Thermal and quantum phase transitions of the ϕ^4 model

István Gábor Márián,[1,2] Andrea Trombettoni,[3,4] and István Nándori[5,1]
[1]HUN-REN Atomki, P.O.Box 51, H-4001 Debrecen, Hungary
[2]University of Debrecen, Institute of Physics, P.O.Box 105, H-4010 Debrecen, Hungary
[3]Department of Physics, University of Trieste, Strada Costiera 11, I-34151 Trieste, Italy
[4]CNR-IOM DEMOCRITOS Simulation Center, Via Bonomea 265, I-34136 Trieste, Italy
[5]University of Miskolc, Institute of Physics and Electrical Engineering, H-3515, Miskolc, Hungary

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In this paper we discuss and revisit the finite temperature extension of the renormalization group (RG) treatment of $T = 0$ field theories, focusing as a case study on the ϕ^4 model. We first discuss the extension of RG equations of the very same model from $T = 0$ to finite T in the usual way by resorting to sums on the Matsubara frequencies and fixing the physical temperature parameter T. We show that this approach, although useful for a variety of applications, may lead to the disappearance of the critical points as extracted from the RG flow. Since the identification of fixed points is key in the study of classical and quantum phase transitions, we propose a modification of the usual finite-temperature RG approach by relating the temperature parameter to the running RG scale, $T \equiv k_T = \tau k$, where k_T is the running cutoff for thermal and k is for the quantum fluctuations. Once this dimensionless temperature τ is introduced, we investigate the consequences on the thermal RG approach for the ϕ^4 model and construct its phase diagram. Finally, we formulate requirements for the phase diagram of the ϕ^4 theory based on known properties of the quantum and classical phase diagrams of the Ising model.

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I. INTRODUCTION

A quantum phase transition (QPT) occurs at the quantum critical point at zero temperature where thermal fluctuations are absent. The QPT is understood as a transition between different quantum phases and driven by quantum fluctuations. Contrary to a classical phase transition (CPT) which is a thermal phase transition, the QPT is accessed at absolute zero temperature by varying a physical parameter such as the magnetic field or the pressure. In this work, we refer to this as the "quantum" parameter. As a major example, the critical behaviour of the CPT of the Ising model is in the same universality class of the QPT of the ϕ^4 quantum field theory (QFT), with this providing a paradigmatic instance of the relation between a CPT and its quantum counterpart QPT. It is well known that a classical system at finite temperature in d dimension is—for short-range interactions—in the same universality class of the same model in $d - 1$ spatial dimension at $T = 0$.

At a finite but suitably low temperature, classical and quantum fluctuations compete with each other. In the quantum critical region near the quantum critical point, quantum fluctuations dominate the system behaviour and one can study properties of the QPT [1]. Thus, a so-called QPT-CPT phase diagram can be drawn where the ordered and the disordered phases are separated by each other by a critical line which connects the critical temperature and the quantum critical point. Around the classical phase transition, in the classical critical region, the system is governed by thermal fluctuations and this region becomes narrower with decreasing temperatures and converges towards the quantum critical point monotonically.

The Ising model in $d = 2$ or $d = 3$ dimensions represents an excellent playground to test and to understand the interplay between quantum and classical phase transitions. The quantum Ising model [1–3] can be realized by adding a transverse, external magnetic field to the usual Hamiltonian of the classical Ising model. This is the so-called transverse field Ising model where the QPT is realized by varying the magnitude of the transverse external magnetic field [2,3]. The QPT-CPT phase diagram of the Ising model is well-known and its interpretation in terms of effective field theories also deeply investigated [1,4]. The classical Ising model with vanishing external magnetic field undergoes a second order classical (thermal) phase transition at a finite critical temperature T_c. The quantum Ising model at zero temperature has a QPT at a critical magnetic field h_c.

It is also well-known that the classical Ising model (without any transverse magnetic field) can be mapped onto the ϕ^4 QFT which has a QPT where its quantum critical behaviour is found to be identical to the classical critical behavior of the Ising model. As a consequence, the two models belong to the same universality class and their critical exponents are identical. Natural questions are what is the QPT-CPT diagram of the ϕ^4 QFT and how is it related to that of the Ising model, an issue that has been investigated using a variety of tools? In this article we aim to clarify this issue using the functional renormalization group (FRG) approach [5], where we need to compare results obtained at zero temperature to those obtained when temperature is introduced. We will argue that a *dimensionless* temperature has to be introduced, as we discuss in the next section.

II. MAIN IDEA

In general, in order to map out the QPT-CPT diagram of the ϕ^4 field theory one has to perform its thermal RG analysis where both quantum and thermal fluctuations are taken into account. This can be done by, e.g., using the finite temperature extension of the zero-temperature FRG method. The standard formulation of the FRG method [5] is written in Euclidean spacetime in the framework of the zero-temperature QFT. Its generalization to finite temperature requires the inclusion of the inverse temperature parameter $\beta = 1/k_B T$ as the upper bound for the imaginary time integral [6–28].

Once β has been introduced, one has to relate the temperature parameter to a momentum scale by using units in which $c = \hbar = k_B = 1$. In the usual perturbative approach the standard choice is $\mu = 2\pi T$ where μ is the perturbative RG scale which is related to the nonperturbative one: $\mu \sim k$, where k is the momentum scale. However, it is known that the k-dependence of the effective action of the FRG equation is only introduced artificially to implement the Wilsonian integration [29] of fluctuating modes and the quantized theory is obtained in the physical limit $k \to 0$. Thus, one has to take the limit $k \to 0$. For this reason, in Refs. [6–17] the running couplings are defined at an arbitrary but fixed intermediate momentum scale $k_\star = 2\pi T = 2\pi\tau\Lambda$ where Λ is the UV initial value of the running momentum cutoff k. So, the temperature parameter is linked to the UV cutoff, i.e., $T = \tau\Lambda$. Once the couplings are defined, the zero-temperature FRG equation is integrated from $k_\star$ up to Λ, and then starting from these bare parameters, the FRG equation is solved down from $k = \Lambda$ to $k = 0$ but this time with the temperature T turned on and the physical quantities obtained at $k = 0$. This investigation is then performed with the usual tools of QFT.

