NON-LOCAL TORSION FUNCTIONS AND EMBEDDINGS

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Abstract. Given $s \in (0,1)$, we discuss the embedding of $\mathcal{D}_0^{s,p}(\Omega)$ in $L^q(\Omega)$. In particular, for $1 \le q < p$ we deduce its compactness on all open sets $\Omega \subset \mathbb{R}^N$ on which it is continuous. We then relate, for all q up the fractional Sobolev conjugate exponent, the continuity of the embedding to the summability of the function solving the fractional torsion problem in Ω in a suitable weak sense, for every open set Ω. The proofs make use of a non-local Hardy-type inequality in $\mathcal{D}_0^{s,p}(\Omega)$, involving the fractional torsion function as a weight.

Keywords: Sobolev embedding; Torsional rigidity; Hardy inequality; Non-local Equations.

2010 Mathematics Subject Classification: 35P15, 46E35, 34K37

1. Introduction

Let Ω be an open set in $\mathbb{R}^N$, let $1 < p < \infty$, let $0 < s < 1$, and let $\mathcal{D}_0^{s,p}(\Omega)$ be the homogeneous fractional Sobolev space obtained by completion of $C_0^\infty(\Omega)$ with respect to the Gagliardo norm

$$\left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy\right)^{\frac{1}{p}}.$$

Let also $1 \le q < p_s^*$ (here $p_s^* = Np/(N-sp)$ if $sp < N$ and $p_s^* = \infty$ otherwise.) The continuity of the embedding of $\mathcal{D}_0^{s,p}(\Omega)$ into $L^q(\Omega)$ is equivalent to condition $\lambda_{p,q}^s(\Omega) > 0$, where

$$\lambda_{p,q}^s(\Omega) = \inf_{u \in C_0^\infty(\Omega)} \left\{ \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy \colon \int_\Omega |u|^q\,dx = 1 \right\}. \tag{1.1}$$

One aim of this short note is to relate this condition to the *compactness* of the embedding. To do so, following [7] we combine variational techniques and comparison principles and in Section 3 we give, for *every* open set Ω, a suitable weak definition of the unique solution $w_{s,p,\Omega}$ of the problem

$$\begin{cases} (-\Delta_p)^s\, w = 1\,, & \text{in } \Omega, \\ w = 0\,, & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{1.2}$$

The (s,p)-*laplacian* $(-\Delta_p)^s$ is the integro-differential operator defined (up to renormalisations) by

$$(-\Delta_p)^s u(x) := -2 \lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}}\,dy\,, \tag{1.3}$$

for all smooth functions u. The function $w_{s,p,\Omega}$ is to be called the (s,p)-torsion function on Ω, since (formally) for $s = 1$ the solution of (1.2) is the p-torsion function on Ω.

First, we have the following result.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N$ be open, $1 < p < \infty$, $1 \leq q < \infty$, and $0 < s < 1$. Then the following holds:*

- *If $1 \leq q < p$, then*

$$\lambda^s_{p,q}(\Omega) > 0 \Longleftrightarrow w_{s,p,\Omega} \in L^{\frac{p-1}{p-q}q}(\Omega)\,. \tag{1.4}$$

- *If $p \leq q < p^*_s$ then*

$$\lambda^s_{p,q}(\Omega) > 0 \Longleftrightarrow w_{s,p,\Omega} \in L^{\infty}(\Omega)\,. \tag{1.5}$$

We also present a consequence of Theorem 1.1, concerning super-homogeneous embeddings. We refer to [14] for a different proof in Sobolev spaces for $s = 1$.

Corollary 1.2. *Let $1 < p < \infty$, let $0 < s < 1$, and let Ω be an open set. Then $\mathcal{D}^{s,p}_0(\Omega) \hookrightarrow L^p(\Omega)$ if and only if $\mathcal{D}^{s,p}_0(\Omega) \hookrightarrow L^q(\Omega)$ for some (hence for all) q with $p < q < p^*_s$.*

Next, we provide a criterion for the compactness of sub-homogeneous Sobolev embeddings.

Theorem 1.3. *Let $1 \leq q < p$, let $0 < s < 1$, and let Ω be an open set in $\mathbb{R}^N$. Then the compactness of the embedding $\mathcal{D}^{s,p}_0(\Omega) \hookrightarrow L^q(\Omega)$ is equivalent both to to the finiteness of $\|w_{s,p,\Omega}\|_{\frac{p-1}{p-q}q}$ and to the positivity of $\lambda^s_{p,q}(\Omega)$.*

The proofs of these results are presented in Section 5 and they rely a new Hardy-type inequality, involving the (s,p)-torsion function, proved in Section 4.

2. Preliminaries

Throughout this note, for every open set Ω in the Euclidean N-space $\mathbb{R}^N$ we will denote by $C^\infty_0(\Omega)$ the set of all C^∞ smooth functions with compact support in Ω. Given $s \in (0,1)$ and $p \in (1,\infty)$, we define $\mathcal{D}^{s,p}_0(\Omega)$ as the completion of $C^\infty_0(\Omega)$ with respect to the norm

$$[u]_{s,p} = \left\{ \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy \right\}^{\frac{1}{p}}\,, \qquad u \in C^\infty_0(\Omega)\,. \tag{2.1}$$

A list of properties of $\mathcal{D}^{s,p}_0(\Omega)$ is given e.g. in [4], see in particular Section 2 and Appendix B therein. We summarise here a couple of facts we shall need in the sequel.

If Ω is bounded in one direction, in view of [2, Lemma 5.2] we get $\mathcal{D}^{s,p}_0(\Omega)$ by completion also starting from the norm

$$\|u\|_{L^p(\Omega)} + [u]_{s,p}\,. \tag{2.2}$$

Instead, for a general open set the two procedures are not equivalent and adding the L^p norm results in a smaller space unless Ω supports a fractional Poincaré inequality, i.e., if there exists $\lambda > 0$ with

$$\lambda \int_\Omega |u|^p\,dx \leq \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy\,, \qquad \text{for all } u \in C^\infty_0(\Omega). \tag{2.3}$$

In fact, in general $\mathcal{D}^{s,p}_0(\Omega)$ is not a space of distributions, either (for some examples, we refer the interested reader, e.g., to [10, 11])

Incidentally, if in addition $sp \neq 1$ and Ω has a Lipschitz regular boundary then $\mathcal{D}^{s,p}_0(\Omega)$ coincides with the subspace $W^{s,p}_0(\Omega)$ of the Sobolev-Slobodeckiĭ space $W^{s,p}(\Omega)$, given by the closure in $W^{s,p}(\Omega)$ of $C^\infty_0(\Omega)$ with respect to a norm different from (2.2), more precisely the following one:

$$\Big(\int_\Omega |u|^p\,dx\Big)^{\frac{1}{p}} + \Big(\int_\Omega\int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy\Big)^{\frac{1}{p}}\,. \tag{2.4}$$

On the contrary, the existence of functions $u \in W_0^{s,p}(\Omega)$ for which the integral

$$\int_\Omega \int_{\mathbb{R}^N \setminus \Omega} \frac{|u(x)|^p}{|x-y|^{N+sp}}\, dx\, dy \tag{2.5}$$

is infinite cannot be ruled out except if the boundary of Ω is smooth, hence in general $\mathcal{D}_0^{s,p}(\Omega)$ is a narrower space than $W_0^{s,p}(\Omega)$, even if Ω is bounded.

