

## ON A FOKKER–PLANCK EQUATION FOR WEALTH DISTRIBUTION

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(Communicated by the associate editor name)

**ABSTRACT.** We study here a Fokker–Planck equation with variable coefficient of diffusion and boundary conditions which appears in the study of the wealth distribution in a multi-agent society [2, 9, 21]. In particular, we analyze the large-time behavior of the solution, by showing that convergence to the steady state can be obtained in various norms at different rates.

**1. Introduction.** Among the mathematical models introduced in recent years to study the evolution of wealth distribution in a multi-agent society [21], Fokker–Planck type equations play an important role. Let  $f(v, t)$  denote the density of agents with personal wealth  $v \geq 0$  at time  $t \geq 0$ . The prototype of these Fokker–Planck equations reads

$$\frac{\partial f}{\partial t} = J(h) = \frac{\sigma}{2} \frac{\partial^2}{\partial v^2} (v^2 f) + \lambda \frac{\partial}{\partial v} ((v-1)f), \quad (1)$$

where  $\lambda$  and  $\sigma$  denote two positive constants related to essential properties of the trade rules of the agents. Equation (1) has been first derived by Bouchaud and Mezard [2] through a mean field limit procedure applied to a stochastic dynamical equation for the wealth density. The same equation was subsequently obtained by one of the present authors with Cordier and Pareschi [9] via an asymptotic procedure from a Boltzmann-type kinetic model for trading agents. This procedure also furnished the existence (without uniqueness) of a weak solution to equation (1).

One of the main features of equation (1) is that it possesses a unique stationary solution of unit mass, given by the (inverse)  $\Gamma$ -like distribution [2, 9]

$$f_\infty(v) = \frac{(\mu-1)^\mu \exp\left(-\frac{\mu-1}{v}\right)}{\Gamma(\mu) v^{1+\mu}}, \quad (2)$$

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2010 *Mathematics Subject Classification.* Primary: 91D10; Secondary: 35Q84, 82B21, 94A17.  
*Key words and phrases.* Wealth distribution; kinetic theory; Fokker–Planck equations, large-time behavior.

This work has been written within the activities of the National Group of Mathematical Physics (GNFM) of INdAM (National Institute of High Mathematics), and partially supported by the MIUR-PRIN Grant 2015PA5MP7 “Calculus of Variations”.

where

$$\mu = 1 + 2\frac{\lambda}{\sigma} > 1.$$

This stationary distribution, as predicted by the analysis of the Italian economist Vilfredo Pareto [22], exhibits a power-law tail for large values of the wealth variable.

Equation (1) differs from the classical Fokker–Planck equation in two important points. First, the domain of the wealth variable  $v$  takes only values in  $\mathbb{R}_+$ . Second, the coefficient of diffusion depends on the wealth variable. This makes the analysis of the large-time behavior of the solution to equation (1) very different from the analogous one studied in [23] for the classical Fokker–Planck equation.

Indeed, Fokker–Planck equations with variable coefficients and in presence of boundary conditions have been rarely studied. Maybe the first result in this direction can be found in a paper by Feller [12], who treated the case  $v \in \mathbb{R}_+$  and coefficient of diffusion  $v$ , with a general drift term (cf. also the book [13] for a general view about boundary conditions for diffusion equations). In particular, the importance of the boundary conditions has been shown in [12] to be related to the action of the drift term.

More recently, Fokker–Planck type equations with almost general coefficient of diffusions have been studied by Le Bris and Lions in [18]. Unlikely, their analysis does not apply to equation (1).

As far as the large-time behavior is concerned, the main argument in the standard Fokker–Planck equation is to resort to entropy decay, and to logarithmic Sobolev inequalities [1]. However, as discussed in [19], this type of inequalities do not seem available in presence of variable diffusion coefficients.

In the following, we will try to give a satisfactory answer to some of the open questions. As we shall see, various properties of the solution to equation (1) can be extracted from the limiting relationship between the Fokker–Planck description and its kinetic level, given by the bilinear Boltzmann-type equation introduced in [9]. We will discuss this aspect in Section 2, by means of a detailed Fourier analysis. In particular, we will show that, at least for some range of the parameters  $\lambda$  and  $\sigma$ , the Fokker–Planck equation (1) can be rigorously obtained in the asymptotic limit procedure known as quasi-invariant trade limit. Then, convergence to equilibrium will be discussed in Section 3. The essential argument here will be to resort to an inequality of Chernoff type [8, 17], that allows to prove convergence with exponential rate in the case of initial data sufficiently close to the steady state, and to a rate at least  $1/t$  for a large class of initial data.

## 2. Kinetic model and Fokker–Planck equation.

**2.1. Main properties of the Fokker–Planck equation.** To start with the analysis of the initial value problem for the Fokker–Planck equation (1), it is essential to consider, together with a suitable decay of the solution at infinity, physical boundary conditions at the point  $v = 0$ . Let  $\phi(v)$  be a smooth function, bounded at  $v = 0$ . Then, a simple computation shows that

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}_+} \phi(v) f(v, t) dv &= \int_{\mathbb{R}_+} \left[ \frac{\sigma}{2} v^2 \phi''(v) - \lambda(v-1)\phi'(v) \right] f(v, t) dv + \\ &\left[ \frac{\sigma}{2} \left( \phi(v) \frac{\partial}{\partial v} (v^2 f(v, t)) - v^2 \phi'(v) f(v, t) \right) + \lambda \phi(v) (v-1) f(v, t) \right]_0^\infty. \end{aligned}$$

While the vanishing of the boundary term at infinity follows by choosing initial data with a smooth and rapid decay, at the boundary  $v = 0$  it is required that

$$v^2 f(v, t)|_{v=0} = 0, \quad t > 0 \quad (3)$$

and

$$\lambda(v-1)f(v, t) + \frac{\sigma}{2} \frac{\partial}{\partial v} (v^2 f(v, t)) \Big|_{v=0} = 0, \quad t > 0. \quad (4)$$

Condition (3) is automatically satisfied for a sufficiently regular density  $f$ . On the contrary condition (4) requires an exact balance between the so-called advective and diffusive fluxes on the boundary  $v = 0$ . This condition is usually referred to as the *no-flux* boundary condition. If both conditions (3) and (4) hold, we can pass from equation (1) to its weak form, given by

$$\frac{d}{dt} \int_{\mathbb{R}_+} \phi(v) f(v, t) dv = (\phi, J(f)) = \int_{\mathbb{R}_+} \left[ \frac{\sigma}{2} v^2 \phi''(v) - \lambda(v-1) \phi'(v) \right] f(v, t) dv. \quad (5)$$

By choosing  $\phi(v) = 1$ ,  $v$  shows that the solution to (1) satisfies

$$\frac{d}{dt} \int_{\mathbb{R}_+} f(v, t) dv = 0, \quad \frac{d}{dt} \int_{\mathbb{R}_+} v f(v, t) dv = \lambda \left( - \int_{\mathbb{R}_+} v f(v, t) dv + \int_{\mathbb{R}_+} f(v, t) dv \right).$$

Therefore, if the (nonnegative) initial value  $\varphi(v)$  of equation (1) is a density function satisfying the normalization conditions

$$\int_{\mathbb{R}_+} \varphi(v) dv = 1; \quad \int_{\mathbb{R}_+} v \varphi(v) dv = 1 \quad (6)$$

the solution to (1) still satisfies conditions (6). In other words, if the initial datum is a probability density with unit mean, then the solution at any subsequent time remains a probability density with unit mean. For  $n \in \mathbb{N}_+$  let us define

$$M_n(t) = \int_{\mathbb{R}_+} v^n f(v, t) dv.$$

An elementary computation shows that, if  $\varphi$  satisfies conditions (6) and its second moment is bounded, the second moment of the solution follows the law

$$\frac{d}{dt} M_2(t) = (\sigma - 2\lambda) M_2(t) + 2\lambda. \quad (7)$$

Hence, the value of the second moment stays bounded when  $\sigma < 2\lambda$ , while it diverges in the opposite case. In the former case, solving equation (7) we obtain

$$M_2(t) = e^{(\sigma-2\lambda)t} \left( M_2(0) + \frac{2\lambda}{\sigma-2\lambda} \right) + \frac{2\lambda}{2\lambda-\sigma}, \quad (8)$$

which implies

$$\lim_{t \rightarrow \infty} M_2(t) = \frac{2\lambda}{2\lambda - \sigma}.$$

It is clear that the principal moments of the solution to the Fokker-Planck equation can be obtained recursively, and explicitly evaluated at the price of increasing length of computations. Since it will be useful in the following, we evaluate here the third moment  $M_3(t)$ . We obtain

$$\frac{d}{dt} M_3(t) = 3(\sigma - \lambda) M_3(t) + 3\lambda M_2(t).$$

Then, if the initial density  $\varphi(v)$  has the third moment bounded, the evolution law for  $M_3(t)$  is given by

$$M_3(t) = e^{3(\sigma-\lambda)t} \left\{ M_3(0) + 3\lambda \int_0^t e^{-3(\sigma-\lambda)r} M_2(r) dr \right\}. \quad (9)$$