However, every approach has advantages and disadvantages. In the fixed T approach the dimensionful temperature is well-defined, the IR limit $k \to 0$ can be taken safely but the RG flow equations have no fixed point solutions, since the explicit k-dependence cannot be removed from the dimensionless RG flow equations. For example, the dimensionless thermal FRG Eq. (3.9) of [9] which is given in the so-called local potential approximation, has an explicit k-dependence and it blows up in the IR limit, i.e., for $k \to 0$ where the attractive fixed points are always given, so it makes no room for nontrivial fixed points such as the Wilson-Fisher (WF) one. As an example, we mention that in Refs. [30–32] the quantum $O(N)$ and the Bose-Hubbard model were studied at finite temperatures and it was shown that in order to properly identify the "quantum WF fixed point" (i.e. at zero temperature) and the "classical WF fixed point" (at finite temperatures, where quantum fluctuations are irrelevant), one needs to use different dimensionless coupling constants; see more in the following.

In this work we deal with the above problems by suggesting the identification

$$T \equiv k_T = \tau k$$

where k_T is the running cutoff for thermal, and k is for the quantum fluctuations. This choice can be supported by the following argument. The Wilsonian approach requires the rescaling of the upper bound of the imaginary time integral in every blocking step, so it has to be linked to the running momentum cutoff k. However, in the FRG method one has to take the limit $k \to 0$; therefore, the upper bound cannot be considered as the (inverse) temperature but can be used as a running momentum cutoff $T \equiv k_T$. Thus, the temperature is related to the dimensionless quantity τ which is kept constant over the RG flow. A dimensionless (reduced) temperature has been used in the framework of FRG [18–28]; however, in these works the typical choice is $\tau \equiv \tau_k = T/k$ which means that the dimensionful parameter T is kept constant over the RG flow, so τ_k has a trivial (nonvanishing) dependence on the RG scale k; see [33] for a few examples. One may summarize that the dimensionless temperature is introduced as a technical tool in the literature [18–28], yet a complete study of consequences and the determination of physical properties from the *constant* dimensionless temperature is lacking, to the best of our knowledge, making it more difficult to understand and to identify thermal and quantum phase transitions and their crossovers, as we are going to argue in the following.

It is important to note that our proposal for fixing $\tau = T/k$ and taking the simultaneous T, $k \to 0$ limit addresses a complementary regime compared to the widely used fixed T approach. At a certain level of approximation the two methods can give qualitatively same results, so one cannot come to the conclusion that previous studies were incorrect. However, it is also important to note that in our fixed τ approach the RG flow equations have real (and not pseudo) fixed points which have fundamental importance since the determination of critical behavior and various phases are strongly related to fixed point solutions of RG flow equations.

Thermal field theory is a very well-developed framework for finite temperature applications [34], and it is employed in many-body physics [34,35] which relates a quantum system in $d-1$ (spatial) dimensions at $T=0$ and the corresponding classic system in d dimensions at finite temperature [1]. It generates the discretization of the imaginary time-integral in momentum space, i.e., summation on Matsubara frequencies [34,35]. At nonzero temperatures, classical fluctuations with an energy scale of $k_B T$ compete with the quantum fluctuations of energy scale $\hbar\omega$, where ω is the frequency of quantum oscillations. By using natural units, one finds that ω must be compared to T and in the RG method one has to relate the frequency ω to the running cutoff k which is used to integrate quantum fluctuations systematically, so one finds $\omega = k$. Thus, one can introduce a dimensionless quantity, $\tau = T/k = T/\omega$ which is used to compare the strength of quantum and thermal fluctuations. If $\tau \lesssim 1$ then quantum fluctuations dominate; if $\tau \gtrsim 1$ then thermal fluctuations dominate. Notice that in the Wilsonian approach fluctuations are taken into account by the successive elimination (integration) of degrees of freedom above these running cutoffs which are chosen to be different in order to make difference between thermal and quantum fluctuations. In other words, For $k_T \sim k$ (i.e., for $\tau \sim 1$), one cannot distinguish between thermal and quantum fluctuations.

We aim to explore the possibility of a thermal RG approach in the framework of the FRG method using the identification $T = \tau k$ where the temperature is related to the dimensionless quantity τ which is kept constant over the RG flow. Once the dimensionless temperature τ has been introduced, we aim at determining the consequences of this introduction in the FRG formalism and how the points previously discussed can dealt with. We apply this new thermal RG approach for the ϕ^4 model. We are motivated in this study by the results presented in Ref. [36] where the thermal RG method for the 4-dimensional ϕ^4 theory has been studied and its consequences on key issues of Inflationary Cosmology investigated.

Our goal here is to map out the QPT-CPT phase diagram of the ϕ^4 theory and compare it to the QPT-CPT diagram of the Ising model. It represents graphically the interplay between the CPT and QPT of the ϕ^4 scalar field theory which has also been investigated in connection to the naturalness/hierarchy problem [37–40]. For example, in [39,40] it was shown that the hierarchy problem as well as the metastability of the electroweak vacuum can be understood as the Higgs potential being near-critical, i.e., close to a QPT. We discuss known properties of the thermal and quantum phase transitions of the Ising model and based on these, we formulate requirements for the QPT-CPT phase diagram of the ϕ^4 theory. We construct the QPT-CPT phase diagram of the ϕ^4 theory (focusing to dimensions below the upper critical dimension) by the thermal RG approach suggested in this work and compare it to the formulated requirements in order to check the viability of the new method.

III. MODIFIED THERMAL RG EQUATION

The FRG equation [5] at zero-temperature is formulated in Euclidean spacetime and reads as

$$k\partial_k \Gamma_k[\phi] = \frac{1}{2}\int \frac{d^d p}{(2\pi)^d} \frac{k\partial_k R_k(p)}{R_k(p) + \Gamma_k^{(2)}[\phi]}, \tag{1}$$

where $\Gamma_k[\phi]$ is the running effective action with its Hessian $\Gamma_k^{(2)}[\phi]$, k is the running momentum cutoff (i.e., the RG scale) and $R_k(p)$ is an appropriately chosen regulator function. In order to handle the FRG equation one has to use approximations. In the so-called gradient expansion, the action is expanded in terms of the derivatives of the field. The lowest order of the gradient expansion is the so-called, local potential approximation (LPA). In LPA when the couplings of the scaling potential $V_k(\phi)$ carry RG scaling only, the Wetterich FRG equation (1) is written as

$$k\partial_k V_k(\phi) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{d^d p}{(2\pi)^d} \frac{k\partial_k R_k}{p^2 + R_k + V_k''(\phi)}, \tag{2}$$

with $V_k'' = \partial_\phi^2 V_k$.

