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REDUCTION OF THE MAXIMUM CUT PROBLEM TO A MAXIMAL FLOW PROBLEM FOR A CLASS OF GRAPHS

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Abstract.

Finding a maximum cut of an arbitrary graph is a problem contained in a list of 21 NP-complete problems (Karp-Cook list). It is unknown whether or not any of these problems can be solved by a polynomial bounded algorithm. Hadlock has found a polynomial bounded algorithm for finding a maximum cut in planar graphs. In this paper it is shown that the maximum cut problem can be solved in polynomial time when there exists at least one node $\bar{x}$ in the planar or non planar graph under consideration such that each odd elementary cycle of the graph contains two edges incident to $\bar{x}$. The overall computation of the proposed translation process is $O(n^3)$ in lenght.

1. Introduction.

Let $G=(N,A_G)$ be an undirected graph, where the edges are weighted with positive integers. Let $S \subset N$ be a subset of nodes in the given graph and denote by A(S) the set of edges that are incident to at least one node in S. Let s and t be two distinct nodes in N, with $s \in S$, $t \in N-S$. Then the edges in the subset $A(S) \cap A(N-S)$ are said to be a cut disconnecting s and t and are denoted by S, N-S. The sum of the weights of the edges in S, N-S is called the weight of the cut, and is denoted by w(S, N-S).

Hadlock [HAD] has found a polynomial bounded algorithm for finding a maximum cut, that is a cut S*, N-S* such that w(S*, N-S*) is maximum, in planar graphs. In this paper it is shown that the maximum cut problem can be translated into the maximal flow problem between a pair of nodes [FOR] when there exists at least one node $\bar{x}$ in the planar or non planar graph under consideration such that each odd elementary cycle of the graph contains two edges incident to $\bar{x}$. Since the maximal flow problem has a polynomial bounded algorithm [EDM 1], a maximum cut of any graph satisfying the property above mentioned can be found in polynomial time.

Before describing the translation process, some properties of hypergraphs need to be proved.

2. The hypergraphical matroid.

A matroid [HAR] $M=(Z,\mathcal{J})$ is a structure which consists of a finite set Z of elements together with a family $\mathcal{J}$ of subsets of Z called independent sets such that:

1) $\emptyset \in \mathcal{J}$;
2) if $L \in \mathcal{J}$ and $L' \subset L$ then $L' \in \mathcal{J}$;
3) for every $Z' \subseteq Z$, if X_1 and X_2 are maximal independent sets contained in Z', then $|X_1|=|X_2|$.

The elements of Z are said to be the cells of M.
Let $Z' \subseteq Z$ be a subset of cells. The rank $r(Z')$ of Z' is defined as follows:

$$r(Z') = \max_{L \in \mathcal{J}} |L \cap Z'|$$

For example, let A_G be the set of edges of a graph G and $\mathcal{J}$ be the family of all acyclic subgraphs of G, then $M_G=(A_G,\mathcal{J})$ is a matroid.

The following theorem gives another, equivalent, definition of matroid:

Theorem 1: [WHI]. Let Z be a finite set of elements and let each subset of Z be or not be a circuit. If:

1) no proper subset of a circuit is a circuit,
2) let P_1 and P_2 be circuits and $e \in P_1 \cap P_2$, then $P_1 \cup P_2 - \{e\}$ contains a circuit,

the resulting system is a matroid.

Let $H=(N, A)$ be a weighted hypergraph. Let $E \subseteq A$ be a subset of edges in the given hypergraph and denote by $N(E)$ the set of nodes that are connected to at least one edge in E. Let $x \in N$ be a node in H and denote by $A(x)$ the set of edges that are incident to x. Then a subset $C \subseteq A$ such that $|A(x) \cap C|$ is even for each $x \in N(C)$ is said to be an h-cycle in H. Let $C \subseteq A$ be an h-cycle, if no proper subset of C is an h-cycle, then C is said to be an elementary h-cycle. A subset of A that is not an h-cycle and does not contain h-cycles is said to be and h-forest in H. Note that, if H is an ordinary graph, definition of elementary h-cycle in H is equivalent to definition of elementary cycle in H.

Lemma 1. Let C_1, C_2 be h-cycles in H. Then $C_3=(C_1 \cup C_2)-(C_1 \cap C_2)$, (shortly $C_3=C_1 \oplus C_2$) is also an h-cycle in H.

Proof. Consider a node $x \in N(C_3)$. Observe that:

$$A(x) \cap C_3 = (A(x) \cap (C_1 \cup C_2))-(A(x) \cap (C_1 \cap C_2)) =$$
$$= (A(x) \cap C_1) \cup (A(x) \cap C_2) - (A(x) \cap (C_1 \cap C_2))$$

Then $A(x) \cap C_3$ contains $|A(x) \cap C_1|+|A(x) \cap C_2|-2|A(x) \cap (C_1 \cap C_2)|$ edges, that is an even number of edges. It follows that C_3 is an h-cycle in H.

Theorem 2. Let H=(N, A) be an hypergraph. Then $M_H=(A,\mathcal{F})$, where $\mathcal{F}$ is the family of all h-forests of A, is a matroid.

Proof. In order to prove this theorem it is sufficient to recall from Whitney [WHI] that a subset of cells is independent if and only if it contains no circuit and to observe that any elementary h-cycle satisfies conditions stated by Theorem 1 for circuits. In fact:

1) no proper subset of an elementary h-cycle is an elementary h-cycle (it is immediate from the definition);
2) if C_1 and C_2 are elementary h-cycles and $e \in C_1 \cap C_2$ then $C_1 \cup C_2-\{e\}$ contains an elementary h-cycle: it follows from Lemma 1 observing that:

   $C_1 \cup C_2-\{e\} \supseteq C_1 \oplus C_2$.

The theorem follows.

