Convexity in nonlinear integer programming *

Paola Favati and Fabio Tardella

Istituto di Elaborazione dell'Informazione - CNR,
Via S.Maria 46
56100 PISA, ITALY.

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Abstract

We introduce a notion of convexity, called integer convexity, for a function defined on a discrete rectangle X of points in $\mathbf{Z}^n$, by means of ordinary convexity of an extended function defined on the convex hull of X in $\mathbf{R}^n$.

One of the most interesting features of integrally convex functions is the coincidence between their local and global minima.

We also analyze some connections between the convexity of a function on $\mathbf{R}^n$ and the integer convexity of its restriction to $\mathbf{Z}^n$, determining some nontrivial classes of integrally convex functions.

Finally, we prove that a submodular integrally convex function can be minimized in polynomial time over any discrete rectangle in $\mathbf{Z}^n$, thereby extending well-known results of Grotschel, Lovasz and Schrijver and of Picard and Ratliff, and we present an algorithm for this problem together with some computational results.

Key words:

Convexity, integer convexity, nonlinear integer programming, submodularity.

1. Introduction and notations

Owing to the importance of convexity in continuous optimization, various attempts have been made to introduce an analogous concept for functions defined over a discrete set (see refs. [5,11,12,13,15,18]).

The interest for discrete convexity was generally motivated by the fact that local (in some sense) minima of a discretely convex function are also global minima. However, in most cases, not much attention was paid to the problem of determining interesting classes of functions which are discretely convex according to the given definitions.

In this paper we present a rather natural way of extending a function f defined over a discrete rectangle X in $\mathbf{Z}^n$ to a piecewise-linear function $\bar{f}$ defined over the convex hull of X in $\mathbf{R}^n$. A notion of convexity for f is then introduced by simply requiring that $\bar{f}$ is convex in the ordinary sense.

Integer convexity is a global property, but we show that it may be characterized by means of local properties and we exploit these characterizations to determine some nontrivial classes of integrally convex functions.

Furthermore, we show that the problem of minimizing a submodular integrally convex function over a discrete rectangle can be solved in polynomial time and we present a practical algorithm for solving this problem. Finally,we give some computational results on the performance of our algorithm on some randomly generated problems.

We shall now introduce some definitions and notations. A discrete rectangle in $\mathbf{Z}^n$ is a set of the type

$$X = \{x \in \mathbf{Z}^n : a_i \leq x_i \leq b_i, \ i=1,2,\dots,n\},$$

where $a_i, b_i \in \mathbf{Z} \cup \{-\infty,+\infty\}$. A discrete unit hypercube is a discrete rectangle where $b_i = a_i+1$, for $i=1,2, \dots , n$. The convex hull of X, denoted coX, is the set

$$\text{co}\, X = \{x \in \mathbf{R}^n : a_i \leq x_i \leq b_i, \ i=1,2,\dots,n\}.$$

The convex hull of a discrete unit hypercube is called a unit hypercube.The dimension of X, denoted dimX, is the dimension of the smallest affine manifold containing X. Given a point $x \in coX$, we call discrete neighborhood of x the set

$$N(x) = \{z \in \mathbf{Z}^n : | x_i - z_i |<1, i=1,2,...,n\}.$$

Clearly, if $x \in X$, $N(x) = \{x\}$. For any point $x \in \mathbf{R}^n$ we define

$$\| x \|_2 = (\sum_{i=1}^{n} x_i^2)^{1/2} \quad \text{and} \quad \| x \|_\infty = \max_i | x_i |.$$

Given $x \in X$ we call 1-neighborhood of x the set

$$N_1(x) = \{ y \in \mathbf{Z}^n : \| x - y \|_\infty = 1 \}.$$

We say that a point $x \in X$ is a local minimum point for a function $f : X \to \mathbf{R}$ iff

$$f(x) \le f(y), \quad \forall\, y \in N_1(x) \cap X .$$

2. Continuous extension of a discrete function.

Several ways of extending a function f defined over a finite subset S of $\mathbf{R}^n$ to a function defined on a bigger set (usually the convex hull of S or the whole $\mathbf{R}^n$) have been proposed in the literature in connection with the problem of minimizing f over S (see refs. [3,10,14,20,22]). The motivation for the extension is that of reducing a combinatorial optimization problem to a continuous one that is easier to solve or provides useful informations for the solution of the original problem.

In this section we define and study an extension of a function defined over a discrete rectangle in $\mathbf{Z}^n$.

Definition 2.1

Given a discrete rectangle X in $\mathbf{Z}^n$ and a function $f : X \to \mathbf{R}$ we call extension of f the function $\overline{f} : co\,X \to \mathbf{R}$ defined as follows

$$\overline{f}(x) = \min\{\sum_{i=1}^{k} \alpha_i f(z^i) : z^i \in N(x), \sum_{i=1}^{k} \alpha_i z^i = x, \sum_{i=1}^{k} \alpha_i = 1, \alpha_i \ge 0 \},$$

where $k = \text{card}(N(x))$.

Note that we trivially have $\overline{f}(x) = f(x)$ for every x in X. It is also easy to check that the restriction of $\overline{f}$ to any unit hypercube B contained in coX coincides with the convex envelope of f over B (see,e.g. [3]). Alternatively, we may think of $\overline{f}$ as the marginal value function of a linear program over any such unit hypercube B. From both interpretations we deduce that $\overline{f}$ is piecewise linear on coX and convex on every unit hypercube contained in coX .

Proposition 2.1

Let $f,g : X \to \mathbf{R}$, then

(1) $$\overline{f}(x) + \overline{g}(x) \leq (\overline{f+g})(x), \qquad \forall x \in \text{co}X .$$

Furthermore, if over any unit hypercube contained in coX at least one of the functions $\overline{f}(x)$ and $\overline{g}(x)$ is linear, then

(2) $$\overline{f}(x) + \overline{g}(x) = (\overline{f+g})(x), \qquad \forall x \in \text{co}X .$$

Proof.

Inequality (1) is a straightforward consequence of the definition of extension. To prove (2) consider any unit hypercube B contained in coX and suppose that $\overline{g}(x)$ is linear over B. Then applying (1) to the functions f+g and -g we obtain

$$\overline{f}(x) = (\overline{f+g-g})(x) \geq (\overline{f+g})(x) - \overline{g}(x), \qquad \forall x \in B,$$

which, combined with (1) , yields

$$\overline{f}(x) + \overline{g}(x) = (\overline{f+g})(x), \qquad \forall x \in B .$$

Since the above equality holds for every B in coX , (2) follows.

Note that a function $f : X \to \mathbf{R}$ is separable , i.e. $f(x_1, \ldots, x_n) = \sum_{i=1}^{n} f_i(x_i)$, if and only if $\overline{f}$ is linear on all unit hypercubes contained in coX . This implies the following corollary :

Corollary 2.1.1

Let $f,g : X \to \mathbf{R}$ and assume that f is separable. Then (2) holds.

Remark 2.1

With a careful analysis of definition 2.1 it may be seen that (2) holds also when $f(x) = \sum_{i=1}^{m} f_i(x_i)$, where $x_i \in X_i$ and $X = X_1 \times X_2 \times \ldots \times X_m$.

Remark 2.2

Another case in which (2) holds is when f and g belong to a class Φ of functions such that for every $x \in X$ it is possible to find $\alpha_i(x) \in [0,1]$ and $z^i(x) \in N(x)$, for $i = 1, 2, \ldots, k = \mathrm{card}(N(x))$, such that

$$x = \sum_{i=1}^{k} \alpha_i(x)\, z^i(x) \qquad \text{and} \qquad \overline{h}(x) = \sum_{i=1}^{k} \alpha_i(x) h(z^i(x)) ,$$

for every $h \in \Phi$. In section 5 we will show that the class of submodular (or unate) functions enjoys this property.

The problem of minimizing f over X is, in a sense, equivalent to that of minimizing $\overline{f}$ over coX. More precisely we have the following results.

Proposition 2.2

A point $x \in X$ is a local minimum point for $\overline{f}$ over coX iff it is a local minimum point for f over X.

Proof:

Let $S = \{ y \in coX : \| x - y \|_\infty \le 1 \}$. Observe that $S = \bigcup_{i=1}^{2^n} B_i$, where $\{ B_i \}$ is the family of all the unit hypercubes having a vertex in x . Since $\bar{f}$ is convex on every B_i and x is a local minimum point for $\bar{f}$ we have that x is a global minimum point for $\bar{f}$ over every B_i and hence also over S. Therefore

(3) $$f(x) = \bar{f}(x) \le \bar{f}(z) = f(z), \quad \forall z \in S \cap X,$$

thus proving that x is a local minimum point for f over X. Conversely, assume that (3) holds. Given any point $y \in S$, we have $N(y) \subset S \cap X$. Then from (3) and the definition of $\bar{f}$ we derive $\bar{f}(x) \le \bar{f}(y)$. Hence x is a global minimum point for $\bar{f}$ over S and thus it is a local minimum point for $\bar{f}$ over coX .