The extension of the zero-temperature FRG method for finite temperatures requires the modification of the imaginary time $\tilde{t}$ integral $\int d^d x \to \int_0^\beta d\tilde{t} \int d^{d-1}x$ where $\beta = 1/T$ and T is the temperature parameter. As a next step, the momentum integral with respect to the imaginary time is modified as $\int d^d p \to T\sum_{\omega_n} \int d^{d-1} p$ by using Matsubara frequencies where the summation is performed over these discrete frequencies; for bosonic degrees of freedom $\omega_n = 2\pi n T$.

A. Thermal RG with the litim regulator

In order to proceed further one has to assume that the regulator function is independent of the Matsubara frequency, but otherwise the same as for the zero-temperature case. This is the case of the "frequency independent regulator". In addition, it is also important to note that once approximations are used, the concrete form of the FRG equation starts to depend on the particular choice of the regulator function. There are two types of regulator functions where the momentum integrals in (2) and its finite-temperature counterpart can be performed analytically in LPA. The first one is the Litim regulator [41], which has the following form at zero-temperature:

$$R_k(p) = (k^2 - p^2)\theta(k^2 - p^2), \tag{3}$$

where $\theta(y)$ is the Heaviside step function. The frequency-independent form of the Litim regulator results in the following thermal RG equation in LPA [9],

$$k\partial_k V_k(\phi) = \frac{2\alpha_{d-1}}{d-1} k^{d-1} T \sum_{n=-\infty}^{\infty} \frac{k^2}{k^2 + \omega_n^2 + \partial_\phi^2 V_k(\phi)}, \tag{4}$$

with $\omega = 2\pi nT$ and $\alpha_d = \Omega_d/(2(2\pi)^d)$, where $\Omega_d = 2\pi^{d/2}/\Gamma(d/2)$ is the d-dimensional solid angle. The summation can be performed [9] which results in an RG equation [identical to Eq. (3.9) of [9] for $N = 1$]:

$$k\partial_k V_k(\phi) = \frac{2\alpha_{d-1}}{d-1} k^{d+1} \frac{2n(\omega_k) + 1}{2\omega_k},$$

where $n(\omega_k) = (\exp(\omega_k/T) - 1)^{-1}$ is the bosonic distribution function with $\omega_k = \sqrt{k^2 + \partial_\phi^2 V_k(\phi)}$ which appears in the so called "thermal contribution" which vanishes in the limit $T \to 0$ and the remaining term is the so-called "vacuum contribution". This equation can be rewritten as

$$k\partial_k V_k(\phi) = \frac{\frac{2\alpha_{d-1}}{d-1} k^{d+1}}{2\sqrt{k^2 + \partial_\phi^2 V_k(\phi)}} \frac{\exp\left(\frac{\sqrt{k^2+\partial_\phi^2 V_k(\phi)}}{T}\right) + 1}{\exp\left(\frac{\sqrt{k^2+\partial_\phi^2 V_k(\phi)}}{T}\right) - 1}$$

and can be further simplified by using the identity $\coth(x/2) = \frac{\exp(x)+1}{\exp(x)-1}$ which results in

$$k\partial_k V_k(\phi) = \frac{2\alpha_{d-1}}{d-1} k^{d+1} \frac{\coth\left(\frac{\sqrt{k^2+\partial_\phi^2 V_k(\phi)}}{2T}\right)}{2\sqrt{k^2 + \partial_\phi^2 V_k(\phi)}}. \tag{5}$$

Let us note that the above thermal RG equation is well-known. For example, in Ref. [9] the authors performed a detailed analysis of the $O(N)$ scalar field theory by the thermal RG Eq. (5) which was also discussed in [10] with the inclusion of pressure and in [17] by taking into account volume fluctuations too where the temperature is linked to the UV cutoff Λ, i.e., $T = \tau\Lambda$.

The schematic thermal RG flow with $T = \tau\Lambda$ was shown in Figs. 6 of [11] and 4 of [12] and is given here in Fig. 1. It is clear from Fig. 1 that the zero-temperature FRG equation is integrated from $k = 0$ up to Λ while the couplings are defined at an arbitrary but fixed intermediate momentum scale $k_\star = 2\pi T = 2\pi\tau\Lambda$. Then starting from these bare parameters, the thermal ($T \neq 0$) FRG equation is solved from $k = \Lambda$ to $k = 0$. It is also clear that RGs flow at zero and at finite temperatures do not differ from each other for $k > k_\star$ which is evident from (5) since in the limit $x \to \infty$

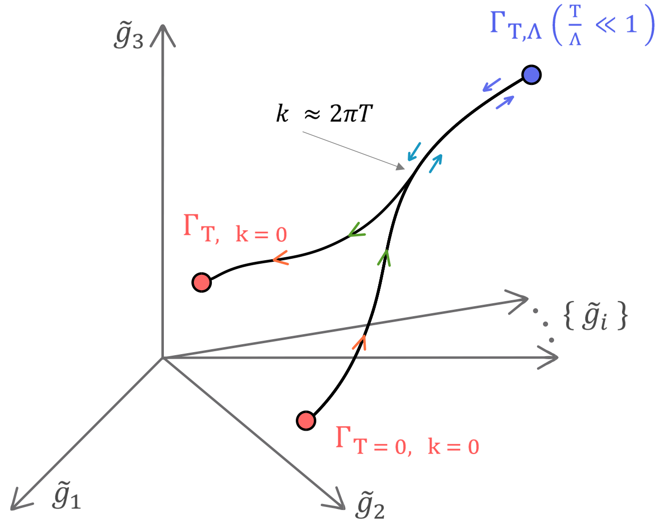

FIG. 1. Schematic thermal RG flow with $T = \tau\Lambda$. The figure is similar to those found in [11,12].

one finds $\coth(x) \to 1$. However, for $k < k_\star$ they start to deviate from each other, so the physical quantities can be obtained at $k = 0$ of the thermal RG flow. Nevertheless, this procedure has a drawback: the explicit k-dependence cannot be removed from the dimensionless RG flow equations.

