Weak convergence of probability-valued solutions of general one-dimensional kinetic equations: Characterizations and Applications

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Abstract: For a general inelastic Kac-like equation recently proposed, this paper studies the long-time behaviour of its probability-valued solution. In particular, the paper provides necessary and sufficient conditions for the initial datum in order that the corresponding solution converges to equilibrium. The proofs rest on the general CLT for independent summands applied to a suitable Skorokhod representation of the original solution evaluated at an increasing and divergent sequence of times. It turns out that, roughly speaking, the initial datum must belong to the standard domain of attraction of a stable law, while the equilibrium is presentable as a mixture of stable laws. An entire section is devoted to an application of these results to the distribution of income. It highlights a strict relationship between the Arrow-de Finetti local risk aversion index, assumed to be the same for all agents, and the inequality (concentration) in the stationary income distribution.

Primary 60F05, 82C40; secondary 91B15.
Keywords and phrases: Central limit theorem, inelastic Kac-like equations, (weak) Pareto laws, Skorokhod representation theorem, stable laws.

1. Introduction

There has been recent interest in some inelastic counterparts of the Kac one-dimensional Boltzmann-like equation. Particular attention has been paid to the convergence to equilibrium as, for example, in [4, 5, 6, 8, 9, 11, 30, 39]. See also [7] and references therein for multidimensional inelastic models. Typically, one considers suitable initial data (in the form of probability laws) and proves that the ensuing solutions converge weakly to some distinguished probability distributions (p.d.'s, for short). Furthermore, remarkable efforts have been made to discover the rate of approach to equilibrium, exactly as in allied works on

*Work partially supported by MIUR-2012AZS52J-003 and by INdAM-GNAMPA Project 2014.
†Also affiliated with CNR-IMATI, Milano, Italy. Work partially supported by MIUR-2012AZS52J-003.

the classical kinetic equations such as [13, 15, 23, 24, 28, 29, 35, 36] for one-dimensional models, and [14, 16, 25] for multidimensional ones. New problems arise in connection with the following natural questions: Can one formulate necessary conditions on the initial data in order that they produce relaxation to equilibrium? Can one determine necessary and sufficient conditions for the same purpose? In the present paper, the former question is solved for a general form of the model introduced in [6], while the latter is addressed with respect to a few specific cases including, in any case, the one that has been considered repeatedly, for example, in connection with applications of an economic nature.

In the aforesaid general model, if (v, w) and (v', w') indicate pre-collisional and post-collisional velocities, respectively, of two colliding particles, one assumes that

$$\begin{cases} v' = \widetilde{L}_1 v + \widetilde{R}_1 w \\ w' = \widetilde{R}_2 v + \widetilde{L}_2 w \end{cases} \tag{1}$$

where $(\widetilde{L}_1, \widetilde{R}_1)$ and $(\widetilde{L}_2, \widetilde{R}_2)$ are random vectors in $\mathbb{R}^2$ with common p.d. τ. In the rest of the paper, it is supposed that:

τ has continuous marginals and

$$(x_0, y_0) \in \operatorname{supp}(\tau) \quad \text{whenever} \quad |x_0|^\alpha + |y_0|^\alpha = 1. \tag{2}$$

Moreover, it is also assumed that the function $\mathcal{S}$ defined by

$$\mathcal{S}(p) = \int_{\mathbb{R}^2} (|l|^p + |r|^p)\tau(dldr) - 1 \qquad (p \geq 0) \tag{3}$$

satisfies the condition:

$$\begin{array}{l}\text{The equation } \mathcal{S}(p) = 0 \text{ admits at least one solution on } (0, +\infty) \\ \text{and } \alpha \text{ will denote the smallest root.}\end{array} \tag{4}$$

It is worth mentioning that both the Kac equation in [32] and its inelastic direct counterpart given in [39] satisfy (4)-(2) for suitable α in $(0, 2]$ and τ. At this stage, the ensuing kinetic equation reads

$$\partial_t \mu_t + \mu_t = Q^+(\mu_t) \qquad (t \geq 0) \tag{5}$$

where μ_t is a time-dependent probability measure (p.m., for short) on the real line and $Q^+(\mu_t)$ is the p.m. specified by the Fourier-Stieltjes transform

$$\widehat{Q^+(\mu_t)}(\xi) := \int_{\mathbb{R}^2} \varphi(t, l\xi)\varphi(t, r\xi)\tau(dldr)$$

where $\varphi(t, \xi) := \int_{\mathbb{R}} e^{i\xi v} \mu_t(dv)$ is the Fourier-Stieltjes transform $\widehat{\mu}_t$ of μ_t. Following the terminology adopted for kinetic equations, $Q^+(\mu_t)$ can be seen as a sort of 2-fold *Wild convolution* of μ_t with itself. See [35]. It is also important to recall that the Cauchy problem, obtained from the combination of (5) with any initial p.d. μ_0, has a unique solution. The proof of this fact is immediate by mimicking the argument used in [6], where the support of τ is assumed to be a subset of $[0, +\infty)^2$.

Returning to the original questions, complete answers have been given, until now, only for the Kac equation [15, 28] and for its direct inelastic counterpart [11, 30]. In both these cases, in order that the solution converge, it is necessary and sufficient that the symmetrized form μ_0^* of the initial p.d. μ_0 characterized, through the probability distribution function (p.d.f., for short), by

$$F_0^*(x) = \mu_0^*(-\infty, x] = \{\mu_0(-\infty, x] + \mu_0[-x, +\infty)\}/2 \quad (x \in \mathbb{R}),$$

belong to the *standard domain of attraction* (s.d.a., for short) of the α-stable distribution having Fourier-Stieltjes transform $\xi \mapsto e^{-k_\alpha |\xi|^\alpha}$, for some k_α in $[0, +\infty)$. This s.d.a. has to be meant as the class of all the p.m.'s with p.d.f.'s F satisfying either

$$\begin{aligned} &\lim_{x\to+\infty} x^\alpha F(-x) = c_1 \text{ and } \lim_{x\to+\infty} x^\alpha [1 - F(x)] = c_2 \\ &\text{for some non-negative } c_1 \text{ and } c_2, \text{ if } 0 < \alpha < 2, \end{aligned} \tag{6}$$

or

$$\int_{\mathbb{R}} x^2 dF(x) < +\infty \qquad \text{if } \alpha = 2. \tag{7}$$

The limiting behaviour of the solution to (5), when (4) is in force for some α greater than 2, is considered, for the first time, in Theorem 2.1 of the present paper. Moreover, in line with the results obtained for the above-mentioned special cases, it will be proved that the symmetrized initial datum μ_0^* for (5) must satisfy one of the two conditions (6)-(7) − with F replaced by F_0^* − in order that the solution of the Cauchy problem converge weakly. As far as sufficiency is concerned, it is shown that situations in which that very same statement turns out to be also sufficient, for example when τ is invariant w.r.t. $(\pi/2)$-rotations, coexist with others in which it is required that μ_0 itself belongs to a specific s.d.a., for example when the support of τ is contained in $[0, +\infty)^2$. Of course, if μ_0 is an element of some s.d.a., then μ_0^* is such, but μ_0^* can belong to some s.d.a. even if μ_0 does not. See the example in Appendix B.

The rest of the paper is organized as follows. In Subsection 2.1, a probabilistic representation of the solution to the general Cauchy problem is recalled. The new results are carefully formulated in Subsection 2.2, while their proofs are deferred to Section 4. From a methodological viewpoint, it seems appropriate to highlight Lemma 4.2, since it represents the key point in the proof of the main results. An application of some of these results is presented in Section 3.

2. Preliminaries and statement of the main results

Before providing a precise formulation for the new results, some facts about a probabilistic representation of the solution of the Cauchy problem are recalled. They are useful, on the one hand, to grasp the connections of the issues raised in the introduction with the central limit problem, and, on the other hand, to cast new light on some aspects of the application developed in Section 3.

2.1. Probabilistic representation of the solution to the Cauchy problem

This representation follows from associating a stochastic model with a system of many molecules colliding in pairs, in such a way that this very same model turns out to be consistent with the Wild-McKean representation of the solution of (5). See (8)-(9) in [6] for this point. All the random elements one is about to consider are supposed to be defined as measurable functions on some measurable space $(\Omega, \mathcal{F})$, in such a way that their p.d.'s turn out to be images of a p.m. $\mathcal{P}$ supported by $(\Omega, \mathcal{F})$. One starts by considering a distinguished molecule, and defines $\tilde{\nu}_t$ to be the random number of particles that "contribute", according to the following scheme, to the velocity V_t of the observed molecule at time t, for any $t > 0$. Cf. [12]. This idea of "contribution" in now illustrated with an example. Consider particles $1, 2, \dots, 7$ with initial velocities $X_1, \dots, X_7$, and assume that 2 and 3 collide before 2 encounters 1, and 3 disappears; moreover, assume that 5 and 6 collide before 5 encounters 4, and 6 disappears; to continue, suppose that 4 collides with 7, 5 disappears and afterwards 4 encounters 1, while 7 disappears. According to (1), due to this specific sequence of collisions, initial velocities $X_1, \dots, X_7$ change as follows: the first collision between 2 and 3 yields post-collisional velocities $X_2' = \widetilde{L}_1 X_2 + \widetilde{R}_1 X_3$ and $X_3' = \widetilde{L}_2 X_3 + \widetilde{R}_2 X_2$. Thus, the velocity of 2 immediately before encountering 1 is given by X_2' and, hence, the post-collisional velocity of 1 is $X_1' = \widetilde{L}_3 X_1 + \widetilde{R}_3 X_2'$. In the meantime, 5 encounters 6 and changes its velocity X_5 into $X_5' = \widetilde{L}_4 X_5 + \widetilde{R}_4 X_6$ so that the velocity of 4 immediately after the collision with 5 is given by $X_4' = \widetilde{L}_5 X_4 + \widetilde{R}_5 X_5'$. At this stage, this velocity changes, due to the collision with 7, into $X_4'' = \widetilde{L}_6 X_4' + \widetilde{R}_6 X_7$. Finally, the collision between 1 and 4 occurs with pre-collisional velocities X_1' and X_4'', respectively, and then the post-collisional velocity of 1 is $X_1'' = \widetilde{L}_7 X_1' + \widetilde{R}_7 X_4'' = \widetilde{L}_7 \widetilde{L}_3 X_1 + \widetilde{L}_7 \widetilde{R}_3 \widetilde{L}_1 X_2 + \widetilde{L}_7 \widetilde{R}_3 \widetilde{R}_1 X_3 + \widetilde{R}_7 \widetilde{L}_6 \widetilde{L}_5 X_4 + \widetilde{R}_7 \widetilde{L}_6 \widetilde{R}_5 \widetilde{L}_4 X_5 + \widetilde{R}_7 \widetilde{L}_6 \widetilde{R}_5 \widetilde{R}_4 X_6 + \widetilde{R}_7 \widetilde{R}_6 X_7$. This formula clarifies the meaning of the aforementioned term "contribution": 1's contribution to V_t is $\widetilde{L}_7 \widetilde{L}_3 X_1$, 2's contribution is $\widetilde{L}_7 \widetilde{R}_3 \widetilde{L}_1 X_2$ and so on up to particle 7, since $\tilde{\nu}_t = 7$. The above description can be visualized through the McKean tree of the following figure: Its leaves are labelled, from left to right, by the labels $1, \dots, 7$ of the seven particles contributing to V_t. This way, the contribution of each particle j can be read as the product of the $\widetilde{L}$'s and $\widetilde{R}$'s one finds on the path which connects j with the root node 0. In general, there is a one-to-one correspondence between McKean trees and systems of many molecules colliding (in pairs) and changing their states according to (1). Therefore, both the number and the entities of the "contributions" to V_t can be easily determined by resorting to trees like in Figure 1.

In point of fact, for the sake of realism, one thinks of these systems as governed by probabilistic, rather than deterministic, laws. In particular, $\tilde{\nu} := (\tilde{\nu}_t)_{t \geq 0}$ is seen as a pure birth (Yule-Furry) process on $\{1, 2, \dots\}$ having birth rate $\lambda_n = n$ for every $n \geq 1$, and the unit mass at 1 as initial distribution. Then, $\mathcal{P}\{\tilde{\nu}_t = n\}$ coincides with the probability that the process starting at 1 will be in state n

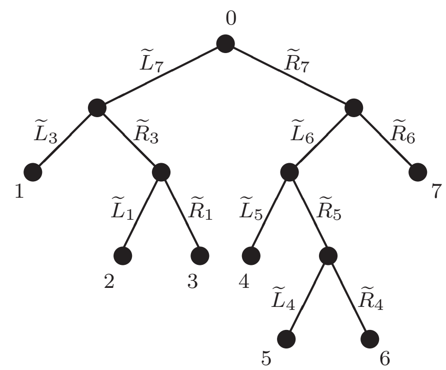

Figure 1.

at time t, that is,

$$\mathcal{P}\{\tilde{\nu}_t = n\} = e^{-t}(1 - e^{-t})^{n-1} \qquad (n = 1, 2, \ldots,\ t \geq 0)$$

with the proviso that $0^0 = 1$. As to the McKean trees, i.e., as to all possible forms of collision systems, one resorts to a discrete-parameter Markov chain $\tilde{a} = \{\tilde{a}_n : n = 1, 2, \ldots\}$ defined as follows. The state space of $\tilde{a}$ is the set of all McKean trees and, for each value n of the parameter, $\tilde{a}_n$ has to be meant as a random tree with n leaves. The initial distribution, that is, the p.d. of $\tilde{a}_1$, is assumed to be the unit mass at the unique tree with one leaf. The transition probabilities are specified as

$$\mathcal{P}\{\tilde{a}_{n+1} = a_{n+1} | \tilde{a}_n = a_n\} = \frac{1}{n} \mathbb{I}_{G(a_n)}(a_{n+1})$$

for every a_n in the set $\mathcal{A}_n$ of all trees with n leaves ($n = 1, 2, \ldots$), $G(a_n)$ denoting the set of the trees in $\mathcal{A}_{n+1}$ obtained by attaching a copy of $\tilde{a}_2$ (which, of course, is non-random) to a specific leaf of a_n. To complete the picture, one introduces a sequence $(\widetilde{L}, \widetilde{R}) = ((\widetilde{L}_n, \widetilde{R}_n))_{n \geq 1}$ of independent and identically distributed (i.i.d., for short) random vectors with τ as common p.d. (see (4)-(2)) and a sequence $\tilde{X} = (\tilde{X}_n)_{n \geq 1}$ of i.i.d. random numbers having the initial datum μ_0 as common p.d.. It remains to specify that $\tilde{\nu}, \tilde{a}, (\widetilde{L}, \widetilde{R}), \tilde{X}$ are supposed to be stochastically independent random elements. Now, the contribution of each particle to V_t can be fully described. One considers the random tree $\tilde{a}_{\tilde{\nu}_t}$, a particle j in $\{1, \ldots, \tilde{\nu}_t\}$ and the product $\tilde{\beta}_{j, \tilde{\nu}_t}$ of the $\widetilde{L}$'s and $\widetilde{R}$'s one encounters on the path that connects j with the root node of $\tilde{a}_{\tilde{\nu}_t}$ (the number of the edges of such a path will be called *depth* of j). Then, in the general case,

$$V_t = \sum_{j=1}^{\tilde{\nu}_t} \tilde{\beta}_{j, \tilde{\nu}_t} X_j \qquad (t \geq 0). \tag{8}$$

The connection between this stochastic model and the Cauchy problem attached to the Boltzmann-like equation (5) lies in the important fact that the p.d. of V_t

satisfies (5) with initial datum μ_0. See Proposition 1 in [6]. Of course, recalling that $\varphi(t,\cdot)$ stands for the Fourier-Stieltjes transform of the solution μ_t, putting $\varphi_0(\cdot) := \varphi(0,\cdot) = \widehat{\mu_0}(\cdot)$ as a consequence one gets

$$\varphi(\xi,t) = \int_\Omega \prod_{j=1}^{\tilde{\nu}_t(\omega)} \varphi_0(\tilde{\beta}_{j,\tilde{\nu}_t(\omega)}(\omega)\xi)\mathcal{P}(d\omega)$$

providing a disintegration of the solution to the Cauchy problem associated with (5) into components which are the p.d.'s of weighted sums of i.i.d. random numbers having common p.d. μ_0. It is just this disintegration which connects the relaxation to equilibrium of the kinetic model with the central limit problem of the probability theory.

To pave the way for next developments it is worthwhile to recall an interesting fact about the $\tilde{\beta}$'s, proved in ([6], Subsection 2.3 and Lemma 2): $\sum_{j=1}^{\tilde{\nu}_t} |\tilde{\beta}_{j,\nu_t}|^\alpha$ *converges, with probability one, to a non-negative random number* $M_\infty^{(\alpha)}$ *satisfying* $\mathbb{E}(M_\infty^{(\alpha)}) = 1$, where $\mathbb{E}$ stands for expectation w.r.t. $\mathcal{P}$.