Proposition 2.3

Let $x \in X$ be a global minimum point for f over X .Then x is a global minimum point for $\bar{f}$ over coX. Furthermore if $y \in coX$ is a global minimum point for $\bar{f}$ over coX , there exists a point x in N(y) which is a global minimum point for f over X and $f(x) = \bar{f}(y)$.

Proof:

Let $x \in X$ be a global minimum point for f over X and let y be any point in coX. Then

$$\bar{f}(y) \ \ge \ \min_{z \in N(y)} f(z) \ \ge \ f(x).$$

Conversely, if y is a global minimum point for $\bar{f}$ over coX then $\bar{f}(y) \le f(x)$, $\forall x \in N(y)$. By definition of $\bar{f}$, this implies $\bar{f}(y) = f(x)$ for at least a point x in N(y).

3. Integer convexity.

A notion of convexity for functions defined over a discrete rectangle in $\mathbb{Z}^n$ may be introduced in a very natural way by means of their continuous extensions .

Definition 3.1.

A function $f : X \to \mathbb{R}$ is called *integrally convex* iff its extension $\overline{f} : \mathrm{co}X \to \mathbb{R}$ is convex .

We have already remarked that the restriction of $\overline{f}$ to any unit hypercube B contained in coX coincides with the convex envelope $\mathrm{co}_B\, f$ of f over B . Exploiting this fact and the properties of convex envelopes (see, e.g., [19] pp. 6-10) it is easy to prove that f is integrally convex on X if and only if $\overline{f}$ coincides with the convex envelope $\mathrm{co}_X\, f$ of f over the whole set X .

Unfortunately, the class of integrally convex functions, as well as the classes of discretely convex functions introduced by Miller [15] and Pieroni- Saviozzi [18] , is not closed under addition . However, if f and g are integrally convex on X and property (2) holds, then f+g is also integrally convex . This implies, e.g., that the sum of two submodular integrally convex functions or of an integrally convex function and a separable integrally convex function is still integrally convex .

In continuous optimization , one of the more useful properties of convex functions is the coincidence between their local and global minima. The notion of integer convexity allows us to extend this property to the case of functions defined on a discrete rectangle. Indeed, if x is a local minimum point for a function f which is integrally convex on X then, by proposition 2.2, x is a local minimum point for $\overline{f}$. Furthermore, since $\overline{f}$ is convex, x is a global minimum point for $\overline{f}$ and hence, a fortiori, for f. We have thus proved the following result:

Proposition 3.1

Let f be an integrally convex function on a discrete rectangle X. If x is a local minimum point for f over X, then x is a global minimum point .

It can be easily verified that if a function f is integrally convex on a discrete rectangle X, then f is convex also according to the definitions given by Miller [15] and Pieroni-Saviozzi [18]. Hence, Proposition 3.1 may be seen as a consequence of the analogous results obtained by these authors.

We shall now present two characterizations of integer convexity that will highlight the local nature of this notion.

Lemma 3.1

Let $a = x_0 < x_1 < .... < x_m = b$ be a partition of the interval $[a,b]$ and let $\alpha_1, \dots, \alpha_m \in [0,1]$ be such that

(5) $$x_i = (1 - \alpha_i)\, x_{i-1} + \alpha_i\, x_{i+1}\ , \qquad i = 1, \dots, m-1.$$

Let $h : [a,b] \to \mathbf{R}$ be a continuous function which is linear on every interval $[x_{i-1}, x_i]$, $i = 1, \dots, m$, and satisfies the condition

(6) $$h(x_i) \le (1 - \alpha_i)\, h(x_{i-1}) + \alpha_i\, h(x_{i+1})\ , \quad i = 1, \dots, m-1.$$

Then h is convex on $[a,b]$.

Proof:

Since h is linear on each interval $[x_{i-1}, x_i]$, there exist $c_i, d_i \in \mathbf{R}$, such that

$$h(x) = c_i x + d_i \qquad \text{on} \qquad [x_{i-1}, x_i], \qquad i = 1, \dots, m.$$

Furthermore from the continuity of h we obtain $c_i x_i + d_i = c_{i+1} x_i + d_{i+1}$, for $i = 1, \dots, m-1$. Hence,

(7) $$d_i - d_{i+1} = (c_{i+1} - c_i)\, x_i\ , \qquad i = 1, \dots, m-1.$$

From (5) and (7) we derive

$$(1-\alpha_i)\,h(x_{i-1}) + \alpha_i\, h(x_{i+1}) = h(x_i) + \alpha_i(c_{i+1} - c_i)(x_{i+1} - x_i) \quad , i = 1, \ldots, m-1.$$

Hence assumption (6) implies $(c_{i+1} - c_i) \geq 0$, $i = 1, \ldots, m-1$.

We have thus proved that $c_1 \leq c_2 \leq \ldots \leq c_m$. This trivially implies that h is convex .

It is a straightforward consequence of the definition that, if a function is integrally convex on a discrete rectangle X , then it is integrally convex on all the discrete rectangles contained in X . Hence, if a function is integrally convex on Z^n, then it is integrally convex on every discrete rectangle in Z^n. Conversely, we will now show that if a function f is integrally convex on all the discrete rectangles with edges of length at most 2 contained in a discrete rectangle X, then f is integrally convex on X .

Proposition 3.2

The following properties are equivalent:

(8a) f is integrally convex on X.

(8b) f is integrally convex on $N_1(x) \cap X$, for every x in X.

Proof:

(8a) $\Rightarrow$ (8b): Obvious. (8b) $\Rightarrow$ (8a) : Given x^1 , $x^2 \in$ co$X$ and $\lambda \in [0,1]$, we have to prove that $\bar{f}(x(\lambda)) \leq (1-\lambda)\ \bar{f}(x^1) + \lambda\ \bar{f}(x^2)$, where $x(\lambda) = (1-\lambda)\, x^1 + \lambda\, x^2$. We shall do so by showing that $\varphi(\lambda) = \bar{f}(x(\lambda))$ is convex on $[0,1]$. Let us observe that $\varphi(\lambda)$ is piecewise linear and continuous and define $I = \{ i:\ x^1_i \neq x^2_i \}$, $A = \{ \lambda \in [0,1] : \exists\, i \in I, x_i(\lambda) \in \mathbb{Z} \}$ and $B = \{ \lambda \in [0,1] : x(\lambda)$ is a breakpoint for $\varphi(\lambda) \}$.

Let $0 = \lambda_0 < \lambda_1 < \ldots < \lambda_m = 1$ be the partition of $[0,1]$ determined by the points of $A \cup B$. Note that $\varphi(\lambda)$ is linear on $[\lambda_{i-1}, \lambda_i]$, for $i = 1, \ldots, m$.

Furthermore, it is possible to find points $v^1, \ldots, v^{m-1}$ in X, not necessarily distinct, such that

$$x(\lambda_{i-1}),\ x(\lambda_i),\ x(\lambda_{i+1}) \in \mathrm{co}(N_1(v^i) \cap X), \qquad i = 1,\ldots,m-1.$$

By assumption $\bar{f}$ is convex on every rectangle $\mathrm{co}(N_1(v^i) \cap X)$ and hence $\varphi(\lambda)$ satisfies condition (6). The convexity of φ is then guaranteed by a straightforward application of lemma 3.1.

In order to provide a second characterization of integer convexity we need to introduce the following lemma.

Lemma 3.2.

Let $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_m = 1$ and let $c_i = (1 - \alpha_i)\, a + \alpha_i\, b$, $i = 0,\ldots, m$, where $a, b \in \mathbb{R}$, $a<b$. Let $h : [a,b] \to \mathbb{R}$ be a function which is convex on each subinterval $[c_{i-1}, c_i]$, $i = 1,\ldots, m$, and satisfies

$$h(c_i) \leq (1 - \alpha_i)\, h(a) + \alpha_i\, h(b), \qquad i = 0,\ldots, m.$$

Then, for every α in $[0,1]$, we have

$$h((1-\alpha)\, a + \alpha\, b) \leq (1-\alpha)\, h(a) + \alpha\, h(b).$$

Proof:

Given $\alpha \in [0,1]$ we can find $\beta \in [0,1]$ and an index k, with $0 \leq k \leq m-1$, such that

$$\alpha = (1 - \beta)\, \alpha_k + \beta\, \alpha_{k+1}.$$

Hence, setting $x(\alpha) = (1-\alpha)\, a + \alpha\, b$, we obtain

$$x(\alpha) = (1-\beta)\, c_k + \beta\, c_{k+1}.$$

From the assumptions on h we then derive

$$h(x(\alpha)) \leq (1 - \beta)\, h(c_k) + \beta\, h(c_{k+1}) \leq$$
$$\leq [(1-\beta)(1 - \alpha_k) + \beta(1 - \alpha_{k+1})]h(a) + [(1-\beta)\, \alpha_k + \beta\, \alpha_{k+1}]h(b) =$$
$$= (1-\alpha)\, h(a) + \alpha\, h(b).$$

Proposition 3.3

The following properties are equivalent:

(9a) f is integrally convex on X.