Using (8), the third moment is evaluated as

$$\begin{aligned} M_3(t) &= M_3(0)e^{3(\sigma-\lambda)t} + \frac{2\lambda^2}{(\sigma-2\lambda)(\lambda-\sigma)} \left(1 - e^{3(\sigma-\lambda)t}\right) \\ &+ \left( \frac{3\lambda}{(\lambda-2\sigma)} M_2(0) + \frac{6\lambda^2}{(\sigma-2\lambda)(\lambda-2\sigma)} \right) \left( e^{(\sigma-2\lambda)t} - e^{3(\sigma-\lambda)t} \right). \end{aligned} \quad (10)$$

Therefore the third moment is uniformly bounded in time if  $\sigma < \lambda$  and it grows to  $+\infty$  in the opposite case.

Last, choosing  $\phi(v) = e^{-i\xi v}$  we obtain the Fourier transformed version of the Fokker–Planck equation (1)

$$\frac{\partial}{\partial t} \widehat{f}(\xi, t) = \widehat{J}(\widehat{f}) = \frac{\sigma}{2} \xi^2 \frac{\partial^2}{\partial \xi^2} \widehat{f}(\xi, t) - \lambda \xi \frac{\partial}{\partial \xi} \widehat{f}(\xi, t) - i\lambda \xi \widehat{f}(\xi, t), \quad (11)$$

where, as usual  $\widehat{g}(\xi)$  denotes the Fourier transform of  $g(v)$ ,  $v \in \mathbb{R}_+$ . In this case

$$\widehat{g}(\xi) = \int_{\mathbb{R}_+} e^{-i\xi v} g(v) dv.$$

**2.2. The kinetic model.** The basic model discussed in this section has been introduced in [9] within the framework of classical models of wealth distribution in economy.

As shown in [9], the Fokker–Planck equation (1) is strongly related to a bilinear kinetic model of Boltzmann type, modelling the evolution of wealth in a multi-agent society in which agents interact through binary trades [21]. This model belongs to a class of models in which the interacting agents are indistinguishable. The agent's *state* at any instant of time  $t \geq 0$  is completely characterized by his current wealth  $v \geq 0$  [10, 11]. When two agents encounter in a trade, their *pre-trade wealths*  $v, w$  change into the *post-trade wealths*  $v^*, w^*$  according to the rule [5, 6, 7]

$$v^* = p_1 v + q_1 w, \quad w^* = q_2 v + p_2 w.$$

The *interaction coefficients*  $p_i$  and  $q_i$  are non-negative random variables. While  $q_1$  denotes the fraction of the second agent's wealth transferred to the first agent, the difference  $p_1 - q_2$  is the relative gain (or loss) of wealth of the first agent due to market risks. It is usually assumed that  $p_i$  and  $q_i$  have fixed laws, which are independent of  $v$  and  $w$ , and of time. This means that the amount of wealth an agent contributes to a trade is (on the average) proportional to the respective agent's wealth.

In [9] the trade has been modelled to include the idea that wealth changes hands for a specific reason: one agent intends to *invest* his wealth in some asset, property etc. in possession of his trade partner. Typically, such investments bear some risk, and either provide the buyer with some additional wealth, or lead to the loss of wealth in a non-deterministic way. An easy realization of this idea consists in coupling the saving propensity parameter [5, 6] with some *risky investment* that

yields an immediate gain or loss proportional to the current wealth of the investing agent. The interactions rules for this model are obtained by fixing

$$\begin{aligned} p_1 &= 1 - \varepsilon\lambda + \eta_\varepsilon, & q_1 &= \varepsilon\lambda \\ p_2 &= \varepsilon\lambda, & q_2 &= 1 - \varepsilon\lambda + \tilde{\eta}_\varepsilon, \end{aligned} \quad (12)$$

where  $0 \leq \lambda \leq 1$  is the parameter which identifies the saving propensity, namely the intuitive behavior which prevents the agent to put in a single trade the whole amount of his money, while  $\varepsilon$  is a small positive parameter, which measures the quantity of money exchanged in a single trade. The coefficients  $\eta_\varepsilon, \tilde{\eta}_\varepsilon$  are independent and identically distributed random parameters, such that always  $\eta_\varepsilon, \tilde{\eta}_\varepsilon \geq \varepsilon\lambda - 1$ . This clearly implies  $v^*, w^* \geq 0$ . Therefore, the collision rule in [9] reads

$$\begin{aligned} v^* &= (1 - \varepsilon\lambda)v + \varepsilon\lambda w + \eta_\varepsilon v, \\ w^* &= (1 - \varepsilon\lambda)w + \varepsilon\lambda v + \tilde{\eta}_\varepsilon w. \end{aligned} \quad (13)$$

**Remark 1.** In the rest of the paper, the mean value of a random quantity  $\theta$  will be denoted by  $\langle \theta \rangle$ . A simple way to characterize the  $\varepsilon$ -dependence of the random parameters is to define  $\eta_\varepsilon$  and  $\tilde{\eta}_\varepsilon$  as independent copies of a random variable  $\eta$  with finite variance  $\sigma$ , with  $\eta_\varepsilon = \tilde{\eta}_\varepsilon = \sqrt{\varepsilon}\eta$ . If the random parameters are even, so that  $\langle \eta_\varepsilon \rangle = \langle \tilde{\eta}_\varepsilon \rangle = 0$

$$\langle v^* + w^* \rangle = (1 + \langle \eta_\varepsilon \rangle)v + (1 + \langle \tilde{\eta}_\varepsilon \rangle)w = v + w, \quad (14)$$

implying conservation of the average wealth. In the remaining cases, it is immediately seen that the mean wealth is not preserved, but it increases or decreases exponentially (see the computations in [9]). Various specific choices for the random parameters have been discussed in [20]. Note that, when  $\langle \eta \rangle = 0$ ,  $\langle \eta_\varepsilon^2 \rangle = \langle \tilde{\eta}_\varepsilon^2 \rangle = \varepsilon\sigma$ .

Owing to classical arguments of kinetic theory [21], it has been shown in [9] that the evolution of the wealth density consequent to the binary interactions (13) obeys to a Boltzmann-type equation. To outline the dependence on  $\varepsilon$ , let us denote with  $h_\varepsilon(v, \tau)$  the distribution of the agents wealth  $v \geq 0$  at time  $\tau > 0$ . Then, the equation for the evolution of  $h_\varepsilon$  can be fruitfully written in weak form. It corresponds to say that, for any smooth function  $\phi$ ,  $h_\varepsilon$  satisfies the equation

$$\begin{aligned} \frac{d}{d\tau} \int_{\mathbb{R}_+} \phi(v) h_\varepsilon(v, \tau) dv = \\ \frac{1}{2} \left\langle \int_{\mathbb{R}_+ \times \mathbb{R}_+} h_\varepsilon(v, \tau) h_\varepsilon(w, \tau) (\phi(v^*) + \phi(w^*) - \phi(v) - \phi(w)) dv dw \right\rangle. \end{aligned} \quad (15)$$

Existence and uniqueness of the solution to equation (15) has been proven in [20]. We will detail later on some of these results for their connection with the Fokker-Planck equation (1). The weak form (15) allows to evaluate moments of the solution in a closed form. The choice  $\phi(v) = 1$  immediately gives mass conservation. In addition, if  $\phi(v) = v$ , in view of (14) one obtains that the mean value of the solution is preserved in time. Therefore, if the initial value satisfies the normalization conditions (6) it follows that the solution  $h_\varepsilon(v, \tau)$  still satisfies the same conditions at any subsequent time  $\tau > 0$ .

