

Periodic systems have new classes of synchronization stability

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The master stability function is a robust and useful tool for determining the conditions of synchronization stability in a network of coupled systems. While a comprehensive classification exists in the case in which the nodes are chaotic dynamical systems, its application to periodic systems has been less explored. By studying several well-known periodic systems, we establish a comprehensive framework to understand and classify their properties of synchronizability. This allows us to define five distinct classes of synchronization stability, including some that are unique to periodic systems. Specifically, in periodic systems, the master stability function vanishes at the origin, and it can therefore display behavioral classes that are not achievable in chaotic systems, where it starts, instead, at a strictly positive value. Moreover, our results challenge the widely held belief that periodic systems are easily put in a stable synchronous state, showing, instead, the common occurrence of a lower threshold for synchronization stability.

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I. INTRODUCTION

Over the last couple of decades, the most successful structural paradigm in the study of complex systems has been that of networks, in which discrete elements called nodes or vertices interact across connections called links or edges [1–4]. Of particular relevance to real-world applications is the case where the nodes are dynamical systems, coupled to each other if they share an edge. However, proper frameworks and techniques are required to operationally define robustness and resilience of networks leading to optimal performance [5]. In dynamical networks, a vast array of phenomena can occur, driven by the collective organization of the individual dynamical systems. A significant one is the emergence of a synchronized state, in which a number of elements that can extend to the entire network eventually converge to the same trajectory in phase space [6–8]. The study of synchronized states holds a special importance across fields, as it has found notable applications such as in modeling the functioning of

neurons and the brain, and in investigating and optimizing the operation of power grids [9–16]. As a result, strong efforts have been directed toward the study of the different forms under which synchronized states appear and of the effects that factors such as network structure and coupling configuration have on their properties [17–38].

A related question, which has generated a large body of work, is how to assess the stability of a synchronized state. A powerful tool to address this problem is the method known as the master stability function (MSF) [39]. The method estimates the stability of the synchronous solution to the dynamics by estimating the largest Lyapunov exponent after the system is perturbed in directions within the subspace transverse to the synchronization manifold. This allows one to evaluate the synchronization stability from the sign of the exponent, so that the trajectory of the perturbed system will converge back onto the synchronized state only if the largest Lyapunov exponent is negative. The elegance of the method, which is equivalent to a decomposition of the dynamics into eigenmodes, has made it a preferred tool for the exploration of the properties of synchronization, so that, over time, it has been extended and applied to a diverse range of complex networks, underscoring its power and versatility [40–48].

The nature of the MSF has also made it a natural choice of method to employ when studying systems whose dynamics is chaotic. In fact, a general classification scheme has also been presented for the synchronization behavior of chaotic systems, based on the positivity regions of the MSF [2]. This has shown

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that each chaotic system belongs to one of three classes, which correspond to a vanishing, unbounded, or bounded region of parameters for which their synchronized state is stable after an initial threshold in coupling strength. However, notwithstanding the numerous successes of this method, no systematic study of the MSF behavior for periodic systems had been carried out so far. Note that ensuring that periodic systems reliably remain in a stable synchronized state is an important task for numerous engineering applications where the precision of timing and the mitigation of jitter, instabilities, and phase noise are of critical relevance. Specific examples of such situations include synchronizing AC power-distribution networks to ensure efficient and coordinated power delivery [49,50], guaranteeing the synchronous operation of digital communication networks to obtain reliable data transmission [51], and maintaining coherence in clock distribution trees within electronic devices and circuit boards, which is essential for optimal performance and functionality [52].

In this article, we close the gap in the synchronizability of periodic systems by a thorough investigation into their synchronization dynamics. We use the MSF method introduced by Pecora and Carroll, focusing on its application to periodic systems, which differs from the predominantly chaotic systems studied in previous research. Chaotic systems are inherently challenging to synchronize due to the butterfly effect, which has driven significant research into their synchronizability. In contrast, it was traditionally believed that identical periodic systems would synchronize at infinitesimal coupling strength. However, our findings demonstrate that the MSF behavior for periodic systems can differ substantially from that of chaotic systems. This difference stems from the initial value of the MSF, which is equal to the maximum Lyapunov exponent of the system. By applying the MSF method to several periodic systems, we reveal the existence of distinct stable synchronization regions and propose a classification scheme for periodic systems. We identify two additional classes of MSF behavior unique to periodic systems. Furthermore, we present examples where periodic systems do not synchronize at infinitesimal coupling. These findings can help in expanding the understanding of synchronization in periodic systems.

II. MODEL AND METHODS

Given a connected network of N diffusively coupled d -dimensional identical systems with weighted adjacency matrix \mathbf{W} , its dynamics is described by the system of equations

$$\dot{\mathbf{x}}_i = \mathbf{F}(\mathbf{x}_i) - \sigma \sum_{j=1}^N L_{i,j} \mathbf{H}(\mathbf{x}_j), \quad (1)$$

where \mathbf{x}_i is a vector with d components representing the state of node i , $\mathbf{F} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\mathbf{H} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ are vector fields describing the internal dynamics of the systems and their mutual coupling, respectively, σ is the coupling strength, and the matrix \mathbf{L} is the graph Laplacian of the network, whose elements are

$$\begin{aligned} L_{i,i} &= \sum_{j=1}^N W_{i,j} \\ L_{i,j} &= -W_{i,j}. \end{aligned} \quad (2)$$