(9b) for every z^1, z^2 in X with $\| z^1 - z^2 \|_\infty = 2$ we have

$$\bar{f}\,((z^1 + z^2)/2) \le (f(z^1) + f(z^2))/2.$$

Proof:

(9a) $\Rightarrow$ (9b): Obvious. (9b) $\Rightarrow$ (9a) : We procede by induction on the dimension of X. If dimX = 1 the thesis follows easily from lemma 3.1. We then assume that (9b) $\Rightarrow$ (9a) if dimX = m-1, and prove that the implication is still true when dimX = m. More precisely, we shall prove the equivalent implication (9b) $\Rightarrow$ (8b). Let $v \in X$ and set

$$S = co(N_1(v) \cap X), \quad S_j = \{x \in S : x_j = v_j\}.$$

We will prove that for all x,y in S and α in [0,1] we have

(10) $$\bar{f}((1-\alpha)\,x + \alpha\, y) \le (1-\alpha)\,\bar{f}(x) + \alpha\,\bar{f}(y).$$

Step1. Assume $x \in S \cap X$ and $y \in S$. Let [x,y] denote the line segment joining x and y and define $J = \{ j : S_j \cap [x,y] \neq \emptyset,\ S_j \neq S\}$. If J is empty , it is easy to see that there is a unit hypercube in S containing x and y . In this case $\bar{f}$ is trivially convex on [x,y] and hence (10) holds. If, for some j in J, $[x,y] \subset S_j$, then (10) holds because $\bar{f}$ is convex on S_j by the induction assumption. Suppose now that $J \neq \emptyset$ and that, for all $j \in J$, $[x,y] \not\subset S_j$. Then, for all $j \in J$, [x,y] and S_j intersect at a single point r^j which can be expressed in the form

$$r^j = (1-\alpha_j)\,x + \alpha_j\, y,$$

where α_j is an appropriate scalar in [0,1]. Let the elements of J be ordered in such a way that $0 < \alpha_{j_1} < \ldots < \alpha_{j_k} < 1$, then, since for every i there is a unit hypercube containing r^{j_i} and $r^{j_{i+1}}$ f is convex on all the intervals $[r^{j_i}, r^{j_{i+1}}]$ for i = 0, ...,k+1, where $r^{j_0} = x$ and $r^{j_{k+1}} = y$. Hence, if we prove that

(11) $$f(r^j) \leq (1-\alpha_j)\,\overline{f}(x) + \alpha_j\,\overline{f}(y), \quad \forall j \in J,$$

condition (10) follows trivially from lemma 3.2.

Given j in J we now prove (11). By definition of $\overline{f}$ we can find scalars β_i in $[0,1]$, for $i = 1, ..., N$, such that

(12) $$y = \sum_{i=1}^{N} \beta_i z^i, \quad \overline{f}(y) = \sum_{i=1}^{N} \beta_i f(z^i), \quad \sum_{i=1}^{N} \beta_i = 1,$$

where $z^1, ..., z^N$ are the points of the discrete neighborhood $N(y)$. Let us consider the epigraph E of $\overline{f}$ over S

$$E = \{ (s,t) \in S\text{x}\mathbf{R} : t \geq \overline{f}(s) \}.$$

Since $\overline{f}$ is convex on S_j we have that the intersection of E with $\overline{S}_j = S_j \times \mathbf{R}$ is convex . From (12) we deduce that the line segment joining $(x, \overline{f}(x))$ and $(y, \overline{f}(y))$ in $S \times \mathbf{R}$ is contained in the convex cone K generated by the following halflines

$$R_i = \{ (x, \overline{f}(x)) + \lambda\ (z^i - x, \ \overline{f}(z^i) - \overline{f}(x)) : \lambda \in \mathbf{R}_+ \}, \ i = 1, ..., N.$$

Note that $R_i \cap \overline{S}_j = \{ u^i \}$, $i = 1, ..., N$, where

$$u^i = \begin{cases} (z^i, \overline{f}(z^i)) & , \text{ if } z^i \in S_j \\ 1/2\,(x + z^i, \overline{f}(x) + \overline{f}(z^i)) & , \text{ if } z^i \notin S_j . \end{cases}$$

If $z^i \in S_j$, then trivially

(13) $$u^i \in E \cap \overline{S}_j.$$

If $z^i \notin S_j$, it may be noticed that $\| x - z^i \|_\infty = 2$ and then (13) is still true because of assumption (9b). Observe finally that, since $[(x, \overline{f}(x)), (y, \overline{f}(y))] \subset K$, we have

$$((1-\alpha_j)\, x + \alpha_j\, y\, ,(1-\alpha_j)\ \overline{f}(x) + \alpha_j\ \overline{f}(y)) \in E \cap \overline{S}_j.$$

Hence, since $r^j = (1-\alpha_j)\, x + \alpha_j\, y$, condition (11) holds.

Step 2. Assume $x,y \in S$.

Note that in step 1 the assumption $x \in X$ has been used only to guarantee the validity of (13) when $z^i \notin S_j$, by using condition (9b). The argument employed in step 1 can therefore be applied to the general case $x,y \in S$, if condition (13) can be shown to hold also in this case. Observe that, if $x,y \in S$ and $z^i \notin S_j$, we have

$$u^i = ((1-\delta)\, x + \delta\ z^i\, , (1-\delta)\ \overline{f}(x) + \delta\ \overline{f}(z^i)),$$

where δ is an appropriate scalar in $[0,1]$. In view of step 1 we can now conclude that

$$\overline{f}\ ((1-\delta)\, x + \delta\ z^i\,) \le (1-\delta)\ \overline{f}(x) + \delta\ \overline{f}(z^i)),$$

so that (13) holds.

Remark 3.1

Note that, if X is bounded, proposition 3.3 allows us to establish whether a given function is integrally convex on X by checking only a finite number of inequalities.

4. Convexity and integer convexity.

In this section we shall examine some relationships between the convexity of a function defined on the convex hull of the discrete rectangle $X=\{x\in \mathbb{Z}^n : a_i \leq x_i \leq b_i, i=1, ..., n\}$ and the integer convexity of its restriction to X. The following examples show that, without additional assumptions, these two properties are not related to each other.

Example 4.1

Let $X = \{(x_1, x_2) \in \mathbb{Z}^2 : 0 \leq x_1 \leq 2, 0 \leq x_2 \leq 1\}$ and $f(x_1, x_2) = 5x_1^2+14x_2^2 - 12x_1x_2$. It is easy to check that f is a strictly convex quadratic function . However the restriction of f to X is not integrally convex. Indeed we have

$$\bar{f}(1/2[(0,0)+(2,1)]) = 1/2\ (f(1,0)+f(1,1)) > 1/2\ (f(0,0)+f(2,1)),$$

where $\bar{f}$ is the extension of f.

Example 4.2

Let X be defined as in the previous example and let $f(x_1, x_2) = 10x_1^2 - x_2^2$. Clearly f is not convex on coX, but its restriction to X is integrally convex .

It is true, however, that a quadratic function which is integrally convex on $\mathbb{Z}^n$ is also convex in the ordinary sense.This fact is proved in the following proposition.

Proposition 4.1

Let $f(x) = x^TCx + d^Tx$ be integrally convex on $\mathbb{Z}^n$. Then f is convex on $\mathbb{R}^n$.

Proof:

Assume ab absurdo that f is not convex. Then there must be a vector $q\in \mathbb{R}^r$ such that $q^TCq < 0$. Because of the density of $\mathbb{Q}^n$ in $\mathbb{R}^n$, we can find $q_0\in \mathbb{Q}$ such that $q_0^T C q_0 < 0$. Clearly, it is possible to multiply q_0 by an appropriate positive integer scalar λ so that $z_0 = \lambda q_0 \in \mathbf{Z}^n$. Observe now that by the integer convexity of f we have $f(z_0) \leq 1/2\,(f(0) + f(2\,z_0))$. Hence $z_0^T C z_0 \geq 0$, contradicting the inequality $z_0^T C z_0 = \lambda^2 q_0^T C q_0 < 0$.

We shall now examine some simple special cases in which the ordinary convexity of a function on coX implies the integer convexity of its restriction to X.

Proposition 4.2

If $\dim X = 1$ and the function $f : \mathrm{co}X \to \mathbf{R}$ is convex, then its restriction $f : X \to \mathbf{R}$ is integrally convex .

Proof:

Given $z^1, z^2 \in X$, with $\| z^1 - z^2 \|_\infty$, we have $1/2(z^1 + z^2) \in X$ and hence $\overline{f}\,(1/2(z^1 + z^2)) = f\,(1/2(z^1 + z^2))$. Since f is convex on coX , condition (9b) holds and thus f is integrally convex on X .

