THE MINIMAL SECTION OF A TRIANGULAR MASONRY DAM

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SUMMARY. A masonry dam with triangular cross section is subjected to its own weight and to hydrostatic pressure in a state of plane strain. Under the constraint that the principal stresses be strictly negative in the interior, it is shown that there is a section of minimal area and, consequently, a dam of lowest weight.

1. INTRODUCTION

There is a renewal of interest for the mechanics of materials weakly reacting to tension, a theme long neglected. Some recent contributions notwitstanding, to determine the stress and deformation fields at equilibrium in elastic solids that do not support tension is still an open problem, even in apparently simple cases (cf., for instance, [1],[2]).
Fortunately, an alternative presents itself to the designer of a masonry structure : *for a linearly elastic material and prescribed external loads, find structural shapes, if any, that give rise to purely compressive states in their interior.* Here we use this approach for the classical problem of a dam with triangular cross section subjected to its own weight and to hydrostatic pressure in a state of plane strain. Among the cross sections that allow for purely compressive equilibrium states, we seek those of minimal area.

2. THE ELASTIC PROBLEM FOR A RIGHT-ANGLED TRIANGLE

A gravity dam has the cross section Ω of the form of a right-angled triangle: H is its height and β $(0 < \beta < \pi/2)$ is the angle formed by the faces. The plane of the section is referred to an orthogonal system x0y with the axes oriented as in fig. 1.
The left side OA is subjected to the hydrostatic pressure

$-\gamma x j$, where γ is the specific weight of the liquid. Besides its own weight, the dam is subjected to the reactions of the supporting plane AB, which, for sake of simplicity, are supposed to be linear in y, in agreement with the design criteria usually employed.

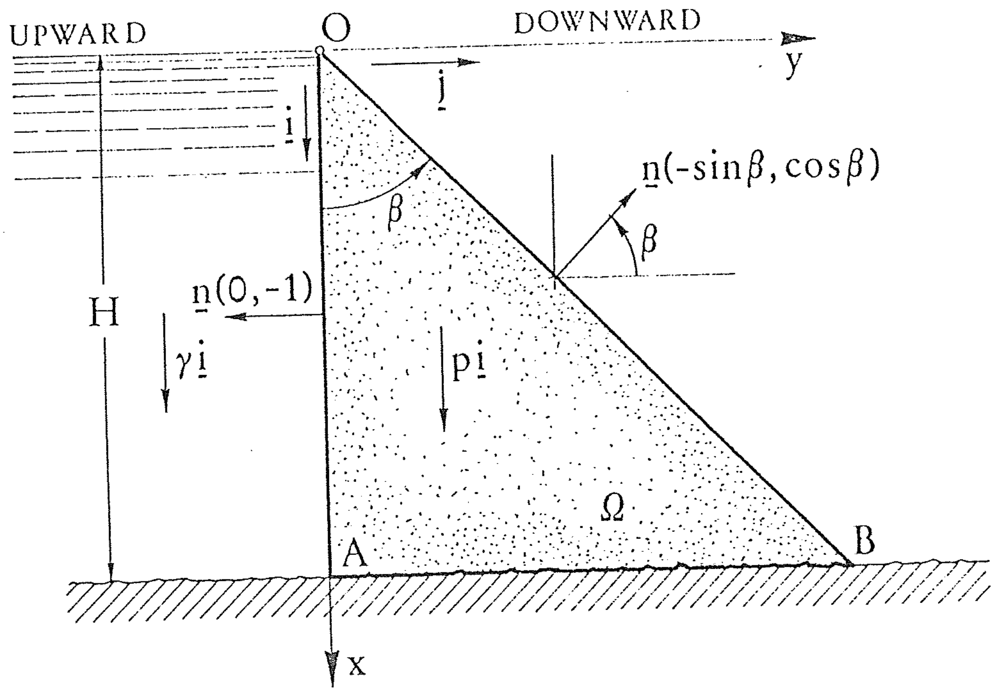

Fig. 1

Assume further that the material of the dam is linearly elastic with constitutive equation

(1) $$T = 2\mu E + \lambda(\mathrm{tr}E)I,$$

where T is the stress tensor, E the strain tensor, $\mathrm{tr}E$ its trace, I the identity tensor, and λ and μ the Lame' moduli. If the dam is infinitely long in the direction orthogonal to its plane section, there results a plane strain problem. The stress components in Ω satisfy the equilibrium equations