Note that, for the sake of brevity, here and in the following we will omit writing explicit time dependencies, except when we wish to draw specific attention to them. The definition of the graph Laplacian in the previous equation makes it a positive semidefinite zero-row-sum matrix. This means that in this case, it has one zero eigenvalue ($\lambda_1 = 0$), while all the others are positive ($\lambda_i > 0$ for $i = 2, \dots, N$). Also, its presence in Eq. (1) guarantees the existence of an invariant synchronous solution of the dynamics $\mathbf{s}(t)$, so that $\mathbf{x}_i(t) = \mathbf{s}(t)$ for all i . In turn, this allows one to introduce the synchronization error vectors $\delta \mathbf{x}_i = \mathbf{x}_i - \mathbf{s}$, which measure the componentwise difference between the state of each node at a given time and the synchronous solution. If \mathbf{F} and \mathbf{H} are at least C^1 , i.e., if they are continuous and differentiable, one can linearize them via a vector equivalent of a first-order Taylor expansion around the synchronous solution, so that

$$\mathbf{F}(\mathbf{s} + \delta \mathbf{x}_i) \approx \mathbf{F}(\mathbf{s}) + \hat{\mathbf{J}}\mathbf{F}(\mathbf{s})\delta \mathbf{x}_i \quad (3)$$

and

$$\mathbf{H}(\mathbf{s} + \delta \mathbf{x}_i) \approx \mathbf{H}(\mathbf{s}) + \hat{\mathbf{J}}\mathbf{H}(\mathbf{s})\delta \mathbf{x}_i, \quad (4)$$

where $\hat{\mathbf{J}}\mathbf{F}(\mathbf{s})$ and $\hat{\mathbf{J}}\mathbf{H}(\mathbf{s})$ are the Jacobians of \mathbf{F} and \mathbf{H} , respectively. Then, substituting $\mathbf{x}_i = \mathbf{s} + \delta \mathbf{x}_i$ into Eq. (1) leads to

$$\dot{\mathbf{s}} + \dot{\delta \mathbf{x}}_i = \mathbf{F}(\mathbf{s}) + \hat{\mathbf{J}}\mathbf{F}(\mathbf{s})\delta \mathbf{x}_i - \sigma \sum_{j=1}^N L_{i,j} (\mathbf{H}(\mathbf{s}) + \hat{\mathbf{J}}\mathbf{H}(\mathbf{s})\delta \mathbf{x}_j). \quad (5)$$

As $\dot{\mathbf{s}} = \mathbf{F}(\mathbf{s})$ and $\sum_{j=1}^N L_{i,j} = 0$, the evolution of the synchronization error vectors simplifies to

$$\dot{\delta \mathbf{x}}_i = \hat{\mathbf{J}}\mathbf{F}(\mathbf{s})\delta \mathbf{x}_i - \sigma \sum_{j=1}^N L_{i,j} \hat{\mathbf{J}}\mathbf{H}(\mathbf{s})\delta \mathbf{x}_j. \quad (6)$$

Finally, the synchronization error vectors can be decomposed along the directions determined by the eigenvectors of the Laplacian, which can be conveniently arranged in an orthogonal matrix \mathbf{V} . This yields a decomposition of the dynamics into N decoupled modes $\boldsymbol{\eta}_i = \mathbf{V}^{-1} \delta \mathbf{x}_i$, whose evolution is given by the set of variational equations

$$\dot{\boldsymbol{\eta}}_i = (\hat{\mathbf{J}}\mathbf{F}(\mathbf{s}) - \sigma \lambda_i \hat{\mathbf{J}}\mathbf{H}(\mathbf{s})) \boldsymbol{\eta}_i. \quad (7)$$

Because of the fact that $\lambda_1 = 0$ and because of the orthogonality of \mathbf{V} , the evolution of these variational equations occurs along the synchronous solution of the dynamics for $i = 1$, and along directions transverse to it for $i > 1$. Then, one can consider the generic equation

$$\dot{\boldsymbol{\eta}} = (\hat{\mathbf{J}}\mathbf{F}(\mathbf{s}) - K \hat{\mathbf{J}}\mathbf{H}(\mathbf{s})) \boldsymbol{\eta} \quad (8)$$

and compute its maximum Lyapunov exponent Λ . The dependence of Λ on the generalized coupling strength K is the master stability function [39]. Given a coupling strength σ , if the synchronized state is stable for that value of σ , then the MSF is negative for all values of $K = \sigma \lambda_i$ with $i > 1$. Note that when $K = 0$, the value of the MSF is the maximum Lyapunov exponent of the uncoupled system. Consequently, for chaotic systems, the MSF has a positive intercept.

Based on the qualitative behavior of the MSF, a general classification for the synchronization stability of chaotic systems was introduced in Ref. [2]. According to it, any system belongs to one of the following three classes:

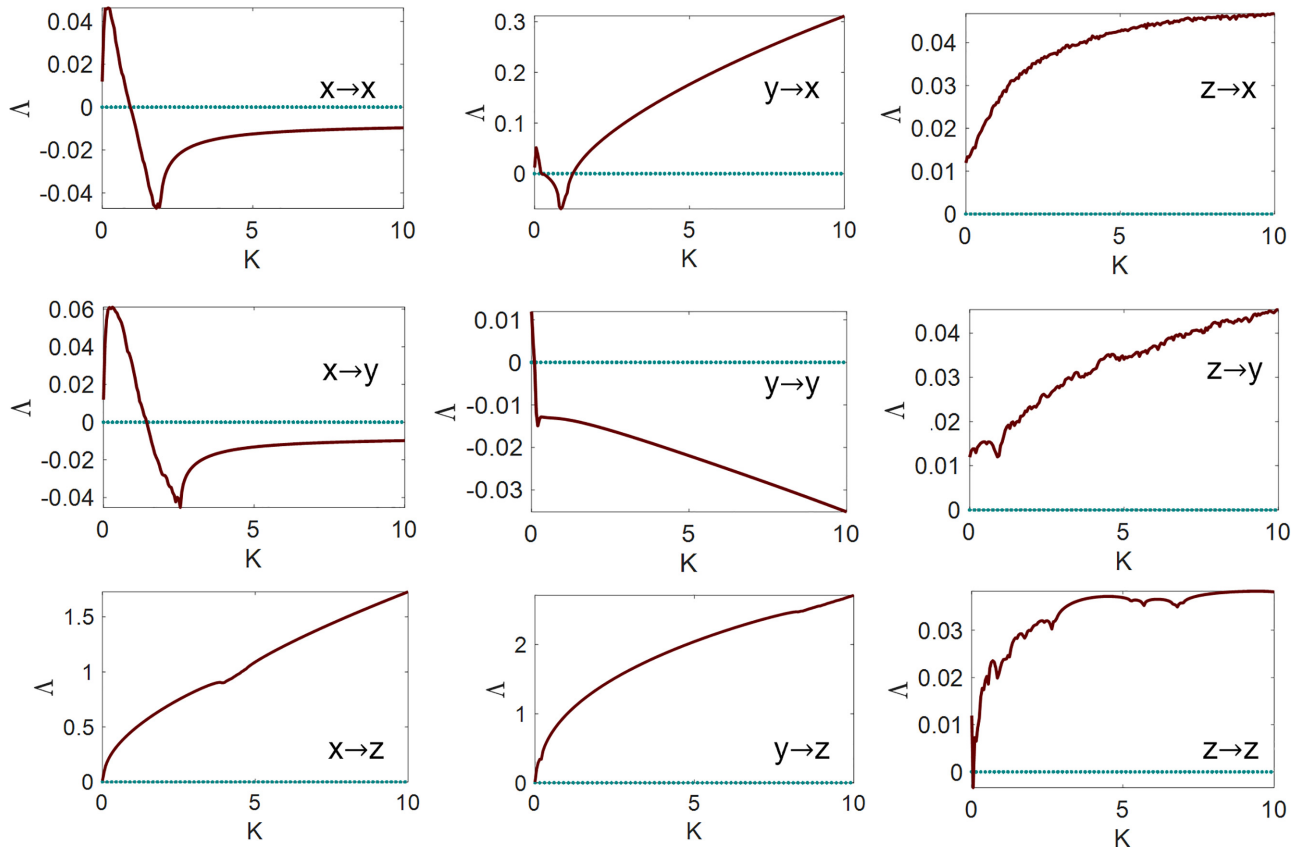


FIG. 1. The chaotic Hindmarsh-Rose system can belong to Classes I, II, and III. The master stability function (Λ) of the chaotic Hindmarsh-Rose system, Eq. (9), plotted as a function of the generalized coupling strength K , is in Class III for the couplings $y \rightarrow x$ and $z \rightarrow z$, in Class II for $x \rightarrow x$, $x \rightarrow y$, and $y \rightarrow y$, and in Class I for all other choices. Thus, depending on the choice of the coupling, the Hindmarsh-Rose system in the chaotic regime can belong to any of the synchronizability classes. The parameter values are $a = 1$, $b = 3$, $I = \frac{16}{5}$, $c = 1$, $d = 5$, $r = 6 \times 10^{-3}$, $s = 4$, and $x_1 = \frac{8}{5}$.

Class I. The MSF is positive for all values of K . Consequently, synchronization is not stable for any coupling strength.

Class II. The MSF is negative for an unbounded interval of values of K . Consequently, there exists a critical value K^* , at which the function intersects the horizontal axis and after which it is always negative.

Class III. The MSF is negative in a bounded interval of values of K . Consequently, there are two intersection points K_1^* and K_2^* , such that the MSF is negative for $K_1^* < K < K_2^*$.

As an example of a chaotic system that can belong to any of the three classes depending on the coupling between elements, consider the Hindmarsh-Rose model, whose system of equations describes the spiking and bursting behavior of a single neuron [53]:

$$\begin{aligned} \dot{x} &= y - ax^3 + bx^2 - z + I, \\ \dot{y} &= c - dx^2 - y, \\ \dot{z} &= -rz + rs(x + x_1). \end{aligned} \tag{9}$$

To obtain the MSF of the HR system, first the perturbed equations are derived according to Eq. (8). Then, the maximum Lyapunov exponent Λ of the perturbed system is calculated as a function of the generalized coupling strength K , and defined as the MSF. Choosing $a = 1$, $b = 3$, $I = \frac{16}{5}$, $c = 1$, $d = 5$,

$r = 6 \times 10^{-3}$, $s = 4$, and $x_1 = \frac{8}{5}$ results in a rich variety of synchronization behaviors corresponding to the nine possible single-variable couplings $i \rightarrow j$, with $(i, j) \in \{x, y, z\} \times \{x, y, z\}$. The MSFs of the chaotic HR system are illustrated in Fig. 1. The figure shows that the MSF can belong to all synchronizability classes of chaotic systems, namely Class III for the couplings $y \rightarrow x$ and $z \rightarrow z$, Class II for $x \rightarrow x$, $x \rightarrow y$, and $y \rightarrow y$, and Class I for all other couplings.