Proposition 4.3

Let $f(x_1,..., x_n) = \sum_{i=1}^{n} f_i(x_i)$ be a separable function from coX to $\mathbf{R}$. If f is convex on coX, then its restriction to X is integrally convex.

Proof:

First of all, observe that f is convex , if and only if f_i is convex for $i = 1, \ldots ,n$.

By proposition 4.2 , this implies that the restriction of f_i to $X_i = [a_i , b_i] \cap \mathbf{Z}$ is integrally convex for $i = 1, \ldots ,n$. In view of remark 2.1 we also have

$$\overline{\sum_{i=1}^{n} f_i(x_i)} = \sum_{i=1}^{n} \overline{f}_i(x_i) .$$

Hence, the restriction of f to X is integrally convex .

We shall now restrict our attention to the class of quadratic functions. Let C be a nxn symmetric matrix and let $d \in \mathbb{R}^n$. Consider the function $f(x) = x^T C x + d^T x$ and the function $\varphi_C : (\mathbb{R}^n)^k \times [0,1]^{k-1} \rightarrow \mathbb{R}$ defined as follows

(14) $$\varphi_C(x^1, ..., x^k ; \alpha_1, ..., \alpha_{k-1}) = \sum_{i=1}^{k} \alpha_i (x^i)^T C x^i - \sum_{i,j=1}^{k} \alpha_i \alpha_j (x^i)^T C x^j ,$$

where $x^i \in \mathbb{R}^n$, $i=1, ..., k$, $\alpha_i \in [0,1]$, $i=1, ..., k-1$ and $\alpha_k = 1 - \sum_{i=1}^{k-1} \alpha_i$.

Observe that, when $\sum_{i=1}^{k-1} \alpha_i \leq 1$, $\varphi_C(x^1, ...,x^k;\alpha_1, ..., \alpha_{k-1})$ may be interpreted as the convexity gap of the function f at the point $x= \sum_{i=1}^{k} \alpha_i x^i$ with respect to the convex combination of values of f determined by $\alpha_1, ... ,\alpha_k$ and $x^1 , ..., x^k$. In fact we have

(15) $$\sum_{i=1}^{k} \alpha_i f(x^i) - f(\sum_{i=1}^{k} \alpha_i x^i) = \varphi_C(x^1, ..., x^k ; \alpha_1, ..., \alpha_{k-1}) .$$

Remark 4.1

Taking into account equality (15) and proposition 3.3, we observe that f is integrally convex on a discrete rectangle X iff for all $z^1, z^2 \in X$, such that $\| z^1 - z^2 \|_\infty = 2$, one has

(16) $$\min \{\varphi_C(y^1, ..., y^k; \alpha_1 , ... , \alpha_{k-1}) : \alpha_i \in [0 , 1] , \sum_{i=1}^{k-1} \alpha_i \leq 1\} \leq \varphi_C(z^1, z^2 ; 1/2),$$

where $y^1, ..., y^k$ denote the points of $N(1/2(z^1 + z^2))$.

Proposition 4.4

Let φ_C be defined as in (14). Then

$$(17) \qquad \varphi_C(x^1, \ldots, x^k ; \alpha_1, \ldots, \alpha_{k-1}) = 1/2 \sum_{i,j=1}^{k} \alpha_i \alpha_j (x^i - x^j)^T C (x^i - x^j) .$$

Proof:

Replacing $\alpha_k = 1 - \sum_{i=1}^{k-1} \alpha_i$ in (14), we find $\varphi_C(x^1, \ldots, x^k; \alpha_1, \ldots, \alpha_{k-1}) =$

$$= \sum_{i=1}^{k-1} \alpha_i (x^i - x^k)^T C (x^i - x^k) - \sum_{i,j=1}^{k-1} \alpha_i \alpha_j ((x^i)^T C x^j + (x^k)^T C x^k - 2 (x^i)^T C x^k) =$$

$$= \sum_{i=1}^{k-1} \alpha_i (x^i - x^k)^T C (x^i - x^k) - \sum_{i,j=1}^{k-1} \alpha_i \alpha_j ((x^i - x^k)^T C (x^i - x^k) + (x^i)^T C (x^j - x^i)).$$

Observing that

$$\sum_{i,j=1}^{k-1} \alpha_i \alpha_j (x^i - x^k)^T C (x^i - x^k) = \sum_{i=1}^{k-1} \alpha_i (1 - \alpha_k)(x^i - x^k)^T C (x^i - x^k) \qquad \text{and}$$

$$\alpha_i \alpha_j (x^i)^T C (x^j - x^i) + \alpha_j \alpha_i (x^j)^T C (x^i - x^j) = - \alpha_i \alpha_j (x^i - x^j)^T C (x^i - x^j)$$

we obtain

$$\varphi_C(x^1, \ldots, x^k; \alpha_1, \ldots, \alpha_{k-1}) =$$

$$= \sum_{i=1}^{k-1} \alpha_i \alpha_k (x^i - x^k)^T C (x^i - x^k) + 1/2 \sum_{i,j=1}^{k-1} \alpha_i \alpha_j (x^i - x^j)^T C (x^i - x^j) =$$

$$= 1/2 \sum_{i,j=1}^{k} \alpha_i \alpha_j (x^i - x^j)^T C (x^i - x^j) .$$

Remark 4.2

From proposition 4.4 it follows that the value of φ_C does not change if we translate all points $x^1, \ldots, x^k$ by a vector x. Hence by (15), the convexity gap of f is also invariant under translation. This observation together with remark 4.1 implies that f is integrally convex on a discrete rectangle X iff for all $z \in \mathbb{Z}^n$ with $\| z \|_\infty = 2$ one has

$$(18) \quad \min \{\varphi_C(y^1, \ldots, y^k ; \alpha_1, \ldots, \alpha_{k-1}) : \alpha_i \in [0, 1], \sum_{i=1}^{k-1} \alpha_i \le 1\} \le \varphi_C(0, z ; 1/2),$$

where $y^1, \ldots, y^k$ denote the points of $N(z/2)$, or equivalently iff

$$(19) \qquad \bar{f}(z/2) \le (f(0) + f(z))/2.$$

We are now ready to prove that convex diagonally dominant quadratic functions are integrally convex on $\mathbb{Z}^n$.

Proposition 4.5

Let $f(x) = x^T C x + d^T x$, where $C \in \mathbf{R}^{n \times n}$ and $d, x \in \mathbf{R}^n$. Assume that C is convex and diagonally dominant, i.e.

$$(20) \qquad \sum_{\substack{j=1 \\ j \ne i}}^{n} | c_{ij} | \le c_{ii}, \quad i = 1, \ldots, n.$$

Then f is integrally convex on $\mathbb{Z}^n$.

Proof:

In view of remark 4.2, we only have to prove that for every $z \in \mathbf{Z}^n$ such that $\| z \|_\infty = 2$, inequality (19) holds. Given any $z \in \mathbf{Z}^n$, with $\| z \|_\infty = 2$, let us consider the sets $I = \{i : z_i = 2\}$ and $J = \{i : z_i = 1\}$. Then

$$N(z/2) = \{ y^K : y^K = \sum_{i \in I \cup K} e^i , K \subset J \} .$$

Let $\wp(J)$ denote the set of all subsets of J and let $\alpha \in [0,1]^{\wp(J)}$, i.e. $\alpha = (\alpha_K)_{K \subset J}$

Consider the set $\Lambda = \{ \alpha \in [0,1]^{\wp(J)} : \alpha_K = \alpha_{J \setminus K} , K \subset J \}$ and define

$$\tilde{f}(z/2) = \min \{ \sum_{K \subset J} \alpha_K f(y^K) : \sum_{K \subset J} \alpha_K = 1 , \alpha \in \Lambda \} .$$

It is easy to see that $\bar{f}\ (z/2) \leq f(z/2)$. In order to complete the proof it is then sufficient to show that

$$2 \tilde{f}(z/2) \leq f(0) + f(z) .$$

Given any $\alpha \in \Lambda$ with $\sum_{K \subset J} \alpha_K = 1$, one has

$$z/2 = \sum_{K \subset J} 2\alpha_K z/2 = \sum_{K \subset J} 2\alpha_K (y^K + y^{J \setminus K})/2 .$$

Observe that

$$\text{(21)} \qquad \tilde{f}(z/2) = \min \{ \sum_{K \subset J} 2\alpha_K (f(y^K) + f(y^{J \setminus K}))/2 : \sum_{K \subset J} 2\alpha_K = 1, \ \alpha \in \Lambda \} .$$

Hence, we can find a subset $K' \subset J$ such that

$$\text{(22)} \qquad \tilde{f}(z/2) = (f(y^{K'}) + f(y^{J \setminus K'}))/2 .$$

After some calculations we obtain

$$f(0) + f(z) - 2 \tilde{f}(z/2) = 2 (\sum_{\substack{i,j \in I \\ j \in J}} c_{ij} + \sum_{\substack{i \in I \\ j \in J \setminus K'}} c_{ij} + \sum_{i \in K'} c_{ij}) .$$

From (20) we deduce

$$\sum_{i,j \in I} c_{ij} + \sum_{\substack{i \in I \\ j \in J}} c_{ij} \geq 0 .$$

Furthermore, form (21) and (22) we derive

$$2 \sum_{\substack{i \in K' \\ j \in J \setminus K'}} c_{ij} = f(y) + f(y^J) - (f(y^{K'}) + f(y^{J \setminus K'})) \geq 0 .$$

This completes the proof .