Let us choose now  $\phi(v) = v^2$ . A simple computation gives

$$\langle v^{*2} + w^{*2} - v^2 - w^2 \rangle = 2(\varepsilon^2\lambda^2 - \varepsilon\lambda)(v - w)^2 + \varepsilon\sigma(v^2 + w^2).$$

Therefore

$$\frac{d}{d\tau} \int_{\mathbb{R}_+} v^2 h_\varepsilon(v, t) dv = \varepsilon [\sigma - 2(\lambda - \varepsilon\lambda^2)] \int_{\mathbb{R}_+} v^2 h_\varepsilon(v, \tau) dv + 2\varepsilon(\lambda - \varepsilon\lambda^2). \quad (16)$$

The evolution law of the second moment of  $h_\varepsilon(v, \tau)$  depends explicitly on  $\varepsilon$ , and clearly changes with  $\varepsilon$ . This is due to the fact that changing the value of  $\varepsilon$  in the binary collision (13) we change the quantity of wealth which is involved into the trade. In the limit case  $\varepsilon \rightarrow 0$ , we have a trade in which the post interaction wealths are left unchanged. This suggests to scale time in such a way to maintain an effective law of evolution of the second moment even in the limit  $\varepsilon \rightarrow 0$ . This can be easily done by setting  $t = \varepsilon\tau$ , while  $h_\varepsilon(v, \tau) = f_\varepsilon(v, t)$  [9]. One then obtains that  $f_\varepsilon(v, t)$  satisfies

$$\frac{d}{dt} \int_{\mathbb{R}_+} \phi(v) f_\varepsilon(v, t) dv = (\phi, Q_\varepsilon(f_\varepsilon, f_\varepsilon)), \quad (17)$$

where we defined

$$(\phi, Q_\varepsilon(f_\varepsilon, f_\varepsilon)) = \frac{1}{2\varepsilon} \left\langle \int_{\mathbb{R}_+ \times \mathbb{R}_+} f_\varepsilon(v, t) f_\varepsilon(w, t) (\phi(v^*) + \phi(w^*) - \phi(v) - \phi(w)) dv dw \right\rangle. \quad (18)$$

In this case, (16) changes into

$$\frac{d}{dt} \int_{\mathbb{R}_+} v^2 f_\varepsilon(v, t) dv = [\sigma - (\lambda - \varepsilon\lambda^2)] \int_{\mathbb{R}_+} v^2 f_\varepsilon(v, t) dv + 2(\lambda - \varepsilon\lambda^2). \quad (19)$$

**Remark 2.** Note that the conservation of the mean value is not modified by this scaling. However, in (19) the dependence on  $\varepsilon$  remains in the factor  $\lambda - \varepsilon\lambda^2$ . One can easily eliminate this dependence by choosing in (13) a different value ( $\varepsilon$ -dependent) of the saving propensity  $\lambda$ . This can be obtained by the choice

$$\tilde{\lambda} = \tilde{\lambda}(\varepsilon) = \frac{2\lambda}{1 + \sqrt{1 - 4\varepsilon\lambda}}, \quad (20)$$

which is such that  $\tilde{\lambda} - \varepsilon\tilde{\lambda}^2 = \lambda$ . Moreover

$$\tilde{\lambda} > \lambda, \quad \lim_{\varepsilon \rightarrow 0} \tilde{\lambda} = \lambda.$$

Clearly, definition (20) requires to choose  $\varepsilon$  small enough.

**2.3. The invariant trade limit of the Boltzmann equation.** The close relation between the kinetic equation (17) and the Fokker–Planck equation (5) has been outlined in [9], where it was shown that in the limit  $\varepsilon \rightarrow 0$  a subsequence of solutions  $f_\varepsilon(v, t)$  to (17) converges to  $f(v, t)$ , solution of (5). In this section we aim in improving these results.

In the rest, we will fix a time  $T > 0$ , and we will consider both equations (17) and (1) in the time interval  $0 \leq t \leq T$ . In addition, let the even random variable  $\eta$  which defines the random part of the trade possess the third moment bounded, and let us set  $\langle |\eta|^3 \rangle = \sigma_3$ . Analogously, let  $\langle |\eta| \rangle = \sigma_1$ . Using a Taylor's formula at the second order, one can write  $\phi(v^*)$  as

$$\langle \phi(v^*) - \phi(v) \rangle = \phi'(v) \langle v^* - v \rangle + \frac{1}{2} \phi''(v) \langle (v^* - v)^2 \rangle + \frac{1}{3!} \phi'''(\bar{v}) \langle (v^* - v)^3 \rangle$$

where, for some  $\alpha$  such that  $0 \leq \alpha \leq 1$ ,  $\tilde{v} = \alpha v + (1 - \alpha)v^*$ . Then, by (13) and the properties of the random variable  $\eta$  it holds

$$\begin{aligned} \langle v^* - v \rangle &= \varepsilon \lambda (w - v), \\ \langle (v^* - v)^2 \rangle &= \varepsilon^2 \lambda^2 (w - v)^2 + \varepsilon \sigma v^2, \\ \langle |v^* - v|^3 \rangle &\leq \varepsilon^{3/2} \left[ \sigma_3 v^3 + 3\varepsilon^{1/2} \sigma \lambda v^2 |w - v| + 3\varepsilon \sigma_1 \lambda^2 v (w - v)^2 + \varepsilon^{3/2} \lambda^3 |v - w|^3 \right]. \end{aligned} \quad (21)$$

In particular, (17) can be rewritten as

$$\begin{aligned} \frac{d}{dt} \int_I \phi(v) f_\varepsilon(v, t) dv = \\ \int_{\mathbb{R}_+} f_\varepsilon(v, t) \left[ \frac{\sigma}{2} v^2 \phi''(v) - \lambda(v-1) \phi'(v) \right] dv + \int_{\mathbb{R}_+} R_\varepsilon(\phi(v), t) f_\varepsilon(v, t) dv, \end{aligned} \quad (22)$$

where the last integral (the remainder) accounts for all the higher order  $\varepsilon$ -dependent terms in the expansion

$$R_\varepsilon(\phi(v), t) = \int_{\mathbb{R}_+} \left[ \frac{1}{2} \varepsilon \lambda^2 (w - v)^2 \phi''(v) + \frac{1}{\varepsilon 3!} \langle \phi'''(\tilde{v}) (v^* - v)^3 \rangle \right] f_\varepsilon(w, t) dw. \quad (23)$$

Therefore, for any given (smooth function)  $\phi(v)$  and density  $f(v)$ ,  $v \in \mathbb{R}_+$ , we have the identity

$$(\phi, Q_\varepsilon(f, f) - J(f)) = \int_{\mathbb{R}_+} R_\varepsilon(\phi(v), t) f(v, t) dv. \quad (24)$$

**Remark 3.** In reason of the fact that the solution to the kinetic model (17) satisfies conditions (6) for all times  $t \geq 0$ , it follows from (22) that

$$\int_{\mathbb{R}_+} R_\varepsilon(\phi(v)) f_\varepsilon(v, t) dv = 0$$

whenever  $\phi(v) = 1, v$ .