The matroid $M_H=(A,\mathcal{F})$ is said to be the hypergraphical matroid of M. Any maximal independent set $F_S \in \mathcal{F}$ is called a spanning h-tree of H.

3. The cut=h-cycle hypergraph.

Let $G=(N, A_G)$ be a weighted graph. Let R_G be a set of cycles of G such that:

a) R_G contains a cycle basis $B_G=\{C_1,C_2,\ldots,C_b\}$ of G, that is a set of elementary cycles such that each other cycle can be written as $C_{i(1)}\oplus C_{i(2)}\oplus\ldots\oplus C_{i(c)}$, where $C_{i(1)},C_{i(2)},\ldots,C_{i(c)}$ are cycles of B_G. A cycle C is said to be an elementary cycle if no proper subset of C is a cycle;

b) R_G contains, also repeated if already contained in B_G, the elementary cycles of the set $D=(C_1\oplus C_2\oplus C_3\oplus\ldots\oplus C_b)$.

It is immediate from the definition that each edge of A_G is contained in at least two cycles of R_G[1]. The set R_G will be called a representation set of G. Note that if G contains at least one odd cycle, that is a cycle containing an odd number of edges, then R_G contains in any case at least two odd cycles. In fact, if G has only one odd cycle, then each cycle basis of G must contain an odd cycle (let C_1 and C_2 be even cycles, that is $|C_1|$ and $|C_2|$ be even: then $|C_1\oplus C_2|$ is also even). On the other hand, if a cycle basis of G contains one odd cycle, then $|D|$ is odd and must contains at least one odd cycle.

Let us define an hypergraph $I=(R, A_I)$ as follows:

1) R is a set of nodes corresponding one-to-one to the elements of R_G;
2) A_I is a set of edges corresponding one-to-one to the edges of A_G;
3) let $R'_G\subseteq R_G$ be a set of cycles, and denote by $s(R'_G)$ the corresponding set of nodes in R. Let $a\in A_G$ be an edge in G, and denote by t(a) the set of the cycles in R_G containing a. Then the edge of A_I corresponding to a is connected to the nodes of s(t(a));
4) each edge of A_I has the weight of the corresponding one in A_G.

[1] If there are cut-edges in G, the whole procedure must be repeated for each connected component yielded removing all cut-edges from G, thus each maximum cut of G consists of maximum cuts of all connected components and of all cut-edges in G.

The hypergraph $I=(R, A_I)$ will be called the cut-h-cycle hypergraph of G.

Note that if G is a planar graph, then the geometric dual graph of G is a cut-h-cycle hypergraph of G. In fact it follows from the definition of R_G that the set D coincides in this case with the contour of the infinite face of G.

Theorem 3. Let $G=(N, A_G)$ be a weighted graph, and $I=(R, A_I)$ be a cut-h-cycle hypergraph of G. Let $T=S_T, N-S_T$ be a cut in G and denote by $C_T \subseteq A_I$ the set of edges of I corresponding to the edges of T. Then C_T is an h-cycle in I.

Proof. If T contains an even number of edges of each cycle disconnected by T in G, it is immediate that C_T is an h-cycle in I. Suppose that there exists a cycle C_0 such that $|C_0 \cap T|$ is an odd number. It follows that, beginning with a node in S_T, the sequence of points and lines composing C_0 must end with a node in $N-S_T$, because it must cross T $|C_0 \cap T|$ times. This contradicts the definition of cycle. The theorem follows.

Theorem 5 states that there exists a one-to-one correspondence between cuts in G and h-cycles in I. In order to prove this result, a rule for finding spanning h-trees of I needs to be found. A spanning h-tree of I could be found by means of the greedy algorithm [EDM2], based on the property that any independent set of cells F_C such that, for each element $a \in A_I - F_C$, $F_C \cup \{a\}$ is not independent, is a maximal independent set of M_I. In this case, the problem of verifying the independency of a set of edges through the definition of h-cycle is very difficult. It is indeed more convenient consider the corresponding set of edges in A, as stated by the following theorem:

Theorem 4: Let $G=(N, A_G)$ be a weighted graph, and $I=(R, A_I)$ be a cut-h-cycle hypergraph of G. Let $M_I=(A_I, \mathcal{J})$ be the hypergraphical matroid of I. Then the subset $F_S \subset A_I$ corresponding to a cotree $\bar{A}$ of G is a spanning h-tree of I.

Proof. In order to prove this theorem it is sufficient to verify that:

1) $F_S \in \mathcal{F}$;

2) $F_S \cup \{a\} \notin \mathcal{F}$, for each $a \in A_I - F_S$.

In order to verify condition 1, consider an edge $\bar{a} \in \bar{A}$. From the definition of cotree it follows that there exists a subset $A^* \subseteq A_G - \bar{A}$ such that $C^* = A^* \cup \{\bar{a}\}$ is an elementary cycle. Obviously $C^* \cap \bar{A} = \{\bar{a}\}$. Consider now a cycle basis $B_G = \{C_1, C_2, C_3, \ldots, C_b\}$ contained in R_G.

Let $\bar{B} = \{C_1, C_2, \ldots, C_i\}$ be the subset of B_G such that $C^* = C_1 \oplus C_2 \oplus C_3 \ldots \oplus C_i$. Then at least one cycle of $\bar{B}$ must contain an odd number of edges of $\bar{A}$, that is, $\bar{A}$ is not an h-cycle. In fact, if it is not true, then:

$\lvert (C_1 \oplus C_2) \cap \bar{A} \rvert$	is even
$\lvert (C_1 \oplus C_2 \oplus C_3) \cap \bar{A} \rvert$	is even
$\lvert (C_1 \oplus C_2 \oplus C_3 \ldots \oplus C_i) \cap \bar{A} \rvert$	is even

thus contradicting the hypothesis that $C^* \cap \bar{A} = \{\bar{a}\}$. As this result holds for each edge of $\bar{A}$, it follows that each subset $F'_S \subseteq F_S$, corresponding to a subset $A' \subseteq \bar{A}$ does not contain h-cycles.

