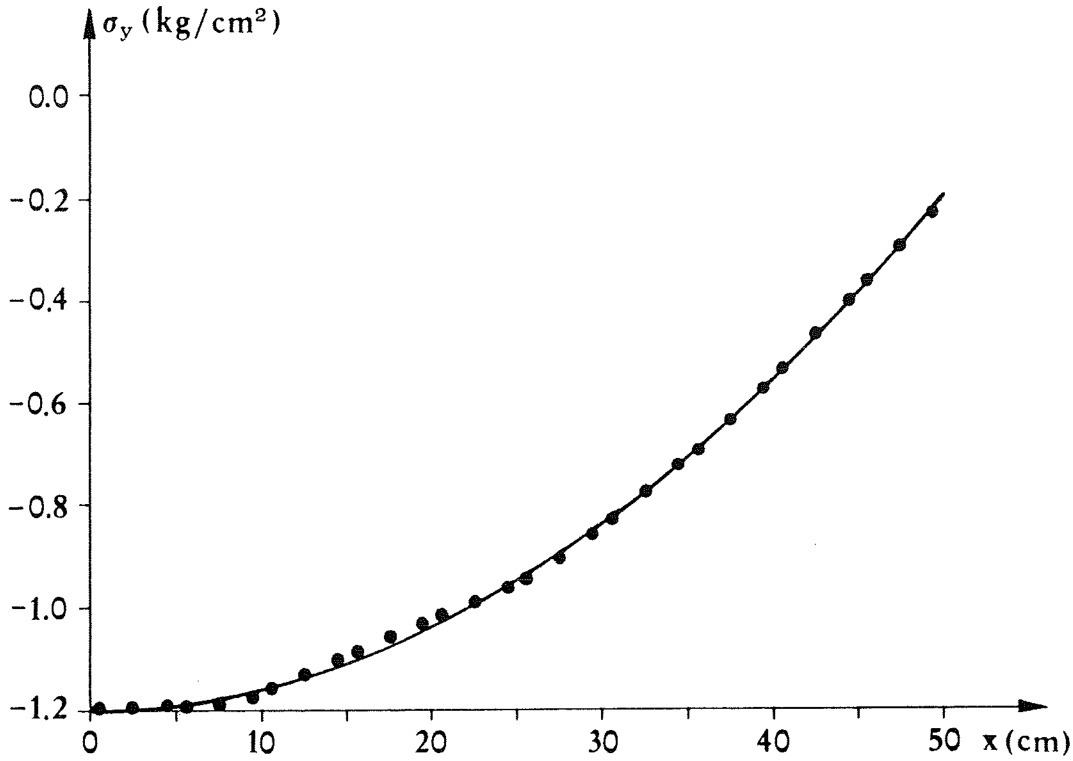

Figure 10. σ_y vs. x for y = 0.56 cm.