III. MSF OF PERIODIC SYSTEMS

The situation is subtly different when one considers periodic systems. In fact, if an isolated system supports a periodic orbit, its corresponding maximum Lyapunov exponent is zero. This means that the MSF in the case of periodic systems does not start from a strictly positive value, but rather it starts from zero. This has two immediate consequences. First, there is always at least one point where the MSF vanishes, namely $K = 0$. Second, an initial discrimination for the synchronizability of a coupled network is determined by the sign of the (right-hand) derivative of the MSF at zero. This suggests the possibility that the properties of synchronizability of a network of periodic oscillators are actually more complex than those of a network of chaotic ones.

To confirm the correctness of this consideration, we computed the MSF for the Hindmarsh-Rose model, using the same

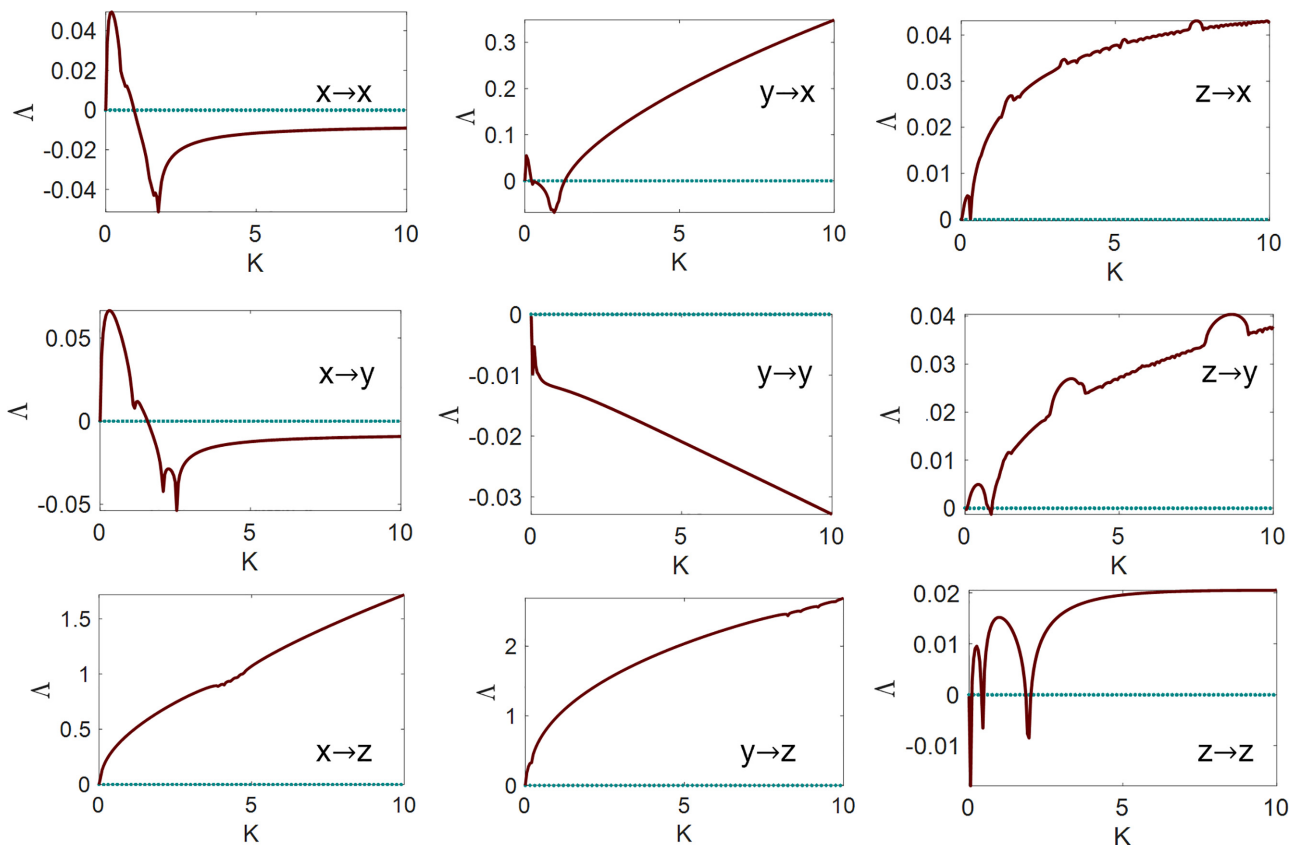


FIG. 2. *New classes of synchronizability for the periodic Hindmarsh-Rose system.* The master stability function (Λ) of the periodic Hindmarsh-Rose system, Eq. (9), plotted as a function of the generalized coupling strength K , is always positive for the couplings $x \rightarrow z$, $y \rightarrow z$, and $z \rightarrow x$; it has bounded intervals of negative values for $y \rightarrow x$ and $z \rightarrow y$; it is negative in an unbounded range for $x \rightarrow x$, $x \rightarrow y$, and $y \rightarrow y$; and it has multiple regions of negativity for $z \rightarrow z$. Note that the $z \rightarrow z$ coupling results in the system belonging to a synchronization class that is exclusive of periodic systems. Also note that, in contrast with conventional belief, a minimum coupling strength is required to achieve synchronization in many cases. The parameter values are $a = 1$, $b = 3$, $I = \frac{16}{5}$, $c = 1$, $d = 5$, $r = 5.6 \times 10^{-3}$, $s = 4$, and $x_1 = \frac{8}{5}$.

