

**MATERIALS WITH ELASTIC RANGE  
AND PLASTIC CHANGE OF VOLUME**

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# MATERIALS WITH ELASTIC RANGE AND PLASTIC CHANGE OF VOLUME

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**Abstract** - Following the lines of [1, 2, 3, 4], we sketch a theory of materials with elastic range that, at variance with the treatment in those papers, does not incorporate the assumption of no plastic change of volume (or any other assumption of a similar kind). We then describe with some detail the constitutive behavior of two important classes of materials with elastic range that, by analogy with the terminology used in classical plasticity, we call  $v$ . Mises materials and ideally plastic materials. Finally, for a specific ideally plastic  $v$ . Mises material, we present a numerical computation that shows how, when plastic change of volume is allowed, the shear stress maintaining a pure shear flow needs not have constant value.

additional hypotheses that, as a rule, enforce plastic hardening of the combined "kinematic" and "isotropic" types; although softening can never occur, the ideally plastic behavior is not excluded. In Section 7 we first introduce the *ideally plastic* material with elastic range, namely, those materials with elastic range whose yield surface is insensitive to deformational vicissitudes; secondly, we restrict our attention to v. Mises materials of the ideally plastic type. Finally, Section 8 features a numerical example illustrating the stress evolution in an ideally plastic v. Mises material during a pure shear process, when plastic change of volume is allowed. A peculiar finding here is that, as the total volume cannot change in a pure shear, whenever plastic flow induces plastic change of volume there must be a compensating elastic change of volume, and therefore the shear stress maintaining the flow must not be constant.

For both hardening and ideally plastic v. Mises materials our main result is a set of constitutive relations having the form of a nonlinear system of differential equations that governs the evolution of stress and other state variables, such as plastic deformation.

## 2. Histories, Constitutive Functional, and Elastic Range.

In this paper we denote by  $\text{Lin}$  the collection of all second-order tensors, and consider the following subcollections of  $\text{Lin}$ :

- $\text{Lin}^+$  = all elements of  $\text{Lin}$  with positive determinant;
- $\text{Sym}$  = all symmetric elements of  $\text{Lin}$ ;
- $\text{Sym}^+$  = all positive-definite elements of  $\text{Sym}$ ;
- $\text{Sym}_0$  = all traceless elements of  $\text{Sym}$ ;
- $\text{Sph}$  = all spherical elements of  $\text{Sym}$ ;
- $\text{Skw}$  = all skew-symmetric elements of  $\text{Lin}$ ;
- $\text{Rot}$  = all orthogonal elements in  $\text{Lin}^+$ .

We make  $\text{Lin}$  into an inner product space by defining, for all  $A, B \in \text{Lin}$ ,  $A \cdot B := \text{tr}(AB^T)$ , with  $\text{tr}$  the trace and  $B^T$  the transpose of  $B$ ; the norm associated to this inner product is the euclidean norm  $\|A\| := (A \cdot A)^{1/2}$ . For  $A \in \text{Lin}$ , we write  $A_0$  for the

traceless (or deviatoric) part of  $A$ :  $A_0 := A - (\frac{1}{3} \text{tr} A)I$ , with  $I$  the identity tensor.

We deal with elastic-plastic solids whose mechanical response is described by a constitutive functional defined over a structured collection of deformation processes.

Deformation processes are here called *histories*. Formally, a history is a continuous and piecewise continuously differentiable mapping  $\hat{F}$  from  $[0,1]$  into  $\text{Lin}^+$ . The value  $\hat{F}(\tau)$  of  $\hat{F}$  at "time"  $\tau$  is interpreted as the deformation gradient from a fixed reference placement, at a fixed material point. A history is *constant* if it has constant value; for  $A \in \text{Lin}^+$  fixed, we denote by  $A^\dagger$  the history with constant value  $A$ . A history  $\hat{Q}$  is *rigid* if it takes its values in  $\text{Rot}$ . For  $\hat{F}$  a history and  $\tau \in [0,1]$ , the  $\tau$ -*section* of  $\hat{F}$  is the history  $\hat{F}_\tau$  such that  $\hat{F}_\tau(\tau') = \hat{F}(\tau\tau')$  for all  $\tau' \in [0,1]$ . A history  $\hat{G}$  is said to be a *continuation* of  $\hat{F}$  if there exists a  $\tau$ -section  $\hat{G}_\tau$  of  $\hat{G}$  such that  $\hat{G}_\tau = \hat{F}$ . For  $A \in \text{Lin}^+$ , a continuation of  $\hat{F}$  up to  $A$  is a continuation  $\hat{G}$  of  $\hat{F}$  such that  $\hat{G}(1) = A$ .

Let  $\mathcal{G}$  be the set of all histories whose initial value  $\hat{F}(0)$  belongs to  $\text{Rot}$ . We denote by  $\tilde{K} : \mathcal{G} \rightarrow \text{Sym}$  a rate-independent, frame-indifferent and isotropic *constitutive functional*,<sup>2</sup> whose value  $\tilde{K}(\hat{F})$  gives the Kirchhoff stress at the end of history  $\hat{F}$ . For each  $\hat{F} \in \mathcal{G}$  and for each  $\tau \in [0,1]$ , we write  $\hat{K}_F(\tau)$  for  $\tilde{K}(\hat{F}_\tau)$ , and we interpret  $\hat{K}_F(\tau)$  as the Kirchhoff stress at time  $\tau$  during history  $\hat{F}$ . We recall that the Kirchhoff and the Cauchy stresses are by definition related as follows:

$$\hat{T}_F(\tau) = (\det(\hat{F}_\tau))^{-1} \hat{K}_F(\tau) .$$

The *elastic range* of a history  $\hat{F} \in \mathcal{G}$  is a set  $\mathcal{E}(\hat{F}) \subset \text{Lin}^+$ , the closure of an arcwise connected open set containing  $\hat{F}(1)$ , whose boundary is attainable from interior points only, and whose points are interpreted as the gradients of all deformations from the reference configuration to configurations which are

<sup>2</sup> The notions of rate independence, frame indifference, and isotropy are fairly standard; for completeness, we recall those notions in the Appendix, where we also collect a number of constitutive properties of materials with elastic range following essentially from frame indifference and isotropy.

elastically accessible from the current one. Each history in  $\mathcal{G}$  is assumed to have an elastic range. An *elastic continuation* of  $\hat{F}$  is a continuation  $\hat{G}$  such that  $\hat{G}_\tau = \hat{F}$  and  $\hat{G}(\tau') \in \mathfrak{E}(\hat{F})$ , for all  $\tau' \in [\tau, 1]$ .