For  $m \in \mathbb{N}_+$ , let  $C^m(\mathbb{R}_+)$  be the set of  $m$ -times continuously differentiable functions, endowed with its natural norm  $\|\cdot\|_m$ . Then for  $f = f(v)$ ,  $v \in \mathbb{R}_+$ , let us define

$$\|f\|_m^* = \sup \{ |(\phi, f)|, \phi \in C^m(\mathbb{R}_+), \|\phi\|_m \leq 1 \}. \quad (25)$$

Let  $m \geq 3$ . Thanks to (21), whenever  $f(v)$  is a probability density with the third moment bounded

$$\|Q_\varepsilon(f, f) - J(f)\|_m^* = \sup_\phi \left| \int_{\mathbb{R}_+} R_\varepsilon(\phi(v)) f(v) dv \right| \leq \varepsilon^{1/2} C_\varepsilon(M_1(f), M_2(f), M_3(f)),$$

and

$$\lim_{\varepsilon \rightarrow 0} \|Q_\varepsilon(f, f) - J(f)\|_m^* = 0. \quad (26)$$

Let the initial datum of the Fokker-Planck equation possess moments bounded up to the order three. Thanks to (10) the moments up to the third order of the solution to the Fokker-Planck equation remain uniformly bounded in the time interval  $0 \leq t \leq T$ . Therefore, if  $f(v, t)$  is a solution to the Fokker-Planck equation (5), for  $0 \leq t \leq T$

$$\|Q_\varepsilon(f, f) - J(f)\|_m^*(t) = \sup_\phi \left| \int_{\mathbb{R}_+} R_\varepsilon(\phi(v)) f_\varepsilon(v, t) dv \right| \leq \varepsilon^{1/2} C_\varepsilon(T), \quad (27)$$

where the constant  $C_\varepsilon(T)$  depends on moments of  $f(v, t)$  up to the order three.

Let us consider a family of metrics that has been introduced in the paper [15] to study the trend to equilibrium of solutions to the space homogeneous Boltzmann equation for Maxwell molecules, and subsequently applied to a variety of problems related to kinetic models of Maxwell type. For a more detailed description, we address the interested reader to the lecture notes [4].

Given  $s > 0$  and two probability densities  $f_1$  and  $f_2$ , their Fourier based distance  $d_s(f_1, f_2)$  is given by the quantity

$$d_s(f_1, f_2) := \sup_{\xi \in \mathbb{R}^n} \frac{|\widehat{f}_1(\xi) - \widehat{f}_2(\xi)|}{|\xi|^s}. \quad (28)$$

The distance is finite, provided that  $f_1$  and  $f_2$  have the same moments up to order  $[s]$ , where, if  $s \notin \mathbb{N}_+$ ,  $[s]$  denotes the entire part of  $s$ , or up to order  $s - 1$  if  $s \in \mathbb{N}_+$ . Moreover  $d_s$  is an ideal metric. Its main properties are the following

1. Let  $X_1, X_2, X_3$ , with  $X_3$  independent of the pair  $X_1, X_2$  be random variables with probability distributions  $f_1, f_2, f_3$ . Then

$$d_s(f_1 * f_3, f_2 * f_3) \leq d_s(f_1, f_2)$$

where the symbol  $*$  denotes convolution;

2. Define for a given nonnegative constant  $a$  the dilation

$$f_a(x) = \frac{1}{a} f\left(\frac{x}{a}\right).$$

Then, given two probability densities  $f_1$  and  $f_2$ , for any nonnegative constant  $a$

$$d_s(aX_1, aX_2) = d_s(f_{1,a}, f_{2,a}) \leq a^s d_s(f_1, f_2) = a^s d_s(X_1, X_2).$$

3. Let  $d_s(f_1, f_2)$  be finite for some  $s > 0$ . Then the following interpolation property holds [4]

$$d_p(f_1, f_2) \leq 2 \left( \frac{s-p}{2p} \right)^{p/s} \frac{s}{s-p} [d_s(f_1, f_2)]^{p/s} = C_{p,s} [d_s(f_1, f_2)]^{p/s}, \quad (29)$$

for any  $0 < p < s$ .

Since the equation for the Fourier transform of the density  $f_\varepsilon(v, t)$ , solution of the kinetic equation (17) takes the form [20]

$$\frac{\partial}{\partial t} \widehat{f}_\varepsilon(\xi, t) = \widehat{Q}_\varepsilon(f_\varepsilon, f_\varepsilon)(\xi, t) = \frac{1}{\varepsilon} [\langle \widehat{f}_\varepsilon((1 - \varepsilon\lambda + \eta_\varepsilon)\xi) \rangle \widehat{f}_\varepsilon(\varepsilon\lambda\xi) - \widehat{f}_\varepsilon(\xi, t)], \quad (30)$$

in reason of (24) we can write

$$\widehat{J}(\widehat{f}) = \widehat{Q}_\varepsilon(\widehat{f}, \widehat{f}) + \widehat{R}_\varepsilon(\widehat{f}), \quad (31)$$

where

$$\widehat{R}_\varepsilon(\widehat{f}) = \widehat{J}(\widehat{f}) - \widehat{Q}_\varepsilon(\widehat{f}, \widehat{f}).$$

As proven in [24], the metric  $d_2$  is equivalent to  $\|\cdot\|_m^*$ ,  $m \in \mathbb{N}_+$ , that is

$$d_2(f, g) \rightarrow 0 \quad \text{if and only if} \quad \|f - g\|_m^* \rightarrow 0.$$

Let the initial datum of the Fokker–Planck equation possess moments bounded up to the order three. Thanks to (27), if  $f(v, t)$  is the corresponding solution to the



Fokker–Planck equation, in the time interval  $0 \leq t \leq T$ , for a suitable constant  $D_\varepsilon$

$$d_2 \left( \widehat{J}(\widehat{f}(t)), \widehat{Q}_\varepsilon(\widehat{f}(t), \widehat{f}(t)) \right) = \sup_\xi \frac{1}{|\xi|^2} |\widehat{R}_\varepsilon(\widehat{f})(t)| \leq r_\varepsilon(T), \quad (32)$$

where

$$\lim_{\varepsilon \rightarrow 0} r_\varepsilon(T) = 0.$$

Using the expression (31) for the Fokker–Planck operator in (5), we obtain that the difference between the solution  $f_\varepsilon$  of (30) and the solution  $f$  to the Fokker–Planck equation (11) satisfies

$$\begin{aligned} & \frac{\partial}{\partial t} \left( \frac{\widehat{f}_\varepsilon - \widehat{f}}{|\xi|^2} \right) + \frac{\widehat{f}_\varepsilon - \widehat{f}}{|\xi|^2} = \\ & \frac{1}{\varepsilon} \frac{\widehat{f}_\varepsilon((1 - \lambda + \eta)\xi) \widehat{f}_\varepsilon(\lambda\xi) - \widehat{f}((1 - \lambda + \eta)\xi) \widehat{f}(\lambda\xi)}{|\xi|^2} + \frac{1}{|\xi|^2} \widehat{R}_\varepsilon(\widehat{f}). \end{aligned} \quad (33)$$

Let  $f_\varepsilon(v, t)$  and  $f(v, t)$  solutions departing from initial value  $\tilde{f}_0$  and  $f_0$  satisfying conditions (6) and such that their distance  $d_2(\tilde{f}_0, f_0)$  is finite. Let us set

$$h_\varepsilon = \frac{\widehat{f}_\varepsilon - \widehat{f}}{|\xi|^2},$$

which is such that  $\|h_\varepsilon(\cdot, t)\|_\infty = d_2(f_\varepsilon, f)$ . Since  $|\widehat{f}_\varepsilon| = |\widehat{f}| = 1$  we obtain for any  $0 \leq t \leq T$

$$\begin{aligned} \left| \frac{\partial}{\partial t} h_\varepsilon + \frac{1}{\varepsilon} h_\varepsilon \right| & \leq \frac{1}{\varepsilon} \left| \frac{\widehat{f}_\varepsilon((1 - \lambda\varepsilon + \eta_\varepsilon)\xi) - \widehat{f}((1 - \lambda\varepsilon + \eta_\varepsilon)\xi)}{\langle |1 - \lambda\varepsilon + \eta_\varepsilon\xi|^2 \rangle} \right| \langle |1 - \lambda\varepsilon + \eta_\varepsilon|^2 \rangle \\ & + \frac{1}{\varepsilon} \left| \frac{\widehat{f}_\varepsilon(\lambda\xi) - \widehat{f}(\lambda\xi)}{|\lambda\varepsilon\xi|^2} \right| (\lambda\varepsilon)^2 + \frac{1}{|\xi|^2} |\widehat{R}_\varepsilon(\widehat{f})| \\ & \leq \frac{1}{\varepsilon} \|h_\varepsilon(\cdot, t)\|_\infty [\langle (1 - \lambda\varepsilon + \eta_\varepsilon)^2 \rangle + (\lambda\varepsilon)^2] + r_\varepsilon(T). \end{aligned} \quad (34)$$