2.2. New results

The new results of the present paper are chiefly concerned with necessary conditions for weak convergence of the solution to the Cauchy problem. The first one provides an essential justification for the restriction of the admissible values of α to the interval $(0,2]$. In point of fact, this restriction has been conventionally, rather than substantially, so far accepted, and the next theorem confirms its reasonableness.

Theorem 2.1. *Suppose τ satisfies (4)-(2) with $\alpha > 2$. If the solution μ_t to the Cauchy problem associated with (5), initial p.d. μ_0, converges weakly to a p.d. μ_∞, as $t \to +\infty$, then both μ_0 and μ_∞ must be unit masses at points x_0 and x_1, respectively.*

Moreover:

- (i) *If $x_0 \neq 0$ and $\tau\{(x,y) \in \mathbb{R}^2 : \ x+y-1 = 0\} = 1$, then $\mu_\infty = \delta_{x_0}$. Conversely, if $\mu_\infty = \delta_{x_1}$ with $x_1 \neq 0$, then $\tau\{(x,y) \in \mathbb{R}^2 : \ x+y-1 = 0\} = 1$ and $x_0 = x_1$.*
- (ii) *If (at least) one of the two following conditions is verified:*
  - (ii_1) $x_0 = 0$
  - (ii_2) $\sum_{j=1}^{\tilde{\nu}_t} \tilde{\beta}_{j,\tilde{\nu}_t}$ *converges in distribution to zero, as t goes to infinity,*

  then $x_1 = 0$. Conversely, if $x_1 = 0$, then at least one of the conditions (ii_1),(ii_2) must be in force.

Proof. See Subsection 4.4. □

The lesson of Theorem 2.1 is that, if $\alpha > 2$, unit masses are the sole admissible stationary distributions. From a logical viewpoint, it is worth noticing that condition (4) may coexist with (i). Consider, for example the case in which

$\widetilde{R}_1 = 1 - \widetilde{L}_1$ almost surely and the p.d. of $\widetilde{L}_1$ is the Gaussian centred at 1/2 with variance σ^2. It is easy to show that there is a value of σ^2, say $\bar{\sigma}^2$, so that $\int_{\mathbb{R}} |x|^\alpha 1/\sqrt{2\pi\bar{\sigma}^2} \exp\{-(x-1/2)^2/2\bar{\sigma}^2\}dx = 1/2$. In the same vein, (4) and (ii_2) may coexist, for example, if $\widetilde{L}_1 = \widetilde{R}_1$ almost surely and the p.d. of $\widetilde{L}_1$ is the Gaussian centred at zero with variance 1/4.

The second, and more significant, group of results is concerned with values of α in $(0, 2]$, when convergence to the steady state occurs in the presence of interesting forms of initial data.

Theorem 2.2. *Suppose τ satisfies both the moment and support conditions (4)-(2) for some α in $(0, 2]$. If the solution μ_t to the Cauchy problem associated with (5), initial datum μ_0, converges weakly to a p.d., as $t \to +\infty$, then*

$$\lim_{x\to+\infty} x^\alpha[1 - F_0^*(x)] \text{ exists and is finite.} \tag{9}$$

Proof. See Subsection 4.1. □

Like the central limit theorem for weighted sums of i.i.d. summands, one could expect that the above condition is also sufficient for the convergence of the solution. On the one hand, this is the case when, for example, τ agrees either with the Kac model or with its direct inelastic counterpart (see [11, 15, 28, 30] and the next more general Theorem 2.5). On the other hand, the conjecture fails, for example, when the support of τ is a subset of $[0, +\infty)^2$, which is the version of (5) most widely studied so far. To deal with this case, some additional notation is needed together with the following reformulation of (6), where F_0 is taken in the place of F and α belongs to $(0, 2]$,

$$\lim_{x\to+\infty} x^\alpha F_0(-x) = c_1 \text{ and } \lim_{x\to+\infty} x^\alpha[1 - F_0(x)] = c_2 \tag{10}$$

where c_1 and c_2 are non-negative numbers. Moreover, let $m_{0,i} := \int_{\mathbb{R}} x^i \mu_0(dx)$ for $i = 1, 2$ and let $\widehat{\psi_\alpha}(\cdot\,; \chi, k_\alpha, \gamma)$ indicate the Fourier-Stieltjes transform of the stable law $\psi_\alpha(\cdot\,; \chi, k_\alpha, \gamma)$, i.e.,

$$\widehat{\psi_\alpha}(\xi; \chi, k_\alpha, \eta) = \exp\left\{i\chi\xi - k_\alpha|\xi|^\alpha\Big(1 - i\gamma\frac{\xi}{|\xi|}\omega(\xi,\alpha)\Big)\right\}$$

where

$$\begin{aligned} \omega(\xi,\alpha) &= \tan(\pi\alpha/2) \quad && \alpha \neq 1 \\ &= 2\pi^{-1}\log|\xi| && \alpha = 1 \end{aligned}$$

with the proviso that:

- $\chi := 0$, $k_\alpha := [2\Gamma(\alpha)\sin(\pi\alpha/2)]^{-1}(c_1 + c_2)\pi$ and $\gamma := \mathbb{I}_{\{c_1+c_2>0\}}(c_2 - c_1)/(c_1 + c_2)$, if $\alpha \in (0, 1) \cup (1, 2)$;

- $\chi := \eta - \int_{\mathbb{R}\setminus\{0\}}[-\mathbb{I}_{(-\infty,-1]}(y) + y\mathbb{I}_{(-1,1]}(y) + \mathbb{I}_{(1,+\infty)}(y) - \sin y]\nu(dy)$, $k_1 := (c_1 + c_2)\pi/2$ and $\gamma := 0$, when $\alpha = 1$, η and ν being characteristic parameters of the Lévy-Khinchin representation of $\widehat{\psi_\alpha}$ according to Proposition 11 in Chapter 17 and Theorem 30 in Chapter 16 of [27], with $Q_{1,n} = Q_{2,n} = \cdots = \mu_0$;
- $\chi := 0$, $k_2 := (m_{0,2} - m_{0,1}^2)/2$ and $\gamma := 0$, if $\alpha = 2$.

Coming back to the discussion about sufficiency, one starts by giving a more complete version of Theorems 1-3 in [6] and Theorem 2.3 in [9], viz

Theorem 2.3. *Suppose τ satisfies (4) for some α in $(0, 2]$, with supp$(\tau) \subset [0, +\infty)^2$, and*

$$\mathcal{S}(p) < 0 \qquad \text{for some } p > 0. \tag{11}$$

Then, the solution μ_t to the Cauchy problem associated with (5), initial datum μ_0, converges weakly to a p.m. μ_∞, as $t \to +\infty$, if:

condition (10) holds whenever $\alpha \in (0, 1)$;
condition (10) along with $c_1 = c_2$ are met whenever $\alpha = 1$;
condition (10) and $m_{0,1} = 0$ are in force whenever $\alpha \in (1, 2)$;
$m_{0,1} = 0$ and $m_{0,2} < +\infty$ are valid whenever $\alpha = 2$.

Furthermore, the Fourier-Stieltjes transform of the limiting p.d. μ_∞ is given by

$$\int_0^{+\infty} \widehat{\psi_\alpha}(\xi m^{1/\alpha}; 0, k_\alpha, \gamma)\nu_\alpha(dm) \qquad \text{whenever } \alpha \in (0,1) \cup (1,2)$$

$$\int_0^{+\infty} \widehat{\psi_1}(\xi m; \chi, k_1, 0)\nu_1(dm) \qquad \text{whenever } \alpha = 1$$

$$\int_0^{+\infty} \widehat{\psi_2}(\xi m^{1/2}; 0, k_2, 0)\nu_2(dm) \qquad \text{whenever } \alpha = 2$$

for every $\xi \in \mathbb{R}$, where ν_α is the p.d. of $M_\infty^{(\alpha)}$ for each α in $(0, 2]$, and χ, k_α, γ are the same as in the itemization preceding the theorem.

Proof. See Subsection 4.2. ☐

Before presenting necessary and sufficient conditions, the following miscellaneous remarks could be in order.

Remark 1. In view of Proposition 2 in [6], one recalls that, when (11) is in force and α is the unique root of equation (4), the p.d. ν_α admits moments of any order. Moreover, if a second root θ exists, the only finite moments of ν_α are those of order strictly smaller than θ/α. Combination of these facts with the well-known moment properties of the stable laws yields: If $\alpha < 2$, then μ_∞ admits the p-th moment if and only if $p < \alpha$. If $\alpha = 2$, then the p-th moment of μ_∞ is finite for $p > 2$ if and only if $p < \theta$.

Remark 2. An interesting problem is the search of conditions under which the limiting law μ_∞ is a (pure) stable law. A complete answer can be obtained by combining Theorems 1, 3 in [6] with Theorem 2.3 in [9] and allied results in [1] on the fixed points of operators like Q^+.

Remark 3. Proposition 3.9 in [9] shows that, whenever α belongs to $(0, 2)$, the tails of non-degenerate limiting p.d.'s behave like $(x_0/|x|)^\alpha$ as $x \to \infty$, for suitable x_0. According to [33], these p.d.f.'s can be called *weak Pareto laws*. Furthermore, limiting point masses arise when $c_1 = c_2 = 0$. These facts are of importance w.r.t. to the application presented in the next section.

Remark 4. As to the case of $\alpha = 2$, in view of Remark 1 it is worth distinguishing the following two subcases. If α is the sole root of equation (4), since in this case μ_∞ has moments of any order, one can conclude that $F_\infty(-x) = 1 - F_\infty(x) = o(1/x^p)$ for every $p > 0$, as $x \to +\infty$. On the other hand, if $\theta > \alpha$ is the second root of the equation in (4), then (see Remark 1) $\int_{\mathbb{R}} |x|^p d\mu_\infty < +\infty$ for $p < \theta$ and $\int_{\mathbb{R}} |x|^\theta d\mu_\infty = +\infty$ so that, from the Markov inequality,

$$F_\infty(-x) = 1 - F_\infty(x) \leq \frac{A_p}{x^p} \qquad (x > 0)$$

for every $p < \theta$ and $A_p := \int_{\mathbb{R}} |x|^p \mu_\infty(dx)/2$. To obtain a lower bound for $F_\infty(-x)$, one notes that, putting $G(y) := [2F_\infty(y^{1/\theta}) - 1]\mathbb{I}_{[0,+\infty)}(y)$, one has $\int_0^{+\infty} y dG(y) = +\infty$. From Proposition 3.3. in [21] one obtains $\lim_{y\to+\infty}(1 - G(y))g(y) = +\infty$ for every function g continuous, strictly increasing and positive on $(a, +\infty)$ for some $a > 0$ and such that $\int_a^{+\infty}\{1/g(y)\}dy < +\infty$. Thus, choosing, for example, $g(y) := y(\log y)^{1+\delta}$ for $y > 1$ and $\delta > 0$, for every positive M there exists $\bar{y}$ for which $1 - G(y) \geq M/[y(\log y)^{1+\delta}]$ holds for every $y \geq \bar{y}$. Finally, from the definition of G, one gets

$$F_\infty(-x) = 1 - F_\infty(x) = \frac{1}{2}(1 - G(x^\theta)) \geq \frac{M}{2\theta^{1+\delta}} \frac{1}{x^\theta(\log x)^{1+\delta}}$$

for every $x \geq \bar{y}^{1/\theta}$.

Resuming the main line of discussion, the way is paved for presenting necessary and sufficient conditions for the relaxation to equilibrium.

Theorem 2.4. *Suppose supp*$(\tau) \subset [0, +\infty)^2$ *and, for some* α *in* $(0, 2]$, (4)-(2) *and* (11) *are in force. Then, the solution* μ_t *to the Cauchy problem associated with* (5), *initial p.d.* μ_0, *converges weakly to a p.m.* μ_∞, *as* $t \to +\infty$, *if and only if*

condition (10) *holds whenever* $\alpha \in (0, 1)$;

condition (10) *along with* $c_1 = c_2$ *are met whenever* $\alpha = 1$;

condition (10) *and* $m_{0,1} = 0$ *are in force whenever* $\alpha \in (1, 2)$;

$m_{0,1} = 0$ *and* $m_{0,2} < +\infty$ *are valid whenever* $\alpha = 2$.

Of course, μ_∞ *is the same as in* Theorem 2.3.

Proof. See Subsection 4.3. □

The next theorem, which includes both the Kac model and its inelastic counterpart, provides an example in which the weaker condition involved in Theorem 2.2 turns out to be sufficient for the convergence.

Theorem 2.5. *Suppose τ is invariant w.r.t. $(\pi/2)$-rotations and, for some α in $(0,2]$, (4)-(2) and (11) are in force. Then, (9) $[m_{0,2} < +\infty$, respectively$]$ is necessary and sufficient whenever α belongs to $(0,2)$ $[\alpha = 2$, respectively$]$ in order that the solution μ_t to the Cauchy problem associated with (5), initial p.d. μ_0, converge weakly to a p.m. μ_∞, as $t \to +\infty$.*

Proof. See Subsection 4.5. □

One notes the redundancy of assumption (2) in the sufficiency part of the last theorem.

3. Application to the distribution of incomes

The purpose of this section is to show that a suitable reinterpretation of (1)-(5) justifies *weak Pareto laws* as distributions for incomes, pointing out, in addition, a strict connection between risk aversion of agents (assumed to be the same for all of them) and inequality (concentration) of incomes. Recall that, according to Mandelbrot [33], weak Pareto laws are those p.d.f.'s in which "the percentage of individuals with an income (over some fixed period of reference) exceeding some number x" behaves like $(x/a_0)^{-\beta}$, as $x \to +\infty$, a_0 and β being positive parameters. It is well-known that Pareto [37] was the first to maintain, on the basis of empirical observations, "that over a certain range of values of income, its distribution is not markedly influenced either by the socio-economic structure of the community under study, or by the definition chosen for income. That is, these two elements may at most influence the values taken by certain parameters" in the expression of a p.d.f. whose tails apparently and invariably meet the ones described above. In particular, it should be recalled that the parameter $1/\beta$ plays the role, within the Pareto laws, of measure of the inequality of incomes, in the sense that negative increments of the value of β go along with positive increments of the inequality of incomes. In this part, the term income is meant as *surplus* or *deficit* with respect to *subsistence* and, consequently, its distribution is seen as resultant of a great number of transfers of surplus/deficit. These transfers are supposed to be governed by the p.d. of a suitable stochastic process, whilst subsistence is assumed to be a quantity (stochastic or not) to be added to the output of the aforesaid process. In any case, only the surplus/deficit process is considered in the rest of this section. General inspiring ideas about surplus theory of social stratification can be found in [2].