We set

$$p_s^* = \begin{cases} \frac{Np}{N-sp}\,, & \text{if } sp < N, \\ \infty\,, & \text{if } sp \geq N. \end{cases}$$

In cases when $sp<N$, $\mathcal{D}_0^{s,p}(\Omega)$ is indeed a function space, thanks to the embedding of $\mathcal{D}_0^{s,p}(\Omega)$ into $L^{p_s^*}(\Omega)$. In these cases, the best constant in the Sobolev embedding, i.e.,

$$\inf\left\{[u]_{s,p}^p \colon \|u\|_{L^{p_s^*}(\Omega)} = 1\right\},$$

is independent of Ω and here will be denoted by $\mathcal{S}(N,s,p)$. We refer, e.g., to [5, 15] for a more detailed account about this constant and the extremals, viz. the functions u for which inequality

$$\mathcal{S}(N,s,p)\left(\int_{\mathbb{R}^N} |u|^{\frac{Np}{N-sp}}\, dx\right)^{\frac{N-sp}{N}} \leq \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\, dx\, dy \tag{2.6}$$

holds as an equality.

The following Lemma contains a well known fact about functions in the Campanato space $\mathcal{L}^{p,\lambda}$ with $\lambda > N$. We include a proof for the convenience of the reader. Given $u \in C_0^\infty(\mathbb{R}^N)$, we denote by $u_{x,r} = ⨍_{B(x,r)} u\, dy$ the average of u on the ball $B(x,r)$ of radius r about $x \in \mathbb{R}^N$.

Lemma 2.1. *Let $C > 0$ and let $u \in C_0^\infty(\mathbb{R}^N)$ with*

$$⨍_{B(x,r)} |u - u_{x,r}|^p\, dy \leq C r^{sp-N}\,, \qquad \text{for all } x \in \mathbb{R}^N \text{ and for all } r > 0. \tag{2.7}$$

Then

$$|u(x) - u_{x,r}| \leq c(N,s,p) \cdot C r^{s-\frac{N}{p}}\,, \quad \text{for all } x \in \mathbb{R}^N \text{ and for all } r > 0. \tag{2.8}$$

Proof. It is enough to show that

$$|u_{x,2^{-k}r} - u_{x,2^{-(k+h)}r}| \leq c(N,s,p) \cdot C\, \frac{1 - 2^{-h(\frac{sp-N}{p})}}{2^{k\left(s-\frac{N}{p}\right)}} r^{s-\frac{N}{p}}\,, \tag{2.9}$$

for all $x \in \mathbb{R}^N$, for all $r > 0$, and for all $k, h \in \mathbb{N}$. Indeed, (2.9) implies that $(u_{x,2^{-h}r})_{h\in\mathbb{N}}$ is a Cauchy sequence. Then, taking $k = 0$ and passing to the limit as $h \to \infty$ in (2.9) we obtain (2.8).

To prove (2.9), we fix k and we denote by u_h the average of u on the ball of radius $2^{-(h+k)}r$ centred at x. Because of triangle inequality, (2.9) holds if for every h we have

$$|u_{j-1} - u_j| \leq \omega_N^{s-\frac{1}{p}} 2^{1+\frac{N}{p}} 2^{-(j-1)\left(s-\frac{N}{p}\right)} \cdot C R^{s-\frac{N}{p}}\,, \qquad \text{for all } j = 1, \dots, h, \tag{2.10}$$

with $R = 2^{-k}r$. To see that (2.10) holds, we observe that for every $y \in B(x, 2^{-j}R)$ we have

$$2^{1-p}|u_{j-1} - u_j|^p \leq |u_{j-1} - u(y)|^p + |u(y) - u_j|^p\,.$$

Then an integration over $B(x, 2^{-j}R)$, together with straightforward estimates, by (2.7) gives

$$|u_{j-1} - u_1|^p \leq \omega_N^{sp-1} 2^{(1-s)p} (2^{-j})^{sp-N} \cdot C^p R^{sp-N}\,,$$

and taking the p-th root we obtain (2.10). ∎

Remark 2.2. The fact that $u \in C^{0,\alpha}(\mathbb{R}^N)$, with $\alpha = s - \frac{N}{p}$, can be deduced with ease from (2.8).

The following Gagliardo-Nirenberg interpolation inequalities will be used a number of times in the rest of the paper. For every $\gamma > 1$ and for every function u, we abbreviate $\|u\|_{L^\gamma(\mathbb{R}^N)}$ to $\|u\|_\gamma$.

Lemma 2.3. *Let $1 \le q \le p < \infty$ and let $0 < s < 1$. Then the following holds:*

- *if $sp \neq N$, for every $r > 0$ with $q < r \le p_s^*$ and for every $u \in C_0^\infty(\mathbb{R}^N)$ we have*

$$\left(\int_{\mathbb{R}^N} |u|^r\,dx\right)^{\frac{1}{r}} \le C_1 \left(\int_{\mathbb{R}^N} |u|^q\,dx\right)^{\frac{1-\vartheta}{q}} \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy\right)^{\frac{\vartheta}{p}} \tag{2.11}$$

  with $\vartheta = \left(1 - \frac{q}{r}\right)\left(1 + \frac{sp-N}{Np}q\right)^{-1}$, for a suitable $C_1 = C_1(N,p,q,r,s) > 0$;

- *if $sp = N$, for every $r \ge N/s$ and for every $u \in C_0^\infty(\mathbb{R}^N)$ we have*

$$\left(\int_{\mathbb{R}^N} |u|^r\,dx\right)^{\frac{1}{r}} \le C_2 \left(\int_{\mathbb{R}^N} |u|^q\,dx\right)^{\frac{1}{r}} \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{\frac{N}{s}}}{|x-y|^{2N}}\,dx\,dy\right)^{\frac{s}{N}\left(1-\frac{q}{r}\right)} \tag{2.12}$$

  for a suitable $C_2 = C_2(N,r,s) > 0$.

Remark 2.4. Since $q \le p$, an inequality of the form

$$\left(\int_{\mathbb{R}^N} |u|^r\,dx\right)^{\frac{1}{r}} \lesssim \left(\int_{\mathbb{R}^N} |u|^q\right)^{\frac{\alpha}{q}} \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy\right)^{\frac{\beta}{p}}, \qquad \text{for all } u \in C_0^\infty(\mathbb{R}^N),$$

can hold for a unique (ordered) pair (α, β) of exponents. This fact is easily checked by the invariance of the inequality under vertical and horizontal scalings.

Proof of Lemma 2.3. In the case $sp < N$, (2.11) is a direct consequence of the fractional Sobolev inequality (2.6) combined with the standard interpolation inequality

$$\|u\|_r \le \|u\|_q^{1-\vartheta} \|u\|_{p_s^*}^{\vartheta}$$

with $\vartheta = \left(1 - \frac{q}{r}\right)\left(1 - \frac{sp-N}{Np}q\right)^{-1}$, and in this case

$$C_1 = [\mathcal{S}(N,s,p)]^{-\frac{\vartheta}{p}}\ . \tag{2.13}$$

To prove (2.11) in the case $sp > N$, we fix $u \in C_0^\infty(\mathbb{R}^N)$, $x \in \mathbb{R}^N$, $r > 0$, and we observe that

$$\int_{B(x,r)} |u - u_{x,r}|^p\,dy \le \frac{1}{\omega_N r^N} \int_{B(x,r)} \int_{B(x,r)} |u(y)-u(z)|^p\,dy\,dz\,,$$

by Jensen inequality. Since $|y - z| < 2r$ for all $y, z \in B(x,r)$, we deduce

$$\int_{B(x,r)} |u - u_{x,r}|^p\,dy \le \frac{2^{N+sp}}{\omega_N} r^{sp} \iint_{\mathbb{R}^{2N}} \frac{|u(y)-u(z)|^p}{|z-y|^{N+sp}}\,dy\,dz\,.$$