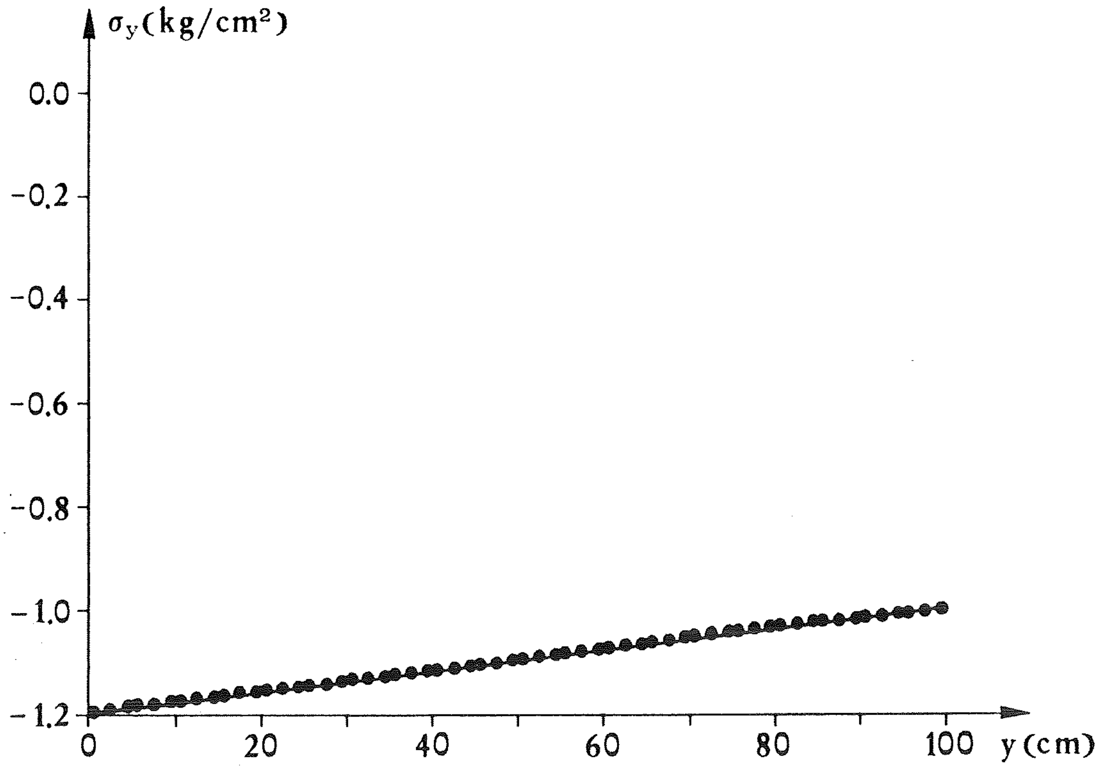

Figure 11. σ_y vs. y when x = 0.56 cm.

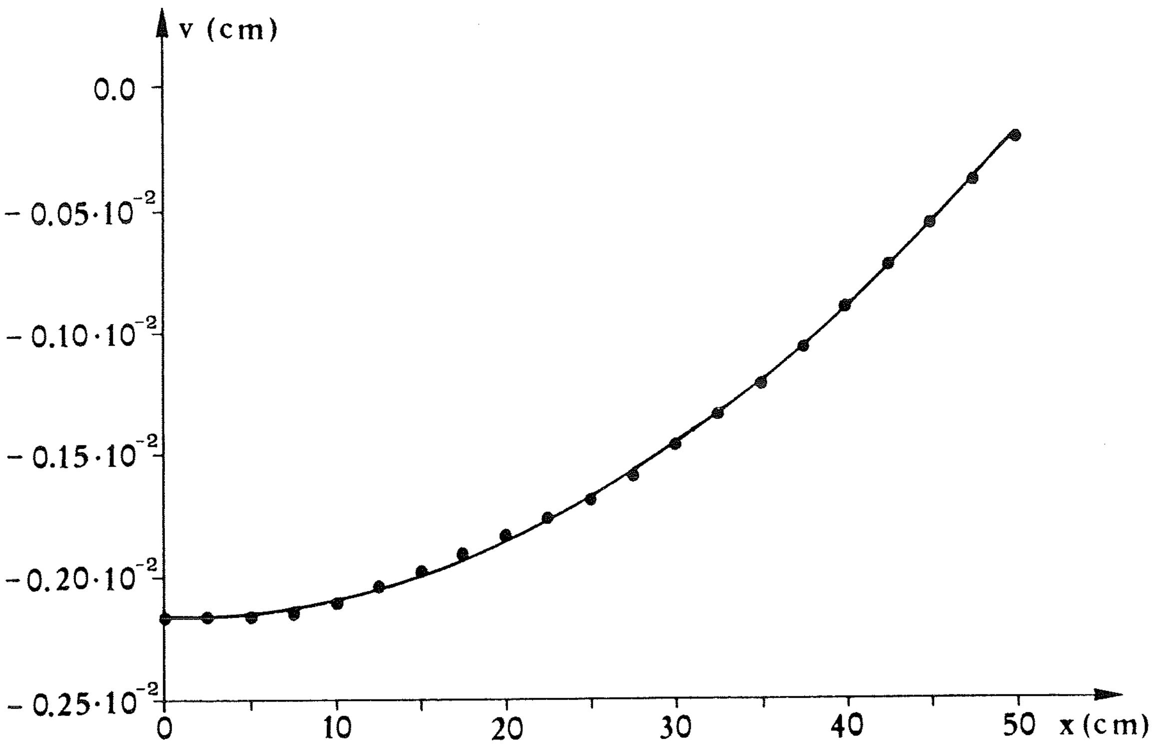

Figure 12. Vertical displacement v vs. x for y = h.