(2) $$\begin{cases} \sigma_{x,x} + \tau_{xy,y} = -p \ , \\ \tau_{xy,x} + \sigma_{y,y} = 0 \ , \end{cases}$$

where p is the specific weight of the material of the dam, together with the compatibility equation

(3) $$\Delta(\sigma_x + \sigma_y) = 0 .$$

In addition, the stress components must satisfy the boundary conditions

(4) $$\begin{cases} \sigma_y(x,0) = -\gamma x , \\ \tau_{xy}(x,0) = 0 , \end{cases}$$

on OA, and

(5) $$\begin{cases} -\sigma_x(x,xtg\beta)\sin\beta + \tau_{xy}(x,xtg\beta)\cos\beta = 0 , \\ -\tau_{xy}(x,xtg\beta)\sin\beta + \sigma_y(x,xtg\beta)\cos\beta = 0 , \end{cases}$$

on OB. Furthermore, as the reactions at the bottom vary linearly, the following boundary conditions must hold on the side AB:

(6) $$\begin{cases} \sigma_x(H,y) = \dfrac{H(\gamma - ptg^2\beta)}{tg^2\beta} + \dfrac{1}{tg^3\beta}(ptg^2\beta - 2\gamma)y , \\ \tau_{xy}(H,y) = -\dfrac{1}{tg^2\beta}\gamma y . \end{cases}$$

The field equations (2) and (3) can be satisfied by choosing a stress function

(7) $$F(x,y) = ax^3 + bx^2y + cxy^2 + dy^3 ,$$

with a, b, c constants, such that

$$(8)\quad \begin{cases} \sigma_x(x,y) = \dfrac{\partial^2 F(x,y)}{\partial y^2}\,, \\[2ex] \sigma_y(x,y) = \dfrac{\partial^2 F(x,y)}{\partial x^2}\,, \\[2ex] \tau_{xy}(x,y) = -\dfrac{\partial^2 F(x,y)}{\partial x \partial y} - py. \end{cases}$$

Substituting (8) into (4) and (5), we obtain

$$\begin{cases} a = -\dfrac{1}{6}\gamma\,, \\ b = 0\,, \\ c = \dfrac{1}{2tg^2\beta}[\gamma - ptg^2\beta]\,, \\[2ex] d = \dfrac{1}{6tg^3\beta}[ptg^2\beta - 2\gamma]\,, \end{cases}$$

and, correspondingly,

$$(9)\quad \begin{cases} \sigma_x = \dfrac{1}{tg^2\beta}[\gamma - ptg^2\beta]\,x + \dfrac{1}{tg^3\beta}[ptg^2\beta - 2\gamma]\,y\,, \\[2ex] \sigma_y = -\gamma x\,, \\[2ex] \tau_{xy} = -\dfrac{1}{tg^2\beta}\gamma y\,, \end{cases}$$

which is also in agreement with (6). The stress components turn out to be linear functions of the variables x and y. This permits easy control of their sign. For example,

$$\sigma_x(x,0) = \frac{\gamma x}{tg^2\beta}[1 - ktg^2\beta]\,,$$

and

$$\sigma_x(x, x\,tg\beta) = - \frac{\gamma x}{tg^2\beta} ,$$

where $k = p/\gamma$. Consequently, $\sigma_x(x,y) < 0$ in $\Omega \setminus \partial\Omega$ whenever

(10) $\quad \beta \geq \mathrm{arctg}\ (1/\sqrt{k})$.

It is very easy to examine the sign of the determinant of the stress matrix T. In fact,

$$\delta(x,\eta) = \det T = - \frac{\gamma x^2}{tg^2\beta} \varphi(\eta) ,$$

where $\eta = y/x\,tg\beta$, $0 < \eta < 1$, and

$$\varphi(\eta) = \eta^2 + (k\,tg^2\beta - 2)\eta + (1-k\,tg^2\beta)$$

is a convex quadratic function of η , vanishing for

$$\eta_1 = 1-k\,tg^2\beta , \quad \eta_2 = 1 .$$

Thus, in order that $\delta > 0$ in $\Omega\setminus\partial\Omega$, it is sufficient to require $\eta_1 \leq 0$, that is, once again, $\beta \geq \mathrm{arctg}(1/\sqrt{k})$. Consequently, the condition (10) garantees that the principal stresses are negative everywhere in Ω, with the possible exception of the boundary points. When condition (10) fails to be satisfied, tensions arise in a region adjacent to the upward side. Indeed, for $\beta < \mathrm{arctg}(1/\sqrt{k})$, $\delta < 0$ whenever $\eta \in [0,\eta_1)$, with $\eta_1 = 1-k\,tg^2\beta > 0$. Thus, the portion of Ω where one of the principal stresses is positive coincides with the triangular region defined by the inequalities $x > 0$, $0 \leq y < x\,tg\beta(1-k\,tg^2\beta)$.