By Lemma 2.1, this implies that

$$|u(x) - u_{x,r}| \le c \left(\iint_{\mathbb{R}^{2N}} \frac{|u(y)-u(z)|^p}{|z-y|^{N+sp}}\,dy\,dz\right)^{\frac{1}{p}} r^{s-\frac{N}{p}}\,,$$

for a suitable constant $c = c(N, s, p) > 0$. By Hölder inequality we have

$$|u_{x,r}| \le \left(\fint_{B(x,r)} |u(y)|^q \, dy \right)^{\frac{1}{q}} .$$

The last two inequalities hold for all $x \in \mathbb{R}^N$ and for all $r > 0$, in particular with $r = 1$. Therefore

$$\|u\|_{L^\infty(\mathbb{R}^N)} \le \left(\int_{\mathbb{R}^N} |u|^q \, dx \right)^{\frac{1}{q}} + c(N, s, p) \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dx \, dy \right)^{\frac{1}{p}} .$$

By a standard homogeneity argument, based on the invariance under horizontal scalings, the latter can be rephrased in the following multiplicative form

$$\|u\|_{L^\infty(\mathbb{R}^N)} \le C(N, s, p) \left(\int_{\mathbb{R}^N} |u|^q \, dx \right)^{\frac{sp-N}{Np+(sp-N)q}} \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dx \, dy \right)^{\frac{N}{Np+(sp-N)q}} .$$

Then (2.11) follows by the obvious estimate $\|u\|_r \le \|u\|_\infty^{1-\frac{q}{r}} \|u\|_q^{\frac{q}{r}}$.

Eventually, to end the proof we assume that $1 \le q \le p = N/s \le r$ and we prove (2.12). Let $\sigma = \frac{3s}{4}$ and set $\theta{=}(1 - \frac{p}{r})\frac{N}{\sigma p}$. Since $\sigma p < N$, applying (2.11) with $q = p$ and s replaced by σ we get

$$\left(\int_{\mathbb{R}^N} |u|^r \right)^{\frac{1}{r}} \le C(N, r, s) \left(\int_{\mathbb{R}^N} |u|^{\frac{N}{s}} \right)^{\frac{s(1-\theta)}{N}} \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{\frac{N}{s}}}{|x - y|^{N(1+\frac{\sigma}{s})}} \right)^{\frac{s\theta}{N}} \tag{2.14}$$

for all $u \in C_0^\infty(\mathbb{R}^N)$. We observe that the inequality

$$\left(\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{\frac{N}{s}}}{|x - y|^{N(1+\frac{\sigma}{s})}} \right)^{\frac{s\theta}{N}} \le C(s) \left(\int_{\mathbb{R}^N} |u|^{\frac{N}{s}} \right)^{\frac{s-\sigma}{\sigma \frac{N}{s}}} \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{\frac{N}{s}}}{|x - y|^{2N}} \right)^{\frac{\sigma}{N}} \tag{2.15}$$

holds for all $u \in C_0^\infty(\mathbb{R}^N)$, too. Indeed, since $\sigma < s$ we have

$$\int_{\mathbb{R}^N} \int_{|y-x|<1} \frac{|u(x) - u(y)|^{\frac{N}{s}}}{|x - y|^{N(1+\frac{\sigma}{s})}} \, dx \, dy \le \int_{\mathbb{R}^N} \int_{|y-x|<1} \frac{|u(x) - u(y)|^{\frac{N}{s}}}{|x - y|^{2N}} \, dx \, dy \, .$$

In addition, we also have that

$$\int_{\mathbb{R}^N} \int_{|x-y|\ge 1} \frac{|u(x) - u(y)|^{\frac{N}{s}}}{|x - y|^{N(1+\frac{\sigma}{s})}} \, dx \, dy \le 2^{\frac{N}{s}} \int_{\mathbb{R}^N} |u(x)|^{\frac{N}{s}} \int_{|y-x|\ge 1} \frac{dy}{|x - y|^{N(1+\frac{\sigma}{s})}} dx \le \frac{2^{\frac{N}{s}+1}}{N} \int_{\mathbb{R}^N} |u|^{\frac{N}{s}} \, dx \, ,$$

where in the last passage we used that $\sigma > s/2$. Then, (2.15) follows by a direct homogeneity argument. Combining (2.14) and (2.15) with standard interpolation in Lebesgue spaces we obtain

$$\|u\|_{L^r(\mathbb{R}^N)} \le C_2(N, r, s) \|u\|_{L^q(\mathbb{R}^N)}^{(1-\lambda)(1-\frac{\theta\sigma}{s})} \|u\|_{L^r(\mathbb{R}^N)}^{\lambda(1-\frac{\theta\sigma}{s})} \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{\frac{N}{s}}}{|x - y|^{2N}} \, dx \, dy \right)^{\frac{\theta\sigma}{s}} \tag{2.16}$$

for all $u \in C_0^\infty(\mathbb{R}^N)$, with $\lambda \in (0, 1)$ being such that $\frac{s}{N} = \frac{1-\lambda}{q} + \frac{\lambda}{r}$. We observe that by definition we have $\frac{\theta\sigma}{s} = 1 - \frac{N}{rs}$. Then (2.12) follows dividing out a term in (2.16). □

Remark 2.5. The proof above works with no difference if $\sigma = \frac{3s}{4}$ is replaced by any other $\sigma \in (\frac{s}{2}, s)$. In this case, the constant appearing in (2.12) will change, going to depend on the choice of σ through the one appearing in (2.14). Note that in view of (2.13) the latter blows up as $\sigma \to s^-$.

3. The Fractional Torsion Function

3.1. **Compact case.** Throughout the present subsection, we shall assume that the embedding of $\mathcal{D}^{s,p}_0(\Omega)$ into $L^1(\Omega)$ is compact, and we list some properties of the fractional torsion function under this assumption.

Definition 3.1. Let Ω be such that the embedding of $\mathcal{D}^{s,p}_0(\Omega)$ into $L^1(\Omega)$ is compact. Then we call the (s,p)-torsion function on Ω, denoted by $w_{s,p,\Omega}$, the unique solution of the minimum problem

$$\min_{u\in\mathcal{D}^{s,p}_0(\Omega)}\left\{\frac{1}{p}\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy-\int_\Omega u\,dx\right\}. \tag{3.1}$$

By a standard homogeneity argument, the minimum value in (3.1) equals $-\frac{p-1}{p}(T_{s,p}(\Omega))^{\frac{1}{p-1}}$, where the (s,p)-*torsional rigidity* is defined by

$$T_{s,p}(\Omega):=\max\left\{\|u\|^p_{L^1(\Omega)}\colon u\in\mathcal{D}^{s,p}_0(\Omega)\,,\ [u]^p_{s,p}=1\right\}. \tag{3.2}$$

We point out that no Lavrentiev's phaenomenon occurs between $\mathcal{C}^\infty_0(\Omega)$ and $\mathcal{D}^{s,p}_0(\Omega)$ in (3.1). More precisely, we get the same value in (3.1) if instead of minimising over $\mathcal{D}^{s,p}_0(\Omega)$ we take the infimum over $C^\infty_0(\Omega)$. Indeed, it is clear that the latter is a quantity greater than or equal to (3.1), due to the inclusion $C^\infty_0(\Omega)\subset\mathcal{D}^{s,p}_0(\Omega)$, and the reverse inequality also holds by the definition of $\mathcal{D}^{s,p}_0(\Omega)$ and by the compactness of its embedding in $L^1(\Omega)$.

Since the (s,p)-torsion function on Ω is obtained by minimizing a convex energy on $\mathcal{D}^{s,p}_0(\Omega)$, it is the unique solution of

$$\iint_{\mathbb{R}^{2N}}\frac{|w(x)-w(y)|^{p-2}(w(x)-w(y))}{|x-y|^{N+sp}}(\varphi(x)-\varphi(y))\,dx\,dy=\int_\Omega\varphi\,dx\,,\quad\text{for all }\varphi\in C^\infty_0(\Omega). \tag{3.3}$$

Simbolically, the Euler-Lagrange equation (3.3) can be written in the form (1.2).