parameter values as before, except for r , which we imposed to be equal to 5.6×10^{-3} . This choice ensures that dynamics of the individual systems is periodic. It should be noted that the steps one has to follow in order to compute the MSF for a periodic system are the same as would be taken in the case of a chaotic system. The only difference is that the periodic synchronous solution is used to obtain the perturbed linear equation. The MSF results, shown in Fig. 2, demonstrate an even broader range of behaviors than observed in the chaotic version of the model. In fact, the couplings $x \rightarrow z$, $y \rightarrow z$, and $z \rightarrow x$ result in a MSF that is always positive, the couplings $y \rightarrow x$, $z \rightarrow y$, and $z \rightarrow z$ yield well-defined ranges of negativity, and the couplings $x \rightarrow x$, $x \rightarrow y$, and $y \rightarrow y$ result in unbounded regions of negative values for the MSF. Therefore, the fundamental effect of the periodicity of the system is on the MSF of the couplings $z \rightarrow y$ and $z \rightarrow z$, with the former that now features a bounded negative region and the latter that includes several negative regions. Moreover, even though $x \rightarrow x$, $x \rightarrow y$, and $y \rightarrow y$ all correspond to unbounded regions of negativity, only in the case of $y \rightarrow y$ does the region start at $K = 0$. Effectively, one could say that the fact that the MSF vanishes at zero has split Class II into two new classes: if the derivative at zero is negative, then the unbounded region of stability starts at zero; if, instead, it is positive, then the interval starts at a value $K^* > 0$. Similarly, $y \rightarrow x$, $z \rightarrow y$,

and $z \rightarrow z$ produce a finite region of negative values, which, however, only starts at zero for the $z \rightarrow z$ coupling. Thus, Class III also undergoes a split that depends on the sign of the derivative at zero, akin to that of Class II. Note that the $z \rightarrow z$ coupling actually produces multiple separate intervals for which the MSF is negative. However, when classifying the stability of synchronized states, one is generally only interested in the interval after the first threshold for stability, which, in this case, is zero.

To further explore this phenomenology, we studied a network of Rössler oscillators [54], whose evolution is given by the system

$$\begin{aligned}\dot{x} &= -y - z, \\ \dot{y} &= x + ay, \\ \dot{z} &= b + (x - c)z.\end{aligned}\quad (10)$$

To ensure periodic dynamics, we chose the parameter values $a = 0.161$, $b = 0.2$, and $c = 9$. The calculation of the MSF for all possible single-variable couplings, illustrated in Fig. 3, shows that in all cases except $x \rightarrow x$, $y \rightarrow y$, and $z \rightarrow x$ the maximum Lyapunov exponent remains positive for all $K > 0$. The $y \rightarrow y$ coupling results in an unbounded negative region starting at zero. For $x \rightarrow x$, the MSF is negative only in a range $0 < K < k^*$. Finally, the $z \rightarrow x$ causes a situation