3. THE CASE OF GENERAL TRIANGULAR SECTION

Let us suppose now that the upward face of the dam, instead of being vertical, makes an angle α $(0 < \alpha < \pi/2)$ with the y-axis (see Fig. 2).

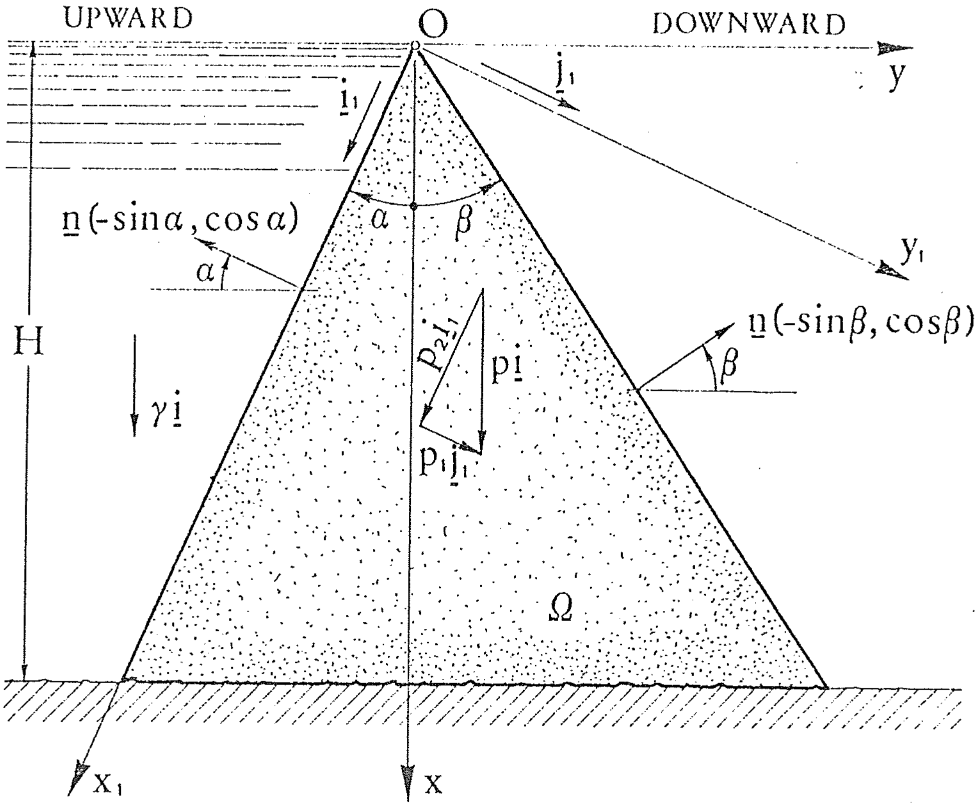

Fig. 2

The elastic solution of this problem, due to Levy [3], can be obtained exactly in the same way, by only modifying the boundary conditions (4), (5) and (6), and by expressing the stress function again with the cubic expression (7). For our purpose, however, it is useful to represent the solution in the reference x_1Oy_1 , which is obtained by rotating the reference xOy clockwise by α in such a manner that x_1 coincides with the upward side of the dam. In this reference the field equations in Ω are

(11) $$\begin{cases} \sigma_{x_1,x_1} + \tau_{x_1y_1,y_1} = -p_2 , \\ \tau_{x_1y_1,x_1} + \sigma_{y_1,y_1} = -p_1 , \end{cases}$$

and

(12) $$\Delta(\sigma_{x_1} + \sigma_{y_1}) = 0 ,$$

where $p_1 = p\sin\alpha$, $p_2 = p\cos\alpha$. The boundary conditions on the sides OA and OB are, respectively ,