Proposition 3.2. *If $\mathcal{D}^{s,p}_0(\Omega)\hookrightarrow L^1(\Omega)$ is compact, then $w_{s,p,\Omega}\in L^\infty(\Omega)$. Moreover, if $sp<N$,*

$$\|w_{s,p,\Omega}\|_{L^\infty(\Omega)}\le\frac{N+sp'}{sp'}\mathcal{S}(N,s,p)^{\frac{N}{Np+sp-N}}\left(\int_\Omega w_{s,p,\Omega}\,dx\right)^{\frac{sp'}{N+sp'}}. \tag{3.4}$$

Proof. Let us abbreviate $w_{s,p,\Omega}$ to w. If $sp>N$, (3.4) is a direct consequence of the Gagliardo-Nirenberg type inequality (2.11), with $q=1$ and $r=\infty$, hence we may assume that $sp\le N$.

We first prove (3.4) in the case when $sp<N$. To do so, we fix $k>0$ and we note that the function defined by truncation setting $\varphi_k(x)=\max\{w(x)-k,0\}$, is an admissible test function for (3.3). We let $A_k=\{x\in\mathbb{R}^N\colon w(x)>k\}$ and we observe that the set $\mathcal{W}_k=A_k\times(\mathbb{R}^N\setminus A_k)$ is contained in $\{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N\colon w(x)-w(y)\ge w(x)-k\ge0\}$. Therefore

$$\iint_{\mathcal{W}_k}\frac{|w(x)-w(y)|^{p-2}}{|x-y|^{N+sp}}(w(x)-w(y))(w(x)-k)\,dy\,dx\ge\iint_{\mathcal{W}_k}\frac{|\varphi_k(x)-\varphi_k(y)|^p}{|x-y|^{N+sp}}\,dx\,dy. \tag{3.5}$$

Moreover we have

$$\iint_{(\mathbb{R}^N\setminus A_k)\times(\mathbb{R}^N\setminus A_k)}\frac{|\varphi_k(x)-\varphi_k(y)|^p}{|x-y|^{N+sp}}\,dx\,dy=0\,, \tag{3.6a}$$

and

$$\iint_{A_k\times A_k}\frac{|w(x)-w(y)|^{p-2}(w(x)-w(y))}{|x-y|^{N+sp}}(\varphi_k(x)-\varphi_k(y))=\iint_{A_k\times A_k}\frac{|\varphi_k(x)-\varphi_k(y)|^p}{|x-y|^{N+sp}}. \tag{3.6b}$$

By the symmetry of the left hand-side in (3.3) with respect to $(x,y)\mapsto(y,x)$, when plug in φ_k into (3.3), combining (3.5) with the identities (3.6) we arrive at

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\varphi_k(x)-\varphi_k(y)|^p}{|x-y|^{N+sp}}\,dx\,dy\le\int_{A_k}(w(x)-k)^p\,dx\,. \tag{3.7}$$

On the other hand, by (2.6), we have that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|\varphi_k(x)-\varphi_k(y)|^p}{|x-y|^{N+sp}}\,dx\,dy\ge\mathcal{S}(N,s,p)|A_k|^{1-p-\frac{sp}{N}}\left(\int_{A_k}(w(x)-k)\,dx\right)^p\,. \tag{3.8}$$

By Fubini's theorem, using the estimates (3.7) and (3.8) and dividing out, we obtain

$$\Big(\int_k^\infty|A_t|\,dt\Big)^{p-1}\le\mathcal{S}(N,s,p)^{-1}|A_k|^{-1+p+\frac{sp}{N}}\,. \tag{3.9}$$

Since $w\in L^1(\Omega)$, $k\mapsto|A_k|$ is a non-increasing function converging to 0 as $k\to\infty$. Thus by (3.9) the function $\varepsilon(k)=\int_k^\infty|A_t|\,dt$ satisfies the differential inequality

$$\varepsilon(k)^{\frac{N}{N+sp'}}\le C(N,s,p)\big(-\varepsilon'(k)\big) \tag{3.10}$$

with $C=\mathcal{S}(N,s,p)^{\frac{-N}{N(p-1)+sp}}$. This gives that $w\in L^\infty(\Omega)$. Indeed, given $k_0>0$ and $k>k_0$ by integration we infer from (3.10) that

$$k-k_0\le C\frac{N+sp'}{sp'}\big(\varepsilon(k_0)^{\frac{sp'}{N+sp'}}-\varepsilon(k)^{\frac{sp'}{N+sp'}}\big)\,. \tag{3.11}$$

To get the quantitative bound (3.4), we observe that (3.11) implies $\varepsilon(k)=0$ whenever

$$k\ge k_0+C\frac{N+sp'}{sp'}\Big(\int_{A_{k_0}}(w-k_0)\,dx\Big)^{\frac{sp'}{N+sp'}}\,. \tag{3.12}$$

Clearly this implies that $|A_k|=0$ for k satisfying (3.12). Since we may take any $k_0>0$ in the lower bound (3.12), this and the definition of C give (3.4).

To end the proof, the only case left to consider is that when $sp=N$. In this case, applying (2.12) with exponents $q=1$ and $r=\frac{tN}{s}$, with $t>1$, and arguing as in the previous case we obtain

$$\varepsilon(k)^{\beta(t)}\le C(N,t,s)(-\varepsilon'(k))\,,\quad\text{with }\beta(t)=\frac{tp(p-1)-(t-1)((t+1)p-1)}{(tp-1)(p-1)}. \tag{3.13}$$

Eventually, we choose $t>1$ so that $\beta(t)=1-\frac{s}{N}$ and arguing as before we get (3.4). ☐

We refer to [13] for the following weak comparison principle. Similar results have been proved in slightly different settings, see [5, 12]

Proposition 3.3. *Let $w_i=w_{s,p,\Omega_i}$ where Ω_i is a bounded open set. If $\Omega_1\subset\Omega_2$ then $w_1\le w_2$.*

Proof. Setting

$$J(w_i,\varphi)=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|w_i(x)-w_i(y)|^{p-2}(w_i(x)-w_i(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dx\,dy$$

clearly we have

$$J(w_2,\varphi)-J(w_1,\varphi)=\int_{\Omega_2\setminus\Omega_1}\varphi\,dx\ge0\,,$$

for all $\varphi\in C_0^\infty(\Omega_2)$ with $\varphi\ge0$. The conclusion then follows arguing as in [13, Lemma 9]. ☐

3.2. **The general case.** In view of Proposition 3.3, we can define the fractional torsion function on arbitrary open sets $\Omega \subset \mathbb{R}^N$ as follows.

Definition 3.4. Given an open set $\Omega \subset \mathbb{R}^N$ the (s,p)-*torsion function* of Ω is defined by

$$w_{s,p,\Omega}(x) = \lim_{r\to\infty} w_r(x)\,, \qquad \text{for every } x \in \Omega, \tag{3.14}$$

where we set

$$w_r(x) = \begin{cases} w_{B_r(0)\cap\Omega}(x)\,, & \text{if } x \in B_r(0)\cap\Omega, \\ 0\,, & \text{otherwise,} \end{cases} \tag{3.15}$$

for all $r > r_0 = \inf\{\rho > 0\colon |B_\rho(0)\cap\Omega| > 0\}$.

We shall often identify w with its extension to the whole space $\mathbb{R}^N$ with $w \equiv 0$ in $\mathbb{R}^N \setminus \Omega$.