In order to verify condition 2, consider an edge $a' \in (A_G - \bar{A})$. The set $\bar{A} \cup \{a'\}$ must contain a cut. Then the corresponding set in A_I must contain an h-cycle. The theorem follows.

Theorem 5. Let $G=(N, A_G)$ be a weighted graph, and $I=(R, A_I)$ be a cut-h-cycle hypergraph of G. Then there is a one-to-one correspondence between cuts in G and h-cycles in I.

Proof. It is immediate from Theorem 4 that each spanning h-tree F_S of I is such that $|F_S| = |A_G| - |N| + 1$. Then, as $F_S \cup \{a\} \notin \mathcal{F}$ for each $a \in A_I - F_S$, there exist a set of at least $|A_I| - |A_G| + |N| - 1 = |N| - 1$ independent elementary h-cycles, (the h-cycles in a set I_H are said to be independent h-cycles if each h-cycle H_i in I_H cannot be written as $H_1 \oplus H_2 \oplus \ldots \oplus H_h$, where $H_1, H_2, \ldots, H_h$ are h-cycles in $I_H - \{H_i\}$).

On the other hand, let C be an elementary h-cycle of I and $B'_H=\{C_1, C_2, \ldots, C_h\}$ a set of independent h-cycles, each one obtained by the union of a spanning h- tree F_S with an edge of $(A_I-F_S) \cap C$. Then $C=C_1 \oplus C_2 \oplus C_3 \ldots \oplus C_h$. In fact the set $C_1 \oplus C_2 \oplus \ldots \oplus C_h$ is the union of $(A_I-F_S) \cap C$ with the unique subset of edges contained in F_S that complete the h-cycle C. (There is only one subset in F_S with such property. In fact, suppose that there are two subsets A_1 and A_2 in F_S with such property; then $A_1 \oplus A_2$ is an h-cycle contained in F_S: this contradicts the definition of spanning h-tree). It follows that any set B_H of $|N|-1$ independent h-cycles, each one obtained by the union of a spanning h-tree F_S with an edge of A_I-F_S is such that each other h-cycle in I can be written as $C_1 \oplus C_2 \oplus \ldots \oplus C_h$, where $C_1, C_2, \ldots, C_h$ are h-cycles of B_H. B_H is said to be an <u>h-cycle basis</u> of I.

Let B_C be a <u>cut basis</u> of G, that is a set of cuts such that each other cut can be written as $T_1 \oplus T_2 \ldots \oplus T_t$ where $T_1, T_2, \ldots, T_t$ are cuts of B_C. Consider the family B_H of the h-cycles corresponding to the cuts of B_C. Obviously $|B_H|=|B_C|=|N|-1$. Moreover the h-cycles in B_H are independent[1], so that B_H is an h-cycle basis of I.

As there is a one-to-one correspondence between cut bases of G and h-cycle bases of I, it is possible to conclude that there is a one-to-one correspondence between cuts in G and h-cycles in I.

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[1]The associated spanning h-tree corresponds in G to the cotree associated to the spanning tree of G defining the cut basis B_C.

4. Reduction of the maximum cut problem to a minimum cut problem for a class of graphs.

The maximum cut problem is then reducible to the problem of finding a maximum h-cycle in I, that is an h-cycle such that the sum of all weights associated to its edges is a maximum in I.

Let C_I^* be a maximum h-cycle in I. Then $F_I^*=A_I-C_I^*$ is a minimum set of edges such that $A_I-F_I^*$ is an h-cycle. Note that $|A(x)\cap F_I^*|$ is even or zero if x is a node with even degree[1] in I, is odd if x has odd degree in I. Remember that if x is a node of I, then A(x) denotes the set of edges that are incident to x.

Let $I=(R, A_I\cup\{d\})$ be the weighted hypergraph obtained by adding to A_I an edge d of weight 1, connecting all nodes with odd degree in I. Then finding a maximum h-cycle in I is reducible to finding in I' a minimum h-cycle containing d. In fact it follows from the definition of d that $F_I^*\cup\{d\}$ is an h-cycle. On the other hand suppose that C^o is a minimum h-cycle containing d. Then $C^o-\{d\}$ is a minimum weighted set of edges such that $|A(x)\cap(C^o-\{d\})|$ is even or zero if x has even degree, is odd if x has odd degree. It follows that $A_I-(C^o-\{d\})$ is a maximum weighted h-cycle in I.

Note that if a representation set of G contains only even cycles, the maximum cut of G consists of all edges of A_G. In fact a graph G is bipartite if and only if G has a cycle basis containing only even cycles [HAR].

The following theorem gives a property P such that, if a graph G satisfies P, then I' is a cut-h-cycle hypergraph of a graph G' that can be easily obtained from G. Then finding in I' a minimum h-cycle containing d is finally reducible to finding a minimum cut in G' disconnecting the two nodes connected by the edge corresponding to d. This problem is equivalent to evaluating the maximal flow between such two nodes and can be solved in polynomial time [EDM1].

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[1] Let H=(N, A) be an hypergraph and x be an arbitrary node in N. Then $|A(x)|$ is said to be the degree of x in H.

Theorem 6. Let $G=(N,A)$ be a weighted graph. Let P be the following property:
P: there exists a representation set $\bar{R}_G$ of G such that:

1) There exists a node $\bar{x}$ such that each odd cycle of $\bar{R}_G$ contains two edges incident to $\bar{x}$;
2) it is possible to order each edge of $A(\bar{x})$ so that the edges of $A(\bar{x}) \cap \bar{C}$, where $\bar{C} \in \bar{R}_G$, have the same direction if $|\bar{C}|$ is odd, have opposite directions if $|\bar{C}|$ is even.