### 3. Unloaded Histories with Change of Volume.

For a given  $\hat{F} \in \mathcal{G}$ , a history  $\hat{S}_F \in \mathcal{G}$  is called an *unloaded history* corresponding to  $\hat{F}$  if, for all  $\tau \in [0, 1]$ , (i)  $\hat{S}_F(\tau) \in \mathfrak{E}(\hat{F}_\tau)$ ; and (ii)  $\tilde{K}(\hat{G}) = 0$  for each elastic continuation  $\hat{G}$  of  $\hat{F}_\tau$  up to  $\hat{S}_F(\tau)$ . The notion of unloaded history is a mathematical formulation of the idea that there should be a stress-free configuration attainable *via* elastic continuation from the current configuration.

As is appropriate for solids [1], we stipulate that each  $\hat{F} \in \mathcal{G}$  can be uniquely decomposed as

$$(3.1) \quad \hat{F}(\tau) = \hat{V}_F(\tau) \hat{P}_F(\tau) ,$$

with  $\hat{V}_F(\tau) \in \text{Sym}^+$  the *elastic stretch* at time  $\tau$  corresponding to history  $\hat{F}$ , and with  $\hat{P}_F$  an unloaded history, whose polar decomposition consists of the unloaded history  $\hat{S}_F^+$  and of the rigid history  $\hat{R}_F$  such that

$$(3.2) \quad \hat{P}_F(\tau) = \hat{R}_F(\tau) \hat{S}_F^+(\tau) \quad \text{with} \quad \hat{R}_F(\tau) \in \text{Rot} , \quad \hat{S}_F^+(\tau) \in \text{Sym}^+ .$$

Employing the same argument as in [1], one finds that  $\hat{S}_F^+(0)$  must be a scalar multiple of the identity tensor  $I$ . Since we now wish to admit plastic changes of volume, we do not accept Axiom 7 of [2], stating that  $\det \hat{S}_F^+(\tau) = 1$  for each  $\tau \in [0, 1]$ ; nevertheless, we assume that  $\hat{S}_F^+(0) = I$  for all  $\hat{F} \in \mathcal{G}$ . Thus, in view of the definition of  $\mathcal{G}$ ,

$$(3.3) \quad \hat{V}_F(0) = I ; \quad \hat{P}_F(0) = \hat{F}(0) \in \text{Rot} .$$

Let time differentiation be denoted by a superimposed dot. We observe [1] that during an elastic continuation  $\hat{G}$  of  $\hat{F}$ ,  $\dot{\hat{S}}_G^+ = 0$ . The symmetric and skew-symmetric parts of  $\dot{\hat{F}} \hat{F}^{-1}$  are the

stretching  $\hat{D}_F$  and the spin  $\hat{W}_F$  associated with  $\hat{F}$ . Similarly, the symmetric part of  $\dot{P}_F \hat{P}_F^{-1}$  is the *plastic stretching*  $\hat{D}_F^p$ ; the plastic spin has no distinguished role in our theory, but we consider the skew symmetric tensor

$$(3.4) \quad \hat{Z}_F := \dot{R}_F \hat{R}_F^T.$$

#### 4. Structural Mapping and Yield Surface.

We assume that, for each  $\hat{F} \in \mathcal{G}$ , the stress at time  $\tau$  is completely determined by the stretch  $\hat{V}_F(\tau)$  through the *structural mapping*  $K^*$ , namely, a frame-indifferent, isotropic  $C^2$ -mapping from  $\text{Lin}^+$  into  $\text{Sym}$ , such that its restriction to  $\text{Sym}^+$  is one-to-one and, moreover,

$$(4.1) \quad \hat{K}_F(\tau) = K^*(\hat{V}_F(\tau)).$$

Under the present assumptions it turns out that the initial configuration is stress free,

$$(4.2) \quad \hat{K}_F(0) = 0,$$

and that

$$(4.3) \quad \dot{K}_F + \hat{K}_F \hat{W}_F - \hat{W}_F \hat{K}_F = DK^*(\hat{V}_F)[\hat{D}_F \hat{V}_F - \hat{V}_F \hat{D}_F^p],$$

with  $DK^*(\hat{V}_F)$  the derivative of  $K^*$  evaluated at  $\hat{V}_F$  (cf. [2]).

For  $\hat{F} \in \mathcal{G}$ , let  $\hat{P}_F$  be the associated unloaded history and define

$$(4.4) \quad \mathcal{E}_R(\hat{F}) := \{A \in \text{Lin}^+ \mid A \hat{P}_F(1) \in \mathcal{E}(\hat{F})\},$$

the *reduced elastic range* corresponding to  $\hat{F}$ .<sup>3</sup> For each history  $\hat{F} \in \mathcal{G}$ , the set

<sup>3</sup> Our present definition is slightly different from the corresponding definition in [1], where we used  $\hat{S}_F^+$  in place of  $\hat{P}_F$ .

$$(4.5) \quad \mathcal{V}(\hat{F}) := \{V \in \text{Sym}^+ \mid V^2 = A^T A, A \in \mathcal{E}_R(\hat{F})\}$$

is called the *reduced elastic range of stretches*. It is easily verified that

$$(4.6) \quad \mathcal{V}(\hat{F}) = \mathcal{E}_R(\hat{F}) \cap \text{Sym}^+ .$$

It follows from (3.1), (4.4) and (4.5), that  $\hat{V}_F(1) \in \mathcal{V}(\hat{F})$  and thus, in view of (4.1),  $\hat{K}_F(1) \in K^*(\mathcal{V}(\hat{F}))$ . The set

$$(4.7) \quad \mathcal{K}(\hat{F}) := K^*(\mathcal{V}(\hat{F}))$$

is called the *stress range*, and its boundary

$$(4.8) \quad \mathcal{Y}(\hat{F}) := \partial\mathcal{K}(\hat{F})$$

is called the *yield surface*. In view of (4.7), for each  $\hat{F} \in \mathcal{G}$  the stress range  $\mathcal{K}(\hat{F})$  is an arcwise connected subset of  $\text{Sym}$ . Guided by common experience with applications, we assume in addition that  $\mathcal{K}(\hat{F})$  is the closure of an open set of  $\text{Sym}$  with locally Lipschitz boundary  $\mathcal{Y}(\hat{F})$ ; and that

$$(4.9) \quad K^*(\partial\mathcal{V}(\hat{F})) = \partial K^*(\mathcal{V}(\hat{F})) ,$$

so that we can write

$$(4.10) \quad \hat{F}(1) \in \partial\mathcal{E}(\hat{F}) \Leftrightarrow \hat{K}_F(1) \in \mathcal{Y}(\hat{F}) .$$

The shape of the stress range, as well as the type of plastic stretching, are restricted by a dissipation axiom on certain cyclic processes. This axiom, an adaptation of Il'yushin postulate [2], states that, for all  $\hat{F} \in \mathcal{G}$  and  $\tau_1, \tau_2 \in [0, 1]$  such that  $0 \leq \tau_1 < \tau_2 \leq 1$  and  $\hat{F}(\tau_1) = \hat{F}(\tau_2) \in \mathcal{E}(\hat{F}_\tau)$  for all  $\tau \in [\tau_1, \tau_2]$ , the work done by the internal forces during the interval  $[\tau_1, \tau_2]$  is nonnegative:

$$(4.11) \quad \int_{\tau_1}^{\tau_2} \widehat{K}_F(\tau) \cdot \widehat{D}_F(\tau) \, d\tau \geq 0 .$$