Remark 3.5. Note that the torsion function is well defined. First of all the limit in (3.14) makes sense by Proposition 3.3. Moreover, for every open set Ω for which the embedding $\mathcal{D}_0^{s,p}(\Omega) \hookrightarrow L^1(\Omega)$ is compact, the function w_r converges, as $r \to \infty$, to the unique solution of (3.1). Indeed, using w_r first as a test function in its equation (i.e., (3.3) with $\Omega \cap B_r(0)$ in place of Ω) and then as a competitor in (3.2) (see [7, Lemma 2.4] where a similar task is carried out in detail) we get

$$[w_r]_{s,p}^p = \|w_r\|_{L^1(\Omega)} \le [w_r]_{s,p} T_{s,p}(\Omega)^{\frac{1}{p}}\,,$$

and we conclude by the reflexivity of $\mathcal{D}_0^{s,p}(\Omega)$ and the compactness of its embedding in $L^1(\Omega)$.

Remark 3.6. We point out that $w_{s,p,\Omega} > 0$ in Ω. To see this we may assume with no restriction Ω to be bounded, since (3.14) is a pointwise monotone limit. Then the embedding $\mathcal{D}_0^{s,p}(\Omega) \hookrightarrow L^1(\Omega)$ is compact, and $w_{s,p,\Omega}$ solves (3.3). Therefore, the conclusion in this case follows by the minimum principle (see, e.g., [3, Appendix A]).

4. Non-local Torsional Hardy inequalities

We begin this section with a fractional Hardy-type inequality involving the torsion function.

Proposition 4.1. *Let $1<p<\infty$, $0<s<1$. Let $\Omega \subset \mathbb{R}^N$ be an open set such that $\mathcal{D}_0^{s,p}(\Omega) \hookrightarrow L^1(\Omega)$ is compact. Then*

$$\int_\Omega \frac{|u|^p}{w_{s,p,\Omega}^{p-1}}\,dx \le \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy\,, \quad \textit{for all } u \in \mathcal{D}_0^{s,p}(\Omega).$$

Proof. We prove the inequality for any fixed $u \in \mathcal{D}_0^{s,p}(\Omega)$ with $u \ge 0$, which is sufficient. To do so, let $\varepsilon > 0$, and let $w = w_{s,p,\Omega}$. Since $f(t) = (t+\varepsilon)^{1-p}$, $t > 0$, is a Lipschitz function, $\varphi = u^p(w+\varepsilon)^{1-p}$ is an admissible test function for equation (3.3) (see, e.g., [2, Lemma 2.4]). Thus, setting $w_\varepsilon = w+\varepsilon$,

$$\int_\Omega \frac{u^{p-1}}{(w+\varepsilon)^{p-1}}\,dx = \iint_{\mathbb{R}^{2N}} \frac{\big|w(x)-w(y)\big|^{p-2}\big(w(x)-w(y)\big)}{|x-y|^{N+sp}} \left(\frac{u(x)^p}{w_\varepsilon(x)^{p-1}} - \frac{u(y)^p}{w_\varepsilon(y)^{p-1}}\right) dx\,dy\,.$$

Hence, thanks to the following discrete Picone-type inequality (see, e.g., [3, Proposition 4.2])

$$|a-b|^{p-2}(a-b)\left(\frac{c^p}{a^{p-1}} - \frac{d^p}{b^{p-1}}\right) \le |c-d|^p\,, \qquad \text{for all } a,b>0 \text{ and } c,d \ge 0, \tag{4.1}$$

we get the conclusion by Fatou's Lemma using the arbitrariness of $\varepsilon > 0$. $\square$

Corollary 4.2. *Let $1<p<\infty$, let $0<s<1$, and let $\Omega \subset \mathbb{R}^N$ be any open set. Then*

$$\int_\Omega \frac{|u|^p}{w_{s,p,\Omega}^{p-1}}\,dx \le \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy\,, \qquad \text{for all } u \in C_0^\infty(\Omega) \tag{4.2}$$

(with the convention that $\frac{c}{\infty}=0$ for all $c \in \mathbb{R}$.)

Proof. We fix $u \in C_0^\infty(\Omega)$ and let $R_0>0$ be such that, for every $R>R_0$, u is supported in the ball B_R of radius R about the origin. Then, setting $\Omega_R = \Omega \cap B_R$, by Proposition 4.1 we have

$$\int_{\Omega_R} \frac{|u|^p}{w_{s,p,\Omega_R}^{p-1}}\,dx \le \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy\,,$$

for all $R>R_0$. Thus, in view of Definition 3.14, the desired inequality follows by Fatou Lemma. □

We end this section with a variation on the torsional Hardy inequality discussed in Proposition 4.1, containing an additional term.

Theorem 4.3. *Let $\Omega \subset \mathbb{R}^N$ be such that the embedding $\mathcal{D}_0^{s,p}(\Omega)$ into $L^1(\Omega)$ is compact and let w be the (s,p)-torsion function on Ω. Then there exists a constant $C>0$, only depending on p, with*

$$\int_\Omega \frac{|u|^p}{w^{p-1}}\,dx + 2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \left|\frac{w(x)-w(y)}{w(x)+w(y)}\right|^p \frac{dy}{|x-y|^{N+sp}}|u(x)|^p\,dx \le C \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy\,,$$

for all $u \in \mathcal{D}_0^{s,p}(\Omega)$.

We skip the proof of Theorem 4.3 because it is completely analogous to that of Proposition 4.1, except that instead of (4.1) one can exploit a Picone-type inequality with a remainder term. More precisely, by [2, Lemma A.5] there exist positive constants C_1, C_2, only depending on p, with

$$|a-b|^{p-2}(a-b)\left(\frac{c^p}{a^{p-1}}-\frac{d^p}{b^{p-1}}\right)+C_1\left|\frac{a-b}{a+b}\right|^p (c^p+d^p) \le C_2|c-d|^p\,, \quad \text{for all } a,b>0 \text{ and } c,d\ge 0.$$

5. Proofs of the main results

For every open set Ω in $\mathbb{R}^N$, for every $1<p<\infty$, $1\le q<p_s^*$, and $0<s<1$, we have

$$\lambda_{p,q}^s(\Omega)\left(\int_\Omega |u|^q\,dx\right)^{\frac{p}{q}} \le \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy\,, \qquad \text{for all } u \in C_0^\infty(\Omega), \tag{5.1}$$

where

$$\lambda_{p,q}^s(\Omega) = \inf_{u \in C_0^\infty(\Omega)} \left\{ \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy \colon \int_\Omega |u|^q\,dx = 1\right\}, \tag{5.2}$$

and $\lambda_{p,q}^s(\Omega)$ is the best possible constant for this inequality to hold.

Remark 5.1. The Poincaré-type inequality (5.1) implies that $\mathcal{D}_0^{s,p}(\Omega)$ is a function space, continuously included in $L^q(\Omega)$, whenever $\lambda_{p,q}^s(\Omega)>0$. Indeed, in this case, if $(u_n)_n \subset C_0^\infty(\Omega)$ is a Cauchy sequence in $\mathcal{D}_0^{s,p}(\Omega)$ then by (5.1) it is a Cauchy sequence in the Banach space $L^q(\Omega)$ as well.

We now prove Theorem 1.1, relating the positivity of $\lambda_{p,q}^s(\Omega)$ to the summability of the (s,p)-torsion function; this is the non-local counterpart of [7, Theorems 1.2, 1.3], and the conclusion is obtained by adapting to the fractional framework the arguments used in [7] in the local setting (for $1\le q\le p$). The proofs are different depending on whether $1\le q<p$, or $p\le q<p_s^*$.