Example 3 . *Supported block subjected to the trapezoidal load p(x).*
A rectangular block with base (a + b) and height h, simply supported on a rigid plane, is subjected to the load

$$(4.9) \qquad p(x) = \begin{cases} p_0 \ , & x \in [0, a] \\ \dfrac{p_0}{b}(a + b - x) \ , & x \in [a, a + b] \end{cases}$$

distributed on its top (Fig. 13).

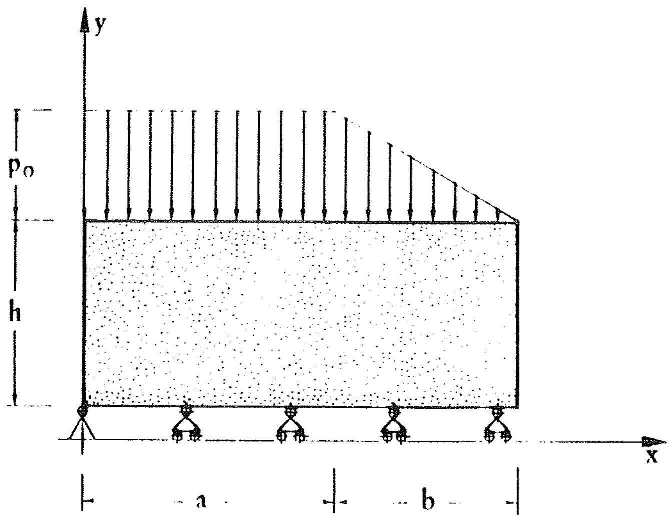

Figure 13. Supported block subjected to the trapezoidal load p(x).

In this case the horizontal displacement, besides not being unique, has a discontinuity in x = a. The vertical displacement is unique and continuous and has the expression

$$(4.10) \qquad v(x, y) = \begin{cases} -\dfrac{1-\nu^2}{E} p_0 \, y \ , & x \in [0, a] \\ -\dfrac{1-\nu^2}{E} \dfrac{p_0}{b} (x - a - b) \, y \ , & x \in [a, a+b] \ . \end{cases}$$

The block is discretized into one hundred and twenty-eight elements and we use the following values of constants

$a = 50$ cm
$b = 50$ cm
$h = 50$ cm
$p_0 = 1$ Kg/cm^2
$\nu = 0.1$
$E = 50000$ Kg/cm^2.

If $\xi_c = 10^{-3}$, the convergence is reached in fifteen iterations and the norm of residual forces is $0.2 \ 10^{-3} \, |f^{(0)}|$.
In Figure 14 the stress σ_y as function of x, for y = 0.70 cm is plotted, and in Figure 15 the vertical displacement v as function of x, when y = h is plotted.