In this work we use the identification $T \equiv k_T = \tau k$ where "τ" plays the role of the (dimensionless) temperature which distinguishes between thermal quantum fluctuations. We suggest a modified thermal RG equation:

$$k\partial_k V_k(\phi) = \frac{2\alpha_{d-1}}{d-1} k^{d+1} \frac{\coth\left(\frac{\sqrt{k^2+\partial_\phi^2 V_k(\phi)}}{2\tau k}\right)}{2\sqrt{k^2 + \partial_\phi^2 V_k(\phi)}}, \tag{6}$$

which was given in Ref. [36], too. The advantage of this modified thermal RG equation is the absence of k-dependence in the dimensionless RG flow equations.

B. Thermal RG with the sharp-cutoff regulator

As we argued, in order to perform the momentum integrals in (2) and its finite-temperature counterpart analytically one has to choose a special type of regulator. In the previous subsection we discussed the case of the Litim regulator [41]. In this subsection we choose the sharp-cutoff which has the following form at zero temperature:

$$R_k(p) = p^2\left(\frac{1}{\theta(p^2 - k^2)} - 1\right), \tag{7}$$

and it is considered as the simplest regulator function. The frequency-independent form of the sharp-cutoff regulator results in the following thermal RG equation in LPA:

$$k\partial_k V_k(\phi) = -\alpha_{d-1} k^{d-1} T \sum_{n=-\infty}^{\infty} \ln\left(\frac{k^2+\omega_n^2+\partial_\phi^2 V_k(\phi)}{k^2+\omega_n^2}\right), \tag{8}$$

where the denominator inside the logarithm can be chosen with some freedom but it must have the dimension of momentum square and it has to be field-independent. For example, it could be k^2 or T^2. Here we take the choice when it becomes identical to the nominator in the absence of fields. In this case, the summation can be performed which results in a thermal RG equation

$$k\partial_k V_k(\phi) = -\alpha_{d-1} k^{d-1} 2T \times \ln\left[\operatorname{csch}\left(\frac{k}{2T}\right)\sinh\left(\frac{\sqrt{k^2+\partial_\phi^2 V_k(\phi)}}{2T}\right)\right], \tag{9}$$

and by using the identification $T \equiv k_T = \tau k$ the following modified thermal RG equation can be obtained:

$$k\partial_k V_k(\phi) = -\alpha_{d-1} k^{d} 2\tau \times \ln\left[\operatorname{csch}\left(\frac{1}{2\tau}\right)\sinh\left(\frac{\sqrt{k^2+\partial_\phi^2 V_k(\phi)}}{2\tau k}\right)\right]. \tag{10}$$

IV. THE ZERO-TEMPERATURE LIMIT

Before we start to apply the modified thermal RG Eqs. (6) and (10) let us discuss in this section an important issue, the zero-temperature limit. As we argued, the finite-temperature approach requires the modification of the imaginary time integral by introducing an upper bound which is the inverse temperature parameter, $\beta = 1/T$. This results in the replacement of the continuous (imaginary time) integral by a sum over discrete Matsubara frequencies in the momentum space in RG Eqs. (1) and (2). Here, the zero-temperature limit, i.e., $\beta \to \infty$ must recover the RG Eqs. (1) and (2).

However, if one would like to perform the momentum integrals (and the summation) the use of the regulator function is unavoidable. The finite-temperature formalism requires a frequency independent regulator which does not regulate the Matsubara sum. This is not true for the zero-temperature case where momentum integrals (including the imaginary time direction) are performed by the regulator function (i.e., with a frequency-dependent regulator). Thus, the zero-temperature limit of RG Eqs. (5) and (9) may differ from their counterparts at zero temperature.

Indeed, by choosing the frequency independent Litim regulator and taking the limit of zero-temperature, the thermal RG Eq. (5) reduces to

$$\mathrm{T} \to 0: \qquad k\partial_k V_k(\phi) = \frac{2\alpha_{d-1}}{d-1}\frac{k^{d+1}}{2\sqrt{k^2+\partial_\phi^2 V_k(\phi)}}, \tag{11}$$

which is not identical to its zero-temperature counterpart

$$\mathrm{T} = 0: \qquad k\partial_k V_k(\phi) = \frac{2\alpha_d}{d}\frac{k^{d+2}}{k^2+\partial_\phi^2 V_k(\phi)}, \tag{12}$$

where the frequency-dependent Litim regulator is used to perform the momentum integrals of (2). The situation is similar for the sharp cutoff case, where the zero-temperature limit of the thermal RG Eq. (9) is not identical to its zero-temperature counterpart, i.e., the Wegner-Houghton equation [42] at LPA.

The essence of this problem is whether the frequency dependence is taken separately or not which is leading to cylindrically (frequency-dependent regulator) or spherically (frequency independent regulator) symmetric geometry [43,44]. The cylinder can have finite or infinite size. An infinite size cylinder with the Litim regulator results in an FRG equation [see Eq. (7) of [44]] identical to (11).

It is clear that Eqs. (11) and (12) have different forms; however, they have the same singularity structure ($k^2 + \partial_\phi^2 V_k = 0$) which results in a convex dimensionful potential in the IR limit ($k \to 0$) in both cases. The position and the critical behavior of the WF fixed point obtained by Eqs. (11) and (12) may differ from each other; i.e., the zero-temperature limit requires attention. Actually, fixed points are not connected to physical observables, but the critical exponents which are used to characterise the critical behavior around the WF fixed point can be measured directly, thus, they can be used to study the zero-temperature limit problem.

One of our goals in this work is to use measurable quantities, i.e., critical exponents to study the zero-temperature limit of the thermal RG approach by using the modified RG Eqs. (6) and (10).