5.1. **Proof of Theorem 1.1 (case $1 \le q < p$).** Let $w_R = w_{s,p,\Omega\cap B_R}$ and $\beta \ge 1$. Since $t \mapsto t^\beta$ is locally Lipschitz continuous and $w_R \in L^\infty(\Omega)$ (by Proposition 3.2), $\varphi = w_R^\beta$ is an admissible test function for equation (3.3). Therefore

$$\iint_{\mathbb{R}^{2N}} \frac{\big|w_R(x) - w_R(y)\big|^{p-2}\big(w_R(x) - w_R(y)\big)\big(w_R(x)^\beta - w_R(y)^\beta\big)}{|x-y|^{N+sp}}\,dx\,dy = \int_\Omega w_R^\beta\,dx\,.$$

Applying the elementary inequality (see [4, Lemma C.1])

$$|a-b|^{p-2}(a-b)(a^\beta - b^\beta) \ge \beta\left[\frac{p}{p+\beta-1}\right]^p \left|a^{\frac{\beta+p-1}{p}} - b^{\frac{\beta+p-1}{p}}\right|^p,$$

with $a = w_R(x)$ and $b = w_R(y)$ and integrating, we deduce that

$$\beta\left[\frac{p}{p+\beta-1}\right]^p \iint_{\mathbb{R}^{2N}} \frac{\left|w_R(x)^{\frac{\beta+p-1}{p}} - w_R(y)^{\frac{\beta+p-1}{p}}\right|^p}{|x-y|^{N+sp}}\,dx\,dy \le \int_\Omega w_R^\beta\,dx\,. \tag{5.3}$$

We observe that

$$\iint_{\mathbb{R}^{2N}} \frac{\left|w_R(x)^{\frac{\beta+p-1}{p}} - w_R(y)^{\frac{\beta+p-1}{p}}\right|^p}{|x-y|^{N+sp}}\,dx\,dy \ge \lambda_{p,q}^s(\Omega)\left(\int_{\Omega\cap B_R} w_R^{\frac{\beta+p-1}{p}q}\right)^{\frac{p}{q}}, \tag{5.4}$$

where we also used the fact that $\lambda_{p,q}^s(\Omega) \le \lambda_{p,q}^s(\Omega\cap B_R)$, in view of the obvious monotonicity of the quantity (5.2) with respect to set inclusion.

Combining (5.4) with (5.3) we get

$$\beta\left[\frac{p}{p+\beta-1}\right]^p \lambda_{p,q}^s(\Omega)\left(\int_{\Omega\cap B_R} w_R^{\frac{\beta+p-1}{p}q}\right)^{\frac{p}{q}} \le \int_\Omega w_R^\beta\,dx\,.$$

Taking $\beta \ge 1$ with $\beta = \frac{\beta+p-1}{p}q$, we obtain

$$\lambda_{p,q}^s(\Omega)\left(\int_{\Omega\cap B_R} w_R^{\frac{p-1}{p-q}q}\right)^{\frac{p-q}{q}} \le \frac{1}{q}\frac{q-1}{p-1}\left(\frac{q-1}{p-q}\right)^{p-1}. \tag{5.5}$$

Recall that $R > 0$ was arbitrary. Hence, if $\lambda_{p,q}^s(\Omega) > 0$, from (5.5) we deduce that

$$\|w_{s,p,\Omega}\|_{L^{\frac{p-1}{p-q}q}(\Omega)} \le \left(\frac{1}{\lambda_{p,q}^s(\Omega)}\frac{q-1}{q(p-1)}\left(\frac{q-1}{p-q}\right)^{p-1}\right)^{\frac{p-1}{p-q}q}, \tag{5.6}$$

by Definition 3.14 and Fatou's Lemma. This concludes the proof. □

5.2. **Proof of Theorem 1.1 (case $q \ge p$).** We first assume that $\lambda_{p,q}(\Omega) > 0$ and we prove that $w := w_{s,p,\Omega}$ belongs to $L^\infty(\Omega)$. More precisely, we show that

$$\|w\|_{L^\infty(\Omega)} \le C\lambda_{p,q}^s(\Omega)^{\frac{1}{1-q}}\,. \tag{5.7}$$

The argument is due to [1, Theorem 9]. Up to an approximation of Ω with an increasing sequence of smooth open sets, while proving (5.7) we may assume without any restriction that Ω is itself smooth and bounded. In particular, in view of Proposition 3.2, we may assume that $\|w\|_{L^\infty(\Omega)} < +\infty$. We shall also require that $w(0) = \|w\|_{L^\infty(\Omega)}$, which again causes no loss of generality (we may assume this up to a translation).

Let $\zeta \in C_0^\infty(\mathbb{R}^N)$ be a cut-off function from $B_{\frac{R}{2}}$ to B_R, with $|\nabla\zeta| \le 2R^{-1}$. Since by our assumptions $w \in L^\infty(\Omega)$, the function $u = w\zeta$ is an admissible competitor for the variational problem (5.2), and we have

$$\lambda^s_{p,q}(\Omega) \le \frac{\displaystyle\iint_{\mathbb{R}^{2N}} \frac{|w(x)\zeta(x) - w(y)\zeta(y)|^p}{|x-y|^{N+sp}}\,dx\,dy}{\displaystyle\int_\Omega w(x)^p\zeta(x)^p\,dx}\,. \tag{5.8}$$

We first estimate the numerator in (5.8). By Proposition 3.2, we can test equation (3.3) with $\varphi = w\zeta^p$ (see, e.g., [2, Lemma 2.4]), so as to get

$$\iint_{\mathbb{R}^{2N}} \frac{|w(x)-w(y)|^{p-2}(w(x)-w(y))}{|x-y|^{N+sp}}(w(x)\zeta(x)^p - w(y)\zeta(y)^p)\,dx\,dy = \int_{B_R} w\zeta^p\,dx\,. \tag{5.9}$$

The double integral appearing in (5.9) splits into its contributions in $\mathcal{C}^+ = \{(x,y) \in \mathbb{R}^{2N} \colon |y| > |x|\}$ and $\mathcal{C}^- = \mathbb{R}^{2N} \setminus \mathcal{C}^+$. Subtracting and adding terms, the two contributions read respectively as

$$\iint_{\mathcal{C}^+} \frac{|w(x)-w(y)|^p}{|x-y|^{N+sp}}\zeta(x)^p + \iint_{\mathcal{C}^+} \frac{|w(x)-w(y)|^{p-2}(w(x)-w(y))}{|x-y|^{N+sp}} w(y)(\zeta(x)^p - \zeta(y)^p) \tag{5.10a}$$

and

$$\iint_{\mathcal{C}^-} \frac{|w(x)-w(y)|^p}{|x-y|^{N+sp}}\zeta(y)^p + \iint_{\mathcal{C}^-} \frac{|w(x)-w(y)|^{p-2}(w(x)-w(y))}{|x-y|^{N+sp}} w(x)(\zeta(y)^p - \zeta(x)^p)\,. \tag{5.10b}$$

Let $\mathcal{A}_1^+ = (B_R \times B_R) \cap \mathcal{C}^+$ and $\mathcal{A}_2^+ = (B_R \times (\Omega \setminus B_R)) \cap \mathcal{C}^+$. We observe that $\zeta(x) \ge \zeta(y)$ in $\mathcal{C}^+$, whence it follows that $\zeta(x)^p - \zeta(y)^p \le p\zeta(x)^{p-1}|x-y|$ for all $(x,y) \in \mathcal{A}_1^+$, provided that we opted for a radially symmetric cut-off with a decreasing radial profile, and clearly we have $\zeta(x)^p - \zeta(y)^p = \zeta(x)$ for all $(x,y) \in \mathcal{A}_2^+$. Therefore

$$\begin{aligned}
&\iint_{\mathcal{C}^+} \frac{|w(x)-w(y)|^{p-2}(w(x)-w(y))}{|x-y|^{N+sp}} w(y)(\zeta(x)^p - \zeta(y)^p)\,dx\,dy \\
&\quad \le \frac{2p}{R}\iint_{\mathcal{A}_1^+} \left|\frac{w(x)-w(y)}{|x-y|^{\frac{N}{p}+s}}\zeta(x)\right|^{p-1} \frac{w(y)\,dx\,dy}{|x-y|^{\frac{N}{p}+s-1}} + \iint_{\mathcal{A}_2^+} \left|\frac{w(x)-w(y)}{|x-y|^{\frac{N}{p}+s}}\right|^{p-1} w(y)\frac{\zeta(x)^p\,dx\,dy}{|x-y|^{\frac{N}{p}+s-1}}
\end{aligned}$$