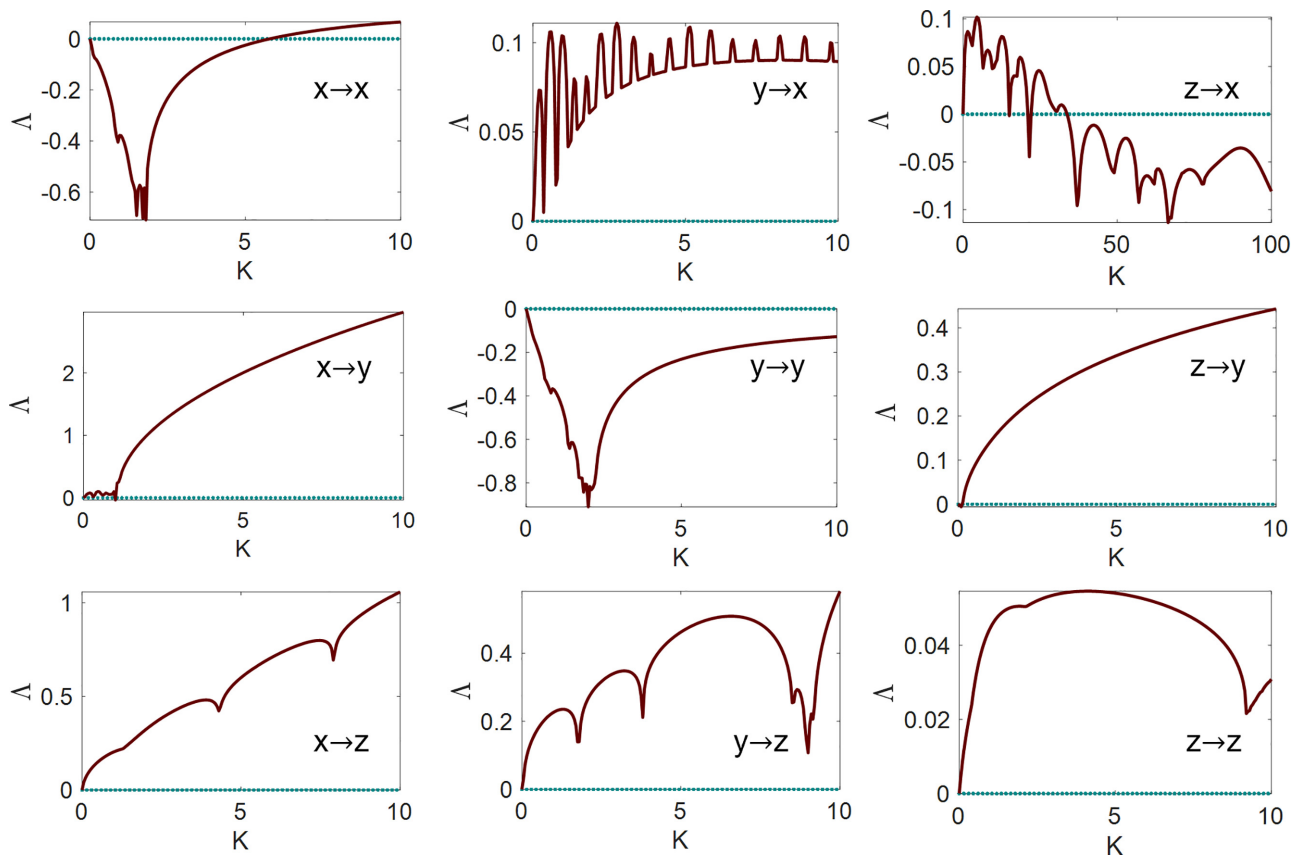


FIG. 3. *New classes of synchronizability for the periodic Rössler oscillator.* The master stability function (Λ) of the periodic Rössler oscillator, Eq. (10), plotted as a function of the generalized coupling strength K , is always positive for all couplings except $x \rightarrow x$, which results in a bounded negative region, $y \rightarrow y$, which yields an unbounded negative region, and $z \rightarrow x$, which causes the appearance of multiple negative intervals. Note how the $z \rightarrow x$ coupling induces a minimum coupling strength after which synchronization is always stable, whereas with the $x \rightarrow x$ coupling a maximum coupling strength emerges after which synchronization is never stable, contrary to the received wisdom about periodic systems. The parameter values are $a = 0.161$, $b = 0.2$, and $c = 9$.

similar to the Hindmarsh-Rose model with $z \rightarrow z$ coupling, with the appearance of multiple finite intervals of stable synchronization, the first of which starts at a positive K_1^* . This confirms the occurrence of a split in Class III, dependent on the sign of the derivative of the MSF at zero: for negative derivatives one obtains a stable region $0 < K < k^*$, whereas for positive derivatives stability happens for $K_1^* < K < K_2^*$, with $K_1^* > 0$.

A similar range of classes of synchronizability also characterizes the behavior of the Lorenz system, which can be always unstable, always stable with a vanishing or nonzero threshold, or with a bounded region of stability starting at a positive coupling strength (see the Supplemental Material [55]).

To check whether the dimensionality of the oscillators plays a role in the emergence of the new synchronizability classes, we studied several two-dimensional systems. We found the most diverse behavior is exhibited by the Brusselator system [56], which is a mathematical model for autocatalytic chemical reactions described by the following system:

$$\begin{aligned} \dot{x} &= a + x^2y - (b + 1)x, \\ \dot{y} &= bx - x^2y. \end{aligned} \tag{11}$$

Choosing $a = 1$ and $b = 3$, we obtain the MSF represented in Fig. 4. Since the model is two-dimensional, we have only four possible single-variable couplings. Notably, each of them produces a different synchronizability behavior: $x \rightarrow x$ results in the MSF being always negative for $K > 0$, $x \rightarrow y$ causes the appearance of an unbounded region of stability after a $K^* > 0$, $y \rightarrow x$ and $y \rightarrow y$ corresponds to a bounded stability region for $0 < K < K^*$, which, in the case of $y \rightarrow x$, is followed by a second one.

A slightly less rich behavior is offered by the unforced undamped Duffing oscillator [57], whose dynamics is given by the system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x - x^3. \end{aligned} \tag{12}$$

In fact, its MSF is either always positive for all $K > 0$, when the coupling is $x \rightarrow y$ or $y \rightarrow x$, or it has multiple intervals of negativity, with the first one starting at zero, when the coupling is $x \rightarrow x$ or $y \rightarrow y$ (Fig. 5).

Similar behaviors are observed in several other two-dimensional periodic systems that we have systematically studied, namely the Lotka-Volterra model, the FitzHugh-Nagumo model, the van der Pol oscillator, the cabbage system, and the Stuart-Landau oscillator (see the

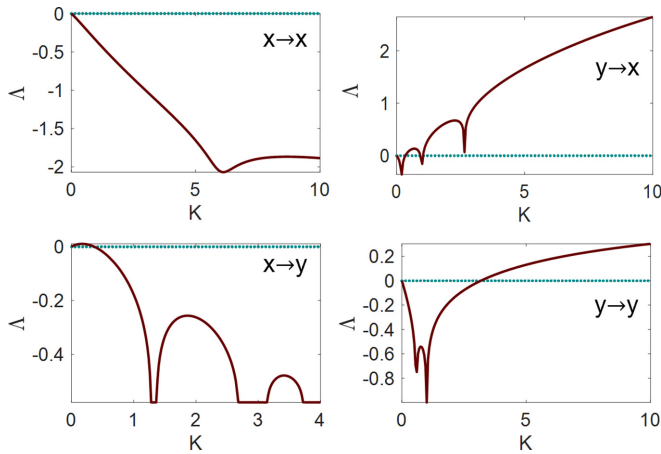


FIG. 4. *New classes of synchronizability for the periodic Brusselator.* The master stability function (Δ) of the periodic Brusselator, Eq. (11), plotted as a function of the generalized coupling strength K , is always negative for $x \rightarrow x$ coupling, negative after a threshold for $x \rightarrow y$, negative before a threshold for $y \rightarrow y$, and has multiple regions of negativity for $y \rightarrow x$. Note that the synchronization class corresponding to the $y \rightarrow x$ coupling is exclusive of a periodic system. Also, the $x \rightarrow y$ coupling requires a minimum coupling strength to achieve stable synchronization, whereas the $y \rightarrow x$ and $y \rightarrow y$ couplings induce a maximum coupling strength, after which synchronization ceases to be stable, in contrast with the current assumptions about periodic systems. The parameter values are $a = 1$ and $b = 3$.