V. THERMAL RG STUDY OF THE ϕ^4 MODEL IN LOWER DIMENSIONS

As a first step we apply the thermal RG Eq. (6) to the ϕ^4 (more precisely, to the ϕ^{2n}) model in low dimensions. For the sake of simplicity we consider the following scalar potential:

$$V_k(\phi) = \sum_{n=1}^{\mathrm{NCUT}} \frac{g_{2n,k}}{(2n)!}\phi^{2n} \to \tilde{V}_k(\tilde{\phi}) = \sum_{n=1}^{\mathrm{NCUT}} \frac{\tilde{g}_{2n,k}}{(2n)!}\tilde{\phi}^{2n}, \tag{13}$$

where the RG scale-dependence is encoded in the couplings and we introduce dimensionless quantities denoted by the tilde superscript. The dimensionful RG flow equations for two couplings, i.e., for NCUT $= 2$ have the following forms:

$$k\partial_k g_{2,k} = \frac{2\alpha_{d-1}}{d-1} k^{d+1} \left[-\frac{g_{4,k}\coth\left(\frac{\sqrt{k^2+g_{2,k}}}{2\tau k}\right)}{4(k^2+g_{2,k})^{3/2}} - \frac{g_{4,k}\mathrm{csch}^2\left(\frac{\sqrt{k^2+g_{2,k}}}{2\tau k}\right)}{8\tau k(k^2+g_{2,k})} \right], \tag{14}$$

$$k\partial_k g_{4,k} = \frac{2\alpha_{d-1}}{d-1} k^{d+1} \left[\frac{9g_{4,k}^2\coth\left(\frac{\sqrt{k^2+g_{2,k}}}{2\tau k}\right)}{8(k^2+g_{2,k})^{5/2}} + \frac{9g_{4,k}^2\mathrm{csch}^2\left(\frac{\sqrt{k^2+g_{2,k}}}{2\tau k}\right)}{16\tau k(k^2+g_{2,k})^2} + \frac{3g_{4,k}^2\coth\left(\frac{\sqrt{k^2+g_{2,k}}}{2\tau k}\right)\mathrm{csch}^2\left(\frac{\sqrt{k^2+g_{2,k}}}{2\tau k}\right)}{16\tau^2k^2(k^2+g_{2,k})^{3/2}} \right], \tag{15}$$

where $g_{4,k}$ and $g_{2,k}$ are dimensionful couplings. One can switch from dimensionful to dimensionless couplings by introducing $g_{2,k} = \tilde{g}_{2,k}k^2$ and $g_{4,k} = \tilde{g}_{4,k}k^{(4-d)}$. The RG flow equations given for dimensionless couplings $\tilde{g}_{2,k}$ and $\tilde{g}_{4,k}$ have no explicit k-dependence. Thus, one can look for nontrivial, WF fixed point solutions. Indeed, Fig. 2 shows the thermal RG flow diagram of the ϕ^4 model in $d = 3$ dimensions. Empty circles show how the position of the WF fixed point changes with τ. The RG trajectory which runs from the Gaussian ($\tilde{g}_2 = 0$ and $\tilde{g}_4 = 0$) fixed point to the WF one separates the phases of the model. If one increases the value of τ, the slope of the critical trajectory and consequently the broken phase is decreased.

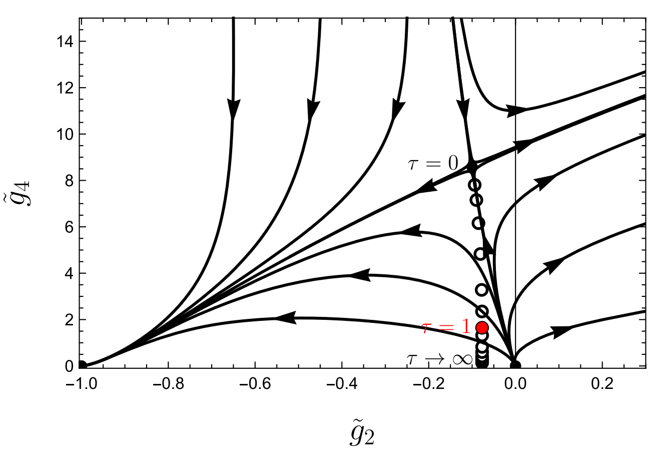

FIG. 2. Thermal RG flow diagram of the ϕ^4 model in $d = 3$ dimensions based on flow Eqs. (14) and (15). Circles show how the position of the WF fixed point changes with τ.

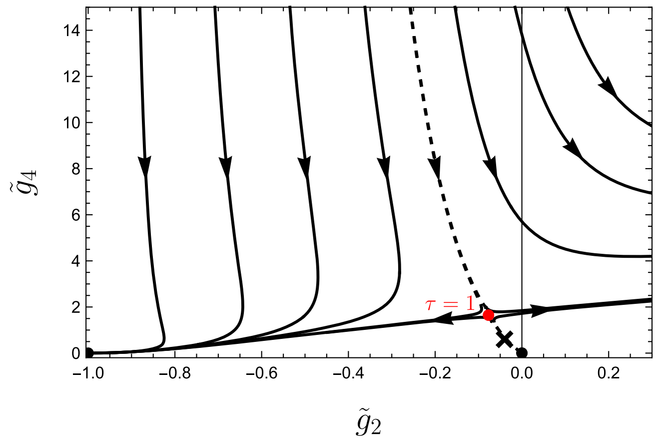

FIG. 3. Thermal RG flow diagram of the ϕ^4 model in $d = 3$ dimensions with $\tau = 1$ based on the RG flow Eqs. (14) and (15). Black cross denotes an initial condition which lies on the separatrix. For $\tau < 1$ the RG trajectory from that starting point runs into the broken (low-temperature) phase, and for $\tau > 1$ the RG trajectory ends up in the symmetric (high-temperature) phase.

For $\tau \to \infty$ it is not possible to find any starting point in the vicinity of the Gaussian fixed point from which an RG trajectory can run into the broken phase. Since τ measures how thermal fluctuations are important compared to quantum fluctuations, one can say that by changing τ a thermal phase transition occurs. For an arbitrary but fixed starting point taken from the vicinity of the Gaussian fixed point (see the black cross on Fig. 3) one can always determine a critical value τ_c: if $\tau < \tau_c$ the RG trajectory from that starting point runs into the broken (low-temperature) phase, and for $\tau > \tau_c$ the RG trajectory ends up in the symmetric (high-temperature) phase.