3.1. Economic reading of (1)-(5)

As said by Mandelbrot, "there is a great temptation to consider the exchanges of money which occur in economic interaction as analogous to the exchanges of energy which occur in physical shocks between gas molecules". In partial support of this, one can mention that in recent years also particular versions of model (1)-(5) have been invoked to explain how and why weak Pareto laws have to do with social sciences. See, e.g. [17, 18, 19, 20, 34]; cf. also the more recent [40] and references therein. One of the major restrictions adopted yet in these works is the supposition that α is equal to 1. This is a real limitation since the model at issue without such a restriction has interesting implications like, for example, the fact − consequence of propositions presented in the last section − that the stationary distributions *must* vary in the family of weak Pareto laws, their parameters β depending, through a precise quantitative relationship, on the agents risk aversion, supposed to be the same for every agent. Of course, these implications might be of some importance with a view to specific policies towards a desirable reduction in the inequality of incomes. Their complete formulation and proof form the core of the rest of this section. At first, one assumes that exchanges are effected according to a fixed scheme, that is, one of the two agents, say I_1, can produce a random income and, at the same time, can extract a random income from the other agent, say I_2. More precisely, due to each exchange, initial incomes, v and w respectively, become

$$\begin{aligned} v' &= \widetilde{L}_1 v + \widetilde{R}_1 w \\ w' &= \widetilde{R}_2 v + \widetilde{L}_2 w \end{aligned} \tag{12}$$

where $\widetilde{L}_j$ represents the positive random unitary gain made by I_j from her/his own income, whilst $\widetilde{R}_j$ indicates the positive random unitary gain drawn by I_j from I_i's income ($i, j = 1, 2$ with $i \neq j$). Random vectors $(\widetilde{L}_1, \widetilde{R}_1)$ and $(\widetilde{L}_2, \widetilde{R}_2)$ are assumed to be identically distributed with common p.d. τ. In existing allied models, specifically in [17], one often adopts $\widetilde{L}_1 = \lambda + (1-\lambda)\tilde{\varepsilon}$, $\widetilde{L}_2 = \lambda + (1-\lambda)(1-\tilde{\varepsilon})$, $\widetilde{R}_1 = (1-\lambda)\tilde{\varepsilon}$, $\widetilde{R}_2 = (1-\lambda)(1-\tilde{\varepsilon})$, with $\tilde{\varepsilon}$ random number whose p.d. is supported by $[0, 1]$ and is symmetrical about $1/2$. This way, the above distributional conditions on $(\widetilde{L}_i, \widetilde{R}_i)$, for $i = 1, 2$, are plainly met. The meaning of λ is that of unitary fraction of income each agent puts aside, while $(1-\lambda)\tilde{\varepsilon}$ [$(1-\lambda)(1-\tilde{\varepsilon})$, respectively] stands for the unitary gain made by I_1 [I_2, respectively] both from her/his initial income and from the initial income of I_2 [I_1, respectively]. Coming back to the general form of the model, it remains to reinterpret (4) and (5). As to the former, one assumes that

$$u(x, y) = |x|^\alpha \text{sign}(x) + |y|^\alpha \text{sign}(y) \tag{13}$$

is the utility associated with any couple (x, y) where x stands for the income produced by an agent on her/his own and y represents the income she/he draws from the other agent. This utility is the sum of two equal utility functions $\bar{u}$ having the sole argument "wealth" expressed in monetary units, i.e.

$$\bar{u}(s) = |s|^\alpha \text{sign}(s) \qquad (s \in \mathbb{R}).$$

For every money utility function $\bar{u}$, $\lambda^* = \lambda^*(s) := -\bar{u}''(s)/(2\bar{u}'(s))$ provides a *local risk aversion index*, independently proposed by de Finetti [22] and Arrow

[3]. In the particular case under consideration, it turns out that

$$\lambda^*(s) = (1-\alpha)/(2s) \qquad (s \in \mathbb{R} \setminus \{0\}). \tag{14}$$

In view of (12) and (13), I_1's [I_2's] utility after an exchange is $u(\widetilde{L}_1 v, \widetilde{R}_1 w) = \widetilde{L}_1^\alpha |v|^\alpha \mathrm{sign}(v) + \widetilde{R}_1^\alpha |w|^\alpha \mathrm{sign}(w)$ $[u(\widetilde{L}_2 w, \widetilde{R}_2 v) = \widetilde{L}_2^\alpha |w|^\alpha \mathrm{sign}(w) + \widetilde{R}_2^\alpha |v|^\alpha \mathrm{sign}(v)]$. Then, assuming strong additivity of individual's utility in expressing group (joint) preferences, one can read

$$u(\widetilde{L}_1 v, \widetilde{R}_1 w) + u(\widetilde{L}_2 w, \widetilde{R}_2 v)$$

as joint welfare function of any pair of individuals exchanging money according to the previous terms. The corresponding utility index (expectation of the above joint welfare function) is then given by

$$u(v,w) \int_{(0,+\infty)^2} (l^\alpha + r^\alpha) \tau(dl dr)$$

in view of the fact that $(\widetilde{L}_1, \widetilde{R}_1)$ and $(\widetilde{L}_2, \widetilde{R}_2)$ are identically distributed with common p.d. τ. Now, the meaning of (4) is clear: it amounts to considering that any exchange preserves the value of the utility index.

It remains to reinterpret (5) from an economic viewpoint, which amounts to giving a sensible explanation for the expression of $\partial_t \mu_t$. To this end consider, conditionally on $\{(\widetilde{L}_1, \widetilde{R}_1) = (l, r)\}$, the incomes $\tilde{v}$ and $\tilde{w}$ of the contracting parts, immediately before the exchange at time t, as i.i.d. random numbers with common p.d. μ_t. Immediately after any exchange, the conditional p.d. of the income of agent I_1 passes from μ_t to the conditional law of the sum $l\tilde{v} + r\tilde{w}$, that is

$$A \mapsto \int_{\mathbb{R}} \mu_t(A_{v,l,r}) \mu_t(dv) \qquad (A \in \mathcal{B}(\mathbb{R}))$$

where $A_{v,l,r} := \{w \in \mathbb{R} : \ lv + rw \in A\}$. Whence, passing from conditional to unconditional distributions,

$$\int_{(0,+\infty)^2} \Big(\int_{\mathbb{R}} \mu_t(A_{v,l,r}) \mu_t(dv) \Big) \tau(dl dr) - \mu_t(A)$$

can be seen as an instantaneous rate of change of μ_t, at each Borel set $A \subset \mathbb{R}$, yielding

$$\partial_t \mu_t(A) = \int_{(0,+\infty)^2} \Big(\int_{\mathbb{R}} \mu_t(A_{v,l,r}) \mu_t(dv) - \mu_t(A) \Big) \tau(dl dr)$$

and its equivalent form (5).

3.2. Significant implications

At this stage, if one adopts the scheme of the previous section to describe the exchange of incomes, then one must conclude that the resulting law of the income of each agent at time t, is the same as the p.d. of the random number V_t in (8). Of course, therein, exchanges take the place of collisions and any tree can be seen as picture of an economic state (situation) relative to a specific agent, say 1. By analogy with the example in Subsection 2.1, one considers the situation in which, during the period $(0, t]$, agents 2 and 3 meet before 2 meets 1 and 3 abandons the market. Suppose that, immediately after, 4 encounters 1 and then abandons the market. Finally, let 5 and 6 exchange before 5 meets 1, and 6 abandons the market. If $X_1, \dots, X_6$ stand for the initial incomes of the agents, then the income of 2 before the exchange with 1 is $X_2' = \widetilde{L}_1 X_2 + \widetilde{R}_1 X_3$. Hence, the income of 1 after the exchange with 2 is given by $X_1' = \widetilde{L}_2 X_1 + \widetilde{R}_2 X_2'$ and, thus, after the encounter with 4, it becomes $X_1'' = \widetilde{L}_3 X_1' + \widetilde{R}_3 X_4$. On the other hand, after the exchange between 5 and 6, the income of the former is $X_5' = \widetilde{L}_4 X_5 + \widetilde{R}_4 X_6$, and the income of 1 at t, that is, after the exchange with 5, turns out to be $X_1''' = \widetilde{L}_5 X_1'' + \widetilde{R}_5 X_5' = \widetilde{L}_5 \widetilde{L}_3 \widetilde{L}_1 X_1 + \widetilde{L}_5 \widetilde{L}_3 \widetilde{R}_2 \widetilde{L}_1 X_2 + \widetilde{L}_5 \widetilde{L}_3 \widetilde{R}_2 \widetilde{R}_1 X_3 + \widetilde{L}_5 \widetilde{R}_3 X_4 + \widetilde{R}_5 \widetilde{L}_4 X_5 + \widetilde{R}_5 \widetilde{R}_4 X_6$. Each summand represents the contribution of each agent to the income of 1 at time t, and gives a precise image of the formation of each of these contributions.

In view of the main subject of the paper, one concludes by mentioning a few implications concerning income laws as stationary distributions. As a preamble, it is useful to dwell upon an economic interpretation, consistent with the rest of the section, of a few aspects of the function $\mathcal{S}$ defined in (3). This function is involved, e.g., in condition (11) and in the discussion of the form of the tails of the limiting p.d. when $\alpha = 2$, etc. In particular, (11) requires that this function be strictly negative at least for all p's in a suitable interval $(\alpha, \alpha + \varepsilon)$. Clearly, this behaviour depends crucially on the choice of τ and states a sort of *Pareto-optimality* of the point $p = \alpha$, thanks to the following reformulation of (11):

(a) For a suitable $\varepsilon > 0$ and for any value $\lambda^*(s)$ of the risk aversion included in $((1 - \alpha - \varepsilon)/(2s), (1 - \alpha)/(2s))$ [in $((1 - \alpha)/(2s), (1 - \alpha - \varepsilon)/(2s))$, respectively] for positive [negative, respectively] s, exchanges of money made according to the rules stipulated in Subsection 3.1 diminish [increase, respectively] the joint expected utility of the agents.

As to the number of roots of equation in (4), assuming that the function $\mathcal{S}$, besides vanishing at α, vanishes also in $\theta > \alpha$, is equivalent to supposing that:

(b) There is a risk aversion $\lambda^*(s) = (1-\theta)/(2s)$, for every s, according to which exchanges preserve joint expected utility, although it is strictly smaller [greater, respectively] than $(1-\alpha)/(2s)$ at positive [negative, respectively] s.

As already mentioned, (a) and (b) may hold or not according to the form of τ. Coming back to the tails of the limiting p.d., from Remark 4, if (b) holds when $\alpha = 2$, then the percentage of individuals with an absolute value of income exceeding x satisfies

$$\frac{K_\theta}{x^\theta (\log x)^{1+\delta}} \leq 1 - F_\infty(x) \leq \frac{A_p}{x^p}$$

for sufficiently large values of x, where $p < \theta$, $\delta > 0$ and K_θ, A_p are suitable positive constants. On the other hand, if $\alpha = 2$ is the sole root of the equation in (4), then $1 - F_\infty(x) = o(1/x^p)$ for every $p > 0$, as $x \to +\infty$. When the local risk aversion $\lambda^*(s)$ takes values in $(-1/(2s), 1/(2s))$ [$(1/(2s), -1/(2s))$, respectively] at positive [negative, respectively] s, Theorems 2.2, 2.4 and Remark 3, show that the limit must be weak Pareto with index equal to $(1 - 2s\lambda^*(s))$, provided that $c_1 + c_2 > 0$ and (a) holds. Finally, Theorem 2.1 states that when agents are "strong risk lovers" ["strong risk averters", respectively] at positive [negative, respectively] s, i.e. when the local risk aversion $\lambda^*(s)$ is strictly smaller [strictly greater, respectively] than $(-1/(2s))$, the sole admissible stationary laws are those which correspond to perfectly egalitarian distributions of income, provided that the initial datum does the same.

4. Proofs

Some crucial parts of the proofs are based on the *Skorokhod representation* for sequences which converge in law. See, e.g., Theorem 6.7 in [10]. It is worth recalling such a representation for the sake of expository clarity, even if analogous descriptions have already been given in [8, 11, 25, 26, 30]. Before proceeding to apply the Skorokhod theorem, it is useful to introduce some slight changes to the presentation in Subsection 2.1. In particular, one replaces the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with the family $\{(\Omega, \mathcal{F}, \mathcal{P}_t) : \ t \geq 0\}$. The random elements $(\widetilde{L}, \widetilde{R})$, $\tilde{X}$ are maintained, while the roles of $\tilde{\nu}$ and $\tilde{a}$ are respectively played by:

- A random number $\tilde{\nu}$ taking values in $\mathbb{N}$ whose p.d., under $\mathcal{P}_t$, is given by $\mathcal{P}_t\{\tilde{\nu} = n\} = e^{-t}(1 - e^{-t})^{n-1}$ for every $n \geq 1$ and $t \geq 0$.
- A sequence $\tilde{i} := (\tilde{i}_n)_{n\geq 1}$ of integer-valued random numbers which, under $\mathcal{P}_t$, are independent, each $\tilde{i}_n$ being uniformly distributed on $\{1, \dots, n\}$, for every $t \geq 0$.

It is easy to verify that each realization of the sequence $\tilde{i}$ specifies a McKean tree, and that the distributional properties of $\tilde{i}$ agree with the Markov structure of the law of $\tilde{a}$.

According to Subsection 2.1, the random elements $\tilde{\nu}, \tilde{i}, (\widetilde{L}, \widetilde{R}), \tilde{X}$ are assumed to be *stochastically independent* under each $\mathcal{P}_t$.

These points accepted, one introduces a random vector W, which contains all the elements that characterize convergence in agreement with the general form of the central limit theorem,

$$W = W(\omega) := (\tilde{\nu}(\omega), \tilde{i}(\omega), (\widetilde{L}(\omega), \widetilde{R}(\omega)), \tilde{\beta}(\omega), \tilde{\lambda}(\omega), \tilde{\Lambda}(\omega), \tilde{M}(\omega), \tilde{u}(\omega))$$

for every ω in Ω, with:

- The same $\tilde{\beta} = (\tilde{\beta}_{j,n} : \ j = 1, \dots, n)_{n\geq 1}$ as in Subsection 2.1, that can now be expressed through the following recursive relation

$$\begin{aligned}
&\tilde{\beta}_{1,1} = 1 \\
&(\tilde{\beta}_{1,n+1}, \dots, \tilde{\beta}_{n+1,n+1}) = (\tilde{\beta}_{1,n}, \dots, \tilde{\beta}_{\tilde{i}_n - 1,n}, \tilde{\beta}_{\tilde{i}_n,n}\widetilde{L}_n, \tilde{\beta}_{\tilde{i}_n,n}\widetilde{R}_n, \\
&\qquad\qquad \tilde{\beta}_{\tilde{i}_n+1,n}, \dots, \tilde{\beta}_{n,n}) \quad (n \geq 1).
\end{aligned}$$

- $\tilde{\lambda} = \tilde{\lambda}(\omega) := (\tilde{\lambda}_1(\omega), \dots, \tilde{\lambda}_{\tilde{\nu}(\omega)}(\omega), \delta_0, \delta_0, \dots)$ where, for each j in $\{1, \dots, \tilde{\nu}(\omega)\}$, $\tilde{\lambda}_j(\omega)$ is the p.d. determined by the characteristic function $\xi \mapsto \varphi_0(\tilde{\beta}_{j,\tilde{\nu}(\omega)}(\omega)\xi)$, $\xi \in \mathbb{R}$.
- $\tilde{\Lambda}$ =convolution of the elements of $\tilde{\lambda}$.
- $\tilde{M}(\omega)$ is the p.d. of $\sum_{j=1}^{\tilde{\nu}(\omega)} |\tilde{\beta}_{j,\tilde{\nu}(\omega)}|^{\alpha}$, where $\tilde{\beta}_{j,\tilde{\nu}(\omega)}$ is the same as $\tilde{\beta}_{j,n}$ with $n = \tilde{\nu}(\omega)$.
- $\tilde{u} := (\tilde{u}_k)_{k\geq 1}$, with $\tilde{u}_k = \max_{1\leq j\leq \tilde{\nu}} \tilde{\lambda}_j([-\frac{1}{k}, \frac{1}{k}]^c)$ for every $k \geq 1$.

Introducing the symbol $\mathcal{P}(M)$ to denote the set of all p.m.'s on the Borel class $\mathcal{B}(M)$ of a metric space M, one can say that the range of W is a subset of

$$S := \overline{\mathbb{N}} \times \overline{\mathbb{N}}^{\infty} \times (\overline{\mathbb{R}}^2)^{\infty} \times \overline{\mathbb{R}}^{\infty} \times (\mathcal{P}(\overline{\mathbb{R}}))^{\infty} \times \mathcal{P}(\overline{\mathbb{R}}) \times \mathcal{P}(\overline{\mathbb{R}}) \times [0,1]^{\infty}.$$

Here, $\mathcal{P}(\overline{\mathbb{R}})$ is metrized consistently with the topology of weak convergence of p.m.'s so that it can be seen as a separable, compact and complete metric space. Thus, S can be metrized so that it results in a separable, compact and complete metric space (Theorems 6.2, 6.4 and 6.5 in Chapter 2 of [38]). Obviously, the family of p.m.'s $\{\mathcal{P}_t W^{-1} : t \geq 0\}$ is *uniformly tight* on $\mathcal{B}(S)$, and any subsequence from this family contains a weakly convergent subsequence $Q_n := \mathcal{P}_{t_n} W^{-1}$ with $0 \leq t_1 < t_2 < \dots$ and $t_n \nearrow +\infty$. Hence, the Skorokhod's representation theorem can be applied to state the existence of a probability space $\left(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathcal{P}}\right)$ and of a sequence of S-valued random elements

$$\widehat{W}_n := (\hat{\nu}^{(n)}, \hat{\imath}^{(n)}, (\hat{L}^{(n)}, \hat{R}^{(n)}), \hat{\beta}^{(n)}, \hat{\lambda}^{(n)}, \widehat{\Lambda}^{(n)}, \hat{M}^{(n)}, \hat{u}^{(n)}), \qquad n \geq 1$$

defined on $\widehat{\Omega}$ so that:

- The p.d. of $\widehat{W}_n$ is Q_n, for every n.
- $\widehat{W}_n$ converges pointwise to a random element $\widehat{W}$ whose p.d. is the weak limit of $(Q_n)_{n\geq 1}$.

From the first point, the equalities

$$\begin{aligned} &\hat{\beta}_{1,1}^{(n)} = 1 \\ &(\hat{\beta}_{1,k+1}^{(n)}, \dots, \hat{\beta}_{k+1,k+1}^{(n)}) = (\hat{\beta}_{1,k}^{(n)}, \dots, \hat{\beta}_{\hat{\imath}_k^{(n)}-1,k}^{(n)}, \hat{L}_k^{(n)} \hat{\beta}_{\hat{\imath}_k^{(n)},k}, \\ &\qquad\qquad \hat{R}_k^{(n)} \hat{\beta}_{\hat{\imath}_k^{(n)},k}^{(n)}, \hat{\beta}_{\hat{\imath}_k^{(n)}+1,k}^{(n)}, \dots, \hat{\beta}_{k,k}^{(n)}) \end{aligned} \tag{15}$$

are met for every k and n, with $\widehat{\mathcal{P}}$-probability 1. This paves the way for two preparatory lemmata in which $\mathcal{M}([1, +\infty))$ represents the set of all the finite measures on $\mathcal{B}([1, +\infty))$.