Consider that, if  $\sigma < 2\lambda$ , for  $\varepsilon$  sufficiently small

$$\langle (1 - \lambda\varepsilon + \eta_\varepsilon)^2 \rangle + (\lambda\varepsilon)^2 = 1 + \varepsilon [\sigma - 2\lambda(1 - \lambda\varepsilon)] \leq 1. \quad (35)$$

If this is the case,  $h_\varepsilon(t)$  satisfies

$$\left| \frac{\partial}{\partial t} h_\varepsilon + \frac{1}{\varepsilon} h_\varepsilon \right| \leq \frac{1}{\varepsilon} \|h_\varepsilon(\cdot, t)\|_\infty + r_\varepsilon(T) \quad (36)$$

Proceeding as in [24], Theorem 5, we conclude by Gronwall inequality that (36) implies

$$\|h_\varepsilon(\cdot, t)\|_\infty \leq \|h_\varepsilon(\cdot, 0)\|_\infty + r_\varepsilon(T) t. \quad (37)$$

Letting  $\varepsilon$  going to 0 we obtain

$$\lim_{\varepsilon \rightarrow 0} d_2(f_\varepsilon, f)(t) \leq d_2(\tilde{f}_0, f).$$

Hence, if we start with the same initial value  $\tilde{f}_0 = f_0$ ,  $\lim_{\varepsilon \rightarrow 0} d_2(f_\varepsilon, f)(t) = 0$  for  $0 \leq t \leq T$ .

Analogous reasoning can be used to prove uniqueness of the solution to the Fokker–Planck equation (5). By resorting to the approximation (31) of the Fokker–Planck operator in (5), we obtain that the difference between two solutions  $f(v, t)$  and  $g(v, t)$  to the Fokker–Planck equation (11), for any given small value of  $\varepsilon$  satisfies

$$\begin{aligned} \frac{\partial}{\partial t} \left( \frac{\widehat{f} - \widehat{g}}{|\xi|^2} \right) + \frac{\widehat{f} - \widehat{g}}{|\xi|^2} = \\ \frac{1}{\varepsilon} \frac{\widehat{f}((1 - \lambda + \eta)\xi)\widehat{f}(\lambda\xi) - \widehat{g}((1 - \lambda + \eta)\xi)\widehat{g}(\lambda\xi)}{|\xi|^2} - \frac{1}{|\xi|^2} \widehat{R}_\varepsilon(\widehat{f}) + \frac{1}{|\xi|^2} \widehat{R}_\varepsilon(\widehat{g}). \end{aligned} \quad (38)$$

Let  $f(v, t)$  and  $g(v, t)$  solutions departing from initial value  $f_0$  and  $g_0$  satisfying conditions (6) and such that their distance  $d_2(f_0, g_0)$  is finite. If the third moments of  $f_0$  and  $g_0$  are finite, proceeding as before we conclude with the bound

$$d_2(f, g)(t) \leq d_2(f_0, g_0),$$

that clearly implies uniqueness of the solution. We can resume the previous result in the following.

**Theorem 2.1.** *Let  $f_0(v)$  be a probability density in  $\mathbb{R}_+$  satisfying conditions (6), and such that its third moment is finite. Assume moreover that the random part  $\eta$  in the binary collision (13) is even, and such that  $\langle |\eta|^3 \rangle$  is finite. Then, for any finite time  $T$ , as  $\varepsilon \rightarrow 0$ , the unique solution  $f_\varepsilon(v, t)$  to the kinetic model (17) with initial datum  $f_0$  converges to the solution  $f(v, t)$  of the Fokker–Planck equation (5) with the same initial datum  $f_0$ , and*

$$d_2(f_\varepsilon, f)(t) \rightarrow 0, \quad 0 < t \leq T.$$

Moreover, the solution to the Fokker–Planck equation is unique.

**2.4. A regularity result.** Theorem 2.1 shows that, for any given datum with a suitable decay at infinity, the Fokker–Planck equation (5) possesses a unique solution. However, if the initial datum belongs to  $\dot{H}_p(\mathbb{R}_+)$ , we can conclude that the solution maintains the same regularity for any subsequent positive time. Let  $\widehat{f}(\xi, t) = a(\xi, t) + ib(\xi, t)$ . Starting from the Fourier transform version of the Fokker–Planck equation (11), let us split it into the real and imaginary part. We obtain

$$\begin{aligned} \frac{\partial}{\partial t} a(\xi, t) &= \frac{\sigma}{2} \xi^2 \frac{\partial^2}{\partial \xi^2} a(\xi, t) - \lambda \xi \frac{\partial}{\partial \xi} a(\xi, t) + \lambda \xi b(\xi, t), \\ \frac{\partial}{\partial t} b(\xi, t) &= \frac{\sigma}{2} \xi^2 \frac{\partial^2}{\partial \xi^2} b(\xi, t) - \lambda \xi \frac{\partial}{\partial \xi} b(\xi, t) - \lambda \xi a(\xi, t). \end{aligned} \quad (39)$$

Multiplying equations (39) respectively by  $2a$  and  $2b$  and summing up, we get

$$\frac{\partial}{\partial t} |\widehat{f}|^2 = \sigma \xi^2 \left[ a \frac{\partial^2}{\partial \xi^2} a + b \frac{\partial^2}{\partial \xi^2} b \right] - \lambda \xi \frac{\partial |\widehat{f}|^2}{\partial \xi}. \quad (40)$$

Hence, multiplying by  $|\xi|^p$  and integrating over  $\mathbb{R}$  with respect to  $\xi$ , we obtain the evolution equation of the  $\dot{H}_{p/2}$ -norm of  $f(v, t)$ , where, as usual, the homogeneous Sobolev space  $\dot{H}_s$ , for  $s > 0$  is defined by the norm

$$\|f\|_{\dot{H}_s} = \int_{\mathbb{R}} |\xi|^{2s} |\widehat{f}|^2(\xi) d\xi.$$

We have

$$\frac{\partial}{\partial t} \int_{\mathbb{R}} |\xi|^p |\widehat{f}|^2 d\xi = \sigma \int_{\mathbb{R}} |\xi|^{2+p} \left[ a \frac{\partial^2}{\partial \xi^2} a + b \frac{\partial^2}{\partial \xi^2} b \right] d\xi - \lambda \int_{\mathbb{R}} \xi |\xi|^p \frac{\partial |\widehat{f}|^2}{\partial \xi} d\xi. \quad (41)$$

Integrating by parts the two integrals, it results

$$\frac{\partial}{\partial t} \int_{\mathbb{R}} |\xi|^p |\widehat{f}|^2 d\xi = (p+1) \left[ \frac{\sigma}{2} (p+2) + \lambda \right] \int_{\mathbb{R}} |\xi|^p |\widehat{f}|^2 d\xi - \sigma \int_{\mathbb{R}} |\xi|^{2+p} \left[ \left| \frac{\partial}{\partial \xi} a \right|^2 + \left| \frac{\partial}{\partial \xi} b \right|^2 \right] d\xi. \quad (42)$$

The last integral in (42) can be estimated from below as follows

$$\int_{\mathbb{R}} |\xi|^{2+p} \left[ \left| \frac{\partial}{\partial \xi} a \right|^2 + \left| \frac{\partial}{\partial \xi} b \right|^2 \right] d\xi \geq \frac{(p+1)^2}{2} \int_{\mathbb{R}} |\xi|^p |\widehat{f}|^2 d\xi.$$

Indeed, for all  $\mu > 0$  it holds

$$0 \leq \int_{\mathbb{R}} |\xi|^p \left( \xi \frac{\partial a}{\partial \xi} + \mu a \right)^2 d\xi = \int_{\mathbb{R}} |\xi|^{2+p} \left| \frac{\partial a}{\partial \xi} \right|^2 d\xi + (\mu^2 - \mu(p+1)) \int_{\mathbb{R}} |\xi|^{2+p} a^2 d\xi.$$

Optimizing over  $\mu$  we get

$$\int_{\mathbb{R}} |\xi|^{2+p} \left| \frac{\partial a}{\partial \xi} \right|^2 d\xi \geq \frac{(p+1)^2}{4} \int_{\mathbb{R}} |\xi|^{2+p} a^2 d\xi.$$

An analogous computation works for  $b$ . Hence

$$\frac{\partial}{\partial t} \int_{\mathbb{R}} |\widehat{f}|^2 |\xi|^p d\xi \leq \frac{p+1}{2} \left[ \sigma + 2\lambda \right] \int_{\mathbb{R}} |\xi|^p |\widehat{f}|^2 d\xi. \quad (43)$$

The inequality (43) implies that if the initial data has finite  $\dot{H}_p$ -norm, then for all finite  $t > 0$ , the  $\dot{H}_p$ -norm of the solution remains finite and it grows up to  $+\infty$  for  $t \rightarrow +\infty$ .