If G satisfies P, then a graph G' can be obtained from G such that I' is a cut-h-cycle hypergraph of G'.

Proof. Suppose that G satisfies P and order the edges of $A(\bar{x})$ like stated by the property P. Consider a graph $G'=(N', A'_G)$ defined as follows:

1) $N' = (N-\{\bar{x}\}) \cup \{\bar{x}_1, \bar{x}_2\}$
2) $A'_G = A_G \cup \{\bar{d}\}$; where $\bar{d}=(\bar{x}_1, \bar{x}_2)$
3) Each edge of $A(\bar{x})$ is connected:
   a) to $\bar{x}_1$ if it ends in $\bar{x}$
   b) to $\bar{x}_2$ otherwise
4) All other connections and weights are as in G.

Consider furthermore a set $\mathcal{C}$ of elementary cycles of G' defined as follows:

a) $\mathcal{C}$ contains each even cycle $\bar{C}'_D$ obtained as follows: let $\bar{C}_D$ be any odd cycle of $\bar{R}_G$; then there exists in G' an even cycle $\bar{C}'_D$ consisting of all edges of $\bar{C}_D \cup \{\bar{d}\}$. In fact the edges of $\bar{C}_D \cap A(\bar{x})$ have the same direction and so in G' one of them is connected to $\bar{x}_1$ and the other to $\bar{x}_2$;
b) $\mathcal{C}$ contains each even cycle $\bar{C}'_E$ obtained as follows: let $\bar{C}_E$ be any even cycle of $\bar{R}_G$; then there exists in G' an even cycle $\bar{C}'_E$ consisting of all edges of $\bar{C}_E$. In fact the edges of $\bar{C}_E \cap A(\bar{x})$, if there are any, have opposite direction and so in G' are both connected to $\bar{x}_1$ or to $\bar{x}_2$.

$\mathcal{C}$ is a representation set of G'. In fact:

a) $\mathcal{C}$ contains a cycle basis $\bar{B}'_G$ of G'. In fact any cycle basis of G' can be obtained considering the elementary cycles yielded by the union of a spanning tree with each other edge in G'. Then, let T_G be the spanning tree corresponding to a cycle basis $\bar{B}_G=(\bar{C}_1,\bar{C}_2,\ldots,\bar{C}_b)$ contained in $\bar{R}_G$. Replacing the node $\bar{x}$ with the pair $(\bar{x}_1, \bar{x}_2)$ connected by the edge $\bar{d}$ transforms T_G into a spanning tree T_G' of G'. The union of T_G' with each edge of $(A_G \cup \{\bar{d}\})-T_G'$ yields a cycle basis $\bar{B}_G'=(\bar{C}_1',\bar{C}_2',\bar{C}_3',\ldots,\bar{C}_b')$ containing the cycles of $\bar{B}_G$ transformed as described in the definition of G'. These cycles have been added to $\mathcal{C}$.

b) $\mathcal{C}$ contains, also repeated if already contained in $\bar{B}_G'$, the elementary cycles of the set $\bar{D}'=(\bar{C}_1'\oplus \bar{C}_2'\oplus\bar{C}_3'\oplus\ldots\oplus\bar{C}_b')$: let $\bar{C}_j'$, $\bar{C}_k'$, $j\neq k$, be cycles of $\bar{B}_G'$ such that $\bar{C}_j'\supset\{\bar{d}\}$ and $\bar{C}_k'\not\supset\{\bar{d}\}$; and $\bar{C}_j$, $\bar{C}_k$ be the corresponding cycles of $\bar{B}_G$. Then it is easily seen that

$$(\bar{C}_j'\oplus \bar{C}_k') = (\bar{C}_j\oplus \bar{C}_k) \cup \{\bar{d}\}$$

If both $\bar{C}_j'$ and $\bar{C}_k'$ contain or not. $\bar{d}$, then $\bar{C}_j'\oplus \bar{C}_k'=\bar{C}_j\oplus \bar{C}_k$. Let $\bar{D}\subseteq A_G$ be a set of edges defined as follows:

$$\bar{D} = (\bar{C}_1\oplus \bar{C}_2\oplus \bar{C}_3\oplus\ldots\oplus\bar{C}_b)$$

Then it follows that

$$\bar{D}'= \bar{D} \quad \text{or} \quad \bar{D}' = \bar{D}\cup\{\bar{d}\} \; .$$

Moreover, as $\bar{D}'$ must be a set of elementary cycles, $\{\bar{d}\}\in\bar{D}'$ if an odd elementary cycle $\bar{C}\in\bar{R}_G$ of the set $\bar{D}$ contains two edges incident to $\bar{x}$. It follows from the definition of $\mathcal{C}$ that there exists in $\mathcal{C}$ a cycle $\bar{C}'=\bar{C}\cup\{\bar{d}\}$.

Consequently , whether $\bar{D}'=\bar{D}$ or $\bar{D}'=\bar{D}\cup\{\bar{d}\}$, it follows that $\mathcal{C}$ contains the elementary cycles of the set $\bar{D}'$.

It is easily seen that the cut-h-cycle hypergraph of G' defined using $\mathcal{C}$ as representation set of G' is exactly I'. The theorem follows.

Note that the graph G' contains only even cycles, because the cut-h-cycle hypergraph of G', obtained as previously described, contains only nodes with even degree. It follows that, if G satisfies P, all odd elementary cycles of G contain two edges incident to $\bar{x}$.

On the other hand the following Theorem 7 states that if each odd elementary cycle of G contains two edges incident to one node $\bar{x}$, then it is possible to order each edge of $A(\bar{x})$ so that the edges of $A(\bar{x}) \cap C$, for each elementary cycle C of G containing two edges incident to $\bar{x}$, have the same direction if $|C|$ is odd, have opposite direction if $|C|$ is even.