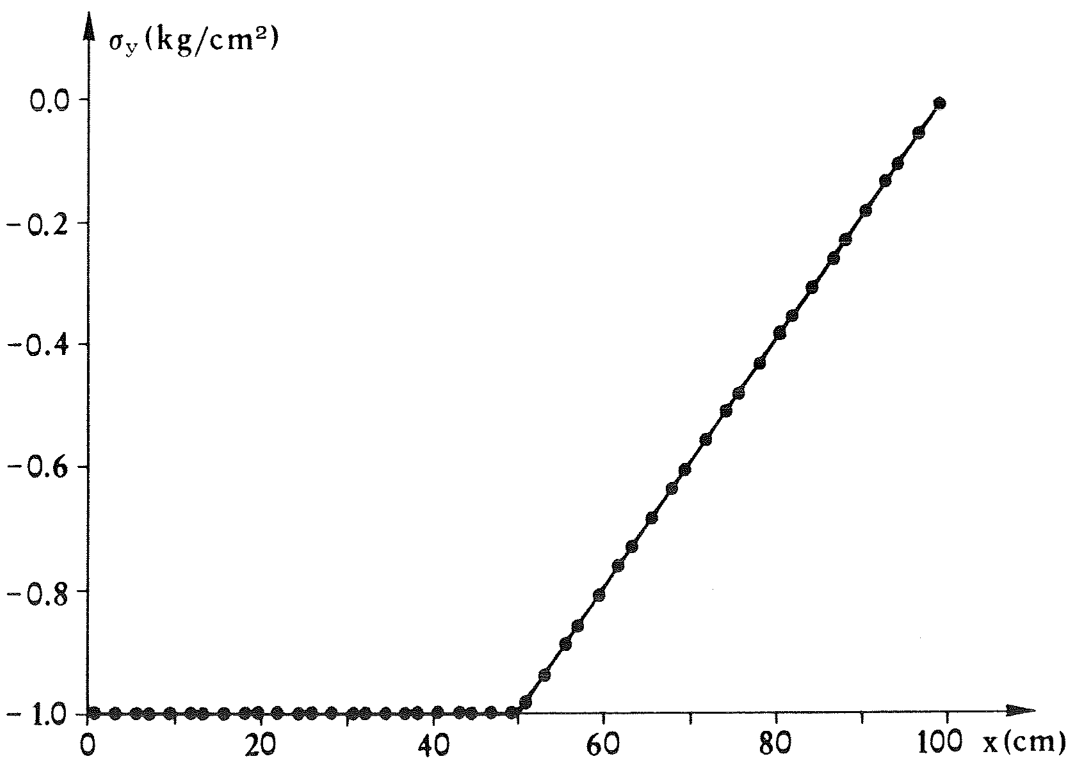

Figure 14. σ_y vs. x for y = 0.704 cm.

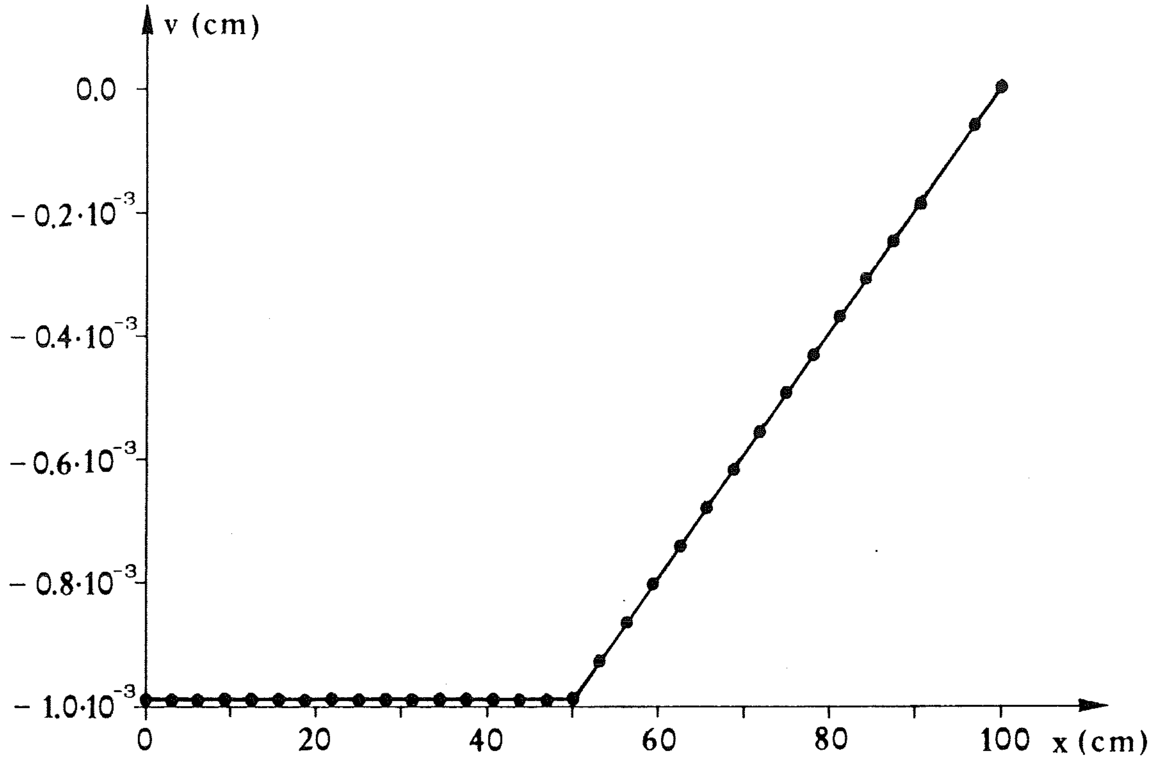

Figure 15. Vertical displacement v vs. x when y = h.