### 3. Convergence to equilibrium.

**3.1.  $L_1$ -convergence.** In this section, we will be concerned with the study of the large-time behaviour of the Fokker-Planck equation (1). The main argument here will be the study of the time evolution of various Lyapunov functionals, starting from Shannon entropy of the solution  $f(v, t)$  relative to the steady state  $f_\infty(v)$ . We recall that the relative Shannon entropy of the two probability density functions  $f$  and  $g$  is defined by the formula

$$H(f, g) = \int_I f(v) \log \frac{f(v)}{g(v)} dv. \quad (44)$$

As a first step in this analysis, we will introduce in the following equivalent formulations of the Fokker-Planck equation, that result to be very useful to justify rigorously the behaviour of these Lyapunov functionals.

Indeed, equation (1) admits many equivalent formulations, each of them well adapted to different purposes [14]. To this extent, recall that the equilibrium distribution  $f_\infty$  defined in (2) satisfies

$$\frac{\partial}{\partial v} (v^2 f_\infty) + (v-1) f_\infty = 0, \quad (45)$$

or, equivalently

$$\frac{\partial}{\partial v} \log(v^2 f_\infty) = -\frac{v-1}{v^2}. \quad (46)$$

Then, for  $v > 0$

$$\begin{aligned} \frac{\partial}{\partial v}(v^2 f) + (v-1)f &= v^2 f \left( \frac{\partial}{\partial v} \log(v^2 f) + \frac{v-1}{v^2} \right) = \\ v^2 f \left( \frac{\partial}{\partial v} \log(v^2 f) - \frac{\partial}{\partial v} \log(v^2 f_\infty) \right) &= v^2 f \frac{\partial}{\partial v} \log \frac{f}{f_\infty} = v^2 f_\infty \frac{\partial}{\partial v} \frac{f}{f_\infty}. \end{aligned}$$

Hence, we can write the Fokker–Planck equation (1) in the equivalent form

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v} \left[ v^2 f \frac{\partial}{\partial v} \log \frac{f}{f_\infty} \right], \quad (47)$$

which enlightens the role of the logarithm of the quotient  $f/f_\infty$ , and

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v} \left[ v^2 f_\infty \frac{\partial}{\partial v} \frac{f}{f_\infty} \right]. \quad (48)$$

In particular, owing to (45), the form (48) allows us to obtain the evolution equation for the quotient  $F = f/f_\infty$ . Indeed

$$\begin{aligned} \frac{\partial f}{\partial t} = f_\infty \frac{\partial F}{\partial t} &= v^2 f_\infty \frac{\partial^2}{\partial v^2} \frac{f}{f_\infty} + \frac{\partial}{\partial v}(v^2 f_\infty) \frac{\partial}{\partial v} \frac{f}{f_\infty} = \\ &= v^2 f_\infty \frac{\partial^2 F}{\partial v^2} - (v-1) f_\infty \frac{\partial F}{\partial v}, \end{aligned}$$

which shows that  $F$  satisfies the equation

$$\frac{\partial F}{\partial t} = v^2 \frac{\partial^2 F}{\partial v^2} - (v-1) \frac{\partial F}{\partial v}. \quad (49)$$

If mass conservation is imposed on equation (1), we obtain at  $v = 0$  the boundary conditions (3) and (4). In analogous way, the boundary conditions of the equivalent form (49) are now written in the form

$$v^2 f_\infty(v) F(v, t) \Big|_{v=0} = 0, \quad (50)$$

and

$$v^2 f_\infty(v) \frac{\partial}{\partial v} \frac{f(v, t)}{f_\infty(v)} \Big|_{v=0} = v^2 f_\infty(v) \frac{\partial F(v, t)}{\partial v} \Big|_{v=0} = 0. \quad (51)$$

In view of the decay property at  $v = 0$  of the steady state  $f_\infty$ , the boundary conditions (50) and (51) are satisfied any time the solution to equation (49) is bounded together with its derivative at zero.

To proceed, and to avoid inessential difficulties, for any given initial density  $f_0(v)$  and positive constant  $\delta \ll 1$ , let us consider a regular approximation  $f_0^\delta(v)$  satisfying the conditions

$$f_0^\delta(v) = f_\infty(v) \quad \text{if } v \leq \delta \text{ and } v \geq 1/\delta, \quad \delta^2 \leq f_0^\delta(v) \leq 1/\delta^2 \quad \text{if } \delta \leq v \leq 1/\delta, \quad (52)$$

while

$$\int_{\mathbb{R}_+} f_0^\delta(v) dv = 1. \quad (53)$$

Then, in a time interval  $(0, T)$  the (unique) solution  $F^\delta(v, t)$  of the initial-boundary value problem for equation (49) corresponding to the initial value  $F_0^\delta = f_0^\delta/f_\infty$  is such that

$$F^\delta(v, t) = 1 \quad \text{if } v \leq \delta \text{ and } v \geq 1/\delta, \quad \delta^2/\Delta_+ \leq F^\delta(v, t) \leq \delta^2/\Delta_- \text{ if } \delta \leq v \leq 1/\delta, \quad (54)$$

where we denoted

$$\Delta_+ = \max_v f_\infty(v), \quad \Delta_- = \min_{\delta \leq v \leq 1/\delta} f_\infty(v) > 0.$$

Indeed, a solution constant in the interval  $v \leq \delta$  satisfies both the boundary conditions (50) and (51), and equation (49), and converges to the right initial value as  $t \rightarrow 0$ . Analogous conclusion can be drawn in the interval  $v \geq 1/\delta$ . Consider now the solution to equation (49) in a bounded interval  $(v_-, v_+)$ , where  $v_- \leq \delta$  and  $v_+ \geq 1/\delta$ , with boundary conditions  $F(v_-, t) = 1$  and  $F(v_+, t) = 1$  for  $t \leq T$ , and initial value  $F_0^\delta = f_0^\delta/f_\infty$ . Since in this interval the coefficient of the second-order term is strictly positive, the second condition in (54) follows from the maximum principle for the solution to a uniformly parabolic equation.

The previous discussion shows that, since the initial datum  $F_0^\delta$  satisfies

$$m_\delta \leq F_0^\delta(v) \leq M_\delta \tag{55}$$

for some positive constants  $m_\delta < M_\delta$ , the same condition holds at any subsequent time  $t \leq T$ , so that

$$m_\delta \leq F^\delta(v, t) \leq M_\delta. \tag{56}$$

As remarked in [14], condition (56) allows to recover rigorously the time decay of various Lyapunov functionals. Indeed, the following holds (cf. Theorem 3.1 of [14])

**Theorem 3.1.** *Let the smooth function  $\Phi(x)$ ,  $x \in \mathbb{R}_+$  be convex. Then, if  $F(v, t)$  is the solution to equation (49) in  $\mathbb{R}_+$ , and  $c \leq F(v, t) \leq C$  for some positive constants  $c < C$ , the functional*

$$\Theta(F(t)) = \int_I f_\infty(v) \Phi(F(v, t)) dv$$

is monotonically decreasing in time, and the following equality holds

$$\frac{d}{dt} \Theta(F(t)) = -I_\Theta(F(t)), \tag{57}$$

where  $I_\Theta$  denotes the nonnegative quantity

$$I_\Theta(F(t)) = \int_{\mathbb{R}_+} v^2 f_\infty(v) \Phi''(F(v, t)) \left| \frac{\partial F(v, t)}{\partial v} \right|^2 dv. \tag{58}$$

We can couple Theorem 3.1 with the so-called Chernoff inequality with weight, recently proven in [14] (cf. Theorem 3.3). In our case, this result reads

**Theorem 3.2.** *Let  $X$  be a random variable distributed with density  $f_\infty(v)$ ,  $v \in \mathbb{R}_+$ , where the probability density function  $f_\infty$  satisfies the differential equality*

$$\frac{\partial}{\partial v} (v^2 f_\infty) + (v - 1) f_\infty = 0, \quad v \in \mathbb{R}_+. \tag{59}$$

If the function  $\phi$  is absolutely continuous on  $\mathbb{R}_+$  and  $\phi(X)$  has finite variance, then

$$\text{Var}[\phi(X)] \leq E \{ X^2 [\phi'(X)]^2 \} \tag{60}$$

with equality if and only if  $\phi(X)$  is linear in  $X$ .