Lemma 4.1. *If $\nu \colon \widehat{\Omega} \to \mathcal{M}([1, +\infty))$ is a random finite measure, then there exists a countable subset $\mathcal{I}$ of $[1, +\infty)$ such that, for every x_0 in $\mathcal{I}^c \cap (1, +\infty)$, $\nu\{x_0\}(\hat{\omega}) = 0$ for every $\hat{\omega}$ in a subset $\widehat{\Omega}_{x_0}$ of $\widehat{\Omega}$ with $\widehat{\mathcal{P}}(\widehat{\Omega}_{x_0}) = 1$.*

Proof. See Section A.1 in Appendix A. □

Lemma 4.2. *If α agrees with (4)-(2), then for every $\varepsilon > 0$ and every strictly increasing and divergent sequence $(y_n)_{n\geq 1}$ such that*

$$y_1^\alpha > \frac{1}{\varepsilon} \qquad \text{and} \qquad y_{n+1}^\alpha > \frac{1}{\varepsilon}\sum_{j=1}^{n}(y_j^\alpha + 1) \qquad (n \geq 1), \tag{16}$$

there exist:

- *a point $\hat{\omega}_0$ in $\widehat{\Omega}$,*
- *an integer-valued, strictly increasing and divergent sequence $(N_n)_{n\geq 0}$, with $N_0 := 1$,*
- *a sequence of sets $(\mathcal{R}_n)_{n\geq 1}$ with $\mathcal{R}_n \subset \{1,\ldots,N_n\}$ and $|\mathcal{R}_n| = N_{n-1}$ for every $n \geq 1$,*
- *an array $(\delta_k^{(n)})_{n\geq 1, k=1,\ldots,n}$ of positive real numbers*

for which $\hat{\nu}^{(n)}(\hat{\omega}_0) = N_n$ and

$$\begin{aligned} |\hat{\beta}^{(n)}_{j,\hat{\nu}^{(k)}(\hat{\omega}_0)}(\hat{\omega}_0)| &\in \Big[\frac{1}{y_k + \delta_k^{(n)}}, \frac{1}{y_k}\Big] \quad \text{if } k \text{ is odd} \\ |\hat{\beta}^{(n)}_{j,\hat{\nu}^{(k)}(\hat{\omega}_0)}(\hat{\omega}_0)| &\in \Big[\frac{1}{y_k}, \frac{1}{y_k - \delta_k^{(n)}}\Big] \quad \text{if } k \text{ is even} \end{aligned} \tag{17}$$

for every $n \geq 1$, $k = 1,\ldots,n$ and for every $j \notin \mathcal{R}_k$. Moreover, for every $n \geq 1$ and every odd number k in $\{1,\ldots,n\}$,

$$\sum_{j\in\mathcal{R}_k} |\hat{\beta}^{(n)}_{j,\hat{\nu}^{(k)}(\hat{\omega}_0)}(\hat{\omega}_0)|^\alpha < \varepsilon. \tag{18}$$

Proof. See Section A.2 in Appendix A. □

4.1. ***Proof of*** Theorem 2.2

With reference to the Skorokhod representation, assuming that μ_t converges weakly as $t \to +\infty$ is equivalent to saying that the p.d. $\widehat{\Lambda}^{(n)}(\hat{\omega})$ converges weakly to a p.d., as $n \to +\infty$, for every $\hat{\omega}$ in $\widehat{\Omega}$. Then, by the central limit theorem (see, for example, (16.36) in [27]), there exists a random Lévy measure $\nu = \nu(\hat{\omega})$ such that

$$\nu(\hat{\omega})[x, +\infty) = \lim_{n\to+\infty} \sum_{j=1}^{\hat{\nu}^{(n)}(\hat{\omega})} \hat{\lambda}_j^{(n)}(\hat{\omega})[x, +\infty)$$

and

$$\nu(\hat{\omega})(-\infty, -x] = \lim_{n\to+\infty} \sum_{j=1}^{\hat{\nu}^{(n)}(\hat{\omega})} \hat{\lambda}_j^{(n)}(\hat{\omega})(-\infty, -x]$$

hold for every $\hat{\omega}$ in $\widehat{\Omega}$ and for every $x > 0$ such that $\nu(\hat{\omega})\{x\} = \nu(\hat{\omega})\{-x\} = 0$. Now, in view of the definitions given at the beginning of this section,

$$\begin{aligned}
&\hat{\lambda}_j^{(n)}(\hat{\omega})[x, +\infty) \\
&\qquad = \Big[1 - F_0\Big(\frac{x}{|\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega})|}\Big) + \mu_0\Big\{\frac{x}{|\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega})|}\Big\}\Big]\mathbb{I}_{\{\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega})>0\}} \\
&\qquad + F_0\Big(-\frac{x}{|\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega})|}\Big)\mathbb{I}_{\{\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega})<0\}}
\end{aligned}$$

and, analogously,

$$\begin{aligned}
&\hat{\lambda}_j^{(n)}(\hat{\omega})(-\infty, -x] \\
&\qquad = F_0\Big(-\frac{x}{|\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega})|}\Big)\mathbb{I}_{\{\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega})>0\}} \\
&\qquad + \Big[1 - F_0\Big(\frac{x}{|\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega})|}\Big) + \mu_0\Big\{\frac{x}{|\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega})|}\Big\}\Big]\mathbb{I}_{\{\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega})<0\}}.
\end{aligned}$$

Hence, recalling the definition of F_0^*,

$$\begin{aligned}
&\hat{\lambda}_j^{(n)}(\hat{\omega})[x, +\infty) + \hat{\lambda}_j^{(n)}(\hat{\omega})(-\infty, -x] \\
&\qquad = 2\Big[1 - F_0^*\Big(\frac{x}{|\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega})|}\Big)\Big] + \mu_0\Big\{\frac{x}{|\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega})|}\Big\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\nu(\hat{\omega})(-\infty, -x] + \nu(\hat{\omega})[x, +\infty) \\
&= \lim_{n\to+\infty}\Big(2\sum_{j=1}^{\hat{\nu}^{(n)}(\hat{\omega})}\Big[1 - F_0^*\Big(\frac{x}{|\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega})|}\Big)\Big] + \sum_{j=1}^{\hat{\nu}^{(n)}(\hat{\omega})}\mu_0\Big\{\frac{x}{|\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega})|}\Big\}\Big)
\end{aligned}$$

is valid for every $\hat{\omega}$ in $\widehat{\Omega}$ and for every $x > 0$ such that $\nu(\hat{\omega})\{x\} = \nu(\hat{\omega})\{-x\} = 0$. At this stage, for every $x > 1$, one defines the random measure $\bar{\nu}\colon \widehat{\Omega} \to \mathcal{M}([1, +\infty))$ by

$$\bar{\nu}(\hat{\omega})[x, +\infty) := \nu(\hat{\omega})(-\infty, -x] + \nu(\hat{\omega})[x, +\infty).$$

This way, one can apply Lemma 4.1 to state the existence of a countable subset $\mathcal{I}$ of $[1, +\infty)$ such that, for every x_0 in $\mathcal{I}^c \cap (1, +\infty)$, there exists a subset $\widehat{\Omega}_{x_0}$ of $\widehat{\Omega}$ with $\widehat{\mathcal{P}}(\widehat{\Omega}_{x_0}) = 1$, such that $\bar{\nu}(\hat{\omega})\{x_0\} = 0$ for every $\hat{\omega}$ in $\widehat{\Omega}_{x_0}$. Now, keeping x_0 fixed, notice that, without real loss of generality (because of (15) combined with the independence of $(\hat{L}_1^{(n)}, \hat{R}_1^{(n)}), (\hat{L}_2^{(n)}, \hat{R}_2^{(n)}), \dots$ and the continuity assumption on the marginals of τ) one can suppose $\widehat{\Omega}_{x_0}$ is contained in $\{\hat{\omega} \in \widehat{\Omega} : \ x \cdot |\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega})|^{-1} \notin D_{F_0},\ \forall j = 1, \dots, \hat{\nu}^{(n)}(\hat{\omega}),\ n \geq 1\}$ where D_f denotes the discontinuity set of the function f. Hence,

$$\bar{\nu}(\hat{\omega})[x_0, +\infty) = \lim_{n\to+\infty} 2 \sum_{j=1}^{\hat{\nu}^{(n)}(\hat{\omega})} \Big[1 - F_0^*\Big(\frac{x_0}{|\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega})|}\Big)\Big] \tag{19}$$

for every $\hat{\omega} \in \widehat{\Omega}_{x_0}$. Going on, one defines $I := \liminf_{x\to+\infty} x^\alpha(1 - F_0^*(x))$ and $S := \limsup_{x\to+\infty} x^\alpha(1 - F_0^*(x))$. It has to be proved that $I = S < +\infty$. Let $(i_m)_{m\geq 1}$ and $(s_m)_{m\geq 1}$ be increasing and divergent sequences of positive real numbers such that

$$\lim_{m\to+\infty} i_m^\alpha(1 - F_0^*(i_m)) = I \quad \text{and} \lim_{m\to+\infty} s_m^\alpha(1 - F_0^*(s_m)) = S.$$

Given any $\varepsilon > 0$, one defines

$$\begin{aligned} x_n \quad &:= i_m \qquad \text{if } n = 2m-1 \\ &:= s_m \qquad \text{if } n = 2m \end{aligned}$$

in such a way that $x_1^\alpha > 1/\varepsilon$ and $x_{n+1}^\alpha > \sum_{i=1}^n x_i^\alpha/\varepsilon$ for every $n \geq 1$. Then, one puts $z_n := x_n/x_0$ for every $n \geq 1$, x_0 being the same as in (19), for the purpose of bounding

$$\sum_{j=1}^{\hat{\nu}^{(n)}(\hat{\omega})} \Big[1 - F_0^*\Big(\frac{x_0}{|\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega})|}\Big)\Big]$$

for every n at the point $\hat{\omega} = \hat{\omega}_0$ determined through the application of Lemma 4.2 to $(\varepsilon, (z_n)_{n\geq 1})$. Along the subsequence $n = 2m-1$, one gets

$$\begin{aligned}
&\lim_{m\to+\infty} \sum_{j=1}^{\hat{\nu}^{(2m-1)}(\hat{\omega}_0)} \Big[1 - F_0^*\Big(\frac{x_0}{|\hat{\beta}^{(2m-1)}_{j,\hat{\nu}^{(2m-1)}(\hat{\omega})}(\hat{\omega})|}\Big)\Big] \\
&\quad = \lim_{m\to+\infty} \Big(\sum_{j\notin\mathcal{R}_{2m-1}} + \sum_{j\in\mathcal{R}_{2m-1}}\Big)\Big[1 - F_0^*\Big(\frac{x_0}{|\hat{\beta}^{(2m-1)}_{j,N_{2m-1}}(\hat{\omega})|}\Big)\Big] \\
&\quad \leq \limsup_{m\to+\infty} \Big(\sum_{j\notin\mathcal{R}_{2m-1}} [1 - F_0^*(x_0 z_{2m-1})] + \sum_{j\in\mathcal{R}_{2m-1}} \Big[1 - F_0^*\Big(\frac{x_0}{|\hat{\beta}^{(2m-1)}_{j,N_{2m-1}}(\hat{\omega})|}\Big)\Big] \\
&\qquad \times \frac{x_0^\alpha}{|\hat{\beta}^{(2m-1)}_{j,N_{2m-1}}(\hat{\omega})|^\alpha} \cdot \frac{|\hat{\beta}^{(2m-1)}_{j,N_{2m-1}}(\hat{\omega})|^\alpha}{x_0^\alpha}\Big) \qquad \text{(in view of (17))} \\
&\quad \leq \limsup_{m\to+\infty} \Big([1 - F_0^*(x_{2m-1})] \cdot |\{1, \dots, N_{2m-1}\} \setminus \mathcal{R}_{2m-1}|\Big) \\
&\qquad + \frac{S}{x_0^\alpha} \limsup_{m\to+\infty} \sum_{j\in\mathcal{R}_{2m-1}} |\hat{\beta}^{(2m-1)}_{j,N_{2m-1}}(\hat{\omega})|^\alpha \\
&\quad \leq \limsup_{m\to+\infty} x_{2m-1}^\alpha [1 - F_0^*(x_{2m-1})] \frac{N_{2m-1} - N_{2m-2}}{x_{2m-1}^\alpha} \\
&\qquad\qquad + \varepsilon \frac{S}{x_0^\alpha} \qquad \text{(in view of (18))}
\end{aligned}$$

$$
\begin{aligned}
&= \limsup_{m\to+\infty} i_m^\alpha [1 - F_0^*(i_m)] \frac{N_{2m-1} - N_{2m-2}}{x_0^\alpha z_{2m-1}^\alpha} + \varepsilon \frac{S}{x_0^\alpha} \\
&\le \frac{I + \varepsilon S}{x_0^\alpha}
\end{aligned}
$$

where the last inequality holds since $(N_n - N_{n-1})/z_n^\alpha \le 1$ for every $n \ge 1$, as shown in the proof of Lemma 4.2. Furthermore,

$$
\begin{aligned}
&\lim_{m\to+\infty} \sum_{j=1}^{\hat\nu^{(2m)}(\hat\omega_0)} \Big[1 - F_0^*\Big(\frac{x_0}{|\hat{\beta}^{(2m)}_{j,\hat\nu^{(2m)}(\hat\omega)}(\hat\omega)|}\Big)\Big] \\
&\ge \limsup_{m\to+\infty} \sum_{j\notin\mathcal{R}_{2m}} \Big[1 - F_0^*\Big(\frac{x_0}{|\hat{\beta}^{(2m)}_{j,N_{2m}}(\hat\omega)|}\Big)\Big] \\
&\ge \limsup_{m\to+\infty} \sum_{j\notin\mathcal{R}_{2m}} [1 - F_0^*(x_0 z_{2m})] \qquad \text{(in view of (17))} \\
&= \frac{(N_{2m} - N_{2m-1})}{x_{2m}^\alpha} x_{2m}^\alpha [1 - F_0^*(x_{2m})] \\
&= \frac{(N_{2m} - N_{2m-1})}{x_0^\alpha z_{2m}^\alpha} s_m^\alpha [1 - F_0^*(s_m)] \\
&\ge \frac{(1-\varepsilon)S}{x_0^\alpha}
\end{aligned}
$$

where the last inequality holds since, as shown in the proof of Lemma 4.2, $(N_n - N_{n-1})/z_n^\alpha > 1 - \varepsilon$ for every $n \ge 1$. Now, as the limit in (19) exists and is finite,

$$
\frac{(1-\varepsilon)S}{x_0^\alpha} \le \frac{I + \varepsilon S}{x_0^\alpha} < +\infty
$$

obtains, implying that both I and S are finite and $(1-\varepsilon)S \le I + \varepsilon S$ for every $\varepsilon > 0$, that is $I = S$.