(13) $$\begin{cases} \sigma_{y_1}(x_1,0) = -\gamma_1 x_1 , \\ \tau_{x_1y_1}(x_1,0) = 0 , \end{cases}$$

and

(14) $$\begin{cases} -\sigma_{x_1}(x_1,x_1 tg\beta^*)\sin\beta^* + \tau_{x_1y_1}(x_1,x_1 tg\beta^*)\cos\beta^* = 0 , \\ -\tau_{x_1y_1}(x_1,x_1 tg\beta^*)\sin\beta^* + \sigma_{y_1}(x_1,x_1 tg\beta^*)\cos\beta^* = 0 , \end{cases}$$

where $\gamma_1 = \gamma\cos\alpha$, $\beta^* = \alpha + \beta$. If we assume

(15) $$\begin{cases} \sigma_{x_1} = \dfrac{\partial^2 F_1(x_1,y_1)}{\partial y_1^2} , \\ \sigma_{y_1} = \dfrac{\partial^2 F_1(x_1,y_1)}{\partial x_1^2} , \\ \tau_{x_1y_1} = -\dfrac{\partial^2 F_1(x_1,y_1)}{\partial x_1 \partial y_1} - p_1x_1 - p_2y_1 , \end{cases}$$

the field equations (11) and (12) are satisfied by choosing for

F_1 the cubic expression $F_1(x_1,y_1) = a_1x_1^3 + b_1x_1^2y_1 + c_1x_1y_1^2 + d_1y_1^3$ where a_1, b_1, c_1, d_1 are constants. By substituting (15) in the boundary conditions (13) and (14), we obtain

$$\begin{cases} a_1 = -\dfrac{1}{6}\gamma_1 , \\ b_1 = -\dfrac{1}{2}p_1 , \\ c_1 = \dfrac{1}{2tg^2\beta^*}[\gamma_1 - p_2tg^2\beta^*] + \dfrac{p_1}{2tg\beta^*} , \\ d_1 = \dfrac{1}{6tg^3\beta^*}[p_2tg^2\beta^* - 2\gamma_1] - \dfrac{p_1}{3tg^2\beta^*} . \end{cases}$$

The corresponding stress components become

$$(16)\quad \begin{cases} \sigma_{x_1}(x_1,\eta_1) = -\dfrac{\gamma_1x_1}{tg^2\beta^*}\{-1+ktg\beta^*(tg\beta^*-tg\alpha)+ \\ \qquad\qquad + (2-ktg^2\beta^*+2ktg\alpha tg\beta^*)\eta_1\} , \\ \sigma_{y_1}(x_1,\eta_1) = -\gamma_1x_1(1-ktg\alpha tg\beta^*\eta_1) , \\ \tau_{x_1y_1}(x_1,\eta_1) = -\dfrac{\gamma_1x_1}{tg\beta^*}(1-ktg\alpha tg\beta^*)\eta_1 , \end{cases}$$

where $\eta_1 = y_1/x_1tg\beta^*$, with $\eta_1 \epsilon [0,1]$. With the aid of (16), it is easy to examine the sign of the principal stresses in the interior of the cross section. For convenience set $u = tg\beta$, $v = tg\alpha$. Then we have

$$(17)\quad \begin{cases} \sigma_{x_1}(x_1,0) = -\gamma_1 x_1\left[-1+k\dfrac{u+v}{1-uv}\left(\dfrac{u+v}{1-uv}-v\right)\right]\left(\dfrac{1-uv}{u+v}\right)^2, \\ \sigma_{x_1}(x_1,1) = -\gamma_1 x_1\left[1+kv\dfrac{u+v}{1-uv}\right]\left(\dfrac{1-uv}{u+v}\right)^2, \\ \sigma_{y_1}(x_1,0) = -\gamma_1 x_1, \\ \sigma_{y_1}(x_1,1) = -\gamma_1 x_1\left[1-kv\dfrac{u+v}{1-uv}\right]. \end{cases}$$

It then follows that $\sigma_{y_1}(x_1,\eta_1)$ is less than zero in $\Omega\backslash\partial\Omega$ whenever

$$(18)\qquad H(u,v) = kv\frac{u+v}{1-uv} - 1 \le 0,$$

whereas $\sigma_{x_1}(x_1,\eta_1)$ is less than zero in $\Omega\backslash\partial\Omega$ whenever

$$(19)\qquad I(u,v) = k\left(\frac{u+v}{1-uv}\right)^2 - kv\frac{u+v}{1-uv} - 1 \ge 0.$$