We write the right hand-side in the form $\mathcal{I}_1^+ + \mathcal{I}_2^+$ and we make repeatedly use of Young inequality $pa^{p-1}b \le (p-1)\frac{a^p}{\tau^{\frac{p}{p-1}}} + \tau^p b^p$, with a suitable $\tau > 0$ to be determined. Estimating $\mathcal{I}_1^+$ we get

$$\begin{aligned}
\mathcal{I}_1^+ &\le \tfrac{p-1}{\tau^{\frac{p}{p-1}}} \iint_{\mathcal{A}_1^+} \frac{|w(x)-w(y)|^p}{|x-y|^{N+sp}}\zeta(x)^p\,dx\,dy + \frac{\tau^p}{R^p}\iint_{\mathcal{A}_1^+} \frac{w(y)^p}{|x-y|^{N+sp-p}}\,dx\,dy \\
&\le \tfrac{p-1}{\tau^{\frac{p}{p-1}}} \iint_{\mathcal{A}_1^+} \frac{|w(x)-w(y)|^p}{|x-y|^{N+sp}}\zeta(x)^p\,dx\,dy + \tau^p w(0)^p\omega_N R^{N-p}\int_0^R \rho^{(1-s)p-1}\,d\rho \\
&\le \tfrac{p-1}{\tau^{\frac{p}{p-1}}} \iint_{\mathcal{A}_1^+} \frac{|w(x)-w(y)|^p}{|x-y|^{N+sp}}\zeta(x)^p\,dx\,dy + \tau^p \tfrac{\omega_N}{(1-s)p} w(0)^p R^{N-sp}\,.
\end{aligned}$$

Similarly,

$$\mathcal{I}_2^+ \le \tfrac{p-1}{\tau^{\frac{p}{p-1}}} \iint_{\mathcal{A}_2^+} \frac{|w(x)-w(y)|^p}{|x-y|^{N+sp}} \zeta(x)^p \,dx\,dy + \tau^p \iint_{\mathcal{A}_2^+} \frac{w(y)^p \zeta(x)^p}{|x-y|^{N+sp}} \,dx\,dy$$
$$\le \tfrac{p-1}{\tau^{\frac{p}{p-1}}} \iint_{\mathcal{A}_2^+} \frac{|w(x)-w(y)|^p}{|x-y|^{N+sp}} \zeta(x)^p \,dx\,dy + \tau^p \tfrac{\omega_N}{(1-s)p} w(0)^p R^{N-sp}\,.$$

Summing up gives

$$\begin{aligned}\tag{5.11a} &\iint_{\mathcal{C}^+} \frac{|w(x)-w(y)|^{p-2}(w(x)-w(y))}{|x-y|^{N+sp}} w(y)(\zeta(x)^p - \zeta(y)^p)\,dx\,dy \\ &\quad\le \tfrac{p-1}{\tau^{\frac{p}{p-1}}} \iint_{\mathcal{C}^+} \frac{|w(x)-w(y)|^p}{|x-y|^{N+sp}} \zeta(x)^p \,dx\,dy + \tau^p C(N,s,p) w(0)^p R^{N-sp}\,.\end{aligned}$$

A similar argument also proves that

$$\begin{aligned}\tag{5.11b} &\iint_{\mathcal{C}^-} \frac{|w(x)-w(y)|^{p-2}(w(x)-w(y))}{|x-y|^{N+sp}} w(x)(\zeta(y)^p - \zeta(x)^p)\,dx\,dy \\ &\quad\le \tfrac{p-1}{\tau^{\frac{p}{p-1}}} \iint_{\mathcal{C}^-} \frac{|w(x)-w(y)|^p}{|x-y|^{N+sp}} \zeta(y)^p \,dx\,dy + \tau^p C(N,s,p) w(0)^p R^{N-sp}\,.\end{aligned}$$

We use the sum of (5.11a) and (5.11b) to estimate from above the sum of (5.10a) and (5.10b). In the inequality which we arrive at, the term divided by $\tau^{\frac{p}{p-1}}$ can be absorbed. Taking into account (5.9), it follows that there exist $C_1, C_2 > 0$, only depending on N, s, p, with

$$\tag{5.12} \int_{B_R} w\zeta^p\,dx \ge (1-C_1) \iint_{\mathbb{R}^{2N}} \frac{|w(x)-w(y)|^p}{|x-y|^{N+sp}} \max\{\zeta(x),\zeta(y)\}^p\,dx\,dy - C_2 w(0)^p R^{N-sp}\,.$$

On the other hand, by standard manipulations we also have

$$[w\zeta]_{s,p}^p \lesssim \iint_{\mathbb{R}^{2N}} \frac{|w(x)-w(y)|^p}{|x-y|^{N+sp}} \max\{\zeta(x),\zeta(y)\}^p + \iint_{\mathcal{C}^+} w(y)^p \frac{(\zeta(x)-\zeta(y))^p}{|x-y|^{N+sp}} + \iint_{\mathcal{C}^-} w(x)^p \frac{(\zeta(y)-\zeta(x))^p}{|x-y|^{N+sp}}$$

where $\lesssim$ means $\le$ up to constants depending only on p. By (5.12) we deduce

$$\tag{5.13} [w\zeta]_{s,p}^p \le C_3(N,s,p)\Big(w(0)R^N + w(0)^p R^{N-sp} + \mathcal{J}_+ + \mathcal{J}_-\Big)\,,$$

where, thanks to the fact that $|\nabla\zeta| \le CR^{-1}$ and $0 \le \zeta \le 1$, we have

$$\begin{aligned}\tag{5.14a} \mathcal{J}_+ &:= \iint_{\mathcal{C}^+} w(y)^p \frac{(\zeta(x)-\zeta(y))^p}{|x-y|^{N+sp}}\,dx\,dy \\ &= \iint_{\mathcal{A}_1^+} w(y)^p \frac{(\zeta(x)-\zeta(y))^p}{|x-y|^{N+sp}}\,dx\,dy + \iint_{\mathcal{A}_2^+} w(y)^p \frac{\zeta(x)^p}{|x-y|^{N+sp}}\,dx\,dy \\ &\le w(0)^p \left[\iint_{B_R\times B_R} \frac{R^{-p}dx\,dy}{|x-y|^{N-(1-s)p}} + \iint_{B_R\times(\mathbb{R}^N\setminus B_R)} \frac{dx\,dy}{|x-y|^{N+sp}}\right] \le Cw(0)^p R^{N-sp}\,,\end{aligned}$$

and similarly

$$\tag{5.14b} \mathcal{J}_+ := \iint_{\mathcal{C}^-} w(x)^p \frac{(\zeta(y)-\zeta(x))^p}{|x-y|^{N+sp}}\,dx\,dy \le Cw(0)^p R^{n-sp}\,,$$

with $C > 0$ depending only on N, s, p. Combining (5.14) with (5.13) we obtain

$$\iint_{\mathbb{R}^{2N}} \frac{|w(x)\zeta(x) - w(y)\zeta(y)|^p}{|x-y|^{N+sp}}\,dx\,dy \le C_4(N,s,p)\Big(w(0)R^N + w(0)^p R^{N-sp}\Big)\,. \tag{5.15}$$