Remark 4.3

When f is a function of two variables, i.e. n=2 , in the previous proof one clearly has Card(J) ≤ 1 and $\tilde{f}(z/2) = \bar{f}(z/2)$. Hence

$$f(0) + f(z) - 2\ \bar{f}(z/2) = 2\ \Big(\sum_{i,j\in I} c_{ij} + \sum_{\substack{i\in I \\ j\in J}} c_{ij} \Big)\ .$$

This implies that a quadratic function is integrally convex on $\mathbf{Z}^2$, if and only if (20) holds. On the other hand, when $n > 2$ condition (20) is only sufficient for integer convexity. To see this, note that the function $f(x) = (x_1 - x_2 - x_3)^2$ does not satisfy condition (20). However, taking into account remark 4.2 , one can verify, by checking only few inequalities, that f is integrally convex on $\mathbf{Z}^3$.

Another case where a convex quadratic function f is also integrally convex is when the ratio between the maximum and the minimum eigenvalue of the hessian matrix of f is not too large. In order to prove this we need the following lemmas.

Lemma 4.6

Let $a \in \mathbf{R}^m$ and consider the set $W = \{-1, 0, 1\}^m$. Then

$$(23) \qquad \sum_{w\in W} (a^T w)^2 = 2 \times 3^{m-1} \sum_{i=1}^{m} a_i^2.$$

Proof:

Given $w \in W$, define $I(w) = \{ i : w_i \neq 0 \}$. Then

$$(a^T w)^2 = \sum_{i\in I(w)} a_i^2 + \sum_{i,j\in I(w)} a_i a_j w_i w_j\ .$$

Observe that the sets

$W^+(r,s) = \{ w \in W : w_r w_s = 1\}$ and $W^-(r,s) = \{ w \in W : w_r w_s = -1\}$

have the same cardinality. Hence, we obtain

$$\sum_{w \in W} (a^T w)^2 = \sum_{w \in W} \sum_{i \in I(w)} a_i^2.$$

Equality (23) follows easily noting that $\mathrm{Card}(\{ w \in W : w_i \neq 0\}) = 2 \times 3^{m-1}$, for $i = 1, \ldots, n$.

Let us consider the function $f(x) = x^T C x + d^T x$ where C is a positive semidefinite matrix. Let $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ be the eigenvalues of C and let $\{v^1, \ldots, v^n\}$ be a corresponding orthonormal system of eigenvectors. Note that we have

$$C = \sum_{i=1}^{n} \lambda_i v^i (v^i)^T . \tag{24}$$

Consider the vector $q = (q_1, \ldots, q_n)$ defined as follows

$$q_i = \sum_{k=1}^{n} \lambda_k (v_i^k)^2$$

and let $\sigma : \{1, \ldots, n\} \rightarrow \{1, \ldots, n\}$ be a bijection such that

$$q_{\sigma(i)} \geq q_{\sigma(i+1)}, \quad i = 1, \ldots, n-1. \tag{25}$$

Lemma 4.7

Let $U = \{ u \in \mathbb{Z}^n : \| u \|_\infty = 2$ and $\exists\, i$ such that $\| u \|_\infty = 1 \}$. If

$$\min_{u \in U} u^T A u / \| u \|_2^2 \geq \max \{1/5\, q_{\sigma(1)} , 1/8\, (q_{\sigma(1)} + q_{\sigma(2)})\},$$

then f is integrally convex on $\mathbb{Z}^n$.

Proof:

Let $z \in \mathbb{Z}^n$ be such that $\| z \|_\infty = 2$. Define $J = \{i : z_i = 1\}$ and $m = \mathrm{card}(J)$. Observe that if $J = \emptyset$, then $z/2 \in \mathbb{Z}^n$. In this case f is trivially integrally convex , since $N(z/2) = \{z/2\}$ and f is convex . Assume now that $J \neq \emptyset$. Then $z \in U$. From proposition 4.4 we obtain

$$\varphi_C(0,z;\ 1/2) = 1/4\ z^T C\, z$$

Hence, since $\| z \|_2^2 \geq 4 + m$,

$$\varphi_C(0,z;\ 1/2) \geq (4 + m)/4 \min_{u \in U} u^T C u / \| u \|_2^2 .$$

Let $y^1, ..., y^k$ denote the elements of the discrete neighborhood $N(z/2)$. Observe that $k = 2^m$ and that $y^i - y^j \in \{-1, 0, 1\}^n$, $i,j = 1, ..., k$. Furthermore

(26) $$y_h^i - y_h^j = 0 \ , \qquad h \notin J .$$

Hence $y^i - y^j$ has at most m nonzero entries for every $i,j = 1, ..., k$. Note that we trivially have

$$\min\{\varphi_C(y^1, ..., y^k ; \alpha_1, ... ,\alpha_{k-1}) : \alpha_i \in [0,1] , \sum_{i=1}^{k-1} \alpha_i \leq 1\} \leq \varphi_C(y^1, ..., y^k; 1/k, ..., 1/k).$$

Furthermore, from proposition 4.4 and relation (24) we obtain

$$\varphi_A(y^1, \ldots, y^k ; 1/k, \ldots, 1/k) = 1/2k^2 \sum_{i,j=1}^{k} (y^i - y^j)^T C (y^i - y^j) =$$

$$= 1/2k^2 \sum_{i,j=1}^{k} (y^i - y^j)^T \left(\sum_{s=1}^{n} \lambda_s v^s (v^s)^T\right) (y^i - y^j) = 1/2k^2 \sum_{s=1}^{n} \lambda_s \sum_{i,j=1}^{k} ((y^i - y^j)^T v^s)^2.$$

Taking into account (26) and lemma 4.6 we find

$$\sum_{i,j=1}^{k} ((y^i - y^j)^T v^s)^2 = 2 \times 3^{m-1} \sum_{h \in J} (v_h^s)^2, \qquad s = 1, \ldots, n.$$

Thus, replacing $k = 2^m$,

$$\varphi_A(y^1, \ldots, y^k ; 1/k, \ldots, 1/k) = 3^{m-1}/2^{2m} \sum_{h \in J} \sum_{s=1}^{n} \lambda_s (v_h^s)^2 = 3^{m-1}/2^{2m} \sum_{h \in J} q_h \leq$$

$$\leq 3^{m-1}/2^{2m} \sum_{h=1}^{m} q_{\sigma(h)} \,.$$

where the last inequality follows from relation (25). Hence a sufficient condition for (18) to hold is that

$$\min_{u \in U} u^T A u / \| u \|_2^2 \geq 3^{m-1}/(4+m)\, 4^{m-1} \sum_{h=1}^{m} q_{\sigma(h)}, \quad m = 1, \ldots, n-1.$$

In order to complete the proof we will now prove that

$$\max_{m} \; 3^{m-1}/(4+m)\, 4^{m-1} \sum_{h=1}^{m} q_{\sigma(h)} = \max \{1/5\, q_{\sigma(1)}, 1/8\, (q_{\sigma(1)} + q_{\sigma(2)})\}.$$

Observe that, given the scalars $c_1 \leq c_2 \leq \ldots \leq c_n$, we have

$$(m+1) \sum_{h=1}^{m} c_h \geq m \sum_{h=1}^{m+1} c_h \,, \qquad m = 1, \ldots, n-1.$$

Hence, the function

$$g(m) = 3^{m-1} / (4+m)\, 4^{m-1} \sum_{h=1}^{m} c_h$$

is decresing for $m \geq 2$. Taking $c_h = q_{\sigma(h)}$, $h = 1, ..., n$, we obtain

$$\max_m \; 3^{m-1} / (4+m)\, 4^{m-1} \sum_{h=1}^{m} q_{\sigma(h)} = \max_m \; g(m) = \max \{g(1), g(2)\} =$$

$$= \max \{1/5\, q_{\sigma(1)},\ 1/8\, (q_{\sigma(1)} + q_{\sigma(2)})\}.$$

Proposition 4.8

If $\lambda_1 \geq 1/4\, \lambda_n$, then f is integrally convex on $\mathbb{Z}^n$.