Choose now  $\Phi(x) = x \log x$ ,  $x \geq 0$ . Then,  $\Theta(F^\delta(t))$  coincides with the entropy of  $f^\delta$  relative to  $f_\infty$ . If the relative entropy is finite at time  $t = 0$ , by Theorem 3.1

it decays, and its rate of decay is given by the expression

$$I(F^\delta(t)) = \int_{\mathbb{R}_+} v^2 f_\infty(v) \frac{1}{F^\delta(v,t)} \left| \frac{\partial F^\delta(v,t)}{\partial v} \right|^2 dv = 4 \int_{\mathbb{R}_+} v^2 f_\infty(v) \left| \frac{\partial \sqrt{F^\delta(v,t)}}{\partial v} \right|^2 dv. \quad (61)$$

If we apply inequality (60) with  $\phi(v) = \sqrt{F^\delta(v,t)}$  with fixed  $t > 0$  we get

$$\begin{aligned} I(F^\delta(t)) &= 4 \int_{\mathbb{R}_+} v^2 f_\infty(v) \left( \partial_v \sqrt{\frac{f^\delta(v,t)}{f_\infty(v)}} \right)^2 dv \geq \\ &4 \left( \int_{\mathbb{R}_+} \frac{f^\delta(v,t)}{f_\infty(v)} f_\infty(v) dv - \left( \int_{\mathbb{R}_+} \sqrt{\frac{f^\delta(v,t)}{f_\infty(v)}} f_\infty(v) dv \right)^2 \right) = \\ &4 \left( 1 - \left( \int_{\mathbb{R}_+} \sqrt{f^\delta(v,t) f_\infty(v)} dv \right)^2 \right). \end{aligned} \quad (62)$$

On the other hand, as remarked in [16], whenever  $f$  and  $g$  are probability density functions, the square of their Hellinger distance

$$d_H(f, g) = \left[ \int_{\mathbb{R}_+} (\sqrt{f} - \sqrt{g})^2 dv \right]^{1/2} \quad (63)$$

satisfies

$$\begin{aligned} &= d_H(f, g)^2 = \int_{\mathbb{R}} (f(v) + g(v) - 2\sqrt{f(v)g(v)}) dv = \\ &2 \left( 1 - \int_{\mathbb{R}} \sqrt{f(v)g(v)} dv \right) \leq 2 \left( 1 - \left( \int_{\mathbb{R}} \sqrt{f(v)g(v)} dv \right)^2 \right). \end{aligned} \quad (64)$$

The last inequality in (64) follows by Cauchy–Schwartz inequality. Finally, for  $t > 0$

$$I(F^\delta(t)) \geq 2d_H(f^\delta(t), f_\infty)^2, \quad t > 0. \quad (65)$$

that implies the differential inequality

$$\frac{d}{dt} H(f^\delta(t)|f_\infty) \leq -2d_H(f^\delta(t), f_\infty)^2, \quad (66)$$

and, consequently, the bound

$$\int_0^\infty d_H(f^\delta(t), f_\infty)^2 dt \leq \frac{1}{2} H(f_0^\delta|f_\infty). \quad (67)$$

Now, let us apply again Theorem 3.1 to the convex function  $\phi(x) = (\sqrt{x} - 1)^2$ . In this case  $\Theta(F^\delta(t))$  coincides with the square of the Hellinger distance (63) between  $f^\delta$  and  $f_\infty$ , which in consequence of (57) is shown to decay in time.

Therefore, inequality (67) coupled with the time decay of the Hellinger distance shows that for large times

$$d_H(f^\delta(t), f_\infty)^2 = o(1/t). \quad (68)$$

Note that to obtain the decay we need the boundedness of the relative entropy  $H(f_0^\delta|f_\infty)$ . Last, by Cauchy–Schwartz inequality we can bound the  $L^1$  distance

between two densities  $f$  and  $g$  in terms of the Hellinger distance  $d_H(f, g)$ . Indeed

$$\begin{aligned} \int_I |f(v) - g(v)| dv &= \int_I \left| \sqrt{f(v)} - \sqrt{g(v)} \right| \left( \sqrt{f(v)} + \sqrt{g(v)} \right) dv \\ &\leq \left( \int_I \left( \sqrt{f(v)} - \sqrt{g(v)} \right)^2 dv \right)^{\frac{1}{2}} \left( \int_I \left( \sqrt{f(v)} + \sqrt{g(v)} \right)^2 dv \right)^{\frac{1}{2}} \\ &= d_H(f, g) \left( \int_I \left( f(v) + g(v) + 2\sqrt{f(v)g(v)} \right) dv \right)^{\frac{1}{2}} \\ &= \sqrt{2} d_H(f, g) \left( 1 + \int_I \sqrt{f(v)g(v)} dv \right)^{\frac{1}{2}}. \end{aligned}$$

Therefore

$$\|f - g\|_{L^1} \leq 2d_H(f, g). \quad (69)$$

Finally, in view of (68), the  $L_1$ -distance between  $f^\delta(t)$  and  $f_\infty$  decays to zero as time goes to infinity,

$$\|f^\delta(t) - f_\infty\|_{L^1}^2 \leq o(1/t), \quad (70)$$

and

$$\int_0^\infty \|f^\delta(t) - f_\infty\|_{L^1}^2 dt \leq 2H(f_0^\delta | f_\infty). \quad (71)$$

Let us now proceed to remove the lifting of the initial value. The following lemma will be useful

**Lemma 3.3.** *Let  $f(v, t)$  be a solution of the initial-boundary value problem for the Fokker-Planck equation (1), corresponding to an initial value  $f_0(v)$  such that, as in Theorem 2.1  $|f_0(v)| \in L_1(\mathbb{R}_+)$  and  $v^3|f_0(v)| \in L_1(\mathbb{R}_+)$ . Then, the  $L_1$ -norm of  $f(v, t)$  is non-increasing for  $t \geq 0$ .*

*Proof.* For a given  $\varepsilon > 0$ , let us consider a regularized increasing approximation of the sign function  $sign_\varepsilon(z)$ , with  $z \in \mathbb{R}$ , and let us define the regularized approximation  $|f|_\varepsilon(z)$  of  $|f|(z)$  via the primitive of  $sign_\varepsilon(f)(z)$ . We now multiply equation (1) by  $sign_\varepsilon(f(t))$  to obtain, after integrating by parts

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}_+} sign_\varepsilon(f(t)) f(t) dv &= - \int_{\mathbb{R}_+} sign'_\varepsilon(f) \left[ \frac{\sigma}{2} \frac{\partial f}{\partial v} \frac{\partial(v^2 f)}{\partial v} + \lambda(v-1) f \frac{\partial f}{\partial v} \right] dv \\ &= - \int_{\mathbb{R}_+} sign'_\varepsilon(f) \frac{\sigma}{2} v^2 \left| \frac{\partial f}{\partial v} \right|^2 dv - \int_{\mathbb{R}_+} \frac{(\lambda + \sigma)v - \lambda}{2} sign'_\varepsilon(f) f \frac{\partial f}{\partial v} dv. \end{aligned} \quad (72)$$