The ensuing restrictions on the stress range and the plastic stretching are stated in the following

**Proposition 1.** *For each  $\widehat{F} \in \mathcal{G}$  and  $\tau \in [0, 1]$ , the stress range  $\mathcal{K}(\widehat{F}_\tau)$  is a convex subset of  $\text{Sym}$ . Moreover, an associate flow rule holds, in the sense that the plastic stretching  $\widehat{D}_F^p(\tau)$ , when it is different from zero, must belong to the normal cone of  $\mathcal{Y}(\widehat{F}_\tau)$  at  $\widehat{K}_F(\tau)$ .*

We skip the proof of this proposition, because it is analogous to the proof of Proposition 7.6 of [2]. The latter describes the particular case that obtains when it is assumed that  $\det \widehat{F}_F(\tau) = 1$ . This assumption implies that  $\text{tr} \widehat{D}_F^p(\tau) = 0$ , and it is then not difficult to prove that  $\mathcal{K}(\widehat{F}_\tau)$  is the cylinder

$$(4.12) \quad \{\alpha I + \mathcal{K}_0(\widehat{F}_\tau), \alpha \in \mathbb{R}\},$$

with  $\mathcal{K}_0(\widehat{F}_\tau)$  a convex subset of  $\text{Sym}_0$ ; in addition, the associated flow rule is the statement that a nonvanishing plastic stretching must belong to the normal cone of  $\partial \mathcal{K}_0(\widehat{F}_\tau)$  in  $\text{Sym}_0$ .

## 5. Ellipticity Index and Plasticity Index.

We call a point  $K \in \mathcal{Y}(\widehat{F}_\tau)$  *regular* if the outward unit normal  $M(K)$  at that point is well-defined; and we let  $\overline{\mathcal{Y}}(\widehat{F}_\tau)$  denote the set of all the regular points of  $\mathcal{Y}(\widehat{F}_\tau)$ . As in [3], we stipulate here that the Kirchhoff structural mapping and the yield surface are such as to satisfy the following material stability condition: there is a positive material constant  $\eta$  such that

$$(5.1) \quad M(K) \cdot DK^*(V)[VM(K)] \geq \eta ,$$

for all histories  $\widehat{F}$  and for all  $K \in \mathcal{Y}(\widehat{F})$ , with  $V = (K^*)^{-1}(K)$ . To put

it in another way, (5.1) is a restricted coercivity requirement for the bilinear form  $\mathbb{K}(V)[A,B] := A \cdot DK^*(V)[VB]$  at each  $V \in (K^*)^{-1}(\overline{\mathcal{Y}}(\hat{F}))$ .

For  $\hat{F}(\tau) \in \partial\mathcal{E}(\hat{F}_\tau)$ , let  $\hat{K}_F(\tau) \in \overline{\mathcal{Y}}(\hat{F}_\tau)$  and let  $M(\tau)$  be the outward unit normal to  $\overline{\mathcal{Y}}(\hat{F}_\tau)$  at  $\hat{K}_F(\tau)$ ; we call

$$(5.2) \quad \eta_F(\tau) := M(\tau) \cdot DK^*(\hat{V}_F(\tau))[\hat{V}_F(\tau)M(\tau)]$$

the *ellipticity index* of history  $\hat{F}$  at time  $\tau$ , and we note that, in virtue of (5.1), the ellipticity index is always bounded away from zero. We also remark that our current definition of the ellipticity index, as well as the definition to come of the plasticity index, coincides formally with the one given in [3], with the important difference that there  $M(\tau)$  is the outward unit normal to  $\partial\mathcal{K}_0(\hat{F}_\tau)$  at  $\hat{K}_F(\tau)_0$  in  $\text{Sym}_0$ .

We say that a history  $\hat{F} \in \mathcal{G}$  satisfies the *alternative* if, whenever  $\tau \in ]0, 1[$  and  $\hat{F}(\tau) \in \partial\mathcal{E}(\hat{F}_\tau)$ , then there exists  $\varepsilon > 0$  such that

either

$$(A)_1 \quad \hat{F}(\bar{\tau}) \in \mathcal{E}(\hat{F}_\tau) \text{ for each } \bar{\tau} \in [\tau, \tau + \varepsilon),$$

or

$$(A)_2 \quad \hat{F}(\bar{\tau}) \notin \mathcal{E}(\hat{F}_\tau) \text{ and } \hat{D}_F^p(\bar{\tau}) \neq 0 \text{ for each } \bar{\tau} \in (\tau, \tau + \varepsilon).$$

Two situations may occur under  $(A)_1$ , usually referred to as *elastic unloading* and *neutral loading*. The situation under  $(A)_2$  is known as *plastic loading*; during plastic loading we have

$$(5.3) \quad \hat{F}(\bar{\tau}) \in \partial\mathcal{E}(\hat{F}_{\bar{\tau}}) \text{ for each } \bar{\tau} \in [\tau, \tau + \varepsilon[.$$

For all  $\tau \in (0, 1)$  and for all  $\bar{\tau} \in [\tau, 1[$  we write

$$(5.4) \quad \begin{aligned} \hat{F}(\bar{\tau}) &= \tilde{V}_F(\tau, \bar{\tau}) \tilde{R}_F(\tau, \bar{\tau}) \hat{S}_F^+(\tau) , \\ \tilde{Z}_F(\tau, \bar{\tau}) &= \left( \frac{\partial}{\partial \tau} \tilde{R}_F(\tau, \bar{\tau}) \right) \tilde{R}_F^T(\tau, \bar{\tau}) , \end{aligned}$$

where  $\tilde{R}_F(\tau, \bar{\tau})$  and  $\tilde{V}_F(\tau, \bar{\tau})$  are the polar factors of  $\hat{F}(\bar{\tau}) \hat{S}_F^+(\tau)^{-1}$ . We observe that  $\tilde{R}_F(\tau, \tau) = \hat{R}_F(\tau)$  and  $\tilde{V}_F(\tau, \tau) = \hat{V}_F(\tau)$ . If  $\hat{F}(\tau) \in \partial \mathcal{E}(\hat{F}_\tau)$  and situation  $(A)_1$  occurs, then  $(5.4)_2$  and (3.4) imply that  $\tilde{Z}_F(\tau, \tau) = \hat{Z}_F(\tau)$ , but in general  $\tilde{Z}_F(\tau, \tau) \neq \hat{Z}_F(\tau)$ ; for simplicity, we shall write  $\tilde{Z}_F(\tau)$  for  $\tilde{Z}_F(\tau, \tau)$ .