4.2. ***Proof of*** Theorem 2.3

It suffices to prove the theorem for $\alpha = 1$, since all the other cases are covered by Theorems 1 and 3 in [6]. Assuming that (10) is in force with $\alpha = 1$ and $c_1 = c_2$, Theorem 2.6.5 in [31] can be invoked to write

$$
\varphi_0(\xi) = \exp\{i\chi\xi - k_1|\xi|(1 + \psi(\xi))\} \qquad (\xi \in \mathbb{R})
$$

where $\psi(\xi) = o(1)$ as $|\xi| \to 0$. It is enough to show that $\varphi_n(\xi) := \mathbb{E}[\exp\{i\xi \times \sum_{j=1}^n \tilde\beta_{j,n}\}]$ converges pointwise, as $n \to +\infty$, to the desired characteristic function. One starts by noting that, from Lemma 3 in [6], given any subsequence (n') of (n), there exists a subsequence (n'') of (n') such that $\max_{j=1,\dots,n''} \tilde\beta_{j,n''} \to$

0 almost surely. Moreover,

$$\varphi_{n''}(\xi) = \mathbb{E}\Big(\prod_{j=1}^{n''} \varphi_0(\xi\tilde{\beta}_{j,n''})\Big)$$
$$= \mathbb{E}\Big(\exp\Big\{i\chi\xi\sum_{j=1}^{n''}\tilde{\beta}_{j,n''} - k_1|\xi|\sum_{j=1}^{n''}\tilde{\beta}_{j,n''} - k_1|\xi|\sum_{j=1}^{n''}\tilde{\beta}_{j,n''}\psi(\xi\tilde{\beta}_{j,n''})\Big\}\Big).$$

Since $\psi(\xi) = o(1)$ as $|\xi| \to 0$, for every $\varepsilon > 0$ there is a positive δ such that $|\psi(\xi)| < \varepsilon$ whenever $|\xi| < \delta$. Now, for every ξ in $\mathbb{R}$, in view of the aforesaid property of the maximum of the $\tilde{\beta}$'s, one can determine the smallest integer $\bar{n} = \bar{n}(\xi,\omega)$ such that $|\xi|\tilde{\beta}_{j,n''}(\omega) < \delta$ holds for every $n'' \geq \bar{n}$ and $j = 1,\dots,n''$, with the exception of a set of points ω of $\mathcal{P}$-probability 0. For such n'' and j,

$$\Big| - k_1|\xi|\sum_{j=1}^{n''}\tilde{\beta}_{j,n''}\psi(\xi\tilde{\beta}_{j,n''})\Big| \leq k_1|\xi|\varepsilon\sum_{j=1}^{n''}\tilde{\beta}_{j,n''}$$

and, then,

$$\lim_{n''\to+\infty}\Big| - k_1|\xi|\sum_{j=1}^{n''}\tilde{\beta}_{j,n''}\psi(\xi\tilde{\beta}_{j,n''})\Big| = 0$$

holds with $\mathcal{P}$-probability 1. Finally, by the monotone convergence theorem, $\lim_{n''\to+\infty}\varphi_{n''}(\xi) = \mathbb{E}(\exp\{i\chi\xi M_\infty^{(1)} - k_1|\xi|M_\infty^{(1)}\})$ completing the proof since the limit is independent of (n').

4.3. ***Proof of*** Theorem 2.4

As for sufficiency, one can refer to Theorem 2.3. As for necessity, arguing as at the beginning of Section 4.1, one has

$$\nu(\hat{\omega})[x_0,+\infty) = \lim_{n\to+\infty}\sum_{j=1}^{\hat{\nu}^{(n)}(\hat{\omega})}\Big[1 - F_0\Big(\frac{x_0}{\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega})}\Big)\Big]$$

where $x_0 \in \mathcal{I}^c \cap (1,+\infty)$ and $\hat{\omega} \in \widehat{\Omega}_{x_0}$, $\mathcal{I}$ be the countable set specified by the application of Lemma 4.1 to the restriction of ν to $[1,+\infty)$. Now, letting

$$I^+ := \liminf_{x\to+\infty} x^\alpha(1 - F_0(x)) \qquad \text{and} \qquad S^+ := \limsup_{x\to+\infty} x^\alpha(1 - F_0(x))$$

and arguing as in the proof of Theorem 2.2, one concludes that $I^+ = S^+ < +\infty$. Analogously, one has

$$\nu(\hat{\omega})(-\infty,-x_0'] = \lim_{n\to+\infty}\sum_{j=1}^{\hat{\nu}^{(n)}(\hat{\omega})} F_0\Big(-\frac{x_0'}{\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega})}\Big)$$

where $x_0' \in \mathcal{I}^c \cap (1, +\infty)$ and $\hat{\omega} \in \widehat{\Omega}_{x_0'}$, $\mathcal{I}$ be the countable set obtained by the application of Lemma 4.1 to the measure $\bar{\nu}_1$ on $\mathcal{B}([1, +\infty))$ such that $\bar{\nu}_1[x, +\infty) := \nu(-\infty, -x]$ for every $x > 1$. Then, resorting once again to the argument developed in the proof of Theorem 2.2,

$$\liminf_{x \to +\infty} x^\alpha F_0(-x) = \limsup_{x \to +\infty} x^\alpha F_0(-x) < +\infty$$

obtains. Thus, weak convergence of μ_t implies (10). At this stage, the rest of the argument is splitted into four points, depending on the value assumed by α.

If α belongs to $(0, 1)$, no further consideration is needed.

Passing to the case of $\alpha = 1$, one has to prove that $c_1 = c_2$ under the assumption that (10) is in force. Resorting to (16.38) in [27] and to the Skorokhod representation (in fact, μ_t converges weakly),

$$\lim_{n \to +\infty} \sum_{j=1}^{\hat{\nu}^{(n)}(\hat{\omega})} \int_{\mathbb{R}} \Big(-\mathbb{I}_{(-\infty,-1]}(x) + x\mathbb{I}_{(-1,1]}(x) + \mathbb{I}_{(1,+\infty)}(x) \Big) dF_0\Big(\frac{x}{\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega})} \Big)$$

exists and is finite for every $\hat{\omega}$ in $\widehat{\Omega}$. Denoting it by $\eta(\hat{\omega})$, by the change of variable $y = x/\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega})$ one gets

$$\begin{aligned}
\eta(\hat{\omega}) = \lim_{n \to +\infty} \sum_{j=1}^{\hat{\nu}^{(n)}(\hat{\omega})} \Big[1 - F_0\Big(\frac{1}{\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega})} \Big) - F_0\Big(- \frac{1}{\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega})} \Big) \\
+ \hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega}) \int_{-1/\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega})}^{1/\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega})} y dF_0(dy) \Big] \\
= \lim_{n \to +\infty} \sum_{j=1}^{\hat{\nu}^{(n)}(\hat{\omega})} \hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega}) \int_0^{1/\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega})} (1 - F_0(x) - F_0(-x)) dx
\end{aligned}$$

where the last equality follows from integration by parts. One proceeds to prove that

$$\lim_{R \to +\infty} \int_{(0,R)} (1 - F_0(x) - F_0(-x)) dx = +\infty \tag{20}$$

implies that μ_t does not converge. Indeed, assuming that (20) is in force, for every positive M there exists $\overline{R}$ such that $\int_{(0,R)} (1 - F_0(x) - F_0(-x)) dx \geq M$ for every $R \geq \overline{R}$. Moreover, since $\max_{j=1,\dots,\hat{\nu}^{(n)}(\hat{\omega})} \hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega}) \to 0$, as $n \to +\infty$, there exists $\bar{n} = \bar{n}(\hat{\omega}, \overline{R})$ such that $1/\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega}) \geq \overline{R}$ for every $n \geq \bar{n}$ and for every $j = 1, \dots, \hat{\nu}^{(n)}(\hat{\omega})$. Thus, putting $\widehat{M}^{(1)}_\infty(\hat{\omega}) = \lim_{n \to +\infty}$

$\sum_{j=1}^{\hat{\nu}^{(n)}(\hat{\omega})} \hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega})$, one has $\eta(\hat{\omega}) \geq M \cdot \widehat{M}^{(1)}_{\infty}(\hat{\omega})$, a contradiction since M is arbitrary and $\eta(\hat{\omega})$ is finite. Analogously, one proves that

$$\lim_{R\to+\infty} \int_{(0,R)} (1 - F_0(x) - F_0(-x))dx = -\infty \tag{21}$$

implies that μ_t does not converge. Finally, one proves that, if $c_1 \neq c_2$, then either (20) or (21) occurs. Indeed, if $c_1 < c_2$, taking ε in $(0, (c_2 - c_1)/2)$, since (10) is in force, there is $\bar{x} > 0$ such that

$$\frac{c_2 - c_1 - 2\varepsilon}{x} \leq 1 - F_0(x) - F_0(-x) \leq \frac{c_2 - c_1 + 2\varepsilon}{x}$$

for every $x \geq \bar{x}$, and

$$\begin{aligned}
&\int_{(0,R)} (1 - F_0(x) - F_0(-x))dx \\
&\quad = \Big(\int_{(0,\bar{x})} + \int_{(\bar{x},R)} \Big)(1 - F_0(x) - F_0(-x))dx \\
&\quad \geq \int_{(0,\bar{x})} (1 - F_0(x) - F_0(-x))dx + \int_{(\bar{x},R)} \frac{c_2 - c_1 - 2\varepsilon}{x} dx \\
&\quad = \int_{(0,\bar{x})} (1 - F_0(x) - F_0(-x))dx + (c_2 - c_1 - 2\varepsilon) \log \frac{R}{\bar{x}}
\end{aligned}$$

which goes to $+\infty$ as $R \to +\infty$. Analogously, one proves that $c_1 > c_2$ entails (21). Combination of these facts with the inconsistency between (20)-(21) and weak convergence of μ_t entails $c_1 = c_2$.

Now, the case of α in $(1, 2)$ is taken into consideration. Condition (10) implies that $m_{0,1} := \int_{\mathbb{R}} x\mu_0(dx)$ is finite. The former summand in the RHS of

$$V_t = \sum_{j=1}^{\tilde{\nu}_t} (X_j - m_{0,1})\tilde{\beta}_{j,\tilde{\nu}_t} + m_{0,1} \sum_{j=1}^{\tilde{\nu}_t} \tilde{\beta}_{j,\tilde{\nu}_t}$$

converges in distribution, as $t \to +\infty$, in view of Theorem 2.3. As to the latter, one notes that

$$\sum_{j=1}^{n} \tilde{\beta}_{j,n} = \sum_{j=1}^{n} \tilde{\beta}^{\alpha}_{j,n} \frac{1}{\tilde{\beta}^{\alpha-1}_{j,n}} \geq \frac{1}{(\max_{j=1,\ldots,n} \tilde{\beta}_{j,n})^{\alpha-1}} \sum_{j=1}^{n} \tilde{\beta}^{\alpha}_{j,n}$$

goes in probability to $+\infty$ as $n \to +\infty$, since $\sum_{j=1}^{n} \tilde{\beta}^{\alpha}_{j,n}$ converges almost surely to the random variable $M^{(1)}_{\infty}$, satisfying $\mathcal{P}\{M^{(1)}_{\infty} > 0\} > 0$, and $\max_{j=1,\ldots,n} \tilde{\beta}_{j,n}$ converges in probability to zero. Then, $m_{0,1} = 0$.

Finally, if $\alpha = 2$, in view of the previous argument, (10) holds and $m_{0,1} = 0$. From the Skorokhod representation combined with (16.37) in [27], there exists $\sigma^2 \colon \widehat{\Omega} \to \mathbb{R}^+$ such that, for every $\hat{\omega}$ in $\widehat{\Omega}$,

$$
\begin{aligned}
\sigma^2(\hat{\omega}) = \lim_{\varepsilon \searrow 0} \limsup_{n \to +\infty} \sum_{j=1}^{\hat{\nu}^{(n)}(\hat{\omega})} \Big[\int_{[-\varepsilon,\varepsilon]} x^2 dF_0\Big(\frac{x}{\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega})} \Big) \\
- \Big(\int_{[-\varepsilon,\varepsilon]} x dF_0\Big(\frac{x}{\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega})} \Big) \Big)^2 \Big] \\
= \lim_{\varepsilon \searrow 0} \liminf_{n \to +\infty} \sum_{j=1}^{\hat{\nu}^{(n)}(\hat{\omega})} \Big[\int_{[-\varepsilon,\varepsilon]} x^2 dF_0\Big(\frac{x}{\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega})} \Big) \\
- \Big(\int_{[-\varepsilon,\varepsilon]} x dF_0\Big(\frac{x}{\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega})} \Big) \Big)^2 \Big].
\end{aligned}
\tag{22}
$$

With the change of variable $y = x/\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega})$,

$$
\begin{aligned}
\liminf_{n \to +\infty} \sum_{j=1}^{\hat{\nu}^{(n)}(\hat{\omega})} \Big[\int_{[-\varepsilon,\varepsilon]} x^2 dF_0\Big(\frac{x}{\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega})} \Big) \\
- \Big(\int_{[-\varepsilon,\varepsilon]} x dF_0\Big(\frac{x}{\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega})} \Big) \Big)^2 \Big]
\end{aligned}
$$

$$
\begin{aligned}
&= \liminf_{n \to +\infty} \sum_{j=1}^{\hat{\nu}^{(n)}(\hat{\omega})} \Big(\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega}) \Big)^2 \Big[\int_{[-\varepsilon/\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega}), \varepsilon/\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega})]} y^2 dF_0(y) \\
&\quad - \Big(\int_{[-\varepsilon/\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega}), \varepsilon/\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega})]} y dF_0(y) \Big)^2 \Big] \\
&\geq \liminf_{n \to +\infty} \sum_{j=1}^{\hat{\nu}^{(n)}(\hat{\omega})} \Big(\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega}) \Big)^2 \int_{[-\varepsilon/\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega}), \varepsilon/\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega})]} y^2 dF_0(y) \\
&\quad - \limsup_{n \to +\infty} \sum_{j=1}^{\hat{\nu}^{(n)}(\hat{\omega})} \Big(\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega}) \Big)^2 \Big(\int_{[-\varepsilon/\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega}), \varepsilon/\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega})]} y dF_0(y) \Big)^2.
\end{aligned}
$$

As for the latter summand of the RHS, since $m_{0,1} = 0$, for every $\delta > 0$ there exists an $\bar{R}$ such that $\Big| \int_{[-R,R]} x dF_0(x) \Big| < \delta$ whenever $R \geq \bar{R}$. Let $\bar{n} = \bar{n}(\hat{\omega}, \bar{R})$ be a strictly positive integer such that $\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega}) \leq \varepsilon/\bar{R}$ for every $n \geq \bar{n}$ and for every $j = 1, \dots, \hat{\nu}^{(n)}(\hat{\omega})$. For such j's and n's one has

$$
-\delta < \int_{[-\varepsilon/\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega}), \varepsilon/\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega})]} y dF_0(y) < \delta
$$

and

$$\sum_{j=1}^{\hat{\nu}^{(n)}(\hat{\omega})} \left(\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega})\right)^2 \Big(\int_{[-\varepsilon/\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega}),\varepsilon/\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega})]} y dF_0(y)\Big)^2 \\ \leq \delta^2 \sum_{j=1}^{\hat{\nu}^{(n)}(\hat{\omega})} \left(\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega})\right)^2.$$

Thus, as the last inequality holds for every $\delta > 0$ and $\sum_{j=1}^{\hat{\nu}^{(n)}(\hat{\omega})} \left(\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega})\right)^2$ converges, as $n \to +\infty$, to a positive $\widehat{M}^{(2)}_\infty(\hat{\omega})$,

$$\limsup_{n\to+\infty} \sum_{j=1}^{\hat{\nu}^{(n)}(\hat{\omega})} \left(\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega})\right)^2 \Big(\int_{[-\varepsilon/\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega}),\varepsilon/\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega})]} y dF_0(y)\Big)^2 = 0.$$

Finally, one proves that $m_{0,2} := \int_{\mathbb{R}} x^2 dF_0(x)$ is finite. In fact, if $m_{0,2} = +\infty$, for every $M > 0$ there is $\bar{R} > 0$ such that $\int_{[-R,R]} x^2 dF_0(x) \geq M$ holds for every $R \geq \bar{R}$. Then, an application of the same argument as in the previous step gives

$$\liminf_{n\to+\infty} \sum_{j=1}^{\hat{\nu}^{(n)}(\hat{\omega})} \left(\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega})\right)^2 \int_{[-\varepsilon/\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega}),\varepsilon/\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega})]} y^2 dF_0(y) \\ \geq M \cdot \widehat{M}^{(2)}_\infty(\hat{\omega})$$

which turns out to be an apparent contradiction since M is arbitrary and $\sigma^2(\hat{\omega})$ is finite.