To estimate the denominator in (5.8), we recall the notation introduced in [9]

$$\mathrm{Tail}(\varphi, x_0, r) = \left(r^{sp}\int_{\mathbb{R}^N\setminus B_r(x_0)} \frac{|\varphi(x_0)|^{p-1}}{|x-x_0|^{N+sp}}\,dx\right)^{\frac{1}{p-1}}$$

for the *non-local tail* and we make use of the fact that for every $\delta > 0$ we have

$$\|w\|_{L^\infty(B_{R/4})} \le C_5(N,s,p)\left[\left(\fint_{B_{R/2}} w^p\,dx\right)^{\frac{1}{p}} + \big(1 + \delta\,\mathrm{Tail}(w,0,\tfrac{R}{4})\big)R^{\frac{sp}{p-1}}\right], \tag{5.16}$$

which follows by the estimate of [6, Theorem 3.8], applied[1] with $F \equiv 1$. Then, choosing $\delta = \delta_R$ so that $\delta\,\mathrm{Tail}(w,0,\frac{R}{4}) \le 1$, we obtain from (5.16) that

$$\|u\|_{L^\infty(B_{R/4})} \le C_5(N,s,p)\left[\left(\fint_{B_{R/2}} w^q\,dx\right)^{\frac{1}{q}} + 2R^{\frac{sp}{p-1}}\right]$$

where we also used Jensen inequality and the fact that $q \ge p$. The latter implies that

$$\int_{B_{R/2}} w^q\,dx \ge \omega_N R^N\left(\frac{w(0)}{C_5} - 2R^{\frac{sp}{p-1}}\right)^q. \tag{5.17}$$

Recalling that $\zeta \equiv 1$ on $B_{R/2}$, with the choice $R = (w(0)/C_5)^{\frac{p-1}{sp}}$ inequality (5.17) yields

$$\int_\Omega w^p\zeta^p\,dx \ge C_6(N,s,p,q)w(0)^{q+\frac{p-1}{sp}N}\,. \tag{5.18}$$

Finally, combining (5.18) with (5.15) we conclude by (5.8) that $\lambda^s_{p,q}(\Omega) \le C_7(N,s,p,q)w(0)^{1-q}$. Since by assumption $w(0) = \|w\|_\infty$, we conclude.

To end the proof, we assume that $w := w_{s,p,\Omega}$ belongs to $L^\infty(\Omega)$. Then condition $\lambda^s_{p,p}(\Omega) > 0$ plainly follows by the torsional Hardy inequality (4.2). Indeed, we have

$$\int_\Omega |u|^p\,dx \le \|w\|^{p-1}_{L^\infty(\Omega)}\int_\Omega \frac{|u|^p}{w^{p-1}}\,dx \le \|w\|^{p-1}_{L^\infty(\Omega)}\iint_{\mathbb{R}^{2N}} \frac{\big|u(x)-u(y)\big|^p}{|x-y|^{N+sp}}\,dx\,dy\,,$$

for all $u \in C^\infty_0(\Omega)$, and in view of (5.2) with $q = p$ this gives the desired conclusion. To deduce (1.5), we observe that $\lambda^s_{p,p}(\Omega) > 0$ implies $\lambda^s_{p,q}(\Omega) > 0$ for $p < q < p^*_s$ as well, by the Gagliardo-Nirenberg inequalities of Lemma 2.3, and this concludes the proof. □

[1] In fact, that estimate implies (5.16) with $\delta = 1$, but a close inspection of its proof at scale 1 reveals that minor arrangements allow for the interpolating parameter δ to appear.

5.3. **Proof of Theorem 1.3.** The proof is analogous to the one presented in [7] in the case $s = 1$. By Theorem 1.1 (see in particular (1.4)) it suffices to show that

$$\lambda^s_{p,q}(\Omega) > 0 \Longleftrightarrow \mathcal{D}^{s,p}_0(\Omega) \hookrightarrow L^q(\Omega) \text{ is compact.}$$

We prove the implication "$\Longrightarrow$", the other one being obvious by (5.2).

We assume $\lambda^s_{p,q}(\Omega) > 0$, and we abbreviate $w_{s,p,\Omega}$ to w. By Theorem 1.1 (case $q < p$), we have

$$w \in L^{\frac{p-1}{p-q}q}(\Omega)\,. \tag{5.19}$$

In addition, in view of Remark 5.1, by (4.2), (5.1), and the density of $C^\infty_0(\Omega)$ in $\mathcal{D}^{s,p}_0(\Omega)$ the assumption also implies that

$$\int_\Omega \frac{|u|^p}{w^{p-1}}\,dx \le \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy\,, \qquad \text{for all } u \in \mathcal{D}^{s,p}_0(\Omega), \tag{5.20}$$

and

$$\lambda^s_{p,q}(\Omega)\left(\int_\Omega |u|^q\,dx\right)^{\frac{p}{q}} \le \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy\,, \qquad \text{for all } u \in \mathcal{D}^{s,p}_0(\Omega). \tag{5.21}$$

Let $(u_n)_n$ be a bounded sequence in $\mathcal{D}^{s,p}_0(\Omega)$. The Gagliardo-Nirenberg inequalities of Lemma 2.3 entail that the sequence is bounded in $L^p(\Omega)$, too. Hence, possibly passing to a subsequence, we may assume that $(u_n)_n$ converges weakly to a function u in $\mathcal{D}^{s,p}_0(\Omega)$ and in $L^p(\Omega)$, since $p > 1$ and both spaces are reflexive. Moreover, by (5.21) the function u belongs to $L^q(\Omega)$.

We prove that the sequence $v_n = u_n - u \in \mathcal{D}^{s,p}_0(\Omega) \cap L^p(\Omega)$ converges to 0 strongly in $L^q(\Omega)$. By Rellich-Kondrašov theorem, this happens strongly in $L^q(\Omega \cap B_R)$, for all $R > 0$. Hence, for every $R > 0$ and for every $\varepsilon > 0$ there exists $n_{R,\varepsilon} \in \mathbb{N}$ with

$$\int_{\Omega\cap B_R} |v_n|^q\,dx \le \varepsilon \tag{5.22}$$

for all indices $n \ge n_{R,\varepsilon}$. If in addition, for every ε there exists $R_\varepsilon > 0$ such that

$$\int_{\mathbb{R}^N\setminus B_{R_\varepsilon}} |v_n|^q\,dx \le C\varepsilon\,, \qquad \text{for all } n \in \mathbb{N}, \tag{5.23}$$

for suitable a constant $C > 0$ independent of ε and n, then the sequence $(v_n)_n$ converges to 0 strongly in $L^q(\Omega)$, as desired.

To prove (5.23) we observe that, for every $R > 1$, by Hölder inequality we have

$$\int_{\Omega\setminus B_R} |v_n|^q\,dx \le \left(\int_\Omega \frac{|v_n|^p}{w^{p-1}}\,dx\right)^{\frac{q}{p}} \left(\int_{\Omega\setminus B_R} w^{\frac{p-1}{p-q}q}\,dx\right)^{\frac{p-q}{q}}.$$

Since the sequence $(v_n)_n$ is bounded in $\mathcal{D}^{s,p}_0(\Omega)$, by (5.20) the first factor in the right hand member is bounded by a constant independent of n. As for the second one, by (5.19) the absolute continuity of the integral implies that for every $\varepsilon > 0$ there exists $R_\varepsilon > 1$ with

$$\left(\int_{\Omega\setminus B_{R_\varepsilon - 1}} w^{\frac{p-1}{p-q}q}\,dx\right)^{\frac{p-q}{q}} \le \varepsilon\,.$$

The last two estimates entail (5.23), which concludes the proof. $\square$

Acknowledgments. The author wishes to thank Prof. Lorenzo Brasco for his useful comments on a preliminary version of the present manuscript, as well as for suggesting the problem, in Osaka in May 2017, during the Workshop "Geometric Properties for Parabolic and Elliptic PDEs", the organisers of which are also gratefully acknowledged. This research is supported by the INdAM FOE 2014 grant "SIES".

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