For  $\hat{F}(\tau) \in \partial \mathcal{E}(\hat{F}_\tau)$  and  $\hat{K}_F(\tau) \in \bar{\mathcal{Y}}(\hat{F}_\tau)$  we call

$$(5.5) \quad \begin{aligned} \gamma_F(\tau) &:= M(\tau) \cdot \{DK^*(\hat{V}_F(\tau))[\hat{D}_F(\tau)\hat{V}_F(\tau)] + \\ &\quad (\hat{W}_F(\tau) - \tilde{Z}_F(\tau))\hat{K}_F(\tau) - \hat{K}_F(\tau)(\hat{W}_F(\tau) - \tilde{Z}_F(\tau))\} \end{aligned}$$

the *plasticity index* of history  $\hat{F}$  at time  $\tau$ . This name is justified by the following proposition.

**Proposition 2.** *For each  $\hat{F} \in \mathcal{G}$  which satisfies the alternative and each  $\tau \in ]0, 1[$  such that  $\hat{F}(\tau) \in \partial \mathcal{E}(\hat{F}_\tau)$  and  $K(\tau) \in \bar{\mathcal{Y}}(\hat{F}_\tau)$ , we have*

$$(5.6) \quad \begin{aligned} (A)_1 &\Rightarrow \gamma_F(\tau) \leq 0 , \quad \gamma_F(\tau) < 0 \Rightarrow (A)_1 ; \\ (A)_2 &\Rightarrow \gamma_F(\tau) \geq 0 , \quad \gamma_F(\tau) > 0 \Rightarrow (A)_2 . \end{aligned}$$

Once again we omit the proof, which is completely analogous to the proof of Proposition 3.1 of [3].

In the rest of this paper we shall consider only histories  $\hat{F} \in \mathcal{G}$  satisfying the alternative, and we shall use the same symbol  $\mathcal{G}$  to denote their collection.

## 6. v. Mises Materials.

A material with elastic range is said to be a *v. Mises material*

if, for each history  $\hat{F} \in \hat{\mathcal{G}}$  and each  $\tau \in [0,1]$ , the yield surface  $\mathcal{Y}(\hat{F}_\tau)$  is the set

$$(6.1) \quad \mathcal{Y}(\hat{F}_\tau) = \{K \in \text{Sym} \mid \|K_0 - \hat{C}_F(\tau)\| = \rho_F(\text{tr}K, \tau)\} .$$

We call  $\hat{C}_F(\tau)$  the *back stress* at instant  $\tau$ . For each fixed value of the real parameter  $\alpha$ , the mappings  $\tau \mapsto \hat{C}_F(\tau) \in \text{Sym}_0$  and  $(\alpha, \tau) \mapsto \rho_F(\alpha, \tau) \in \mathbb{R}^+ := [0, +\infty[$  deliver, respectively, the centre  $\hat{C}_F(\tau)$  and the radius  $\rho_F(\alpha, \tau)$  at time  $\tau$  of the sphere

$$(6.2) \quad \mathcal{Y}_0(\hat{F}_\tau; \alpha) := \{K_0 \in \text{Sym}_0 \mid \|K_0 - \hat{C}_F(\tau)\| = \rho_F(\alpha, \tau)\} ;$$

and it is assumed that

$$(6.3) \quad \rho_F(0, \tau) > 0$$

and

$$(6.4) \quad \|\hat{C}_F(\tau)\| \leq \rho_F(0, \tau)$$

for each  $\tau \in [0,1]$ . Inequalities (6.3) and (6.4) guarantee that the null stress belongs to the current stress range; this situation is consistent with our notion of an unloaded history  $\hat{S}_F$ , according to which  $\hat{S}_F(\tau) \in \mathcal{E}(\hat{F}_\tau)$  for all  $\tau$ .

For a v. Mises material it is possible to prove, by the use of the representation theorem for isotropic functions, that

$$(6.5) \quad K^*(V) \in \text{Sph} \Leftrightarrow V \in \text{Sph} \cap \text{Sym}^+ .$$

It can also be proved that

$$(6.6) \quad \hat{C}_F(0) = 0 ,$$

and that

$$(6.7) \quad \dot{\hat{C}}_F(\bar{\tau}) + \hat{C}_F(\bar{\tau})\hat{Z}_F(\bar{\tau}) - \hat{Z}_F(\bar{\tau})\hat{C}_F(\bar{\tau}) = 0 , \quad \bar{\tau} \in [\tau, \tau + \varepsilon] ,$$

in two distinct circumstances: for each  $\hat{F} \in \mathcal{G}$  and  $\tau \in [0,1]$  such that  $\hat{F}(\tau)$  belongs to the interior part of  $\Xi(\hat{F}_\tau)$ ; and for each  $\hat{F} \in \mathcal{G}$  and  $\tau \in ]0,1[$  such that  $\hat{F}(\tau) \in \partial\Xi(\hat{F}_\tau)$  and situation  $(A)_1$  of the alternative occurs.

Relations (6.6) and (6.7) partially specify the evolution of the back stress. With a view to prescribing hardening rules, we introduce the *Odqvist function* of a history  $\hat{F}$ , namely,

$$(6.8) \quad \tau \mapsto \xi_F(\tau) := \int_0^\tau \|\hat{D}_F^p(\tau')\| \, d\tau'.$$

The value  $\dot{\xi}_F(\tau) = \|\hat{D}_F^p(\tau)\|$  measures the current plastic flow, and thus  $\xi_F(\tau)$  is an overall measure of the plastic deformation up to time  $\tau$  during the history  $\hat{F}$ .

A v. Mises material has a *hardening* behavior if, for each history  $\hat{F} \in \mathcal{G}$ , its stress range evolves as prescribed by two constitutive mappings  $\hat{\rho}$  and  $\hat{\Gamma}$ , whose definitions and properties are:

$(H)_1$  the mapping  $\hat{\rho}$  from  $\mathbb{R} \times \mathbb{R}^+$  into  $\mathbb{R}^+$  is twice piecewise-differentiable, and such that the mappings  $(\alpha, \cdot) \mapsto \hat{\rho}(\alpha, \cdot)$ , for each  $\alpha \in \mathbb{R}$ , and  $(\cdot, \beta) \mapsto \hat{\rho}(\cdot, \beta)$ , for each  $\beta \in \mathbb{R}^+$ , are nondecreasing and concave, respectively; moreover,

$$(6.9) \quad \rho_F(\alpha, \tau) = \hat{\rho}(\alpha, \xi_F(\tau)), \quad \text{for each } \tau \in [0,1];$$