It follows that, if each odd elementary cycle of G contains two edges incident to one node $\bar{x}$, then G satisfies P and any representation set of G can be used for obtaining a cut-h-cycle hypergraph of G'.

Then the reduction of the maximum cut problem to a minimum cut problem, as described in this paragraph, holds for the class of all graphs containing at least one node $\bar{x}$ such that each odd elementary cycle contains two edges incident to $\bar{x}$.

Theorem 7. Let $G=(N, A_G)$ be a graph. If each odd elementary cycle of G contains two edges incident to one node $\bar{x}$, then it is possible to order each edge of $A(\bar{x})$ so that the edges of $A(\bar{x}) \cap C$, for each elementary cycle C of G containing two edges incident to $\bar{x}$, have the same direction if $|C|$ is odd, have opposite directions if $|C|$ is even; that is G satisfies P.

Proof. The proof is constructive. Let us define a graph $P_G=(X, C_G)$ as follows:

1) X is a set of nodes corresponding one-to-one to the edges of $A(\bar{x})$;
2) C_G is a set of edges corresponding one-to-one to all elementary cycles of G that contain two edges incident to $\bar{x}$;
3) each edge $g \in C_G$ corresponding to a cycle C of G connects in P_G the two nodes corresponding to the edges of $A(\bar{x}) \cap C$.

The nodes of X will be labeled '+' or '-' so that the corresponding edges of $A(\bar{x})$ will be directed away from or toward $\bar{x}$, respectively.
Let us suppose, without loss of generality, P_G be a connected graph[1]. Let

[1] If P_G is not connected, the proof must be repeated for each connected component.

$\mathcal{C}_D$ be the set of all edges of C_G corresponding to odd cycles in G and $\mathcal{C}_E$ the set of all edges of C_G corresponding to even cycles in G.
Consider the partial graph $D_G=(X, \mathcal{C}_D)$. Let

$$D_G^1 = (X^1, \mathcal{C}_D^1),\ D_G^2 = (X^2, \mathcal{C}_D^2), \ldots\ ,\ D_G^s = (X^s, \mathcal{C}_D^s)$$

be the connected components of D_G and consider at first $D_G^1=(X^1, \mathcal{C}_D^1)$.
Let T_D^1 be a spanning tree of D_G^1. Label the nodes of X^1 with '+' or '-' so that each edge of T_D^1 connects a node labeled with '+' and a node labeled with '-'. Then each other edge of $\mathcal{C}_D^1$ must connect a node labeled '+' and a node labeled '-'. In fact, suppose that there exists an edge C_D^1 connecting two nodes with the same label. It follows that $C_D^1 \cup T_D^1$ yields an odd cycle K_D in P_G corresponding in G to an odd number of odd cycles $(C_1, C_2, C_3, \ldots, C_d)$. Then $(C_1 \oplus C_2 \oplus \ldots \oplus C_d)$ is a set of elementary cycles of G containing at least one odd cycle C_o, as $|C_1 \oplus C_2 \oplus \ldots \oplus C_d|$ is odd. Moreover $C_o \cap A(\bar{x})=\emptyset$ as K_D is a cycle in P_G. This contradicts the hypothesis that each odd cycle of G contains two edges incident to $\bar{x}$. On the other hand, let $C_E^1 \in \mathcal{C}_E$ be an edge connecting two nodes x_1^1 and x_2^1 of X^1, then x_1^1 and x_2^1 have the same label. In fact, suppose that x_1^1 and x_2^1 have different labels. Then $C_E^1 \cup T_D^1$ yields an even cycle K_E in P_G corresponding in G to a set containing an odd number of odd cycles. This, as previously seen, contradicts the hypothesis that each odd cycle of G contains two edges incident to $\bar{x}$.

This result holds, obviously, for each connected component of D_G.

Consider the subset $\mathcal{C}_I$ of all chains contained in $\mathcal{C}_E$ and connecting two nodes of different connected components. If $\mathcal{C}_I$ contains only cut-edges of P_G, the initial labeling of the nodes of each connected component subsequent the first considered is assigned such that the nodes connected by each cut--edge have the same label. Otherwise, provided that a proper initial labeling of each spanning tree T_D^i, $1 \leq i \leq s$, is performed such that at least one chain in $\mathcal{C}_I$ connecting T_D^i to one spanning tree already labeled connects nodes with the same label, then each other chain in $\mathcal{C}_I$ connects two nodes with the same label. In fact, let $\bar{\mathcal{C}}_I \subset \mathcal{C}_I$ be the set of chains in $\mathcal{C}_I$ connecting nodes with the same label, and suppose that there exists a chain $C_I \varepsilon \mathcal{C}_I$ connecting nodes with different labels.

Then there exists a cycle K_I in P_G consisting of C_I, of some chains in $\bar{\mathcal{C}}_I$, of some edges of some spanning trees of $\{T_D^i, 1 \leq i \leq s\}$. It follows that, as K_I corresponds in G to a set containing an odd number of odd cycles, the chain C_I cannot exists in $\mathcal{C}_I$. Label the intermediate nodes of each chain as the end-points. The theorem follows, as each edge corresponding to an even cycle has the nodes labeled with the same label and each edge corresponding to an odd cycle has the nodes labeled with different labels.

The proof of Theorem 7 gives a polynomial bounded procedure for verifying if any graph G satisfies property P. In fact, let R_G be any arbitrary representation set of G. If a node x' such that each odd cycle of R_G contains two edges incident to x' does not exist, G does not satisfy P. If there exists a set $\{x_i'\}$ of nodes like x', let us define a set of graphs $P_R^i = (X^i, C_R^i)$ as follows:

1) X^i is a set of nodes corresponding one-to-one to the edges of $A(x_i')$;
2) C_R^i is a set of edges corresponding one-to-one to the cycles of R_G that contains two edges incident to x_i';
3) each edge $g \varepsilon C_R^i$, corresponding to a cycle C of R_G, connects in P_G the two nodes corresponding to the edges of $A(x_i') \cap C$

If G satisfies P (that is if there exists in G at least one node $\bar{x}$ such that each odd elementary cycle contains two edges incident to $\bar{x}$), then it is possible to label the nodes of at least one graph P_R^i as described in the proof of Theorem 7 for the graph P_G. On the other hand, if such a labeling is possible, then G satisfies P.