Let us consider the RG flow diagram shown in Fig. 3 with respect to the change in the "quantum" parameter which results in a QPT. The "quantum" parameter is the slope of RG trajectories taken at the vicinity of the Gaussian fixed point. For a fixed temperature, i.e., for fixed τ, one can determine a critical value which is the slope of the separatrix; see dashed line of Fig. 3. If the slope of an RG trajectory is smaller or larger then this critical value, the corresponding trajectory, runs into the broken or the symmetric phase, respectively, which signals the QPT. A good approximation for the critical value of the "quantum" parameter is the ratio $(\tilde{g}_4/|\tilde{g}_2|)_{\mathrm{WF}}$ where $\tilde{g}_4$ and $\tilde{g}_2$ are the coordinates of the WF fixed point. Thus, the critical line on the QPT-CPT diagram of the ϕ^4 model can be given in terms of τ_c and $(\tilde{g}_4/|\tilde{g}_2|)_{\mathrm{WF}}$ which are related to each other.

VI. QPT-CPT DIAGRAM OF THE ϕ^4 MODEL

As a next step we discuss some key issues regarding the comparison of the QPT-CPT diagrams of the Ising model (which is well-known) and the ϕ^4 theory (which we would

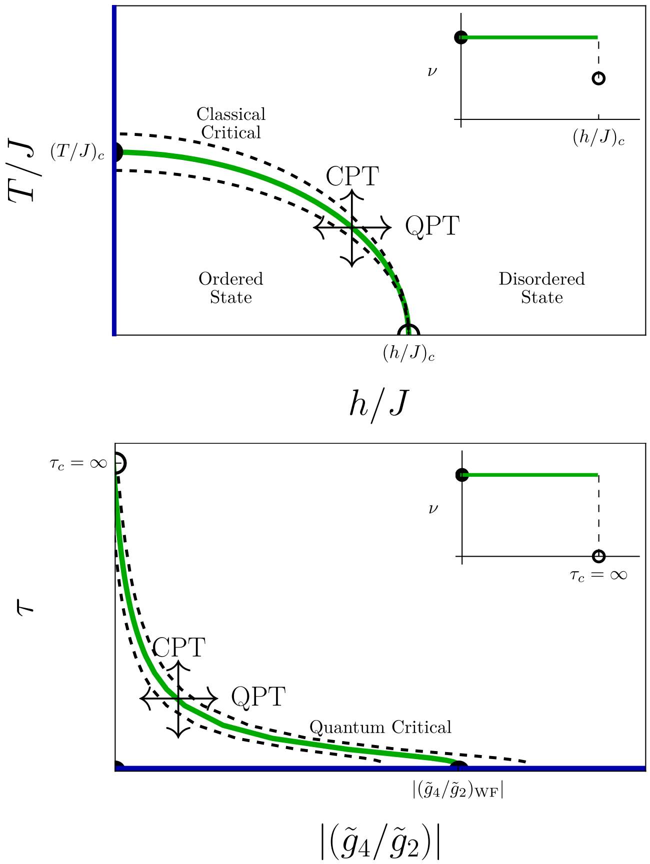

FIG. 4. Schematic QPT-CPT diagram of the Ising model (top) and the ϕ^4 theory (bottom) which are "dual to each other", i.e., (T/J) corresponds to $(\tilde{g}_4/\tilde{g}_2)$ while τ is related to (h/J). Thus, the critical behavior on the blue lines is known to be identical and expected to be the same along the green critical lines but has a discontinuity at their endpoints.

like to map out here). The schematic pictures of these QPT-CPT diagrams are shown in Fig. 4.

It is known that the classical Ising model is equivalent to the zero-temperature ϕ^4 QFT. Thus, the vertical axis of the QPT-CPT diagram and the critical temperature of the Ising model can be mapped onto the horizontal axis of the QPT-CPT diagram and the critical "quantum" parameter of the (zero-temperature) ϕ^4 QFT; see the blue lines of Fig. 4. In addition, the thermal critical behavior of the classical Ising model must be identical to the quantum critical behavior of the ϕ^4 QFT.

It is clear from Fig. 4 that the "quantum" parameter of the Ising model is h/J which is the strength of the transverse-field in terms of J. The quantum critical behavior of the ϕ^4 QFT is given in terms of the irrelevant coupling around the WF saddle point; however, this irrelevant coupling is related to the ratio $(\tilde{g}_4/|\tilde{g}_2|)_{\mathrm{WF}}$ and in the previous section we argued, that this ratio can serve as a good "quantum" parameter for ϕ^4 QFT. Thus, if the temperature is zero, both the quantum Ising model and the ϕ^4 QFT are nontrivial, and they undergo a QPT with a finite h_c and finite ratio $(\tilde{g}_4/|\tilde{g}_2|)_{\mathrm{WF}}$.

For a vanishing quantum parameter, i.e., $h/J = 0$, the Ising model becomes the well-known classical one, which is nontrivial; it has a classical (thermal) phase transition with a finite transition temperature $T_c < \infty$. On the contrary, for a vanishing "quantum" parameter, i.e., for $(\tilde{g}_4/|\tilde{g}_2|)_{\mathrm{WF}} = 0$, the ϕ^4 model becomes a trivial massive free field theory (with thermal fluctuations), which has no phase transition, so there is no room for any finite transition temperature. On the one hand, the critical temperature of the ϕ^4 model must be zero if $(\tilde{g}_4/|\tilde{g}_2|)_{\mathrm{WF}} = 0$. On the other hand, the critical line must be a strictly monotonic function, so one expects $T_c \to \infty$ for the ϕ^4 in the limit $(\tilde{g}_4/|\tilde{g}_2|)_{\mathrm{WF}} \to 0$. Therefore, one expects a discontinuity in the classical limit of the ϕ^4 theory; this is why we use an empty circle at $\tau_c = \infty$ on the lower panel of Fig. 4 and in its inset. In addition, one can conclude that the Ising model has a thermal (classical) and a quantum critical behavior, too but the ϕ^4 model has a quantum critical behavior, only.