$(H)_2$  the mapping  $\hat{\Gamma}$  from  $\mathbb{R}^+ \times \text{Sym}_0 \times \text{Sym}_0$  into  $\text{Sym}_0$  is such that, for each  $(\alpha, A, B) \in \mathbb{R}^+ \times \text{Sym}_0 \times \text{Sym}_0$ ,

$$(6.10) \quad \|\hat{\Gamma}(\alpha, A, B)\| < +\infty,$$

$$(6.11) \quad \hat{\Gamma}(\alpha, QAQ^T, QBQ^T) = Q\hat{\Gamma}(\alpha, A, B)Q^T \quad \text{for all } Q \in \text{Rot};$$

moreover, if the mapping  $\tau \mapsto \Gamma_F(\tau)$  is defined by

$$(6.12) \quad \Gamma_F(\tau) := \begin{cases} 0 & \text{if } \dot{\xi}_F = 0 \\ \hat{\Gamma}(\xi_F(\tau), \hat{C}_F(\tau), M_*(\tau)) & \text{if } \dot{\xi}_F > 0, \end{cases}$$

with  $M_*(\tau)$  the outward unit normal to  $\mathcal{Y}_0(\hat{F}_\tau; \text{tr} \hat{K}_F(\tau))$  at  $\hat{K}_F(\tau)_0$ :

$$(6.13) \quad M_*(\tau) := \frac{1}{\hat{\rho}(\text{tr} \hat{K}_F(\tau), \xi_F(\tau))} [\hat{K}_F(\tau)_0 - \hat{C}_F(\tau)],$$

then

$$(6.14) \quad \dot{C}_F(\tau) + \hat{C}_F(\tau) \tilde{Z}_F(\tau) - \tilde{Z}_F(\tau) \hat{C}_F(\tau) = \dot{\xi}_F(\tau) \Gamma_F(\tau),$$

and

$$(6.15) \quad \Gamma_F(\tau) \cdot M(\tau) \geq 0, \quad \text{for all } \tau \in (0, 1) \text{ such that } \hat{K}_F(\tau) \in \mathcal{Y}(\hat{F}_\tau).$$

We have from (6.1) and (H)<sub>1</sub> that either  $\mathcal{Y}(\hat{F}_\tau)$  is a cylinder whose axis is parallel to Sph (in which case  $\hat{\rho}$  does not depend on  $\alpha$ ) or there exists  $\bar{\alpha} \in \mathbb{R}$  such that  $\mathcal{Y}(\hat{F}_\tau)$  is entirely contained in one of the two half-spaces into which the hyperplane  $\text{tr} K = \bar{\alpha}$  divides Sym.

The following proposition establishes a useful relation between  $M$  and  $M_*$ .

**Proposition 3.** *For each  $\hat{F} \in \mathcal{G}$  and  $\tau \in ]0, 1[$  such that  $\hat{F}(\tau) \in \partial \mathcal{E}(\hat{F}_\tau)$ ,*

$$(6.16) \quad M_*(\tau) - [\partial_1 \hat{\rho}(\text{tr} \hat{K}_F(\tau), \xi_F(\tau))] I = \chi_F(\tau) M(\tau),$$

where

$$(6.17) \quad \underline{[\chi_F(\tau)]^2} := 1 + 3[\partial_1 \hat{\rho}(\text{tr} \hat{K}_F(\tau), \xi_F(\tau))]^2. \quad ^4$$

<sup>4</sup> Here and henceforth we denote differentiation with respect to the  $\alpha$ -th variable by  $\partial_\alpha$ ,  $\alpha = 1, 2$ .

**Proof.** The yield surface  $\mathcal{Y}(\hat{F}_\tau)$  can be seen as the zero level set of the function  $\varphi(\hat{F}_\tau; \cdot)$  defined by

$$\varphi(\hat{F}_\tau; K) := \|K_0 - \hat{C}_F(\tau)\|^2 - \hat{\rho}^2(\text{tr}K, \xi_F(\tau)) .$$

For  $\hat{F}_\tau$  fixed, the gradient of  $\varphi(\hat{F}_\tau; \cdot)$  at  $K$  is given by

$$D\varphi(\hat{F}_\tau; K) = 2 \{K_0 - \hat{C}_F(\tau) - [\hat{\rho}(\text{tr}K, \xi_F(\tau)) \partial_1 \hat{\rho}(\text{tr}K, \xi_F(\tau))]I\} .$$

Thus,

$$\|D\varphi(\hat{F}_\tau; K)\| = 2\hat{\rho}(\text{tr}K, \xi_F(\tau)) \{1 + 3[\hat{\rho}(\text{tr}K, \xi_F(\tau)) \partial_1 \hat{\rho}(\text{tr}K, \xi_F(\tau))]\}^{1/2},$$

and, since the outward unit normal at a point  $\hat{K}_F(\tau)$  of  $\bar{\mathcal{Y}}(\hat{F}_\tau)$  is

$$M(\tau) = \frac{1}{\|D\varphi(\hat{F}_\tau; \hat{K}_F(\tau))\|} D\varphi(\hat{F}_\tau; \hat{K}_F(\tau)) ,$$

recalling (6.17) we arrive at (6.16).  $\square$

For each  $\hat{F} \in \mathcal{G}$  and  $\tau \in ]0, 1[$  such that  $\hat{F}(\tau) \in \partial\mathcal{E}(\hat{F}_\tau)$ , we put

$$(6.18) \quad \xi_F(\tau) := \eta_F(\tau) + \Gamma_F(\tau) \cdot M(\tau) + (\chi_F(\tau))^{-1} \partial_2 \hat{\rho}(\text{tr}K, \xi_F(\tau)) ,$$

where  $\eta_F$  is the ellipticity index defined by (5.2) and  $\chi_F$  is defined by (6.17). In view of (6.15), (6.17) and the fact that  $\hat{\rho}$  is a nondecreasing function of its second argument, one has

$$(6.19) \quad \xi_F(\tau) - \eta_F(\tau) \geq 0 .$$

**Proposition 4.** For each  $\hat{F} \in \mathcal{G}$  and  $\tau \in ]0, 1[$  such that  $\hat{F}(\tau) \in \partial\mathcal{E}(\hat{F}_\tau)$ ,

$$(i) \quad (A)_1 \Rightarrow \dot{\xi}_F(\tau) = 0 ;$$

$$(ii) \quad (A)_2 \Rightarrow \dot{\xi}_F(\tau) = \frac{\delta_F(\tau)}{\xi_F(\tau)} .$$

**Proof.** If  $(A)_1$  holds, the first implication follows from (6.8) and the fact that, if  $\hat{G}$  is an elastic continuation of  $\hat{F}_\tau$ , such that  $\hat{G}_{\tau^*} = \hat{F}_\tau$ , then  $\hat{D}_G^p(\tau) = 0$  for all  $\tau \in [\tau^*, 1[$ . As to the second implication, we note first that, in view of (5.1), (5.2) and (6.19),  $\xi_F(\tau) \geq \eta > 0$ . Moreover, if  $(A)_2$  holds, (5.3), (4.10) and (6.2) imply that

$$(6.20) \quad \|\hat{K}_F(\bar{\tau})_0 - \hat{C}_F(\bar{\tau})\| = \hat{\rho}(\text{tr} \hat{K}_F(\bar{\tau}), \xi_F(\bar{\tau})) \quad \text{for all } \bar{\tau} \in [\tau, \tau + \varepsilon[ ;$$

thus,

$$(6.21) \quad (\hat{K}_F(\bar{\tau})_0 - \hat{C}_F(\bar{\tau})) \cdot (\dot{\hat{K}}_F(\bar{\tau})_0 - \dot{\hat{C}}_F(\bar{\tau})) = \\ \hat{\rho}(\text{tr} \hat{K}_F(\tau), \xi_F(\tau)) \{ [\partial_1 \hat{\rho}(\text{tr} \hat{K}_F(\tau), \xi_F(\tau))] \text{tr} \dot{\hat{K}}_F(\tau) + \\ [\partial_2 \hat{\rho}(\text{tr} \hat{K}_F(\tau), \xi_F(\tau))] \dot{\xi}_F(\tau) \} .$$