As a conclusion, the translation process consists of two cascaded phases:

1) Verifying if G satisfies P;
2) Construction of G'.

In order to execute phase 1, it is necessary to find all x_i' and, for each P_R^i, to find a spanning tree in each connected component, to label its nodes and verifying the labels for each other edge in C_R^i. Since the nodes in $\{x_i'\}$, as well as the associated graphs, the nodes in X^i and the edges in C_R^i are at most $n=|N|$, the computation required for phase 1 is evaluated as $O(n^3)$.

In order to execute phase 2, it is necessary to scan the edges of $A(\bar{x}_i')$, where $\bar{x}_i'$ is the node selected in phase 1. As $A(\bar{x}_i') \leq n-1$, it is concluded that the overall computation for the translation process is $O(n^3)$ in lenght.

An Example.

Consider the weighted undirected graph $G=(N, A_G)$ in Fig. 1. Select the spanning tree T_G consisting of the edges A, L, M, N, F, H, O. T_G defines a cycle basis

$B_G = \{C_1,C_2,C_3,C_4,C_5,C_6\}$ where

$C_1 = \{A,B,O,L\}$ $C_2 = \{C,O,M\}$ $C_3 = \{D,F,N,L\}$

$C_4 = \{E,M,L\}$ $C_5 = \{G,O,N,H\}$ $C_6 = \{I,M,N\}$.

The set

$$D = (C_1 \oplus C_2 \oplus C_3 \oplus C_4 \oplus C_5 \oplus C_6) = \{I,M,N,O,G,H,F,D,L,A,B,C,E\}$$

contains the cycles

$C_7 = \{I,M,N\}$ $C_8 = \{O,G,H,F,D,L\}$ $C_9 = \{A,B,C,E\}$

Then the set of cycles

$R_G = \{C_1,C_2,C_3,C_4,C_5,C_6,C_7,C_8,C_9\}$

is a representation set of G.
By inspection in G, it is easily seen that all cycles in R_G, excepting C_9, contain two edges incident to h. Then the graph $P_R=(X, C_R)$ is defined as shown in Fig. 2, where $X=\{x_L,x_M,x_N,x_O\}$ are the nodes corresponding to A(h);

$C_R = \{A_{C1},A_{C2},A_{C3},A_{C4},A_{C5},A_{C6},A_{C7},A_{C8}\}$

are the edges corresponding to the cycles containing two edges incident to h and the nodes in X are labeled considering the spanning tree $T_D=\{A_{C2},A_{C4},A_{C6}\}$.

As each edge in C_R corresponding to an odd cycle of R_G connects two nodes with different label and each edge in C_R corresponding to an even cycle of R_G connects two nodes with the same label, the graph G satisfies the property P and the graph $G'=(N', A_G')$ can be defined as shown in Fig. 3.

The minimum weighted cut disconnecting the two nodes h_1 and h_2 is

$$T_m = \{h_1\}, N'-\{h_1\}, w(\{h_1\}, N'-\{h_1\}) = 2$$

and consists of the edges M and P.

It follows that the maximum weighted cut of G is

$$T_M = \{b,c,g\}, \quad N-\{b,c,g\} \quad , w(\{b,c,g\}, N-\{b,c,g,\}) = 64$$

and consists of the edges of $A_G-\{M\}$.

Note that the graph of Fig. 1 is a non-planar graph, because, as shown in Fig. 4, removing the nodes a, d, f yields a clique with 5 nodes that is a non-planar graph.

Finally Fig. 5 shows the cut-h-cycle hypergraph $I'=(R, A_I \cup \{d\})$ of G'.