Let us note for reader that the terminology "classical" has to be used with great care. In this work when we use "classical Ising model" we refer to the Ising model with thermal fluctuations; however, when we use "classical" ϕ^4 theory it is understood as a model with no quantum and thermal fluctuations. The change in the sign of the mass parameter of the classical ϕ^4 theory results in two different (single and double well) forms for the potential but actually it is not considered as a phase transition. As we argued, the "quantum" parameter of the ϕ^4 QFT is the irrelevant direction around the WF fixed point which is a combination of the quartic and quadratic couplings. Thus, if one takes the limit of vanishing "quantum" parameter (in some sense the classical limit) one arrives at a free field theory (with no quartic term) which is not equivalent to the "classical" ϕ^4 model.

It is also known that the critical exponents of the finite temperature quantum Ising model are equal to the critical exponents of the classical Ising model; see the inset of the upper panel of Fig. 4. Since the Ising model and the ϕ^4 theory are "dual to each other", i.e., (T/J) corresponds to $(\tilde{g}_4/\tilde{g}_2)$ while τ is related to (h/J), the critical behavior is expected to be the same along the green critical lines of Fig. 4 and the critical exponents of the finite temperature ϕ^4 QFT must be equal to their zero-temperature values; see the inset on the lower panel of Fig. 4. Thus, for finite temperatures and finite "quantum" parameters, the Ising and the ϕ^4 models are expected to belong to the same universality class.

However, it is well-known that the critical behavior of the Ising model has a discontinuity in the limit $T \to 0$; see the empty circle at $T = 0$ (i.e., at h_c) in the upper panel of Fig. 4. So, critical lines (green lines on the upper and lower panels of Fig. 4) have a discontinuity at their endpoints:

for the Ising model it happens at $h/J = (h/J)_c$, and for the ϕ^4 theory it situates at $\tau = \tau_c = \infty$.

These conclusions can be drawn without performing any explicit calculations on the QPT-CPT diagram of the ϕ^4 model. In other words, one can assume the above statements which can be used to test the thermal RG Eqs. (6) and (10) suggested by us.

The QPT-CPT diagram of the three-dimensional ϕ^4 model is constructed by the numerical solution of the thermal RG Eqs. (6) and (10); see the solid (Litim cutoff) and the dashed (sharp-cutoff) lines of Fig. 5. The QPT-CPT diagram based on thermal RG equations with the Litim and the sharp-cutoff regulators are similar but not identical: between $\tau = 1$ and $\tau = \infty$ they coincide but differ from each other between $\tau = 0$ and $\tau = 1$.

The critical exponent ν of the ϕ^4 model is expected to show a small dependence on the parameter τ; practically they must be constant along the critical lines which are see the solid (Litim regulator) and dashed (sharp cutoff regulator) lines of Fig. 5. Indeed, the solid and the dashed lines of the inset in Fig. 5 confirms this. Between $\tau = 1$ and $\tau = \infty$ the solid line is practically constant and it has a relatively small dependence between $\tau = 0$ and $\tau = 1$. The dashed line in the inset of Fig. 5 shows that by using the thermal RG equation with the sharp-cutoff regulator, the critical exponent ν has a larger dependence on the parameter τ; indeed ν varies with τ even in the range $\tau > 1$ which was not true for the Litim case (solid line). Nevertheless, one can argue that the Litim regulator always provides us a better convergence and, as a consequence, more reliable results in the zero-temperature limit, i.e., for $\tau \to 0$. However, this requires further investigations by going beyond LPA and by the study of different models, such as the two-dimensional sine-Gordon theory. In both cases, i.e., for the Litim and the sharp-cutoff regulators the critical exponent ν tends to a constant value in the limit $\tau \to \infty$ which is identical to that of obtained by the zero-temperature FRG equation. Of course, this constant value depends on the choice of the regulator. Let us note for the reader that the Litim regulator is known to give the closest result among all regulators to the "exact" value ($\nu = 0.629971$).

It is also expected that the critical line of the ϕ^4 model must tend to infinity in the limit of vanishing quantum parameter which is clearly supported by the thermal RG studies, see the solid and dashed lines of Fig. 5. Our results are tested for larger values of NCUT, and we have found the same qualitative behavior.

Finally let us discuss an interesting property of the infinity temperature limit. We showed, that in the limit $\tau \to \infty$ the WF fixed point tends to the horizontal axis. In other words, the quartic coupling $\tilde{g}_4^{\mathrm{WF}}$ of the WF fixed point tends to zero for $\tau \to \infty$. However, one can show that their product $\tau\tilde{g}_4^{\mathrm{WF}}$ tends to a constant value. Thus, the QPT-CPT diagram where the vertical axis is chosen to be $\tau\tilde{g}_4^{\mathrm{WF}}$ (see Fig. 6) becomes identical to the well-known QPT-CPT diagram of the Ising model obtained by Monte Carlo simulation (see Fig. 7). It is important to note that real (and not pseudo) fixed points can only be determined by keeping $\tau = T/k$ fixed over the RG flow in the simultaneous $T, k \to 0$ limit. This provides us the tool to draw the QPT-CPT diagram of a QFT, such as the ϕ^4 model; see Fig. 6. The critical line which separates the broken and symmetric phases is determined by these (real) fixed points which can be directly compared to simulation results and

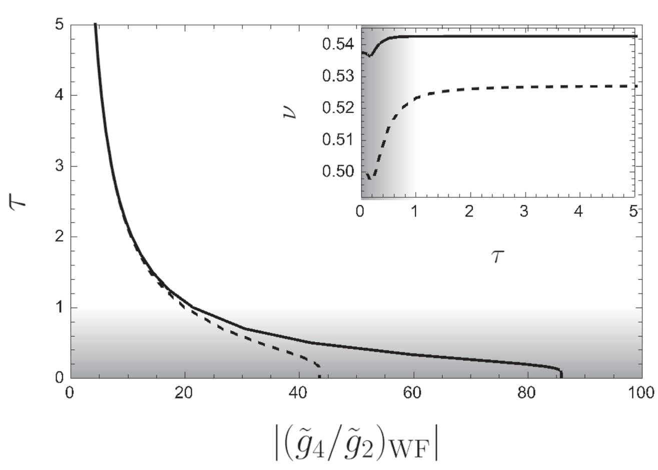

FIG. 5. QPT-CPT diagram of the ϕ^4 model in $d = 3$ dimensions based on the thermal RG Eqs. (6) and (10). The solid (dashed) line stands for results obtained by the Litim (sharp-cutoff) regulator. The inset shows the critical exponent ν as a function of τ which is related to the temperature. The shaded areas stand for $\tau < 1$.