4.4. ***Proof of*** Theorem 2.1

The argument to prove Theorem 2.2 can be plainly extended to the case of $\alpha > 2$ to state that $\lim_{x\to+\infty} x^\alpha(1 - F_0^*(x))$ exists and is finite, which implies $m_{0,2} < +\infty$. An integration by parts followed by the change of variable $y = x/|\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega})|$ transforms the sum in the RHS of (22) into

$$\sum_{j=1}^{\hat{\nu}^{(n)}(\hat{\omega})} \left|\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega})\right|^2 \Big[\int_{[-\varepsilon/|\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega})|,\varepsilon/|\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega})|]} y^2 dF_0(y) \\ - \Big(\int_{[-\varepsilon/|\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega})|,\varepsilon/|\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega})|]} y dF_0(y)\Big)^2\Big].$$

For every $\delta > 0$ there is $\bar{R} > 0$ such that $m_{0,i} - \delta < \int_{[-R,R]} x^i dF_0(x) < m_{0,i} + \delta$ holds for every $R \geq \bar{R}$ and $i = 1, 2$. Moreover, let $\bar{n} = \bar{n}(\hat{\omega}, \bar{R})$ be a strictly positive integer such that $\varepsilon/|\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega})| > \bar{R}$ for every $n \geq \bar{n}$ and $j = 1, \dots, \hat{\nu}^{(n)}(\hat{\omega})$. Then,

$$\sum_{j=1}^{\hat{\nu}^{(n)}(\hat{\omega})} \left|\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega})\right|^2 \Big[\int_{[-\varepsilon/|\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega})|,\varepsilon/|\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega})|]} y^2 dF_0(y)$$
$$- \Big(\int_{[-\varepsilon/|\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega})|,\varepsilon/|\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega})|]} y dF_0(y)\Big)^2 \Big]$$
$$\geq \sum_{j=1}^{\hat{\nu}^{(n)}(\hat{\omega})} \left|\hat{\beta}^{(n)}_{j,\hat{\nu}^{(n)}(\hat{\omega})}(\hat{\omega})\right|^2 \Big[m_{0,2} - \delta - \Big(m_{0,1} + \delta\Big)^2 \Big].$$

Taking $\liminf_{n\to+\infty}$ in both sides of the above inequality one gets

$$\sigma^2(\hat{\omega}) \geq \widehat{M}^{(2)}_{\infty}(\hat{\omega})(m_{0,2} - m_{0,1}^2).$$

Since $\alpha > 2$ implies that $\widehat{M}^{(2)}_{\infty}(\hat{\omega}) = +\infty$, and $\sigma^2(\hat{\omega})$ is finite, the last inequality holds only if $m_{0,2} - m_{0,1}^2 = 0$, i.e. only if μ_0 is the point mass at some x_0 in $\mathbb{R}$. Conversely, if $\mu_0 = \delta_{x_0}$,

$$\sum_{j=1}^{\tilde{\nu}(\omega)} \left|\tilde{\beta}_{j,\tilde{\nu}(\omega)}(\omega)\right|^2 \Big[\int_{[-\varepsilon/|\tilde{\beta}_{j,\tilde{\nu}(\omega)}(\omega)|,\varepsilon/|\tilde{\beta}_{j,\tilde{\nu}(\omega)}(\omega)|]} y^2 dF_0(y)$$
$$- \Big(\int_{[-\varepsilon/|\tilde{\beta}_{j,\tilde{\nu}(\omega)}(\omega)|,\varepsilon/|\tilde{\beta}_{j,\tilde{\nu}(\omega)}(\omega)|]} y dF_0(y)\Big)^2 \Big]$$
$$= \sum_{j=1}^{\tilde{\nu}(\omega)} \left|\tilde{\beta}_{j,\tilde{\nu}(\omega)}(\omega)\right|^2 x_0^2 \Big[\mathbb{I}_{[-\varepsilon/|\tilde{\beta}_{j,\tilde{\nu}(\omega)}(\omega)|,\varepsilon/|\tilde{\beta}_{j,\tilde{\nu}(\omega)}(\omega)|]}(x_0)$$
$$- \mathbb{I}^2_{[-\varepsilon/|\tilde{\beta}_{j,\tilde{\nu}(\omega)}(\omega)|,\varepsilon/|\tilde{\beta}_{j,\tilde{\nu}(\omega)}(\omega)|]}(x_0) \Big]$$
$$= 0$$

which entails

$$\sigma^2(\omega) = \lim_{\varepsilon\searrow 0} \limsup_{n\to+\infty} \sum_{j=1}^{\tilde{\nu}(\omega)} \Big[\int_{[-\varepsilon,\varepsilon]} x^2 dF_0\Big(\frac{x}{\tilde{\beta}_{j,\tilde{\nu}(\omega)}(\omega)}\Big)$$
$$- \Big(\int_{[-\varepsilon,\varepsilon]} x dF_0\Big(\frac{x}{\tilde{\beta}_{j,\tilde{\nu}(\omega)}(\omega)}\Big)\Big)^2 \Big]$$
$$= \lim_{\varepsilon\searrow 0} \liminf_{n\to+\infty} \sum_{j=1}^{\tilde{\nu}(\omega)} \Big[\int_{[-\varepsilon,\varepsilon]} x^2 dF_0\Big(\frac{x}{\tilde{\beta}_{j,\tilde{\nu}(\omega)}(\omega)}\Big)$$
$$- \Big(\int_{[-\varepsilon,\varepsilon]} x dF_0\Big(\frac{x}{\tilde{\beta}_{j,\tilde{\nu}(\omega)}(\omega)}\Big)\Big)^2 \Big]$$
$$= 0$$

for every ω in Ω. To complete the proof that μ_∞ degenerates at some x_1, one can resort to the central limit theorem (cf., e.g., (16.36) in [27]) according to which one has to check that

$$\nu(I) = \lim_{n\to+\infty} \sum_{j=1}^{n} \delta_{x_o \tilde{\beta}_{j,n}}(I) = 0$$

holds for every $I \in \{(-\infty, a],\ [b, +\infty):\ \ a < 0, b > 0\}$, which can be plainly verified by recalling that $\max_{j=1,\dots,n} \tilde{\beta}_{j,n}$ goes to zero in probability.

It remains to characterize the point x_1 at which μ_∞ degenerates. From $Q^+(\mu_\infty) = \mu_\infty = \delta_{x_1}$ one has

$$e^{i\xi x_1} = \widehat{Q^+}(\delta_{x_1})(\xi) = \mathbb{E}\Big(\hat{\delta}_{x_1}(\xi\widetilde{L}_1)\hat{\delta}_{x_1}(\xi\widetilde{R}_1)\Big) = \mathbb{E}\Big(e^{i\xi(\widetilde{L}_1+\widetilde{R}_1)x_1}\Big)$$

which implies that $\widetilde{L}_1 + \widetilde{R}_1 = 1$ almost surely. Moreover, in this case, V_t turns out to be equal to x_0 with probability one since all the X_j's are degenerate at x_0, and the condition $\mathcal{P}\{\widetilde{L}_1 + \widetilde{R}_1 = 1\} = 1$ implies that $\sum_{j=1}^{\tilde{\nu}_t} \tilde{\beta}_{j,\tilde{\nu}_t} = 1$ almost surely. Then, $x_0 = x_1$. Conversely, this very same argument proves that $\mathcal{P}\{\widetilde{L}_1 + \widetilde{R}_1 = 1\} = 1$ together with $x_0 \neq 0$ imply that $x_1 = x_0$. Finally, in order that the solution μ_t converge weakly to the point mass at zero, it is necessary and sufficient that $V_t = x_0 \sum_{j=1}^{\tilde{\nu}_t} \tilde{\beta}_{j,\tilde{\nu}_t}$ converge in law to zero, which happens when (at least) one of the conditions (ii_1), (ii_2) is verified.

4.5. *Proof of* Theorem 2.5

The necessity is just Theorem 2.2. In order to prove sufficiency, one should recall that the Fourier-Stieltjes transform of the solution of (5) has the *Wild series representation*

$$\varphi(t, \xi) = \sum_{n\geq 1} e^{-t}(1 - e^{-t})^{n-1} \hat{q}_n(\xi, \varphi_0)$$

where $\hat{q}_1(\xi; \varphi) := \varphi_0(\xi)$ and, for every $n \geq 2$,

$$\hat{q}_n(\xi; \varphi) := \frac{1}{n-1} \sum_{j=1}^{n-1} \mathbb{E}[\hat{q}_j(\widetilde{L}_1\xi; \varphi)\hat{q}_{n-j}(\widetilde{R}_1\xi; \varphi)].$$

In fact, if $\Re z$ $[\Im z]$ denotes the real [imaginary] part of a complex number z, one shows that

$$\varphi(t, \xi) = e^{-t} \sum_{n\geq 1} (1 - e^{-t})^{n-1} \hat{q}_n(\xi, \Re\varphi_0) + i\Im\varphi_0(\xi)e^{-t} \tag{23}$$

and, thus, in order to study the limiting behaviour of $\varphi(t, \cdot)$, it suffices to study the Cauchy problem associated with (5) with initial datum given by $\varphi_0^*(\cdot) := \Re\varphi_0(\cdot)$, that is the Fourier-Stieltjes transform of μ_0^*. Assuming (23), in view of the symmetry of φ_0^*, without loss of generality one can think of $\widetilde{L}_1$ and $\widetilde{R}_1$

as positive random variables, so that the present theorem would be complete thanks to Theorem 2.3. In point of fact, it remains to prove (23) that, in turn, is implied by

$$\hat{q}_n(\xi;\varphi_0) = \hat{q}_n(\xi;\Re\varphi_0) \text{ for every } n \geq 2. \tag{24}$$

Proceeding by mathematical induction, one first proves (24) when $n = 2$. Write

$$\begin{aligned}\hat{q}_2(\xi;\varphi) &= \mathbb{E}[\Re\varphi_0(\widetilde{L}_1\xi)\Re\varphi_0(\widetilde{R}_1\xi)] + i\mathbb{E}[\Re\varphi_0(\widetilde{L}_1\xi)\Im\varphi_0(\widetilde{R}_1\xi)] \\ &\quad + i\mathbb{E}[\Im\varphi_0(\widetilde{L}_1\xi)\Re\varphi_0(\widetilde{R}_1\xi)] - \mathbb{E}[\Im\varphi_0(\widetilde{L}_1\xi)\Im\varphi_0(\widetilde{R}_1\xi)] \\ &=: A_1 + iA_2 + iA_3 - A_4.\end{aligned}$$

Now,

$$\begin{aligned}A_2 &= \mathbb{E}\Big[\Re\varphi_0(\widetilde{L}_1\xi)\Im\varphi_0(\widetilde{R}_1\xi)\mathbb{I}_{\{\widetilde{R}_1>0\}}\Big] + \mathbb{E}\Big[\Re\varphi_0(\widetilde{L}_1\xi)\Im\varphi_0(-(-\widetilde{R}_1)\xi)\mathbb{I}_{\{-\widetilde{R}_1>0\}}\Big] \\ &= \mathbb{E}\Big[\Re\varphi_0(\widetilde{L}_1\xi)\Im\varphi_0(\widetilde{R}_1\xi)\mathbb{I}_{\{\widetilde{R}_1>0\}}\Big] - \mathbb{E}\Big[\Re\varphi_0(\widetilde{L}'_1\xi)\Im\varphi_0(\widetilde{R}'_1\xi)\mathbb{I}_{\{\widetilde{R}'_1>0\}}\Big] \\ &\qquad\qquad (\text{where } (\widetilde{L}'_1,\widetilde{R}'_1) := (-\widetilde{L}_1,-\widetilde{R}_1)) \\ &= 0\end{aligned}$$

the last equality being a consequence of the fact that $(\widetilde{L}'_1,\widetilde{R}'_1)$ and $(\widetilde{L}_1,\widetilde{R}_1)$ have the same distribution because of the invariance w.r.t. $(\pi/2)$-rotations. In the same way, one can prove that $A_3 = 0$. As for A_4, recalling the above definitions of $(\widetilde{L}'_1,\widetilde{R}'_1)$ and putting $(\widetilde{L}''_1,\widetilde{R}''_1) := (\widetilde{R}_1,-\widetilde{L}_1)$ – which, in view of invariance w.r.t. $(\pi/2)$-rotation, is distributed like $(\widetilde{L}_1,\widetilde{R}_1)$ – one has

$$\begin{aligned}A_4 &= \mathbb{E}\Big[\Im\varphi_0(\widetilde{L}_1\xi)\Im\varphi_0(\widetilde{R}_1\xi)\mathbb{I}_{\{\widetilde{R}_1>0\}}\Big] + \mathbb{E}\Big[\Im\varphi_0(-\widetilde{L}'_1\xi)\Im\varphi_0(-\widetilde{R}'_1\xi)\mathbb{I}_{\{\widetilde{R}'_1>0\}}\Big] \\ &= 2\mathbb{E}\Big[\Im\varphi_0(\widetilde{L}_1\xi)\Im\varphi_0(\widetilde{R}_1\xi)\mathbb{I}_{\{\widetilde{R}_1>0\}}\Big] \\ &= 2\mathbb{E}\Big[\Im\varphi_0(\widetilde{L}_1\xi)\Im\varphi_0(\widetilde{R}_1\xi)\mathbb{I}_{\{\widetilde{R}_1>0,\widetilde{L}_1>0\}}\Big] \\ &\qquad + 2\mathbb{E}\Big[\Im\varphi_0(-\widetilde{R}''_1\xi)\Im\varphi_0(\widetilde{L}''_1\xi)\mathbb{I}_{\{\widetilde{R}''_1>0,\widetilde{L}''_1>0\}}\Big] \\ &= 0.\end{aligned}$$

Verified that (24) holds true for $n = 2$, one assumes its validity for every $n \leq m-1$ $(m \geq 3)$ and proves it for $n = m$. From the definition of $\hat{q}_n$ in conjunction with the inductive hypothesis,

$$\begin{aligned}\hat{q}_m(\xi,\varphi_0) &= \frac{1}{m-1}\Big(\mathbb{E}\Big[\hat{q}_{m-1}(\widetilde{L}_1\xi,\Re\varphi_0)\varphi_0(\widetilde{R}_1\xi) \\ &\qquad + \sum_{j=2}^{m-2}\hat{q}_{m-j}(\widetilde{L}_1\xi;\Re\varphi)\hat{q}_j(\widetilde{R}_1\xi;\Re\varphi_0) + \varphi_0(\widetilde{L}_1\xi)\hat{q}_{m-1}(\widetilde{R}_1\xi;\Re\varphi_0)\Big]\Big) \\ &= \hat{q}_m(\xi;\Re\varphi_0) + \frac{i}{m-1}\Big(\mathbb{E}\Big[\hat{q}_{m-1}(\widetilde{L}_1\xi;\Re\varphi_0)\Im\varphi_0(\widetilde{R}_1\xi) \\ &\qquad + \Im\varphi_0(\widetilde{L}_1\xi)\hat{q}_{m-1}(\widetilde{R}_1\xi;\Re\varphi_0)\Big]\Big).\end{aligned}$$

For every $k \geq 1$, $\xi \mapsto \hat{q}_k(\xi; \Re\varphi_0)$ is an even function and then, arguing as for A_2 and A_3, one gets $\mathbb{E}\Big[\hat{q}_{m-1}(\widetilde{L}_1\xi; \Re\varphi_0)\Im\varphi_0(\widetilde{R}_1\xi)\Big] = \mathbb{E}\Big[\hat{q}_{m-1}(\widetilde{R}_1\xi; \Re\varphi_0)$ $\Im\varphi_0(\widetilde{L}_1\xi)\Big] = 0$ and hence $\hat{q}_m(\xi; \varphi_0) = \hat{q}_m(\xi; \Re\varphi_0)$. This completes the inductive argument and the proof of (24) and hence (23).

Appendix A: Proofs of the lemmata

This Appendix contains the proofs of Lemmata 4.1 and 4.2 which are crucial for the arguments developed in Section 4.