Proof:

Observe that , with the notations of the previous lemma, we have

$$\min_{u \in U} \; u^T A u / \| u \|_2^2 \geq \lambda_1$$

Furthermore, since $q_i = \sum_{k=1}^{n} \lambda_k (v_i^k)^2$ and $\sum_{k=1}^{n} (v_i^k)^2 = 1$, $i = 1, ..., n$, q_i is a convex combination of the eigenvalues $\lambda_1, ..., \lambda_n$, for $i = 1, ..., n$. Hence $q_i \leq \lambda_n$, for $i = 1, .., n$ and thus

$$\max \{1/5\, q_{\sigma(1)},\ 1/8\, (q_{\sigma(1)} + q_{\sigma(2)})\} \leq 1/4\, \lambda_n .$$

5. Submodularity and integer convexity : a polynomial algorithm.

It is well known that the problem of minimizing a function over a discrete rectangle is computationally difficult even in very special cases. The problem of minimizing a quadratic function over $X = \{0,1\}^n$, e.g., is NP-hard (see [4]).

Also, it may be remarked that every function is vacuously integrally convex over any discrete unit hypercube. Therefore, we certainly cannot hope to minimize an integrally convex function over a discrete rectangle in polynomial time unless we make some additional assumption.

In this section, extending the well-known result [7] that a submodular function can be minimized in polynomial time over $X = \{0,1\}^n$, we show that an integrally convex submodular function can be minimized in polynomial time over any bounded discrete rectangle X in $\mathbf{Z}^n$. Furthermore, when f is quadratic, a similar result holds without the boundedness assumption. Our results also improve on the pseudopolynomial-time solvability result established by Picard and Ratliff [17] for the problem of minimizing a quadratic convex diagonally dominant submodular function on X.

Let us recall some definitions . Given $x,y \in \mathbf{R}^n$, the points $x \vee y$ and $x \wedge y$ are defined as follows

$$(x \vee y)_i = \max \{x_i ,y_i \} \qquad \text{and} \qquad (x \wedge y)_i = \min \{x_i ,y_i \} .$$

A real function f defined on a discrete rectangle X in $\mathbf{Z}^n$ is called submodular iff

$$f(x \vee y) + f(x \wedge y) \leq f(x) + f(y) , \qquad \forall\ x,y \in X .$$

The papers by Topkis [23] and Lovasz [14] contain a very good introduction to the theory of submodularity.

An interesting feature of submodular functions is the possibility of computing their convex hull over any discrete unit hypercube B in a very simple way. Indeed, every point $x \in \mathrm{co}B$ determines in unique way the points $y^1(x) , \ldots , y^k(x) \in B$ and the scalars $\alpha^1(x) , \ldots , \alpha^k(x) \in [0,1]$, with $k \leq n$, such that $y^i(x) \leq y^{i+1}(x)$, $y^i(x) \neq y^{i+1}(x)$, for $i = 1 , \ldots ,k-1$, $\sum_{i=1}^{k} \alpha^i(x) = 1$ and

$$\sum_{i=1}^{k} \alpha^i(x)\ y^i(x) = x \quad .$$

Furthermore, given any $x \in B$ the values of $y^i(x)$ and $\alpha^i(x)$ may be computed by a polynomial algorithm in a straightforward way. Lovasz [14] and Singer [22] established the following result.

Proposition 5.1

Let f be a submodular function on a discrete unit hypercube B. Then, for any $x \in \text{co}B$ the value of the convex hull $\text{co}_B f$ of f at x is given by

$$\text{(27)} \qquad \text{co}_B f(x) = \sum_{i=1}^{k} \alpha^i(x)\ f(y^i(x)) \ .$$

Since the extension of a function f defined on a discrete rectangle X coincides with the convex hull of f on every discrete hypercube contained in X, proposition 5.1 has the following important consequences.

Corollary 5.1.1

Let f be a submodular function on a discrete rectangle X. Then, the value of its extension $\bar{f}$ at any point of $\text{co}X$ may be computed in polynomial time.

Corollary 5.1.2

A submodular function f is integrally convex on a discrete rectangle X if and only if for every $z^1, z^2 \in X$ such that $\| z^1 - z^2 \|_\infty = 2$ one has

$$\text{(28)} \qquad f(\lfloor (z^1 + z^2)/2 \rfloor) + f(\lceil (z^1 + z^2)/2 \rceil) \le f(z^1) + f(z^2) \ ,$$

where , for a given $y \in \mathbf{R}^n$, $\lfloor y \rfloor = (\lfloor y_1 \rfloor, \dots, \lfloor y_n \rfloor)$, $\lceil y \rceil = (\lceil y_1 \rceil, \dots, \lceil y_n \rceil)$ and $\lfloor y_i \rfloor, \lceil y_i \rceil$ denote the lower and upper integer part of y_i respectively.

Proof.

Set $x = (z^1 + z^2)/2$. Then, $y^1(x) = \lfloor (z^1 + z^2)/2 \rfloor$, $y^2(x) = \lceil (z^1 + z^2)/2 \rceil$ and $\alpha^1(x) = \alpha^2(x) = 1/2$. Hence, (28) follows from proposition 3.3 and proposition 5.1.

Remark 5.1

Note that the class Φ of submodular functions satisfies the conditions of remark 2.2. Hence, the sum of two integrally convex submodular functions is still integrally convex.

Proposition 5.2

The problem of minimizing a submodular integrally convex function f over a bounded discrete rectangle X in $\mathbf{Z}^n$ can be solved in polynomial time.

Proof:

Since $\overline{f}$ is piecewise linear and convex, it can be minimized in polynomial time over coX (see [8 p.188]). In order to complete the proof, recall that $\overline{f}$ may be computed in polynomial time from f and observe that if $\overline{x}$ is a global minimum point for $\overline{f}$ on coX, then at least one of the points $y^1(\overline{x}), \dots, y^k(\overline{x})$ must be a global minimum point for f on X.

When f is quadratic and strictly convex the boundedness assumption in the above proposition can be omitted.

Proposition 5.3

Let $f(x) = x^TCx + d^Tx$, where C is a $n \times n$ positive definite matrix and $d, x \in \mathbf{R}^n$. If f is submodular and integrally convex, then the problem of minimizing f over any discrete rectangle X in $\mathbf{Z}^n$ can be solved in polynomial time.

Proof:

let $\overline{x} \in \mathbf{R}^n$ denote the global minimum point of f over $\mathbf{R}^n$. Clearly, $\overline{x} = -1/2C^{-1}d$.

Note that $f(x) = (x - \bar{x})^T C(x - \bar{x}) - \bar{x}^T C \bar{x}$ and define

$$E(r) = \{ x \in \mathbf{R}^n : (x - \bar{x})^T C(x - \bar{x}) \leq r \} .$$

Let x' denote any point of X. Then, setting $r' = (x' - \bar{x})^T C(x' - \bar{x})$, one has $E(r') \cap X \neq \emptyset$. This trivially implies that $E(r')$ contains all global minimum points of f over X. Furthermore, it is easy to verify that $E(r')$ is contained in the rectangle

$$R = \{ x \in \mathbf{R}^n : |(x - \bar{x})_i| \leq (r'c'_{ii})^{1/2} , i = 1 , \ldots , n \} ,$$

where c'_{ii} is the i^{th} diagonal element of C^{-1}. Hence, all global minimum points of f over X must belong to the intersection $X \cap X'$, where

$$X' = \{ z \in \mathbf{Z}^n : |(z - \bar{x})_i| \leq \lceil (r'c'_{ii})^{1/2} \rceil , i = 1 , \ldots , n \} .$$

Since X' is bounded the thesis follows from proposition 5.3..

6. A pseudopolynomial algorithm .

The polynomial algorithm described in section 5 for minimizing an integrally convex submodular function on a discrete rectangle, like all known polynomial methods for minimizing a submodular function on the 0-1 hypercube, relies on the ellipsoid method (see [7]) and is not suitable for practical computation.

However, in the 0-1 case, efficient combinatorial algorithms are known that minimize in polynomial time submodular functions belonging to some special classes, e.g. quadratic or cubic functions, positive-negative functions, etc. (see [1,2,10,21]).

In this section we describe an algorithm for solving the nonlinear integer program

(NIP) $\qquad \min f(x) \qquad$ s.t. $\qquad x \in X = \{x \in \mathbf{Z}^n : a \leq x \leq b\}$,

where $f : X \to \mathbf{R}$ is submodular and integrally convex . Our algorithm, which is similar to one proposed by Picard and Ratliff for a more restricted class of problems, is only pseudopolynomial in the worst case. However, since it solves NIP by performing a sequence of one-dimensional minimization and of 0-1 submodular minimizations, it may be fruitfully employed when these subproblems can be solved efficiently. This is the case, e.g., when f is the sum of a quadratic and a separable integrally convex function. In particular, for the class of problems considered by Picard and Ratliff the computational experience reported in the next section shows that the number of 0-1 minimizations required tends to be very small. In all our tests it has never been necessary to solve more than thirteen 0-1 minimization problems in order to solve an istance of NIP.

Our algorithm is based on the following properties of submodular functions.