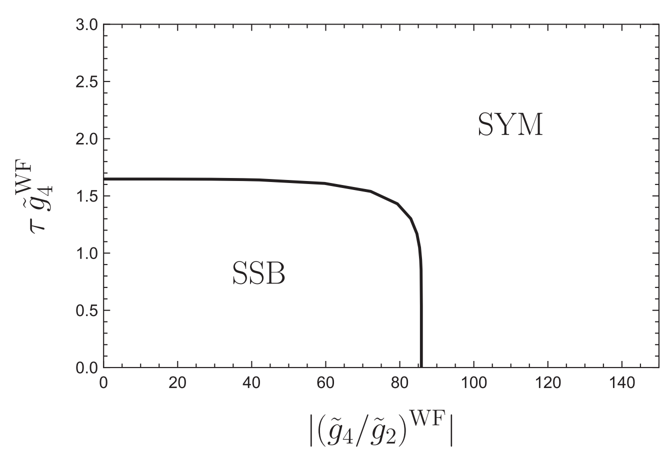

FIG. 6. QPT-CPT diagram of the ϕ^4 model in $d = 3$ dimensions in terms of $\tau\tilde{g}_4^{\mathrm{WF}}$ and $(\tilde{g}_4/\tilde{g}_2)^{\mathrm{WF}}$, where $\tilde{g}_4^{\mathrm{WF}}$ and $\tilde{g}_2^{\mathrm{WF}}$ are the coordinates of the WF fixed point. The black critical line which separates the spontaneous symmetry broken (SSB) and symmetric (SYM) phases terminates at finite value in the limit of $(\tilde{g}_4/\tilde{g}_2)^{\mathrm{WF}} \to 0$ which represents an important difference compared to Fig. 5 where the critical line runs to infinity.

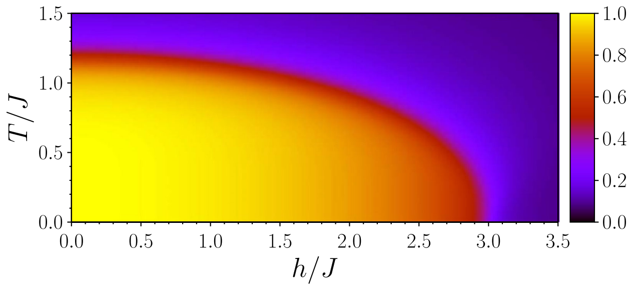

FIG. 7. QPT-CPT diagram of the transverse field Ising model obtained by Monte Carlo simulation and taken from [45]. This can be directly compared to Fig. 6.

measurements performed on the corresponding spin model, in our case on the transverse field Ising model; see Fig. 7. If one keeps T constant over the RG flow no such comparison can be done since in this case the thermal FRG has pseudo (and not real) fixed points. So, the direct comparison between the line of real fixed points of quantum-thermal spin models and their thermal QFT counterparts can only be done in the modified thermal FRG method proposed here.

VII. CONCLUSIONS

In this work, we proposed a modified finite-temperature FRG approach by relating the temperature parameter to the running RG scale, $T \equiv k_T = \tau k$, where k_T is the running cutoff for thermal and k is for the quantum fluctuations. In this case, the temperature must be related to the dimensionless quantity τ. This choice is supported by the idea of the Wilsonian approach where fluctuations are taken into account by the integration of degrees of freedom above these running cutoffs which are chosen to be different in order to make difference between thermal and quantum fluctuations. For $k_T = k$ (i.e., for $\tau = 1$), one cannot distinguish between thermal and quantum fluctuations. We applied this new thermal RG approach for the ϕ^4 model in lower dimensions and constructed its QPT-CPT phase diagram which was used to test the new thermal RG approach. Indeed, we formulated requirements for the QPT-CPT phase diagram of the ϕ^4 theory based on known properties of the QPT-CPT phase diagram of the Ising model and we used these requirements to prove the viability of the thermal RG method proposed in this work.

It is important to note that previous studies based on the finite-temperature FRG method discussed findings with special emphasis on the interplay of the temperature T and the running RG scale k. With Eq. (58) of [17] one finds, for example, the RG flow equation of the infinite volume Stefan–Boltzmann pressure $\partial_k P(T) = -k^4/12\pi^2[\coth(k/2T) - 1]$. To draw conclusions, one has to discuss how the running RG scale k is related to the temperature which is fixed to the UV cutoff, i.e., $T = \tau\Lambda$. For large cutoff scales, $k \gg T$ the flow decays exponentially with $\exp(-k/T)$ which is given by Eq. (58) of [17]. By using our suggestions where the temperature parameter is related to the running scale $T = \tau k$, and the dimensionless parameter τ plays the role of the temperature, this flow equation is simplified as $\partial_k P(\tau) = -k^4/12\pi^2[\coth(1/2\tau) - 1]$ and conclusions can be drawn based on the change of the dimensionless parameter τ only. Thus, the modified thermal RG method presented in this work finds application in a natural way in all previous thermal RG studies of quantum models, and it can be used both to simplify the treatment and to find new results in a variety of quantum models exhibiting quantum phase transitions.

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DATA AVAILABILITY

The data are not publicly available. The data are available from the authors upon reasonable request.

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[33] For example, in [26] at page 42, in Eq. (132) one finds $\partial_t \tau = -\tau$ where $t = \ln(k/\Lambda)$ and in [28] at the top of the page 12 one can read $\partial_t T/k = -T/k$ which result in the trivial RG scale dependence for τ_k. Another example is found in [24] at page 4 above equation (21), $\tilde{T}(\Lambda^\star) = 1$ where the intermediate momentum scale $\Lambda^\star$ is fixed by the running dimensionless temperature $\tilde{T}$ which is identical to τ. Finally, we refer to [32] at page 7 right below equation (50), where one finds $\tilde{T}_k = T/(c_k k)$.
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