A.1. Proof of Lemma 4.1

Consider the following p.d.f.'s

$$F_t(x) := \mathbb{P}\{\nu[t, +\infty) \leq x\}, \quad F_{t,k}(x) := \mathbb{P}\Big(\nu[t, +\infty) \leq x \Big| \nu[1, +\infty) \leq k\Big)$$
$$G_t(x) := \mathbb{P}\{\nu(t, +\infty) \leq x\}, \quad G_{t,k}(x) := \mathbb{P}\Big(\nu(t, +\infty) \leq x \Big| \nu[1, +\infty) \leq k\Big)$$

at each x in $\mathbb{R}$, for every $t \geq 1$ and for every k in $\mathbb{N}$. Since $\nu[1, +\infty)$ is almost surely finite,

$$\lim_{k \to +\infty} \mathbb{P}\{\nu[1, +\infty) \leq k\} = 1$$

and then

$$\lim_{k \to +\infty} F_{t,k}(x) = F_t(x) \quad \text{and} \quad \lim_{k \to +\infty} G_{t,k}(x) = G_t(x)$$

for every $t \geq 1$ and x in $\mathbb{R}$. Moreover, $G_{t,k}(x) \geq F_{t,k}(x)$ for every $t \geq 1$, x in $\mathbb{R}$ and k in $\mathbb{N}$. Now, fix k and suppose that there exists an uncountable subset H of $(1, +\infty)$ such that, for every t in H,

$$G_{t,k} \gneqq F_{t,k} \tag{25}$$

which means that for every t in H there exists $x_{t,k} \geq 1$ such that $G_{t,k}(x_{t,k}) > F_{t,k}(x_{t,k})$. The p.d.f.'s $F_{t,k}$ and $G_{t,k}$ are right-continuous and then for every t in H there exists a proper interval $\Delta_{t,k}$ containing $x_{t,k}$ such that

$$G_{t,k}(x) > F_{t,k}(x) \qquad \text{for every } x \text{ in } \Delta_{t,k}.$$

One proves that the intersection of any uncountable family of elements of $(\Delta_{t,k})_{t \in H}$ is empty. Suppose, for the moment, that there is an uncountable subset J of H such that $\bigcap_{t \in J} \Delta_{t,k}$ is non-empty; it will be shown that this leads to a contradiction. If $\bar{x}$ is an element of such an intersection, then $G_{t,k}(\bar{x}) > F_{t,k}(\bar{x})$ for every t in J. Since $G_{t,k} \leq F_{s,k}$ whenever $s > t$, the class $\Big((F_{t,k}(\bar{x}), G_{t,k}(\bar{x})]\Big)_{t \in J}$ consists of pairwise disjoint proper intervals contained in $[0, 1]$, which contradicts, recalling that J is uncountable, the countability of the rationals. Verified that $\bigcap_{t\in J}\Delta_{t,k}=\emptyset$ for every uncountable $J\subset H$, one shows that (25) may be satisfied only on countable sets of t's. In the beginning, one notes that, since

$$\begin{array}{ll} F_{t,k}(x)=G_{t,k}(x)=0 & (x<0) \\ F_{t,k}(x)=G_{t,k}(x)=1 & (x>k) \end{array}$$

for every t in H, all the elements of the family $(\Delta_{t,k})_{t\in H}$ are proper sub-intervals of $[0,k]$ such that $\bigcap_{t\in J}\Delta_{t,k}=\emptyset$ for every uncountable $J\subset H$. This last statement implies that with each rational q in $[0,k]$ one can associate a countable (possibly empty) set $H_q\subset H$ such that $q\in\Delta_{t,k}$ for every t in H_q. Then, the family $\{\Delta_{t,k}:\ t\in\bigcup_{q\in\mathbb{Q}\cap[0,k]}H_q\}$ is countable and, of course, it is included in $(\Delta_{t,k})_{t\in H}$. So, to complete the argument, it is enough to show that these families coincide. In point of fact, if there exists some $t\in H\setminus\bigcup_{q\in\mathbb{Q}\cap[0,k]}H_q$, then $\Delta_{t,k}\cap\mathbb{Q}=\emptyset$, a patent contradiction. Whence, one can say there is a countable subset $\mathcal{I}_k$ of $(1,+\infty)$ such that, for every t in $\mathcal{I}_k^c\cap(1,+\infty)$, $G_{t,k}\equiv F_{t,k}$. Denoting the countable set $\bigcup_{k\geq 1}\mathcal{I}_k$ by $\mathcal{I}$, the identity $G_{t,k}\equiv F_{t,k}$ holds for every t in $\mathcal{I}^c\cap(1,+\infty)$ and for every k. Moreover, for all of these t's,

$$F_t(x)=\lim_{k\to+\infty}F_{t,k}(x)=\lim_{k\to+\infty}G_{t,k}(x)=G_t(x)$$

obtains for every x in $\mathbb{R}$, which is tantamount to stating that $\nu[t,+\infty)$ and $\nu(t,+\infty)$ are equally distributed. Then, since two positive and equally distributed random numbers X and Y such that $X\geq Y$ must coincide almost surely, one concludes that $\nu[t,+\infty)=\nu(t,+\infty)$ almost surely, which amounts to $\nu\{t\}=0$ almost surely.

A.2. Proof of Lemma 4.2

Following the argument used in the proof of Proposition 1 in [11], there is a subset $\widehat{\Omega}'$ of $\widehat{\Omega}$, $\widehat{\mathcal{P}}(\widehat{\Omega}')=1$, such that the recursive relation (15) holds true at each point of $\widehat{\Omega}'$. Then, without altering the distribution of $\widehat{W_n}$, one can use (15) to redefine the $\hat{\beta}^{(n)}$'s outside $\widehat{\Omega}'$. Let M be the compact space defined by

$$M:=\overline{\mathbb{N}}^{\infty}\times\Big(\times_{j\geq 1}\overline{\mathbb{N}}_j^{\infty}\Big)\times\Big(\times_{j\geq 1}(\overline{\mathbb{R}}_j^2)^{\infty}\Big)$$

where $\overline{\mathbb{N}}_1,\overline{\mathbb{N}}_2,\dots$ are copies of $\overline{\mathbb{N}}:=\{1,2,\dots,+\infty\}$ and $\overline{\mathbb{R}}_1,\overline{\mathbb{R}}_2,\dots$ are copies of $\overline{\mathbb{R}}$. Introduce the mapping $\hat{Y}$ from $\widehat{\Omega}$ to M

$$\hat{Y}:=\Big((\hat{\nu}^{(n)})_{n\geq 1},(\hat{i}^{(n)})_{n\geq 1},((\hat{L}^{(n)},\hat{R}^{(n)}))_{n\geq 1}\Big)$$

and put

$$\begin{aligned} &f_k(\hat{Y}) \\ &\quad:=\Big((\hat{\nu}^{(1)},\hat{i}^{(1)}),\dots,(\hat{\nu}^{(k)},\hat{i}^{(k)}),(\hat{L}^{(1)},\hat{R}^{(1)}),(\hat{L}^{(2)},\hat{R}^{(2)}),\dots,(\hat{L}^{(k)},\hat{R}^{(k)})\Big)\end{aligned}$$

$k = 1, 2, \ldots$. Recall that the pair $(\hat{\nu}^{(k)}, \hat{\imath}^{(k)})$ is enough to single out a specific McKean tree, say $\hat{a}_k$. The proof aims at the definition of a non-increasing sequence $(A_n)_{n\geq 1}$ of non-empty compact subsets of M such that, if $\hat{Y}$ belongs to A_n, then (17)-(18) hold simultaneously for every $n \geq 1$. The expression "weight of a leaf" will be used during the proof to designate the value of the β associated with that leaf. Since both hypothesis (2) and the thesis of the present lemma are concerned with the absolute value of the $\widetilde{L}_i$'s, $\widetilde{R}_i$'s, $\tilde{\beta}$'s, with a view to simplifying the notation in the various steps of the proof, these random elements will be supposed to be positive.

Step 1. This step shows that a node weighted by $1/c$ (for some $c \geq 1$) can be the root node of a tree with a certain number N of leaves in such a way that each of $(N-1)$ of them is weighted by $1/x$ (for some fixed $x > c$) and the remaining one has weight not greater than $1/x$. Thus, define $N := \lfloor (x/c)^\alpha \rfloor + \mathbb{I}_{\{(x/c)^\alpha \notin \mathbb{N}\}}$ and construct the tree of N leaves in such a way that the depth of the leaf 1 is $(N-1)$ and the depth of the leaf j $(j = 2, \ldots, N)$ is equal to $(N+1-j)$. This amounts to the tree constructed by taking $i_1 = i_2 = \cdots = i_{N-1} = 1$. Moreover, for every $k = 1, \ldots, N-1$, one sets

$$R_k := \frac{c}{(x^\alpha - (k-1)c^\alpha)^{1/\alpha}} \quad \text{and} \quad L_k := (1 - R_k^\alpha)^{1/\alpha} = \Big(\frac{x^\alpha - kc^\alpha}{x^\alpha - (k-1)c^\alpha}\Big)^{1/\alpha}.$$

It is easy to verify that $R_k = c/(x\prod_{j=1}^{k} L_j)$, for every $k = 1, \ldots, N-1$, with the proviso that $\prod_{j=1}^{0} L_j = 1$. This way,

$$\beta_{1,N} = \frac{1}{c}\prod_{j=1}^{N-1} L_j, \qquad \beta_{k,N} = \frac{1}{c} R_{N-k+1} \prod_{j=1}^{N-k} L_j \quad (k = 2, \ldots, N)$$

and, by the definition of $(L_1, R_1), \ldots, (L_{N-1}, R_{N-1})$,

$$\beta_{1,N} = \frac{(x^\alpha - (N-1)c^\alpha)^{1/\alpha}}{cx}, \quad \beta_{k,N} = \frac{1}{x} \quad (k = 2, \ldots, N).$$

It should be noted that if $(x/c)^\alpha$ is an integer, then $N = (x/c)^\alpha$ and $\beta_{1,N} = 1/x$, whilst $\beta_{1,N} < 1/x$ whenever $(x/c)^\alpha$ is not an integer: In both cases, $\beta_{1,N} \leq 1/x$.

Step 2. In this step one describes the construction of the sequences $(N_n)_{n\geq 1}$ and $(\mathcal{R}_n)_{n\geq 1}$ by a recursive procedure. For $n = 1$, by applying Step 1 with $c = 1$ and $x = y_1$, one obtains a tree a_1 with $N_1 = \lfloor y_1^\alpha \rfloor + \mathbb{I}_{\{y_1^\alpha \notin \mathbb{N}\}}$ leaves such that

$$\beta_{j,N_1} \leq \frac{1}{y_1} \text{ if } j \in \mathcal{R}_1, \quad \beta_{j,N_1} = \frac{1}{y_1} \text{ if } j \notin \mathcal{R}_1$$

with: $|\mathcal{R}_1| = 1$ (since $\mathcal{R}_1 = \{1\}$), $\sum_{j=1}^{N_1} \beta_{j,N_1}^\alpha = 1$ and $\sum_{j\in\mathcal{R}_1} \beta_{j,N_1}^\alpha < \varepsilon$. Given the tree a_{n-1} $(n \geq 2)$ with N_{n-1} leaves such that

$$\beta_{j,N_{n-1}} \leq \frac{1}{y_{n-1}} \text{ if } j \in \mathcal{R}_{n-1}, \quad \beta_{j,N_{n-1}} = \frac{1}{y_{n-1}} \text{ if } j \notin \mathcal{R}_{n-1}$$

and

$$|\mathcal{R}_{n-1}| = N_{n-2}, \quad \sum_{j=1}^{N_{n-1}} \beta^{\alpha}_{j,N_{n-1}} = 1, \quad \sum_{j\in\mathcal{R}_{n-1}} \beta^{\alpha}_{j,N_{n-1}} < \varepsilon$$

one obtains a_n by applying the construction presented in Step 1 to each leaf of a_{n-1}. More precisely, for each leaf j of a_{n-1}, with $j = 1, \dots, N_{n-1}$, one implements Step 1 with $c = 1/\beta_{j,N_{n-1}}$ and $x = y_n$, where, without loss of generality, y_n is assumed to be strictly greater than c. Thus, the tree appended to leaf j has exactly one leaf with weight not greater than $1/y_n$, and each of the remaining leaves with weight equal to $1/y_n$. Iterating the procedure for $j = 1, \dots, N_{n-1}$, one obtains the tree denoted by a_n. The symbol N_n stands for the number of the leaves of a_n: There are $(N_n - N_{n-1})$ leaves weighted by $1/y_n$ and N_{n-1} leaves with a weight not greater than $1/y_n$. This is equivalent to saying that there exists $\mathcal{R}_n \subset \{1, \dots, N_n\}$ such that $|\mathcal{R}_n| = N_{n-1}$ and

$$\beta_{j,N_n} \le \frac{1}{y_n} \text{ if } j \in \mathcal{R}_n, \quad \beta_{j,N_n} = \frac{1}{y_n} \text{ if } j \notin \mathcal{R}_n$$

where, by construction, $\sum_{j=1}^{N_n} \beta^{\alpha}_{j,N_n} = 1$. To conclude with this step, it remains to prove that $\sum_{j\in\mathcal{R}_n} \beta^{\alpha}_{j,N_n} < \varepsilon$. To this end, it is enough to show that

$$N_n \le \sum_{i=1}^{n} (y_i^{\alpha} + 1) \qquad \text{for every } n \ge 1 \tag{26}$$

since, if (26) holds, then

$$\begin{aligned} \sum_{j\in\mathcal{R}_n} \beta^{\alpha}_{j,N_n} &\le \frac{N_{n-1}}{y_n^{\alpha}} \le \frac{1}{y_n^{\alpha}} \sum_{i=1}^{n-1} (y_i^{\alpha} + 1) \\ &< \varepsilon \qquad\qquad\qquad \text{(since } (y_n)_{n\ge1} \text{ satisfies (16))}. \end{aligned}$$

Coming back to (26), one proceeds by mathematical induction. From the definition of N_1 one gets $N_1 \le y_1^{\alpha} + 1$, that is the claim for $n = 1$. One now supposes that (26) is satisfied for every $n \le m$. Since, for every $k \ge 1$, $\beta_{j,N_k} = 1/y_k$ for every $j \notin \mathcal{R}_k$, $|\mathcal{R}_k| = N_{k-1}$ and $\sum_{j=1}^{N_k} \beta^{\alpha}_{j,N_k} = 1$, then

$$\frac{N_k - N_{k-1}}{y_k^{\alpha}} = \sum_{j\in\mathcal{R}_k} \beta^{\alpha}_{j,N_k} \le 1$$

obtains, entailing $N_k - N_{k-1} \le y_k^{\alpha}$. Combination of this with the inductive hypothesis yields

$$N_{m+1} = N_{m+1} - N_m + N_m \le y_{m+1}^{\alpha} + \sum_{i=1}^{m} (y_i^{\alpha} + 1) \le \sum_{i=1}^{m+1} (y_i^{\alpha} + 1)$$

which is (26) for $n = m + 1$.

Step 3. After constructing sequences $(N_n)_{n\ge1}$, $(\mathcal{R}_n)_{n\ge1}$, $(a_n)_{n\ge1}$ according to

Step 2, one now determines suitable proper intervals included in the ranges of the random elements $\hat{L}_i^{(n)}$'s, $\hat{R}_i^{(n)}$'s for every $n \geq 1$. As far as $n = 1$ is concerned, one considers the tree $a_1 = (\hat{\nu}^{(1)}, \hat{\imath}^{(1)})$. To satisfy (17), one can impose that

$$\hat{R}_1^{(1)} \in B_1^{(1)}(\delta_1^{(1)}) := \Big[\frac{1}{y_1 + \delta_1^{(1)}}, \frac{1}{y_1}\Big]$$

where the strictly positive $\delta_1^{(1)}$ is determined at a latter time. To specify an interval $C_1^{(1)}(\delta_1^{(1)})$ for $\hat{L}_1^{(1)}$, with a view to (2) one forces $C_1^{(1)}(\delta_1^{(1)})$ to satisfy

$$\begin{aligned}\hat{L}_1^{(1)} \in C_1^{(1)}(\delta_1^{(1)}) &:= \Big[\Big(1 - \frac{1}{y_1^\alpha}\Big)^{1/\alpha}, \Big(1 - \frac{1}{(y_1 + \delta_1^{(1)})^\alpha}\Big)^{1/\alpha}\Big] \\ &= \text{range of } (1 - x^\alpha)^{1/\alpha} \text{ as } x \text{ varies in } B_1^{(1)}(\delta_1^{(1)}).\end{aligned}$$

To single out an interval $B_2^{(1)}(\delta_1^{(1)})$ for $\hat{R}_2^{(1)}$, with a view to (17), $\hat{L}_1^{(1)} \cdot \hat{R}_2^{(1)}$ must belong to $[1/(y_1 + \delta_1^{(1)}), 1/y_1]$ for any value of $\hat{L}_1^{(1)}$ in $C_1^{(1)}(\delta_1^{(1)})$, which is granted if

$$\hat{R}_2^{(1)} \in B_2^{(1)}(\delta_1^{(1)}) := \Big[\frac{y_1}{(y_1 + \delta_1^{(1)})(y_1^\alpha - 1)^{1/\alpha}}, \frac{y_1 + \delta_1^{(1)}}{y_1[(y_1 + \delta_1^{(1)})^\alpha - 1)]^{1/\alpha}}\Big]$$

and this, in turn, allows the following specification

$$\hat{L}_2^{(1)} \in C_2^{(1)}(\delta_1^{(1)}) := \text{range of } (1 - x^\alpha)^{1/\alpha} \text{ as } x \text{ varies in } B_2^{(1)}(\delta_1^{(1)}).$$

The procedure can be iterated to yield intervals $C_1^{(1)}(\delta_1^{(1)}), B_1^{(1)}(\delta_1^{(1)}), \dots, C_{N_1-1}^{(1)}(\delta_1^{(1)})$, $B_{N_1-1}^{(1)}(\delta_1^{(1)})$ in such a way that (17) is met, with $n = 1$, whenever

$$(\hat{L}_k^{(1)}, \hat{R}_k^{(1)})_{k=1,\dots,N_1-1} \in \times_{k=1}^{N_1-1}\Big(C_k^{(1)}(\delta_1^{(1)}) \times B_k^{(1)}(\delta_1^{(1)})\Big) =: I_1(\delta_1^{(1)}).$$

Note that the validity of this last claim is granted by the way followed so far to construct the above intervals. To complete the construction, it remains to specify admissible values of $\delta_1^{(1)}$ in such a way that (18) holds for $n = 1$. One starts by noticing, in the notation of Step 1, that $1 = (\sum_{j\in\mathcal{R}_1} + \sum_{j\notin\mathcal{R}_1})\beta_{j,N_1}^\alpha < \varepsilon + (N_1 - 1)/y_1^\alpha$, which entails $(N_1 - 1)/y_1^\alpha > 1 - \varepsilon$. Taking h_1 and η_1 such that

$$0 < h_1 < \frac{N_1 - 1}{y_1^\alpha} - 1 + \varepsilon \qquad \text{and} \qquad 0 < \eta_1 < (1 + h_1)^{1/N_1} - 1$$

one gets $(\hat{L}_k^{(1)})^\alpha + (\hat{R}_k^{(1)})^\alpha \leq 1 + \eta_1$ for every $k = 1, \dots, N_1 - 1$ and hence $\sum_{j=1}^{N_1}(\hat{\beta}_{j,N_1}^{(1)})^\alpha \leq (1 + \eta_1)^{N_1}$ whenever $\delta_1^{(1)}$ is sufficiently small and, in any case, satisfies

$$\delta_1^{(1)} \in \Big(0, \Big(\frac{N_1 - 1}{(1 + \eta_1)^{N_1} - \varepsilon}\Big)^{1/\alpha} - y_1\Big). \tag{27}$$