On the other hand, it follows from (4.3) that

$$(6.22) \quad M_* \cdot \{ DK^*(\hat{V}_F) [ \hat{D}_F \hat{V}_F - \dot{\xi}_F \hat{V}_F M ] + \\ \hat{W}_F \hat{K}_F - \hat{K}_F \hat{W}_F - (\dot{\hat{C}}_F + \hat{C}_F \tilde{Z}_F - \tilde{Z}_F \hat{C}_F) \} = \\ \text{tr}(\dot{\hat{K}}_F) \partial_1 \hat{\rho}(\text{tr} \hat{K}_F, \xi_F) + \dot{\xi}_F \partial_2 \hat{\rho}(\text{tr} \hat{K}_F, \xi_F) .$$

Combining (6.21) and (6.22) we have that

$$(6.23) \quad M \cdot \{ (\hat{K}_F - \hat{C}_F) \tilde{Z}_F - \tilde{Z}_F (\hat{K}_F - \hat{C}_F) \} = 0 ;$$

but (6.7), (6.22), (6.23) and (6.16) imply (ii).  $\square$

The following corollary of Propositions 2 and 4 allows us to compute the plastic flow during a given history.

**Corollary.** For each  $\hat{F} \in \mathcal{G}$  and  $\tau \in [0, 1[$ ,

$$(6.24) \quad \dot{\xi}_F(\tau) = \begin{cases} 0 & \text{if } \|\hat{K}_F(\tau)_0\| < \hat{\rho}(\text{tr} \hat{K}_F(\tau), \xi_F(\tau)) \\ 0 & \text{if } \|\hat{K}_F(\tau)_0\| = \hat{\rho}(\text{tr} \hat{K}_F(\tau), \xi_F(\tau)) , \gamma_F(\tau) \leq 0 \\ \frac{\gamma_F(\tau)}{\xi_F(\tau)} & \text{if } \|\hat{K}_F(\tau)_0\| = \hat{\rho}(\text{tr} \hat{K}_F(\tau), \xi_F(\tau)) , \gamma_F(\tau) > 0. \end{cases}$$

Moreover, from the second statement of Proposition 1 it follows that, for  $\|\hat{K}_F(\tau)_0\| = \hat{\rho}(\text{tr} \hat{K}_F(\tau), \xi_F(\tau))$ ,

$$(6.25) \quad \hat{D}_F^P(\tau) = \dot{\xi}_F(\tau) M(\tau) .$$

Relation (6.19) and the second statement in Proposition 4 show that the rate of plastic deformation  $\dot{\xi}_F$  decreases as the difference  $(\xi_F(\tau) - \eta_F(\tau))$  increases (for a fixed value of the plasticity index  $\gamma_F(\tau)$ ); the largest rate of plastic deformation pertain to the ideally plastic materials that we shall study in the next section, for which  $\xi_F(\tau) - \eta_F(\tau) \equiv 0$ .

## 7. Ideally Plastic Materials.

A material with elastic range is said to be *ideally plastic* if

$$(i) \quad \text{for each } \hat{F} \in \mathcal{G}, \quad \mathcal{E}_R(\hat{F}) = \mathcal{E}_R(\Gamma^\dagger) ;$$

$$(ii) \quad \text{for all } \alpha \in \mathbb{R},$$

$$\mathcal{K}(\Gamma^\dagger) \cap \{A \in \text{Sym} \mid \text{tr} A = \alpha\} \text{ is a bounded subset of Sym.}$$

It is easy to show that, for each  $\hat{F} \in \mathcal{G}$ ,

$$(7.1) \quad \mathcal{K}(\hat{F}) = \mathcal{K}(I^\dagger) \quad \text{and} \quad \mathcal{Y}(\hat{F}) = \mathcal{Y}(I^\dagger).$$

Employing the same argument as in [2], Prop. 8.1, one can prove that for an ideally plastic material,  $K$  and  $M(K)$  commute for  $K \in \overline{\mathcal{Y}}(I^\dagger)$ ; thus, the plasticity index has an expression simpler than (5.5), namely

$$(7.2) \quad \delta_F(\tau) = M(\tau) \cdot DK^*(\hat{V}_F(\tau)) [\hat{D}_F(\tau) \hat{V}_F(\tau)].$$

For each  $\hat{F} \in \mathcal{G}$  and  $\tau \in [0, 1)$  one can prove that

$$(7.3) \quad \dot{\xi}_F(\tau) = \begin{cases} 0 & \text{if } \hat{K}_F(\tau) \notin \mathcal{Y}(I^\dagger) \\ 0 & \text{if } \hat{K}_F(\tau) \in \overline{\mathcal{Y}}(I^\dagger), \delta_F(\tau) \leq 0 \\ \frac{\delta_F(\tau)}{\eta_F(\tau)} & \text{if } \hat{K}_F(\tau) \in \overline{\mathcal{Y}}(I^\dagger), \delta_F(\tau) > 0. \end{cases}$$

Moreover, for  $\hat{K}_F(\tau) \in \overline{\mathcal{Y}}(I^\dagger)$ , one has that

$$(7.4) \quad \hat{D}_F^p(\tau) = \dot{\xi}_F(\tau) M(\tau).$$

Relations (7.3) and (7.4) have formally identical counterparts in the case studied in [2] where the assumption of no plastic change of volume is accepted, and the yield surface is defined only in terms of the deviatoric part of the Kirchhoff stress  $K$ . Here, instead, yielding is influenced by the spherical part of  $K$  as well.

A *v. Mises* ideally plastic material is an ideally plastic material such that, for each  $\hat{F} \in \mathcal{G}$  and  $\tau \in [0, 1]$ , the yield surface  $\mathcal{Y}$  is the set

$$(7.5) \quad \mathcal{Y} = \{K \in \text{Sym} \mid \|K_0\| = \tilde{\rho}(\text{tr}K)\}, \quad \tilde{\rho}(\alpha) := \hat{\rho}(\alpha, 0).$$

It follows from (5.2), (7.2), (7.3) and (7.5) that, for each  $\tau \in [0, 1)$ ,

$$(7.6) \quad \dot{\xi}_F(\tau) = \begin{cases} 0 & \text{if } \|\hat{K}_F(\tau)_0\| < \tilde{\rho}(\text{tr} \hat{K}_F(\tau)) \\ 0 & \text{if } \|\hat{K}_F(\tau)_0\| = \tilde{\rho}(\text{tr} \hat{K}_F(\tau)) , \gamma_F(\tau) \leq 0 \\ \frac{\gamma_F(\tau)}{\eta_F(\tau)} & \text{if } \|\hat{K}_F(\tau)_0\| = \tilde{\rho}(\text{tr} \hat{K}_F(\tau)) , \gamma_F(\tau) > 0. \end{cases}$$

Moreover, for  $\|\hat{K}_F(\tau)_0\| = \tilde{\rho}(\text{tr} \hat{K}_F(\tau))$ , (7.4) holds.