Proposition 6.1

Let f be a submodular function on the discrete rectangle $X = \{x \in \mathbf{Z}^n: a \le x \le b\}$ and define, for every $y \in X$, $X^{\le}(y) = \{x \in X: x \le y\}$ (resp. $X^{\ge}(y) = \{x \in X: x \ge y\}$). Let X^* denote the set of global minimum points for f over X. Then the following properties hold:

(i) If $x' \in X$ and $f(x') < f(x)$ for every $x \in X^{\le}(x') \setminus \{x'\}$ (resp. $x \in X^{\ge}(x') \setminus \{x'\}$), then $x' \le x^*$ (resp. $x' \ge x^*$), for every $x^* \in X^*$.

(ii) If $x' \in X$ and $f(x') \le f(x)$ for every $x \in X^{\le}(x')$ (resp. $x \in X^{\ge}(x')$), then $\exists x^* \in X^*$ such that $x' \le x^*$ (resp. $x' \ge x^*$).

Proof:

The proofs for the cases $X^{\le}(x')$ and $X^{\ge}(x')$ are symmetric. Hence, we prove only the case $X^{\le}(x')$.

(i) Let $x \in X$ and assume that $x_i < x_i'$ for at least an index i. Then, from assumption (i) and from the submodularity of f we deduce

$$f(x) \ge f(x \wedge x') - f(x') + f(x \vee x') > f(x \vee x') .$$

Hence, x cannot be a global minimum point for f.

(ii) Let $x^* \in X^*$. Clearly, one has $x^* \vee x' \geq x'$. Furthermore,

$$f(x^*) \geq f(x^* \wedge x') - f(x') + f(x^* \vee x') > f(x \vee x') .$$

Hence, $x^* \vee x' \in X^*$.

In the sequel e^i denotes the i^{th} unit vector in $\mathbf{R}^n$.

Algorithm NIPMIN :

Step 0 : $x^0 := a$, $x^1 := b$, $y^0 := a - e^1$, $y^1 := b + e^1$.

Step 1 : **While** $y^0 \neq x^0$ **do**

begin

$y^0 := x^0$;

for $i := 1$ **to** n **do**

begin

find $t^* \in [\, 0 , x_i^1 - x_i^0 \,] \cap \mathbf{Z}$ that minimizes $f(x^0 + t\, e^i)$;

if $t^* > 0$ **then** $x^0 := x^0 + t^* e^i$

end

end

if $x^0 = x^1$ then **stop** (x^0 is a global minimum point).

Step 2 : **While** $y^1 \neq x^1$ **do**

begin

$y^1 := x^1$;

for $i := 1$ **to** n **do**

begin

find $t^* \in [\, 0 , x_i^1 - x_i^0 \,] \cap \mathbf{Z}$ that minimizes $f(x^1 - t\, e^i)$;

if $t^* > 0$ **then** $x^1 := x^1 - t^* e^i$

end

end

if $x^0 = x^1$ **then** **stop** (x^0 is a global minimum point).

Step 3 : find $z^* \in \{0,1\}^n$ that minimizes $f(x^0 + z^*)$;
if $f(x^0) = f(x^0 + z^*)$ **then stop** (x^0 is a global minimum point) ;
$x^0 := x^0 + z^*$;
if $x^0 = x^1$ **then stop** (x^1 is a global minimum point) ;

Step 4 : find $z^* \in \{0,1\}^n$ that minimizes $f(x^1 - z^*)$;
if $f(x^1) = f(x^1 - z^*)$ **then stop** (x^1 is a global minimum point) ;
$x^1 := x^1 + z^*$;
if $x^0 = x^1$ **then stop** (x^1 is a global minimum point) **else** go to Step 1.

Analysis of the algorithm.

In steps 1 through 4 the algorithm tries to shrink the current domain $X^C = \{ x \in \mathbf{Z}^n : x^0 \leq x \leq x^1 \}$, on which the minimum of f is sought, by successively increasing x^0 or decreasing x^1 through minimizations of f on an edge of X or on a discrete unit hypercube contained in X. The shrinking is performed in such a way that at least one global minimum point of f on the original domain $X = \{x \in \mathbf{Z}^n : a \leq x \leq b\}$ is always contained in the shrunken domain . This property follows from proposition 6.1, observing that the points $x^0 + t^* e^i$, $x^1 - t^* e^i$, $x^0 + z^*$ and $x^1 - z^*$, computed by the algorithm , are global minimum points for f over the sets $X^{\leq}(x^0 + t^* e^i)$, $X^{\geq}(x^1 - t^* e^i)$, $X^{\leq}(x^0 + z^*)$ and $X^{\geq}(x^1 - z^*)$ respectively.

The algorithm has two different stopping criteria . The first one requires that $x^0 = x^1$, so that the current domain contains only the point x^0. In this case, x^0 is a global minimum point for f. The second criterion is satisfied whenever x^0 (resp. x^1) is a global minimum point for f over the discrete unit hypercube

$x^0+\{0,1\}^n$ (resp. $x^1-\{0,1\}^n$) . This clearly implies that x^0 (resp. x^1) is a local minimum point for f over X^C and hence, by the integer convexity of f, a global minimum point for f over X^C. We can then conclude that x^0 (resp. x^1) is a global minimum point for f over X .

In order to prove the convergence of NIPMIN observe that after every complete iteration of the algorithm at least one component of x^0 must have been increased and one component of x^1 must have been decreased by one or more units. Therefore, the algorithm must terminate after at most

$$\left\lfloor 1/2 \sum_{i=1}^{n} (b_i - a_i) \right\rfloor + 1$$

iterations.

Remark 6.1

Note that the convergence of NIPMIN to an optimal solution of NIP is preserved if the integer convexity assumption is replaced by any other assumption that guarantees the coincidence between local and global minima of f over X. An example of alternative assumption is the discrete convexity introduced by Miller [15]. Hence, NIPMIN may be employed, e.g., to solve the inventory problem considered by Miller .

Remark 6.2

If, at any step of the algorithm , $x_i^0= x_i^1$ for some index i ,then the common value of x_i^0 and x_i^1 must coincide with the i-th component of the optimal solution. Hence, we may reduce the dimension of the problem by fixing the value of x_i . This remark has been employed in our implementation of the algorithm discussed in section 7.

Remark 6.3

If we do not know a priori that the local minima of f are also global minima, we may modify algorithm NIPMIN replacing the second line of step 3 by the following :

if $z^* = 0$ **then** go to step 4 ;

and the second line of step 4 by

if $z^* = 0$ **then stop** (there is a global optimum point x^* satisfying $x^0 \leq x^* \leq x^1$).

This modified version of the algorithm may be used for minimizing any submodular function f on X, but its convergence to a global minimum point is not guarateed. However, even if it does not find an optimal solution , NIPMIN provides a new discrete rectangle, usually smaller than the original one, that certainly contains a global minimum point for f over X.

Complexity of the algorithm .

The complexity of NIPMIN is clearly related to the complexity of the algorithms available for solving the problems of minimizing f over a discrete segment of length m and over a discrete hypercube of dimension n . We shall dentoe by $s(m)$ and $h(n)$ the complexities of these algorithms.

At each iteration of the **while** cycle in steps 1 and 2, the algorithm performs the minimization of f over n discrete segments of length at most b_1-a_1 , ... , b_n-a_n , respectively. Hence, each such iteration requires at most $\sum_{i=1}^{n} s(b_i-a_i)$ time . Let k denote the total number of iterations of **while** cycles in the whole algorithm and observe that after each such iteration either the algorithm moves to the next step or at least one component of x^0 is increased or one component of x^1 is decreased by one or more units. Furthermore, every time step 3 or 4 is performed, at least one component of x^0 is increased or one component of x^1 is decreased by one unit, unless x^0 or x^1 is a global minimum point , in which case the algorithm terminates. Clearly, both step 3 and step 4 have complexity h(n). Let p denote the number of iterations of steps 1 through 4 of the algorithm. Then, in view of the above remarks, the following inequalities hold:

$$0 \leq 2p \leq k \leq \sum_{i=1}^{n} (b_i - a_i) .$$

Hence, the complexity of NIPMIN is bounded by

(29) $$\sum_{i=1}^{n} (b_i - a_i) \left(\sum_{i=1}^{n} s(b_i - a_i) + h(n) \right) .$$

When f is a submodular integrally convex quadratic function, the problem of minimizing f over an n-dimensional discrete unit hypercube may be reduced to a min-cut problem on an associated network with n+2 nodes (see [16]) . Furthermore, in this case the minimization of f on any discrete segment requires time o(n). Hence, in the quadratic case the complexity of NIPMIN is $\sum_{i=1}^{n} (b_i - a_i)\, o(n^3)$.