Moreover, thanks to (17),

$$\sum_{j\notin\mathcal{R}_1}(\hat{\beta}^{(1)}_{j,N_1})^\alpha \geq \frac{|\{1,\dots,N_1\}\setminus\mathcal{R}_1|}{(y_1+\delta_1^{(1)})^\alpha} = \frac{N_1-1}{(y_1+\delta_1^{(1)})^\alpha}$$

whence

$$\sum_{j\in\mathcal{R}_1}(\hat{\beta}^{(1)}_{j,N_1})^\alpha = \Big(\sum_{j=1}^{N_1-1} - \sum_{j\notin\mathcal{R}_1}\Big)(\hat{\beta}^{(1)}_{j,N_1})^\alpha \leq (1+\eta_1)^{N_1} - \frac{N_1-1}{(y_1+\delta_1^{(1)})^\alpha} < \varepsilon$$

where the last inequality follows from (27). At this stage, one can say that if

$$f_1(\hat{Y}) \in \{a_1\}\times I_1(\delta_1^{(1)})\times(\overline{\mathbb{R}}^2)^\infty$$

then (17) and (18) hold simultaneously with $n = 1$. Now, to verify that the above assumption is non-empty, it is enough to use independence to prove that

$$\begin{aligned}
&\widehat{\mathcal{P}}\Big(\bigcap_{k=1}^{N_1-1}\Big\{(\hat{L}^{(1)}_k,\hat{R}^{(1)}_k)\in C^{(1)}_k(\delta_1^{(1)})\times B^{(1)}_k(\delta_1^{(1)})\Big\}\Big)\\
&\quad = \prod_{k=1}^{N_1-1}\widehat{\mathcal{P}}\Big\{(\hat{L}^{(1)}_k,\hat{R}^{(1)}_k)\in C^{(1)}_k(\delta_1^{(1)})\times B^{(1)}_k(\delta_1^{(1)})\Big\}\\
&\quad \geq \prod_{k=1}^{N_1-1}\widehat{\mathcal{P}}\Big\{(\hat{L}^{(1)}_k,\hat{R}^{(1)}_k)\in B_{\rho_k}(x_k,y_k)\Big\}\\
&\quad > 0 \qquad \text{(in view of (2))}
\end{aligned}$$

holds whenever $B_{\rho_k}(x_k,y_k)$ is a suitable neighbourhood of radius ρ_k and center (x_k,y_k) in $\Gamma\cap(C^{(1)}_k(\delta_1^{(1)})\times B^{(1)}_k(\delta_1^{(1)}))$, with $\Gamma := \{(x,y)\in\mathbb{R}^2 :\ |x|^\alpha+|y|^\alpha = 1\}$ and $k = 1,\dots,N_1-1$. Coming back to the aim expressed at the beginning of this section, in view of the previous arguments, the first term of the sequence $(A_n)_{n\geq1}$ can be defined to be

$$A_1 := f_1^{-1}\Big(\{a_1\}\times I_1(\delta_1^{(1)})\times(\overline{\mathbb{R}}^2)^\infty\Big)$$

which is a closed subset of M since f_1 is continuous.

Step 4. This step deals with the case of $n = 2$. Starting from the same N_1, N_2, $\mathcal{R}_1, \mathcal{R}_2$, a_1 and $a_2 = (\hat{\nu}^{(2)}, \hat{\imath}^{(2)})$ as in Step 1, one repeats for a_1, seen as a subtree of a_2, the same intervals construction made in Step 2 with a suitable $\delta_1^{(2)}$, to be determined at a latter time, in the place of $\delta_1^{(1)}$ in such a way that $\delta_1^{(2)} \leq \delta_1^{(1)}$. As a consequence, one has

$$(\hat{L}^{(2)}_k,\hat{R}^{(2)}_k)_{k=1,\dots,N_1-1}\in I_1(\delta_1^{(2)}).$$

Thus, in view of (17), $\hat{R}^{(2)}_{N_1}$ has to satisfy

$$\hat{R}^{(2)}_1\cdot\hat{R}^{(2)}_{N_1}\in\Big[\frac{1}{y_2},\frac{1}{y_2-\delta_2^{(2)}}\Big]$$

for every value of $\hat{R}_1^{(2)}$ in $B_1^{(1)}(\delta_1^{(2)})$, with a positive $\delta_2^{(2)}$ to be determined later. This condition is satisfied if

$$\hat{R}_{N_1}^{(2)} \in B_{N_1}^{(2)}(\delta_1^{(2)}, \delta_2^{(2)}) := \Big[\frac{y_1 + \delta_1^{(2)}}{y_2}, \frac{y_1}{y_2 - \delta_2^{(2)}}\Big]$$

holds together with $\delta_1^{(2)} < \delta_2^{(2)} y_1/(y_2 - \delta_2^{(2)})$. Like in Step 2 one forces $\hat{L}_{N_1}^{(2)}$ to satisfy

$$\hat{L}_{N_1}^{(2)} \in C_{N_1}^{(2)}(\delta_1^{(2)}, \delta_2^{(2)}) := \text{range of } (1 - x^\alpha)^{1/\alpha} \text{ as } x \text{ varies in } B_{N_1}^{(2)}(\delta_1^{(2)}, \delta_2^{(2)}).$$

One can proceed this way to obtain a family of intervals $\Big\{C_k^{(2)}(\delta_1^{(2)}, \delta_2^{(2)}), B_k^{(2)}(\delta_1^{(2)}, \delta_2^{(2)}) : k = N_1, \dots, N_2 - 1\Big\}$ such that (17) is met for $n = 2$ if

$$(\hat{L}_k^{(2)}, \hat{R}_k^{(2)})_{k=1,\dots,N_2-1} \in I_1(\delta_1^{(1)}) \times I_2(\delta_1^{(2)}, \delta_2^{(2)})$$

with $I_2(\delta_1^{(2)}, \delta_2^{(2)}) := \times_{k=N_1}^{N_2-1}\Big(C_k^{(2)}(\delta_1^{(2)}, \delta_2^{(2)}) \times B_k^{(2)}(\delta_1^{(2)}, \delta_2^{(2)})\Big)$. Moreover, arguing as in the previous step, one sees that

$$\widehat{\mathcal{P}}\Big\{f_2(\hat{Y}) \in \{a_1\} \times \{a_2\} \times \Big(I_1(\delta_1^{(1)}) \times (\overline{\mathbb{R}}^2)^\infty\Big) \\ \times \Big(I_1(\delta_1^{(2)}) \times I_2(\delta_1^{(2)}, \delta_2^{(2)}) \times (\overline{\mathbb{R}}^2)^\infty\Big)\Big\}$$

is strictly positive and whence one can set $A_2 := f_2^{-1}\Big(\{a_1\} \times \{a_2\} \times \Big(I_1(\delta_1^{(1)}) \times (\overline{\mathbb{R}}^2)^\infty\Big) \times \Big(I_1(\delta_1^{(2)}) \times I_2(\delta_1^{(2)}, \delta_2^{(2)}) \times (\overline{\mathbb{R}}^2)^\infty\Big)\Big)$. This way, A_2 turns out to be a closed subset of A_1.

Step 5. This step extends the procedure to find A_n for any n. Here, one confines oneself to analysing the case of odd n's. In fact, when n is even, the way of reasoning reduces to a simplified form of the odd case. Hence, let m be an odd number and assume that (17) is satisfied for $n = m - 1$ if

$$f_{m-1}(\hat{Y}) \in \{a_1\} \times \cdots \times \{a_{m-1}\} \times \Big[\times_{k=1}^{m-1}\Big(\times_{i=1}^{k} I_i(\delta_1^{(k)}), \dots, \delta_i^{(k)}) \\ \times (\overline{\mathbb{R}}^2)^\infty\Big)\Big]. \tag{28}$$

Starting from the same $N_1, \dots, N_m, \mathcal{R}_1, \dots, \mathcal{R}_m, a_1, \dots, a_{m-1}$ and $a_m = (\hat{\nu}^{(m)}, \hat{i}^{(m)})$ as in Step 1, one replaces $\delta_1^{(m-1)}, \dots, \delta_{m-1}^{(m-1)}$ with smaller $\delta_1^{(m)}, \dots\ \delta_{m-1}^{(m)}$ in such a way that, for a suitable $\delta_m^{(m)} > 0$, one may determine intervals $C_{N_{m-1}}^{(m)}(\delta_1^{(m)}, \dots, \delta_m^{(m)})$, $B_{N_{m-1}}^{(m)}(\delta_1^{(m)}, \dots, \delta_m^{(m)}), \dots, C_{N_m-1}^{(m)}(\delta_1^{(m)}, \dots, \delta_m^{(m)})$, $B_{N_m-1}^{(m)}(\delta_1^{(m)}, \dots, \delta_m^{(m)})$ for which (17) holds true also for $n = m$, whenever

$$f_m(\hat{Y}) \in \{a_1\} \times \cdots \times \{a_m\} \times \Big[\times_{k=1}^{m}\Big(\times_{i=1}^{k} I_i(\delta_1^{(k)}), \dots, \delta_i^{(k)}) \times (\overline{\mathbb{R}}^2)^\infty\Big)\Big]$$

with $I_m(\delta_1^{(m)},\dots,\delta_m^{(m)}) := \times_{k=N_{m-1}}^{N_m-1}\Big(C_k^{(m)}(\delta_1^{(m)},\dots,\delta_m^{(m)})\times B_k^{(m)}(\delta_1^{(m)},\dots,\delta_m^{(m)})\Big)$. At this stage, $\delta_m^{(m)}$ has to be determined so that (18) is met. From Step 1, one has $1 = \Big(\sum_{j\in\mathcal{R}_m} + \sum_{j\notin\mathcal{R}_m}\Big)\beta_{j,N_m}^{\alpha} < \varepsilon + (N_m - N_{m-1})/y_m^{\alpha}$. Reasoning like in Step 2, one considers positive numbers h_m and η_m satisfying

$$h_m < \frac{N_m - N_{m-1}}{y_m^{\alpha}} - 1 + \varepsilon \quad \text{and} \quad \eta_m < (1+h_m)^{1/N_m} - 1$$

and chooses

$$\delta_m^{(m)} < \Big(\frac{N_m - N_{m-1}}{(1+\eta_m)^{N_m} - \varepsilon}\Big)^{1/\alpha} - y_m.$$

One can get $(\hat{L}_k^{(m)})^{\alpha} + (\hat{R}_k^{(m)})^{\alpha} \le 1+\eta_m$ for every $k = 1,\dots,N_m - 1$ by reducing $\delta_1^{(m)},\dots,\delta_m^{(m)}$ if needed, and then

$$\sum_{j=1}^{N_m} (\hat{\beta}_{j,N_m}^{(m)})^{\alpha} \le (1+\eta_m)^{N_m}.$$

In view of (17),

$$\sum_{j\notin\mathcal{R}_m} (\hat{\beta}_{j,N_m}^{(m)})^{\alpha} \ge \frac{|\{1,\dots,N_m\}\setminus\mathcal{R}_m|}{(y_m+\delta_m^{(m)})^{\alpha}} = \frac{N_m - N_{m-1}}{(y_m+\delta_m^{(m)})^{\alpha}}$$

and hence, by definition of $\delta_m^{(m)}$,

$$\sum_{j\in\mathcal{R}_m} (\hat{\beta}_{j,N_m}^{(m)})^{\alpha} \le (1+\eta_m)^{N_m} - \frac{N_m - N_{m-1}}{(y_m+\delta_m^{(m)})^{\alpha}} < \varepsilon.$$

Thus, (17)-(18) hold for $n = m$ if

$$f_m(\hat{Y}) \in \{a_1\}\times\cdots\times\{a_m\}\times\Big[\times_{k=1}^{m}\Big(\times_{i=1}^{k} I_i(\delta_1^{(k)}),\dots,\delta_i^{(k)})\times(\overline{\mathbb{R}}^2)^{\infty}\Big)\Big].$$

After noting that

$$\widehat{\mathcal{P}}\Big\{f_m(\hat{Y}) \in \{a_1\}\times\cdots\times\{a_m\}\times\Big[\times_{k=1}^{m}\Big(\times_{i=1}^{k} I_i(\delta_1^{(k)},\dots,\delta_i^{(k)})\times(\overline{\mathbb{R}}^2)^{\infty}\Big)\Big]\Big\} > 0$$

one can choose

$$A_n := f_n^{-1}\Big(\{a_1\}\times\cdots\times\{a_n\}\times\Big[\times_{k=1}^{n}\Big(\times_{i=1}^{k} I_i(\delta_1^{(k)}),\dots,\delta_i^{(k)})\times(\overline{\mathbb{R}}^2)^{\infty}\Big)\Big]\Big)$$

which is a closed subset of A_{n-1}.

Conclusion. The decreasing sequence $(A_n)_{n\ge1}$ constructed in the previous steps is formed of non-empty closed subsets of the compact set M. Hence, as granted by the finite intersection principle, $\bigcap_{n\ge1} A_n$ is non-empty, and the proof is completed by noting that $\hat{Y}^{-1}\Big(\bigcap_{n\ge1} A_n\Big) \ne \emptyset$ and that (17)-(18) hold for every $\hat{\omega}_0$ in $\hat{Y}^{-1}\Big(\bigcap_{n\ge1} A_n\Big)$.

Appendix B: Probability measures with symmetrized forms attracted by a stable law

As told in the second last paragraph of Section 1, here is an example of p.d.f. which does not belong to the s.d.a. of any α-stable law, whilst its symmetrized form does. Let I and S be two positive real numbers such that $I < S$ and let $c := (S + I)/2$. One puts

$$\begin{aligned} G_I(x) &:= 1 - Ix^{-\alpha}\mathbb{I}_{(1,+\infty)}(x) \\ G_S(x) &:= 1 - Sx^{-\alpha}\mathbb{I}_{(1,+\infty)}(x) \end{aligned}$$

and chooses $s_1 > 1$. A continuous p.d.f. F is now defined as follows. At first, one sets $F(s_1) := G_S(s_1)$. Then, one considers the derivative function f of F defined by $f(x) = k\alpha/x^{\alpha+1}$ for every x in (s_1, i_1), where k is a fixed number in $(S, S+I)$ and i_1, greater than s_1, satisfies $F(i_1) = G_I(i_1)$. Let s_2 be the number, greater than i_1, which meets $F(i_1) = G_S(s_2)$, and let $f(x) = 0$ on (i_1, s_2). After $2(m-1)$ repetitions of the process, one gets the point s_m and one defines the derivative on (s_m, i_m) to be $f(x) = k\alpha/x^{\alpha+1}$, where $i_m > s_m$ satisfies $F(i_m) = G_I(i_m)$. In the next repetition, one sets $f(x) = 0$ on (i_m, s_{m+1}), where s_{m+1} $(> i_m)$ meets $F(i_m) = G_S(s_{m+1})$. This way, $F(x)$ is specified at every x in $[s_1, +\infty)$ and $i_m^\alpha(1 - F(i_m)) = I$, $s_m^\alpha(1 - F(s_m)) = S$ for every $m \geq 1$, so that

$$\liminf_{x\to+\infty} x^\alpha(1 - F(x)) = I < S = \limsup_{x\to+\infty} x^\alpha(1 - F(x)). \tag{29}$$

Now, one extends F to $(-\infty, -s_1]$ by setting $F(x) := 2c|x|^{-\alpha} - 1 + F(-x)$ for every $x \leq -s_1$, and one completes the definition of F by interpolating linearly on $(-s_1, s_1)$. The resulting function F is a p.d.f. since the derivative of its restriction to $(-\infty, -s_1]$ is $(2c\alpha/(-x)^{\alpha+1} - f(-x))$ which is always positive. Indeed, by construction, $f(-x) \leq (S + I)\alpha/(-x)^{\alpha+1} = 2c\alpha/(-x)^{\alpha+1}$ for every $x < -s_1$. On the one hand, gathering up all these remarks one can say that F is a p.d.f. which, by virtue of (29), cannot belong to the s.d.a. of any α-stable distribution. On the other hand, the symmetrized form F^* of F satisfies

$$x^\alpha(1 - F^*(x)) = 2c \qquad \text{for every } x \in [s_1, +\infty)$$

so that, on the difference of F, F^* belongs to the s.d.a. of an α-stable distribution.

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