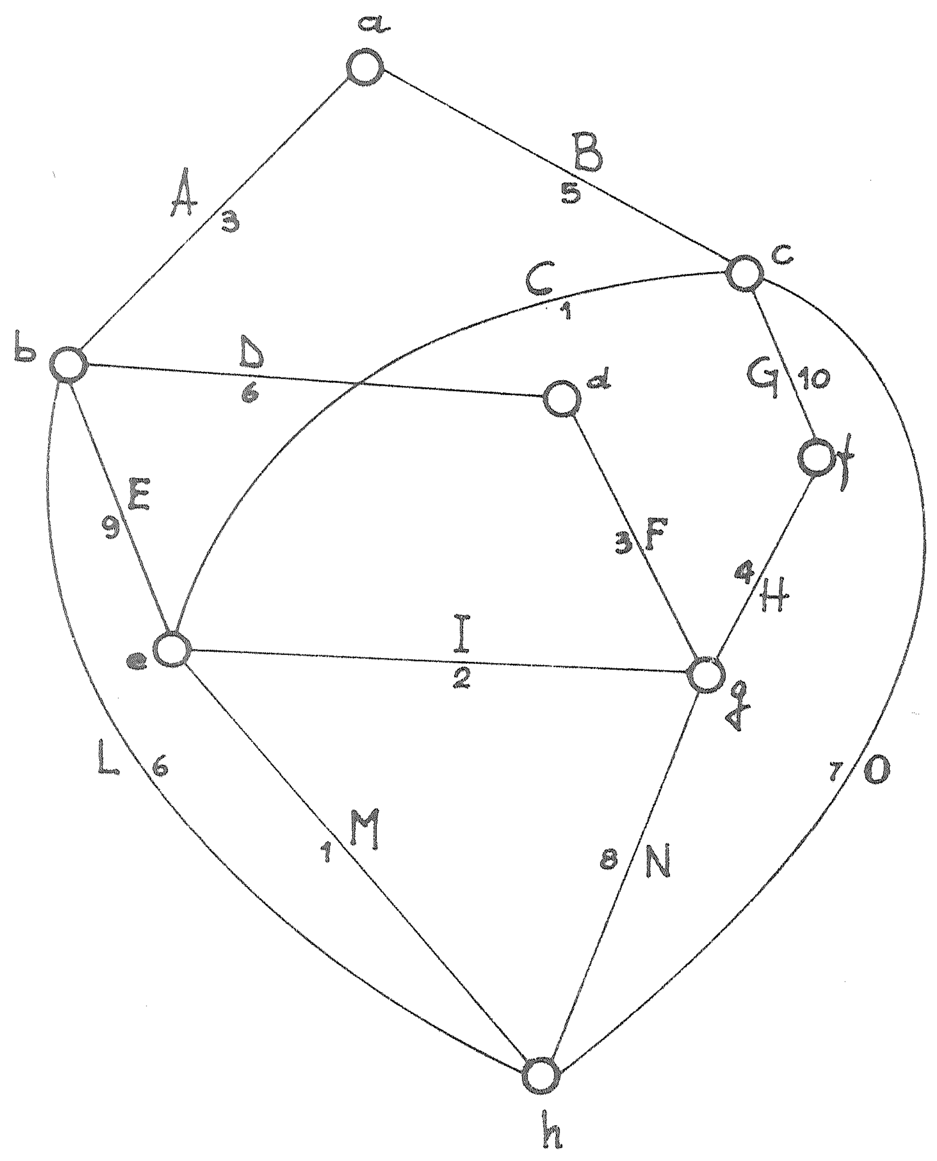

Fig. 1

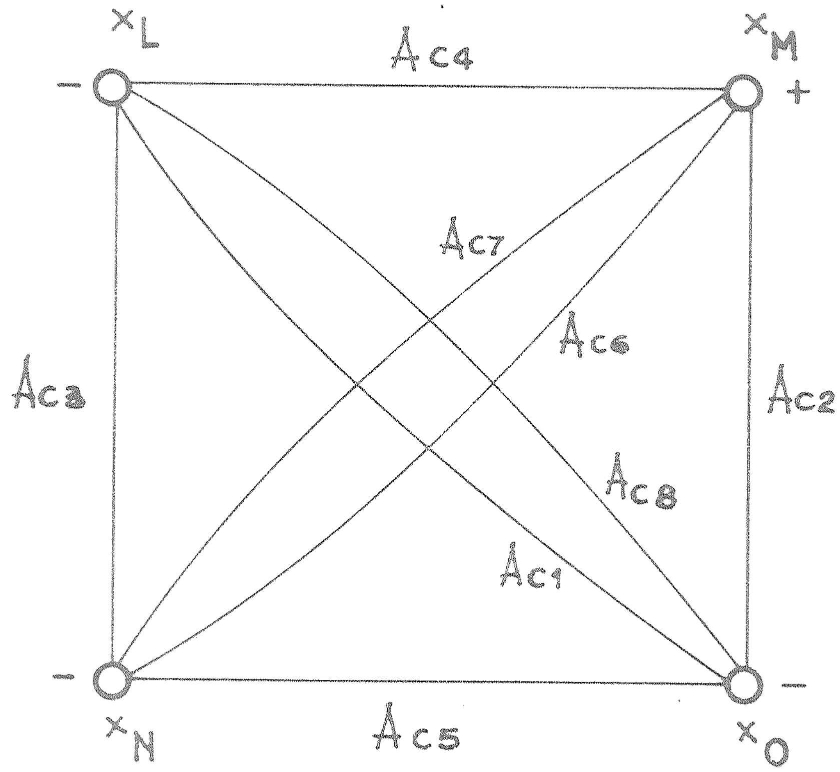

Fig.2

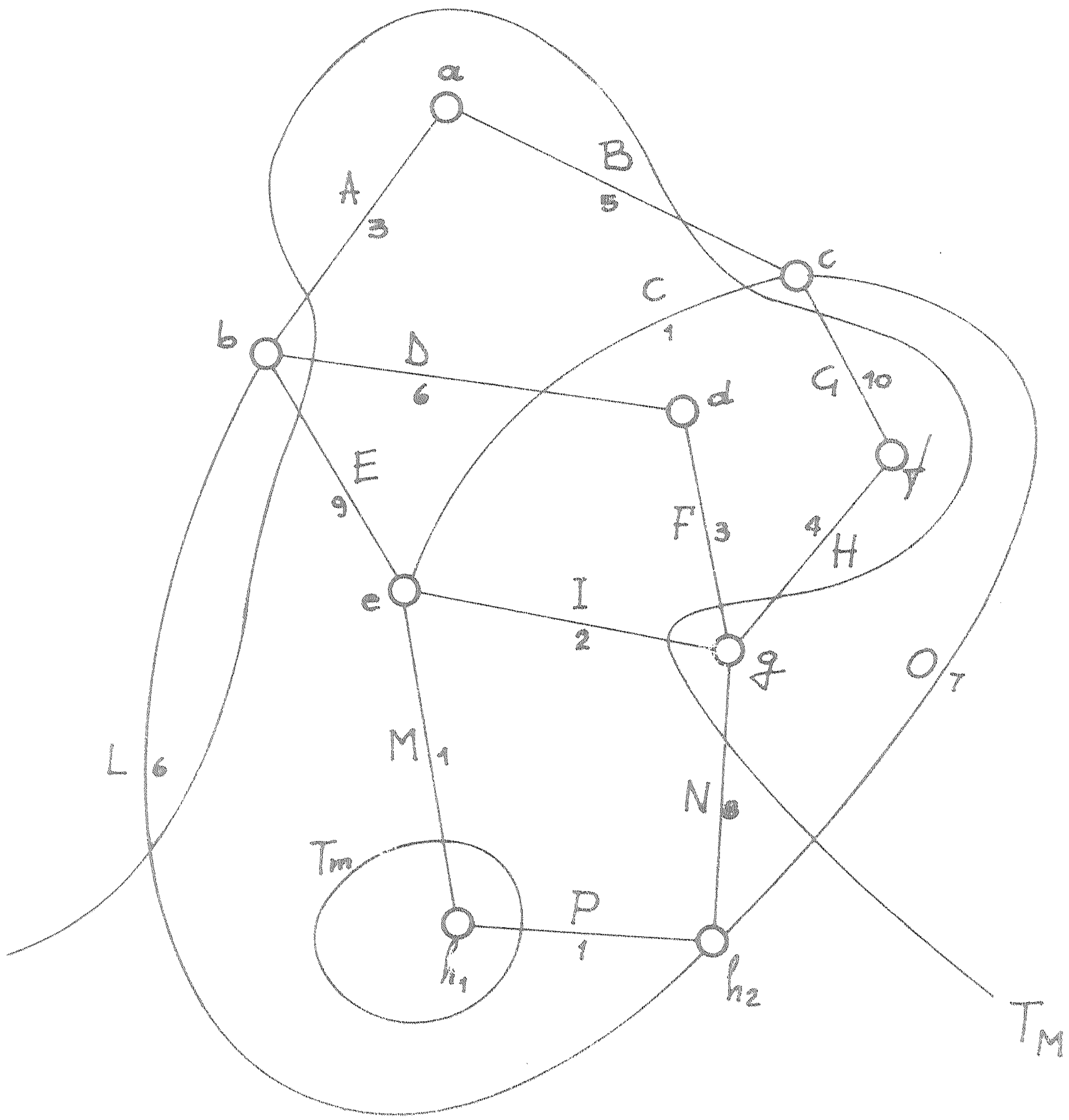