### 8. A Numerical Example.

As an example of the constitutive behaviour we wish to model, we consider a v. Mises ideally plastic material whose Kirchhoff structural mapping is given by

$$(8.1) \quad K^*(N) = \mu_0(B - I) + \frac{1}{2}\lambda_0(\det B - 1)I, \quad B := NN^T,$$

with  $\lambda_0$  and  $\mu_0$  two positive moduli.

For  $\hat{F} \in \mathcal{G}$  and  $\tau \in [0, 1]$ , let

$$(8.2) \quad \hat{B}_F(\tau) := (\hat{V}_F(\tau))^2.$$

An easy but important consequence of (3.1) and the definition of plastic stretching is

$$(8.3) \quad \dot{V}_F \cdot V_F^{-1} = \text{tr} \hat{D}_F - \text{tr} \hat{D}_F^P.$$

If we substitute (8.1), (8.2) in (4.3) and take (8.3) into account, we obtain

$$(8.4) \quad \dot{B}_F + \hat{B}_F \hat{W}_F - \hat{W}_F \hat{B}_F = \hat{B}_F \hat{D}_F + \hat{D}_F \hat{B}_F - 2\hat{V}_F \hat{D}_F^P \hat{V}_F.$$

We remark that the same equation was obtained in [6] under the hypothesis that plastic changes of volume cannot occur, and thus  $\text{tr} \hat{D}_F^P = 0$ ; this comes to no surprise if we consider that the form taken by (4.3) depends only on our choice of the structural mapping.

From (8.1) and (7.2) we have that

$$(8.5) \quad \gamma_F = \mu_0 M \cdot [\hat{B}_F \hat{D}_F + \hat{D}_F \hat{B}_F + \frac{\lambda_0}{\mu_0} (\det \hat{B}_F) (\text{tr} \hat{D}_F) I],$$

whereas (5.2) implies that

$$(8.6) \quad \eta_F = \mu_0 M \cdot [2 \hat{V}_F M \hat{V}_F - \frac{3\lambda_0}{\chi_F \mu_0} (\det \hat{B}_F) \tilde{\rho}'(\text{tr} \hat{K}_F) I],$$

where  $\chi_F$  is defined by (6.17) and we choose the function  $\tilde{\rho}$  in (7.5)<sub>2</sub> to be

$$(8.7) \quad \tilde{\rho}(\alpha) = \rho_0 (1 - \beta_0 \alpha), \quad \beta_0 > 0.$$

For  $\{e_1, e_2, e_3\}$  an orthonormal basis, we consider the pure shear history

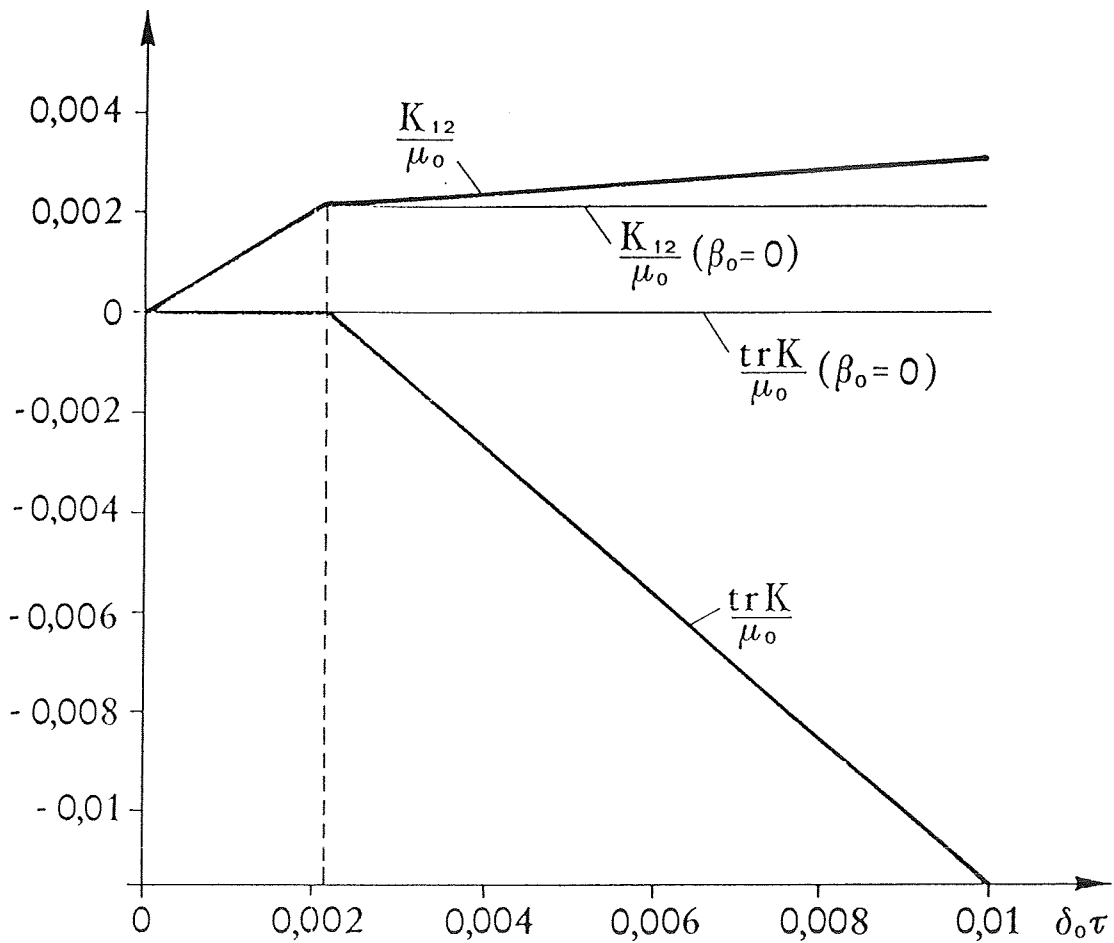
$$(8.8) \quad \hat{F}(\tau) = I + \tau (\delta_0 e_1 \otimes e_2), \quad \delta_0 > 0,$$

and perform a numerical integration of equations (8.4), (7.3) and (7.4) along this history for the following set of values of the parameters:

$$\lambda_0/\mu_0 = 1.5, \quad \rho_0/\mu_0 = 3 \times 10^{-3}, \quad \beta_0 = 5 \times 10^{-5}, \quad \delta_0 = 1 \times 10^{-2}.$$

This choice of parameters is realistic for metals, although it is not meant to describe any special material.