The class of unate functions, introduced in [9] , is a natural extension of the class of submodular functions. We recall that a function $f : X \rightarrow \mathbf{R}$ is called unate if there exists a suitable switch on the positive orientation of the coordinate axes, i.e. a mapping $\sigma : \mathbf{R}^n \rightarrow \mathbf{R}^n$ defined by $\sigma(x) = \alpha^T x$, where $\alpha \in \{-1,1\}^n$, such that $f \circ \sigma$ is submodular on $\sigma^{-1}(X)$. It is not difficult to see that the results of this and the previous section may be straightforwardly extended to the case where the submodularity assumption is replaced by unateness . Furthermore, Hansen and Simeone [9] have shown that unateness may be recognized in linear time in the class of quadratic functions.

7. Computational experience.

We realized, on an IBM 3081 running VM/CMS, a **FORTRAN77** implementation of algorithm **NIPMIN**, for solving problem NIP in the case

where $f(x) = x^T C x + d^T x$, $c_{ij} \leq 0$ for every $i \neq j$ and $c_{ii} \geq | \sum_{\substack{j=1 \\ j \neq i}}^{n} c_{ij} |$.

In this case, following an idea of Picard and Ratliff [16] , the subproblem of minimizing f on a discrete unit hypercube was reduced to the problem of finding a min-cut on an associated network . The latter problem was then solved using Goldfarb and Grigoriadis implementation of the Dinic algorithm for max-flow problems [6].

To our knowledge no previous computational experience has been reported for the problem considered here. We tested our algorithm on a set of randomly generated problems. The controlling parameters were **n**, the size of the problem, **den** , the density of the matrix C (i.e. **den** /100 is the probability that a nonzero off-diagonal element is generated), the upper and lower bounds **a** and **b** on the vector x and the measure **DD** of *diagonal dominance* of C, meaning that the diagonal element c_{ii} is randomly chosen with uniform distribution from the interval $[D_i , \mathbf{DD} \cdot D_i]$, where $D_i = | \sum_{\substack{j=1 \\ j \neq i}}^{n} c_{ij} |$ .The off-diagonal elements c_{ij} , with $i \neq j$, are randomly chosen with uniform distribution from the interval [-100 , 0] and the components of d are randomly chosen with uniform distribution from the interval [-p , p] , where $p = 10 \cdot \mathbf{n} \cdot \mathbf{den}$. This choice of p may be a priori justified with the need of having a balance between the linear and the quadratic part in the objective function . Furthermore, we observed that in practice a smaller value of p resulted in obtaining optimal solutions very close to zero, while a greater value of p often led to optimal solutions at a vertex of the feasible domain . In both cases problem NIP seems to be easier to solve than when p is chosen equal to $10 \cdot \mathbf{n} \cdot \mathbf{den}$. Note that the coefficient 10 in our choice of p was determined taking into account the range of the c_{ij} and the upper and lower bounds **a** and **b** .

Table 1

n	den	DD	One-dimensional min. min	max	avg	Hypercube min. min	max	avg	t
200	25	1	1400	43200	13350	1	2	1.6	5.8
200	25	1.1	11400	13600	12020	2	2	2	5.7
200	25	2	2400	3400	2820	1	2	1.4	1.6
200	25	5	1400	1800	1560	1	2	1.2	1.0
200	50	1	2600	23200	14380	1	2	1.3	11.7
200	50	1.1	11600	12800	12140	2	2	2	11.0
200	50	2	2400	3600	2920	1	2	1.6	3.2
200	50	5	1400	2000	1640	1	2	1.1	1.9
200	100	1	2000	42800	10720	1	2	1.2	17.2
200	100	1.1	9400	13000	11400	1	2	1.8	19.6
200	100	2	2400	3200	2820	1	2	1.3	5.8
200	100	5	1400	2200	1700	1	2	1.1	3.7
500	25	1	5000	262500	90700	1	2	1.6	87.9
500	25	1.1	31500	37500	34000	1	2	1.9	38.5
500	25	2	6500	9000	7950	1	2	1.6	11.5
500	25	5	3500	5000	4300	1	2	1.3	7.0
500	50	1	6500	261000	120850	1	2	1.5	226.4
500	50	1.1	30000	37500	34100	1	2	1.7	73.6
500	50	2	6500	10000	7700	1	2	1.2	20.0
500	50	5	3500	5000	4100	1	1	1	11.8
500	100	1	5500	262000	79500	1	13	2.5	311.4
500	100	1.1	29000	35500	32300	1	2	1.8	140.3
500	100	2	6500	9000	7800	1	2	1.2	38.9
500	100	5	3500	4500	4050	1	1	1	22.4
1000	25	1	18000	538000	173800	1	2	1.8	332.7
1000	25	1.1	64000	76000	72300	2	2	2	168.8
1000	25	2	15000	20000	17000	1	2	1.4	47.9
1000	25	5	7000	10000	8600	1	2	1.2	28.2
1000	50	1	32000	940000	300300	1	2	1.2	1084.0
1000	50	1.1	59000	77000	69300	1	2	1.8	305.9
1000	50	2	16000	18000	16400	1	2	1.4	88.7
1000	50	5	8000	11000	9100	1	1	1	51.2

Table 2

n	den	DD	One-dimensional min. min	max	avg	Hypercube min. min	max	avg	t
200	25	1	1800	42600	14540	1	2	1.7	6.3
200	25	1.1	15200	18800	17100	2	2	2	7.5
200	25	2	3200	4400	3540	1	2	1.6	2.0
200	25	5	1800	2000	1880	1	2	1.2	1.2
200	50	1	3400	44400	26920	1	3	1.7	21.1
200	50	1.1	15200	18800	16640	1	2	1.9	14.0
200	50	2	3200	4200	3700	1	2	1.3	3.6
200	50	5	1600	2400	1900	1	1	1	2.0
200	100	1	2200	44400	23020	1	2	1.5	35.0
200	100	1.1	15000	18600	16380	1	2	1.8	26.4
200	100	2	3000	4200	3640	1	2	1.2	6.7
200	100	5	1600	1800	1760	1	1	1	3.7
500	25	1	7000	260000	116700	1	13	6.2	126.7
500	25	1.1	40500	49500	45350	2	2	2	49.4
500	25	2	8500	10500	9700	1	2	1.5	13.0
500	25	5	4500	6000	4950	1	2	1.4	7.8
500	50	1	5000	35500	21400	1	13	6	72.7
500	50	1.1	42500	52000	46400	1	2	1.9	97.4
500	50	2	8500	11500	9500	1	2	1.2	23.2
500	50	5	4500	5000	4850	1	2	1.3	14.3
500	100	1	7000	16000	12000	1	2	1.4	57.4
500	100	1.1	41000	50500	45850	1	2	1.8	66.8
500	100	2	8500	10500	9650	1	2	1.1	44.7
500	100	5	4500	5500	4950	1	2	1.3	27.7
1000	25	1	120000	999000	322500	1	2	1.6	631.3
1000	25	1.1	88000	114000	98400	1	2	1.9	204.9
1000	25	2	17000	23000	19600	1	2	1.6	52.9
1000	25	5	9000	11000	10000	1	2	1.1	29.5
1000	50	1	16000	987000	332700	1	2	1.4	1196.3
1000	50	1.1	90000	103000	97000	1	2	1.8	404.8
1000	50	2	19000	23000	21500	1	2	1.4	99.5
1000	50	5	9000	11000	10000	1	2	1.1	55.4

In tables 1 and 2 we list the data obtained from a set of randomly generated problems where $a_i = -100$, $b_i = 100$ and $a_i = -1000$, $b_i = 1000$, $i = 1, \dots, \mathbf{n}$, respectively . For each combination of the controlling parameters, 5 to 10 test problems were generated and we reported the average CPU time t expressed in seconds, the minimum , maximum and average number of one-dimensional minimizations of f and the minimum , maximum and average number of minimizations of f over a discrete unit hypercube.

The computational results suggest that the worst case complexity bound (29) tends to be much higher than the average complexity of the algorithm in practice. Indeed, in almost every test only one or two minimizations of f on a discrete unit hypercube have been necessary. Furthermore, the average CPU time and the average number of one dimensional minimizations of f seem to be almost independent of the values of **a** and **b** . Note also that the performance of the algorithm improves considerably when the *diagonal dominance* **DD** increases. On the other hand, for any fixed value of **DD** , with the partial exception of **DD**=1, the average CPU time and the average number of one-dimensional minimizations tend to increase almost linearly with $m = \mathbf{n}^2 \cdot \mathbf{den}$ = expected number of nonzero c_{ij} .

Finally, let us remark that in all our tests the CPU time employed to solve the nonlinear integer problem NIP never exceded by more than 20 times the CPU time required minimize f on a discrete unit hypercube.

8. Concluding remarks.

We introduced and analyzed a new notion of convexity for functions defined over the integers. Clearly, an important issue to consider when introducing a new notion is that of determining nontrivial interesting classes of functions satisfying it. We partially solved this problem, specially in the quadratic and in the submodular case. However, much work should still be done in this direction.

We think that the interest for integer convexity is justified both from the coincidence of local and global minima of integrally convex functions and from the existence of efficient theoretical and practical algorithms for minimizing a submodular integrally convex function on a discrete rectangle.

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