Given k, the relations (18) and (19) define a region R of the plane u,v where $\sigma_{x_1} < 0$ and $\sigma_{y_1} < 0$ in the interior of Ω. Thus, the region S, where the solutions are purely compressive, is a subset of R, which is not empty, since it contains the straight line originating from the point $L(1/\sqrt{k},0)$ and with equation $v = 0$, $u \ge 1/\sqrt{k}$. In addition, it is easy to verify that the interior of S is not empty. In fact

$$\delta_1(x_1,\eta_1) = \det T = -\gamma_1^2 x_1^2\left[G(u,v)\eta_1^2 + E(u,v)\eta_1 + F(u,v)\right]\left(\frac{1-uv}{u+v}\right)^2,$$

with

$$(20)\quad \begin{cases} G(u,v) = kv\,\dfrac{u+v}{1-uv}\left[2-k\left(\dfrac{u+v}{1-uv}\right)^2 + 2kv\,\dfrac{u+v}{1-uv}\right] + \left[1-kv\,\dfrac{u+v}{1-uv}\right]^2, \\ E(u,v) = -\,2+k\left(\dfrac{u+v}{1-uv}\right)^2 - kv\,\dfrac{u+v}{1-uv}\left\{3-k\,\dfrac{u+v}{1-uv}\left[\dfrac{u+v}{1-uv} - v\right]\right\}, \\ F(u,v) = +\,1 - k\,\dfrac{u+v}{1-uv}\left[\dfrac{u+v}{1-uv} - v\right], \end{cases}$$

is a quadratic function of η_1, always vanishing for $\eta_1 = 1$. Thus in the subset S^* of R, defined by $G(u,v)>0$ or, equivalently, by

$$(21)\qquad kv\,\frac{u+v}{1-uv}\left[k\left(\frac{u+v}{1-uv}\right)^2 - 3kv\,\frac{u+v}{1-uv}\right] - 1 \leq 0\ ,$$

$-\,\delta_1$ is a convex function of η_1, and, consequently, S^* is a subset of S also.