It is instructive to compare the results with the corresponding ones for the case of no plastic volume change, when the stress range has the shape of a cylinder ( $\beta_0 = 0$ ). The accompanying figure shows the ( $\mu_0$ -scaled) shear stress  $K_{12}$  and  $\text{tr}K$  as functions of the (adimensional, time-like) variable  $\delta_0 \tau$ , both when  $\beta_0 = 5 \times 10^{-5}$  and when  $\beta_0 = 0$ .



For small values of  $\tau$  the behavior is the same as that of a hyperelastic solid in a pure shear deformation: the volume does not change and, thus, due to (8.1),  $\text{tr}K \equiv 0$  and  $K_{12}$  increases linearly; moreover, by (7.5), (6.2) and (8.7), the sphere  $\mathcal{Y}_0$  has constant radius. When the boundary of the stress range is reached and plastic deformation occurs<sup>5</sup>, the plastic stretching  $\hat{D}_F^P$ , due to (7.4) and (8.7), is directed in such a way as to produce plastic growth of volume ( $\text{tr} \hat{D}_F^P > 0$ ). Since the total

<sup>5</sup> The value  $\delta_0 \tau_0$  at which the boundary of the stress range is reached can be explicitly calculated, by solving the equation  $\|\hat{K}_F(\tau)_0\| = \rho_0$ ; the latter turns out to be a quartic in  $\delta_0 \tau$  having one real positive root,  $\delta_0 \tau_0 = 0.002121$ .

volume cannot change in pure shear, an elastic decrease in volume must occur, and thus  $\text{tr}K$  must decrease as well; this fact, in turn, implies that the radius of  $\mathcal{U}_0(\text{tr}K)$  increases, and so does then  $K_{12}$ .

## Appendix

### Rate Independence, Frame Indifference and Isotropy of the Constitutive Functional, and their Consequences.

Materials with elastic range may be rate-independent or not. A deformation process of duration  $\delta > 0$  is a smooth mapping  $\hat{F}$  from the interval  $[0, \delta]$  into  $\text{Lin}^+$ . For  $\delta_0 > 0$ , a *time-rescaling* is a smooth, non-decreasing mapping  $\gamma$  of  $[0, \delta]$  onto  $[0, \delta_0]$ . The constitutive functional  $\tilde{K}$  is *rate-independent* if it is invariant under time-rescaling, in the sense that

$$\tilde{K}(\hat{F}) = \tilde{K}(\hat{F} \circ \gamma) ,$$

for all rescalings  $\gamma$  and for all deformations processes  $\hat{F}$ . As a consequence, any dependence of the stress response on the duration of processes is ruled out. It is precisely because we assume that  $\tilde{K}$  is rate-independent that we can think of histories as having one common domain  $[0, 1]$ , and we can interpret  $\tilde{K}(\hat{F}_\tau)$ , the Kirchhoff stress associated to the  $\tau$ -section of  $\hat{F}$ , as the stress attained at time  $\tau$  during history  $\hat{F}$ .

We say that the constitutive functional  $\tilde{K} : \mathcal{G} \rightarrow \text{Sym}$  is *frame-indifferent* if, for  $\hat{F} \in \mathcal{G}$  and for  $\hat{Q}$  a rigid history

$$\tilde{K}(\hat{Q}\hat{F}) = \hat{Q}(1)\tilde{K}(\hat{F})\hat{Q}(1)^T$$

identically in  $\hat{F}$  and  $\hat{Q}$ . Moreover,  $\tilde{K}$  is *isotropic* if

$$\tilde{K}(\hat{F}Q^t) = \tilde{K}(\hat{F}) ,$$

for all  $\hat{F} \in \mathcal{G}$  and for all  $Q \in \text{Rot}$ . In the theory of materials with elastic range as developed so far, the constitutive functional delivering the stress is assumed to be both frame-indifferent and isotropic. This assumption has manifold consequences, some of which we now list both for rendering this paper reasonably self-

contained and for providing items of information needed to establish some of the statements left without proof in Sections 2-5.

For each  $\hat{F} \in \mathcal{G}$ , the elastic range  $\mathfrak{E}(\hat{F})$ , the reduced elastic range  $\mathfrak{E}_R(\hat{F})$  and the stress range  $\mathfrak{K}(\hat{F})$  satisfy the following properties [1,2]:

for all rigid histories  $\hat{Q}$  and for all  $Q \in \text{Rot}$ .

$$(i) \quad \mathfrak{E}(\hat{F}) = \text{Rot } \mathfrak{E}(\hat{F}) \quad , \quad \mathfrak{E}(\hat{Q}\hat{F}) = \mathfrak{E}(\hat{F}) \quad , \quad \mathfrak{E}(\hat{F}Q^\dagger) = \mathfrak{E}(\hat{F})Q \quad ;$$

$$(ii) \quad \text{Rot} \subset \mathfrak{E}_R(\hat{F}) = \text{Rot } \mathfrak{E}_R(\hat{F}) \quad , \quad \mathfrak{E}_R(\hat{Q}\hat{F}) = \mathfrak{E}_R(\hat{F})\hat{Q}(1)^T \quad , \\ \mathfrak{E}_R(\hat{F}Q^\dagger) = \mathfrak{E}_R(\hat{F}) \quad ;$$

$$(iii) \quad \mathfrak{K}(\hat{Q}\hat{F}) = \hat{Q}(1)\mathfrak{K}(\hat{F})\hat{Q}(1)^T \quad , \quad \mathfrak{K}(\hat{F}Q^\dagger) = \mathfrak{K}(\hat{F}) \quad .$$

As to the definition of a v. Mises material given in Section 6, the assumption of frame-indifference of  $\tilde{K}$  is obviously important to prove that the yield surface transforms as follows under change in observer [3]:

$$\mathfrak{Y}(\hat{Q}\hat{F}_\tau) = \{K \in \text{Sym} \mid \|K_0 - \hat{C}_{QF}(\tau)\| = \rho_{QF}(\text{tr}K, \tau)\} = \\ = \hat{Q}(\tau)\mathfrak{Y}(\hat{F}_\tau)\hat{Q}(\tau)^T = \{K \in \text{Sym} \mid \|K_0 - \hat{Q}(\tau)C_F(\tau)\hat{Q}(\tau)^T\| = \rho_F(\text{tr}K, \tau)\}$$

for all  $\hat{F} \in \mathcal{G}$ , all rigid history  $\hat{Q}$  and all  $\tau \in [0,1]$ ; and that, moreover,  $\mathfrak{s}_F$  and  $\dot{\mathfrak{s}}_F$  are frame-indifferent [3], in the sense that, for all rigid histories  $\hat{Q}$ ,

$$\mathfrak{s}_{QF} = \mathfrak{s}_F \quad \text{and} \quad \dot{\mathfrak{s}}_{QF} = \dot{\mathfrak{s}}_F \quad .$$

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