Fig. 3

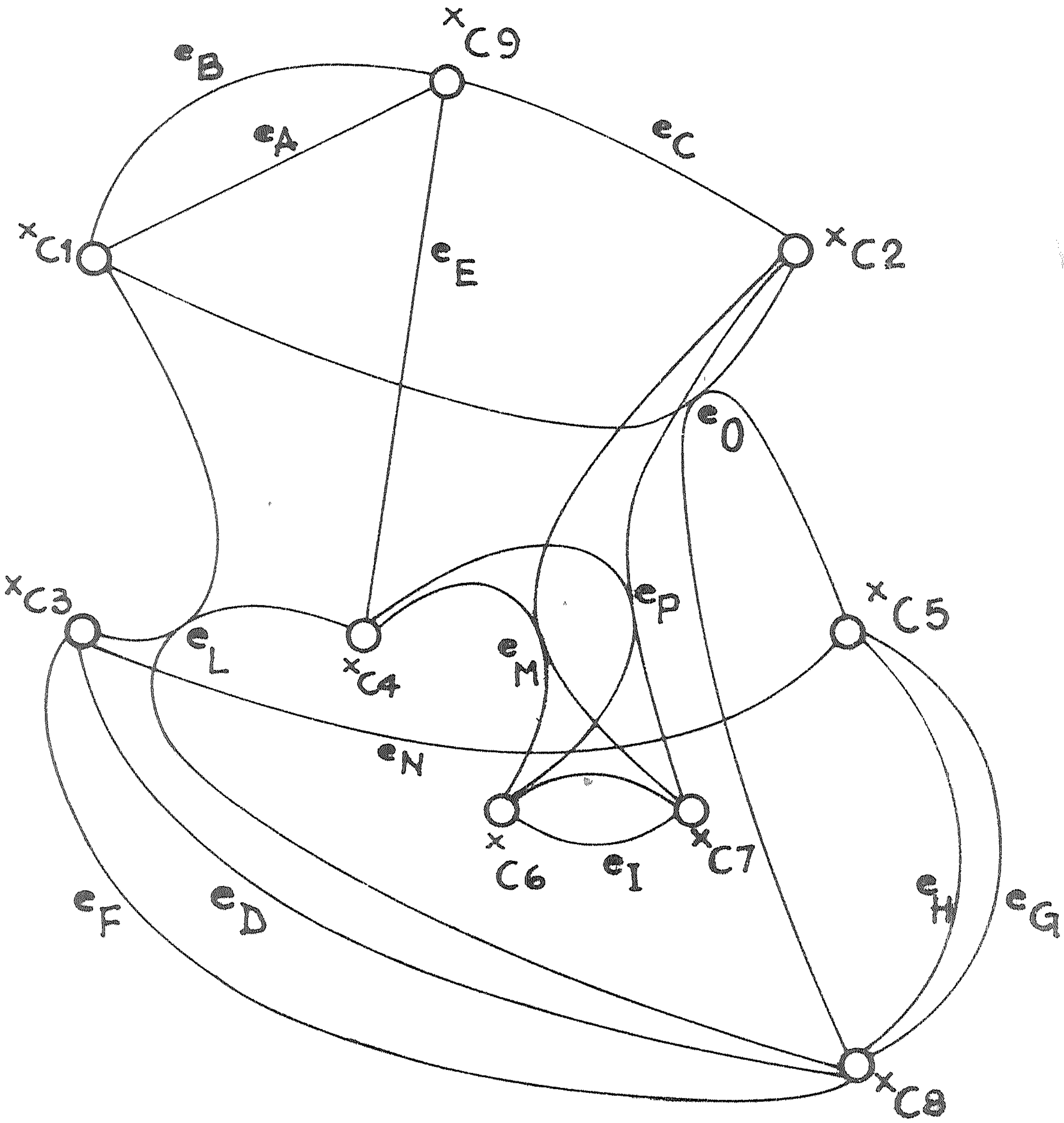

Fig. 5

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