Supplemental Material [55]). In all these cases, we have found the appearance of different synchronizability classes, including split ones.

Note that, in all the cases considered, it is to be expected that, for a given type of coupling, different parameter values will result in a different synchronizability profile. As an example, consider again the Hindmarsh-Rose system. Its

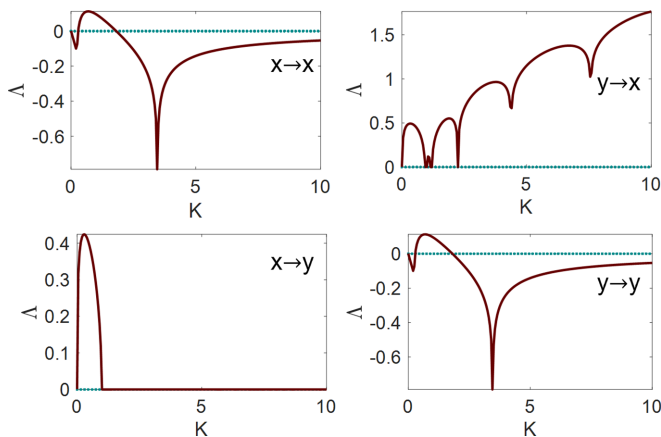


FIG. 5. *New classes of synchronizability for the unforced undamped Duffing oscillator.* The master stability function (Δ) of the unforced undamped Duffing oscillator, defined in Eq. (12), plotted as a function of the generalized coupling strength K , has multiple negative regions for self-couplings, and it is always positive otherwise. Thus, the self-couplings correspond to a new synchronizability class, which is exclusive for periodic systems, and the others contradict the current general assumption that periodic systems synchronize in a stable way for any positive coupling strength.

bifurcation diagram, illustrated in Fig. 6 for the same parameter values as used before and using r as control, shows the existence of multiple transitions between periodic and chaotic dynamics. Studying the MSF for different values of r , one can observe that its qualitative behavior remains unchanged under $x \rightarrow x$, $y \rightarrow x$, $x \rightarrow y$, $x \rightarrow z$, and $y \rightarrow z$ couplings. However, for other coupling schemes, the behavior of the MSF depends on the value of r . Most clearly, in the $y \rightarrow y$ coupling, the MSF can be either always negative, or it can first take on positive values and then turn permanently negative, as synchronization becomes stable (Fig. 6), explicitly showing how a parameter change can switch the sign of the derivative of the MSF at zero and, consequently, alter the stability properties of the synchronized state.

IV. DISCUSSION AND CONCLUSIONS

In summary, we have explicitly shown how the master stability function for periodic networked systems can have a wealth of different behaviors. In particular, class II and class III for chaotic systems, corresponding to unbounded and bounded regions of negativity of the MSF, respectively, split each into two different classes when the systems considered are periodic. This symmetry breaking is caused by the fact that, when the coupling is zero, the MSF is the largest Lyapunov exponent of the uncoupled system, which, in the periodic case, is zero. Thus, there is always at least one point at which the MSF touches the horizontal axis, namely, the point at zero. In turn, this means that it is always possible that $K = 0$ is a threshold value for the MSF, whether the unique one, like in class II, or the lower one, like in class III. The sign of the derivative of the MSF at the origin determines, however, whether, for very small values of the coupling, the function is negative or positive. Therefore, if the derivative is negative, a region of coupling strengths for which the synchronous state is stable starts immediately, whereas synchronization is otherwise unstable for low coupling strengths. Based on these considerations, and in analogy with chaotic systems, we propose the following classification of synchronizability of periodic systems:

Class I. The MSF is positive for all $K > 0$. Thus, synchronization is never stable for any coupling strength.

Class II. The MSF is negative for all $K > 0$. This class, corresponding to class II of chaotic systems with $K^* = 0$, contains systems whose synchronous state is stable for any coupling strength.

Class III. The MSF is negative for $0 < K < K^*$. This class, corresponding to class III of chaotic systems with $K_1^* = 0$ and $K_2^* = K^*$, contains systems whose synchronous state is stable only for nonzero couplings smaller than a threshold.

Class IV. The MSF is negative for $K > K^*$, with $K^* > 0$. This class is exclusive to periodic systems. In fact, even though it resembles class II of chaotic systems, it is to be noted that, in that case, the first point at which the MSF vanishes must have a negative derivative, whereas here the derivative at the first root of the MSF is positive.

Class V. The MSF is negative in a range $K_1^* < K < K_2^*$. Similar to the previous case, this class is typical of periodic systems even though it resembles class III of chaotic ones.