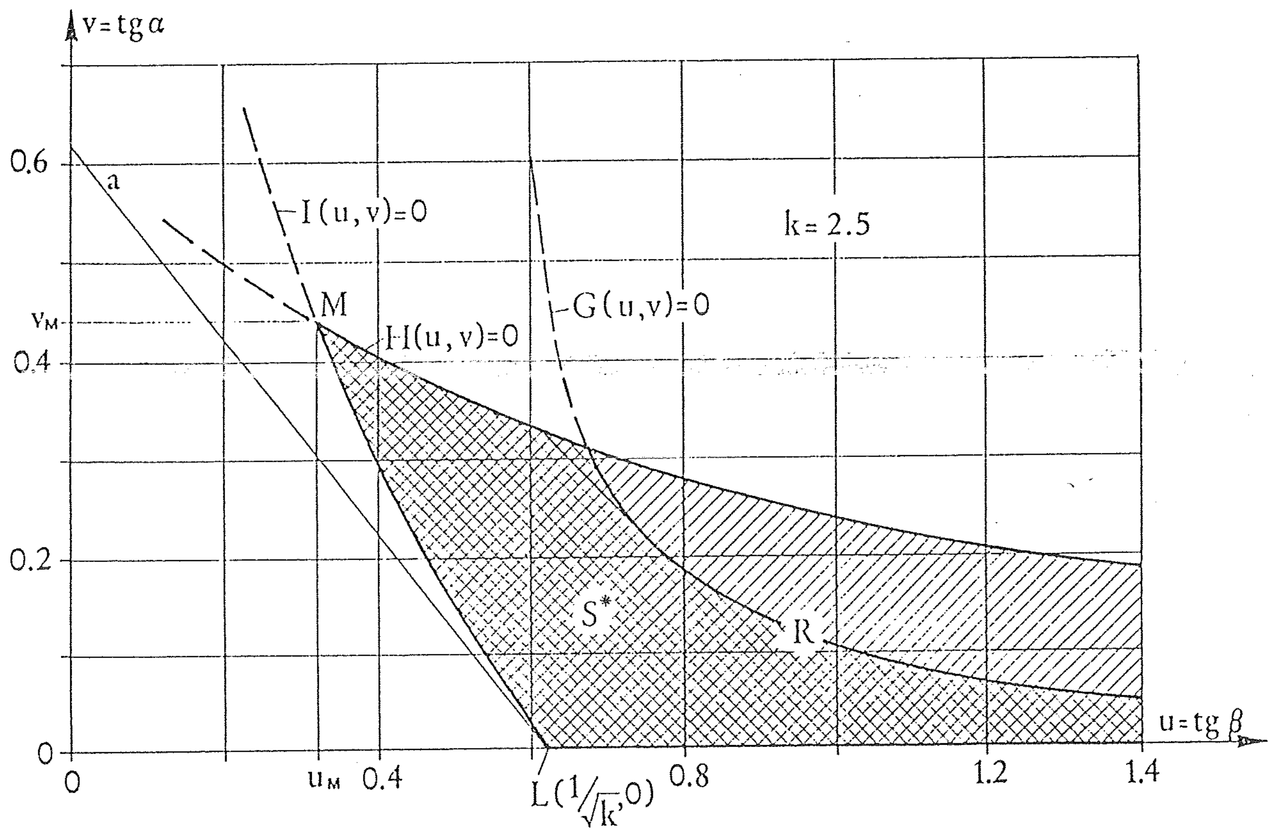

Fig. 3

The Fig.3 shows the regions R and S^* of the plane u,v when k=2.5. We can see that, among the points of S, M is the one which corresponds to the minimal value of β compatible with the property that the principal stresses are negative at each internal point of Ω . The coordinates of M satisfy the equations

$$(22)\qquad \begin{cases} kv(u+v) - (1-uv) = 0 , \\ k(u+v)^2 - kv(u+v)(1-uv) - (1-uv)^2 = 0. \end{cases}$$

Thus,

$$u_M = \sqrt{\frac{k}{2(k+1)^2}} , \quad v_M = \sqrt{\frac{1}{2k}} .$$

When k = 2.5 β_M = arctg $u_M = 17.7^0$ and α_M = arctg $v_M = 24.1^0$.

For each point of S, the elastic solutions can be regarded as solutions of the corresponding problem for a material that does not support tension. Therefore, for any slope of the upward side between 0 and α_M , with α_M dependent on k, it is always possible to determine the minimal slope of the downward side in such a way that the resulting cross section be compatible with the use of materials that support only weak tensions. Conversely, for sufficiently small values of β there is no value of α for which the cross section belongs to S. *That cross sections with small angles α are to be preferred, is confirmed by the fact that, as we will show next, among all the sections of S, the one having the minimal area is obtained on setting α = 0.*

4. THE MINIMAL SECTION

Let us consider in the plane u,v the straight line a, passing trough point $L(1/\sqrt{k},0)$, belonging to S, and with equation

$$(23)\qquad u + v = 1/\sqrt{k} .$$

Along a the area A of the cross section of the dam is constant and equal to $A^* = H^2/2\sqrt{k}$.

The arch LM of the boundary of R (see fig. 3) has equation

$$(24) \qquad I(u,v) = (1-uv)^2 - k(u+v)^2 + kv(u+v)(1+uv) = 0.$$

From the system defined by (23) and (24) we obtain the equation

$$v[v^3 + v^2(k-2)/\sqrt{k} + v(1+1/\sqrt{k}) + (k-2)/\sqrt{k}] = 0 .$$

Since $v \geq 0$, this can be solved, if $k > 2$, only when $v = 0$, value to which the point L, alone, corresponds. Since L belongs to S, in order to prove that it is the point of absolute minimum we are searching for, it is sufficient to verify that R is entirely contained in the semiplane to the right of the straight line a, where $A > A^*$. On the other hand, according to the implicit function theorem

$$\left.\frac{\partial I(u,v)}{\partial v}\right|_{\substack{u = 1/\sqrt{k} \\ v = 0}} \neq 0.$$

Therefore, in a neighbourhood of L the curve LM is the graph of a function $v=V(u)$, with

$$\left.\frac{dV(u)}{du}\right|_{u = 1/\sqrt{k}} = - \left.\frac{\dfrac{\partial I(u,v)}{\partial u}}{\dfrac{\partial I(u,v)}{\partial v}}\right|_{\substack{u = 1/\sqrt{k} \\ v = 0}} = - \frac{2k}{2+k} < - 1 ,$$

provided that $k > 2$. Thus, *when* $k > 2$, *among all the dams whose internal points are in a condition of pure compression, the triangular dam with*

$$\begin{cases} \alpha = 0 , \\ \beta = \operatorname{arctg} (1/\sqrt{k}) , \end{cases}$$

has the cross section of minimal area. The minimum is

$A^* = H^2/2\sqrt{k}$. *In other words, if* H *is fixed and the material does not support tension,* pA^* *is the minimal weight per unit thickness of a gravity dam with triangular cross section.*

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