The previous examples show that the numerical solution is in a good agreement with the exact one in those cases where the displacement is unique (Example 1). When the displacement is not unique (Examples 2 and 3), a considerably higher number of iterations are needed in order to obtain a sufficiently accurate solution for the stress.

One difficulty, which is common to all numerical methods of this kind, can be met when, during iterations, a total strain belonging to region $\mathcal{S}_2$ is calculated. In fact, in this region the stress tensor is null, therefore in the tangent stiffness matrix a null diagonal element appears and this element makes it impossible to solve the system (3.1). In particular, the method is not applicable in those cases in which the linear elastic solution implies the existence of regions where principal stresses are both tractions.

This difficulty did not allow us to numerically solve the problem of a weighted block supported on an elastic cantilever, the exact solution of which is known [6].

Appendix

The matrix D of the engineering components of $D_E T$, obtained starting from the expression of the derivative of the stress calculated in (2.16), (2.17), (2.24), (2.27), (2.28) and (2.29), are

$$D_{11} = \alpha_1 + 2\alpha_2 E_{11} + \alpha_3 E_{11}^2 + \alpha_4 ,$$

(A.1)

$$D_{12} = \alpha_1 + \alpha_2 (E_{11} + E_{22}) + \alpha_3 E_{11} E_{22} ,$$

$$D_{13} = \alpha_2 E_{12} + \alpha_3 E_{11} E_{12} ,$$

$$D_{22} = \alpha_1 + 2\alpha_2 E_{22} + \alpha_3 E_{22}^2 + \alpha_4 ,$$

$$D_{23} = \alpha_2 E_{12} + \alpha_3 E_{22} E_{12} ,$$

$$D_{33} = \alpha_3 E_{12}^2 + \frac{\alpha_4}{2} ,$$

where E_{ij} are the components of **E** with respect to the basis $\{\mathbf{g}_1, \mathbf{g}_2\}$ and the scalar functions $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are

(A.2)

$$\alpha_1 = \lambda, \quad \alpha_2 = \alpha_3 = 0, \quad \alpha_4 = 2\mu \qquad \text{if } \mathbf{E} \in \mathcal{S}_1$$

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0 \qquad \text{if } \mathbf{E} \in \mathcal{S}_2$$

$$\alpha_1 = \frac{\varphi I_1 (I_1^2 - 3I_2)}{(I_1^2 - 4I_2)^{3/2}}, \quad \alpha_2 = -\frac{\varphi (I_1^2 - 2I_2)}{(I_1^2 - 4I_2)^{3/2}}, \quad \alpha_3 = \frac{\varphi I_1}{(I_1^2 - 4I_2)^{3/2}}, \quad \alpha_4 = -\varphi \frac{I_1 - \sqrt{I_1^2 - 4I_2}}{2\sqrt{I_1^2 - 4I_2}} \qquad \text{if } \mathbf{E} \in \mathcal{S}_3$$

for the plane strain and

(A.3)

$$\alpha_1 = 2\lambda/(2 + \alpha), \quad \alpha_2 = \alpha_3 = 0, \quad \alpha_4 = 2\mu \qquad \text{if } \mathbf{E} \in \tilde{\mathcal{S}}_1$$

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0 \qquad \text{if } \mathbf{E} \in \tilde{\mathcal{S}}_2$$

$$\alpha_1 = \frac{\varphi_1 I_1 (I_1^2 - 3I_2)}{(I_1^2 - 4I_2)^{3/2}}$$

$$\alpha_2 = - \frac{\varphi_1 (I_1^2 - 2I_2)}{(I_1^2 - 4I_2)^{3/2}}$$

$$\alpha_3 = \frac{\varphi_1 I_1}{(I_1^2 - 4I_2)^{3/2}} \qquad \text{if } E \in \tilde{\mathcal{S}}_3$$

$$\alpha_4 = - \varphi_1 \frac{I_1 - \sqrt{I_1^2 - 4I_2}}{2\sqrt{I_1^2 - 4I_2}}$$

for the plane stress.

Acknowledgements. This research has been supported by the 'Progetto Finalizzato Materiali Speciali per Tecnologie Avanzate' of CNR. This support is gratefully acknowledged.

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