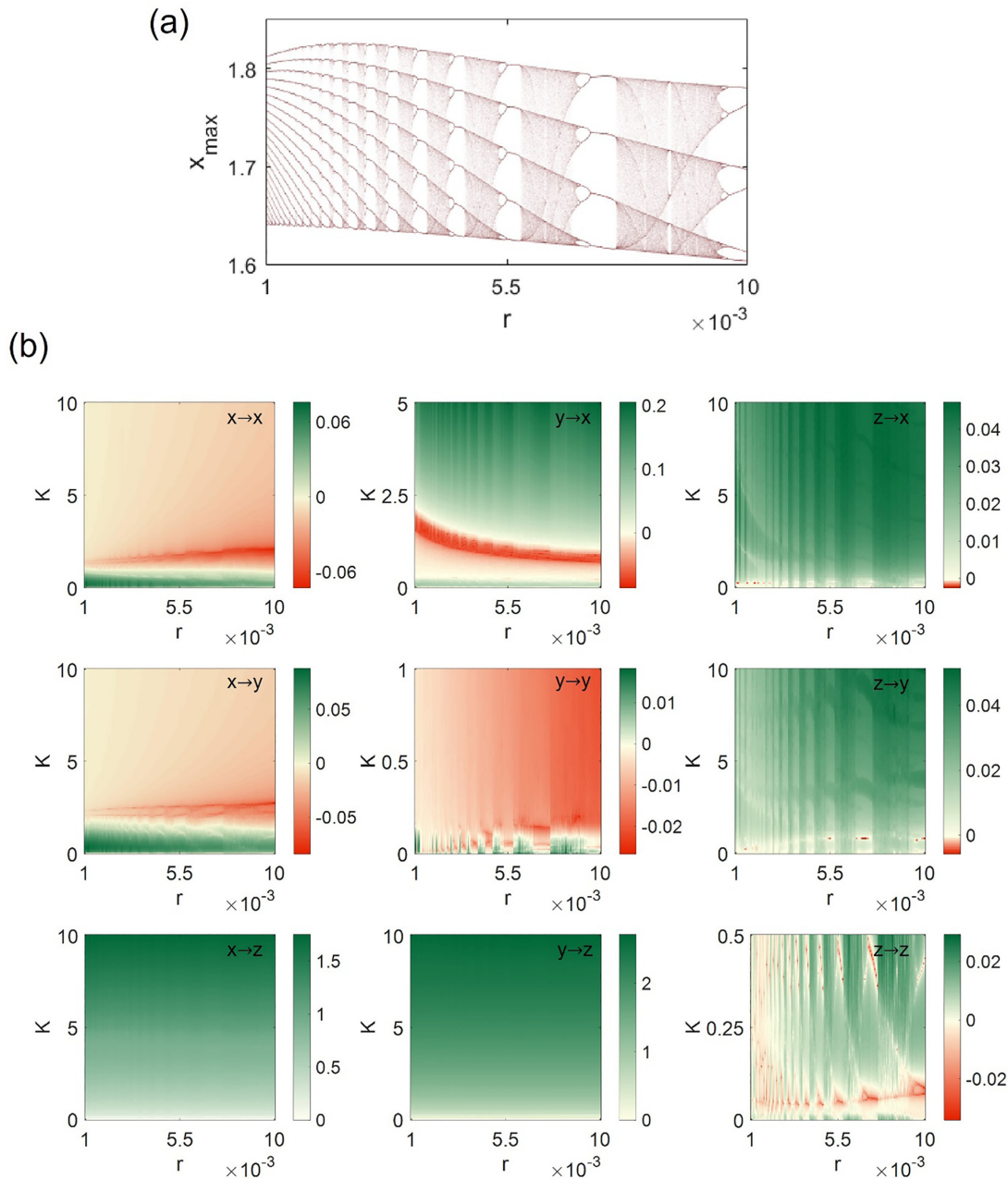


FIG. 6. System parameters can cause a switch in synchronizability properties. (a) The bifurcation diagram of the Hindmarsh-Rose system, Eq. (9), with $a = 1$, $b = 3$, $I = \frac{16}{5}$, $c = 1$, $d = 5$, $s = 4$, and $x_1 = \frac{8}{5}$, plotting the largest value of x (x_{\max}) for different values of r , shows numerous transitions between periodic and chaotic behavior. (b) The MSF (Δ), plotted for different values of r and K in different coupling schemes, shows that in some cases, such as $y \rightarrow y$, the stable synchronizability region can change its profile.

Note that, since the value of the MSF at zero is always zero for periodic systems, this classification is exhaustive, because of the dependence of the new classes on the positivity of the derivative at zero.

Additionally, our results challenge some of the received wisdom about periodic systems. In fact, it was generally believed that periodic systems can achieve a stable synchronized state even for small coupling strengths. However, we have demonstrated that in some cases, such as those falling into class III and class V, too strong a coupling can destroy the stability of synchronization. Even more to the point, the existence of class IV and, again, class V shows that,

sometimes, there is indeed even a nonzero lower threshold for stability. Moreover, classes III and V have some fascinating implications, especially when they feature multiple stability regions with an unbounded final one. In fact, in these cases, systems have to admit a synchronous state that is definitively stable for large enough coupling. However, at the same time, the stability of synchronization may be temporarily lost as the coupling strength increases, before reaching a final threshold. While this is not too surprising in chaotic systems, these behaviors, which we have clearly identified, were believed not to occur in periodic systems, highlighting the value of the master stability function approach in studying

the synchronization of nonlinear systems, regardless of their nature.

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