Indeed, the border term contribution vanishes in view of condition (4). Moreover, since we have the equality

$$sign'_\varepsilon(f) f \frac{\partial f}{\partial v} = \frac{\partial}{\partial v} [f sign_\varepsilon(f) - |f|_\varepsilon]$$

after another integration by parts in the last term of the right-hand side of (72) we obtain

$$\begin{aligned} &- \int_{\mathbb{R}_+} \frac{(\lambda + \sigma)v - \lambda}{2} sign'_\varepsilon(f) f \frac{\partial f}{\partial v} dv = \\ &\frac{\lambda}{2} \left[ (f sign_\varepsilon(f) - |f|_\varepsilon)(v=0) + \int_{\mathbb{R}_+} (f sign_\varepsilon(f) - |f|_\varepsilon) dv \right], \end{aligned}$$

and this contribution, in the limit  $\varepsilon \rightarrow 0$  vanishes. Hence

$$\frac{d}{dt} \int_{\mathbb{R}_+} |f(v, t)| dv \leq 0.$$

□

In reason of lemma 3.3, for any given initial datum  $f_0(v)$  satisfying the hypotheses of theorem 2.1, and its modification (52), we have that, at any subsequent time  $t > 0$

$$\|f(v, t) - f^\delta(v, t)\|_{L_1} \leq \|f_0(v) - f_0^\delta(v)\|_{L_1}.$$

Hence, since  $f_0^\delta(v)$  converges to  $f_0(v)$  in  $L_1$ -norm and in relative entropy, letting  $\delta \rightarrow 0$  inequality (70) implies

$$\|f(t) - f_\infty\|_{L_1}^2 \leq o(1/t). \quad (73)$$

Moreover, for each finite time  $T$ , inequality (71) yields

$$\begin{aligned} \int_0^T \|f(t) - f_\infty\|_{L_1}^2 dt &\leq 2 \int_0^T \|f^\delta(t) - f_\infty\|_{L_1}^2 dt + 2 \int_0^T \|f^\delta(t) - f(t)\|_{L_1}^2 dt \leq \\ &2 \int_0^T \|f^\delta(t) - f_\infty\|_{L_1}^2 dt + 2T \|f_0^\delta - f_0\|_{L_1}^2 \leq 4H(f_0^\delta|f_\infty) + 2T \|f_0^\delta - f_0\|_{L_1}^2. \end{aligned}$$

Hence, letting  $\delta \rightarrow 0$  we obtain, for each  $T > 0$  the inequality

$$\int_0^T \|f(t) - f_\infty\|_{L_1}^2 dt \leq 4H(f_0|f_\infty). \quad (74)$$

We proved

**Theorem 3.4.** *Let  $f(v, t)$  be a solution of the initial-boundary value problem for the Fokker–Planck equation (1), corresponding to an initial density  $f_0(v)$  such that, as in Theorem 2.1  $f_0(v) \in L_1(\mathbb{R}_+)$  and  $v^3 f_0(v) \in L_1(\mathbb{R}_+)$ . Then, if the relative entropy between  $f_0$  and  $f_\infty$  is bounded,  $f(v, t)$  converges in  $L_1(\mathbb{R}_+)$  towards the steady state  $f_\infty$ , and both (73) and (74) hold.*

**3.2. Further convergence results.** The analysis of the previous section shows that the solution to the Fokker–Planck equation converges towards the stationary state in the  $L_1$ -norm for a large class of initial data, but with a polynomial rate of decay. However, stronger results of convergence can be obtained by suitably restricting the allowed initial data, or, in alternative, by relaxing the distance in which the decay holds.

Let us apply Theorem 3.1 to the convex function  $\phi(x) = (x - 1)^2$ . In this case  $\Theta(F^\delta(t))$  coincides with the weighted (with weight  $f_\infty$ )  $L_2$ -norm between  $f^\delta$  and  $f_\infty$

$$\Theta(F^\delta(t)) = \int_{\mathbb{R}_+} |f^\delta(v, t) - f_\infty(v)|^2 f_\infty^{-1}(v) dv.$$

Then

$$I_\Theta(F^\delta(t)) = 2 \int_{\mathbb{R}_+} v^2 f_\infty(v) \left| \frac{\partial F^\delta(v, t)}{\partial v} \right|^2 dv,$$

and application of Chernoff inequality with weight (60) gives

$$I_\Theta(F^\delta(t)) \geq 2\Theta(F^\delta(t)).$$



Hence, exponential decay follows, and

$$\int_{\mathbb{R}_+} |f^\delta(v, t) - f_\infty(v)|^2 f_\infty^{-1}(v) dv \leq e^{-2t} \int_{\mathbb{R}_+} |f_0^\delta(v) - f_\infty(v)|^2 f_\infty^{-1}(v) dv. \quad (75)$$

Proceeding as before, and removing the lifting on the initial data we obtain the following

**Theorem 3.5.** *Let  $f(v, t)$  be a solution of the initial-boundary value problem for the Fokker-Planck equation (1), corresponding to an initial density  $f_0(v)$  such that, as in Theorem 2.1  $f_0(v) \in L_1(\mathbb{R}_+)$  and  $v^3 f_0(v) \in L_1(\mathbb{R}_+)$ . Then, if the the weighted  $L_2$ -norm*

$$\|f_0 - f_\infty\|_2^* = \int_{\mathbb{R}_+} |f_0(v) - f_\infty(v)|^2 f_\infty^{-1}(v) dv$$

is bounded,  $f(v, t)$  converges in  $L_2^*(\mathbb{R}_+)$  towards the steady state  $f_\infty$ , and

$$\|f(t) - f_\infty\|_2^* \leq e^{-2t} \|f_0 - f_\infty\|_2^*. \quad (76)$$

Last, we analyze the rate of decay towards equilibrium in the weak  $d_2$ -distance defined in (28). Proceeding as in Section 2.3, it is immediate to compute the rate of convergence of two different solution of Boltzmann equation (15). Let  $f_\varepsilon(t)$  and  $g_\varepsilon(t)$  denote two solutions of the kinetic model (15), departing from initial data  $\tilde{f}_0$  and  $\tilde{g}_0$  respectively. Let us suppose in addition that the distance in the Fourier metric  $d_3(f_\varepsilon, g_\varepsilon)$  is initially bounded, and let us define

$$h_\varepsilon(\xi, t) = \frac{\widehat{f}_\varepsilon(\xi, t) - \widehat{g}_\varepsilon(\xi, t)}{|\xi|^3}.$$

Then, proceeding as in section 2.3, we obtain

$$\left| \frac{\partial}{\partial t} h_\varepsilon + \frac{1}{\varepsilon} h_\varepsilon \right| \leq \frac{1}{\varepsilon} \|h_\varepsilon\|_\infty [\langle |1 - \lambda\varepsilon + \tilde{\eta}_\varepsilon|^3 \rangle + |\lambda\varepsilon|^3].$$

Since by construction the quantity  $1 - \lambda\varepsilon + \tilde{\eta}_\varepsilon$  is nonnegative, and  $\langle \tilde{\eta}_\varepsilon^3 \rangle = 0$

$$\langle |1 - \lambda\varepsilon + \tilde{\eta}_\varepsilon|^3 \rangle + |\lambda\varepsilon|^3 = 1 - 3\varepsilon(\lambda - \sigma)(1 - \lambda\varepsilon).$$

Therefore Gronwall inequality yields

$$d_3(f_\varepsilon(t), g_\varepsilon(t)) \leq d_3(\tilde{f}_0, \tilde{g}_0) e^{-3(\lambda - \sigma)(1 - \lambda\varepsilon)t}. \quad (77)$$

Clearly, if  $\lambda > \sigma$ , and  $\varepsilon \ll 1$  so that  $\lambda\varepsilon \leq \delta < 1$ , the distance between the solutions  $d_3(f_\varepsilon, g_\varepsilon)$ , for each  $\varepsilon > 0$  decays exponentially with a rate bigger than  $3(\lambda - \sigma)(1 - \delta)$ . The decay of two different solutions to the Kinetic Boltzmann-type equation (15) allows to prove a similar result for the Fokker-Planck equation. To this aim, we recall a result on the Fourier distance proven in [3], adapted to the present situation

**Lemma 3.6.** *Let  $\{f_n(v)\}_{n \geq 0}$  and  $\{g_n(v)\}_{n \geq 0}$ ,  $v \in \mathbb{R}_+$ , be two sequences of probability density functions with moments bounded up to the second order, such that  $f_n \rightharpoonup f$  and  $g_n \rightharpoonup g$ . Suppose in addition that, for some  $r > 2$*

$$\int_{\mathbb{R}_+} |v|^r f(v) < +\infty, \quad \int_{\mathbb{R}_+} |v|^r g(v) < +\infty.$$

If

$$d_r(f_n, g_n) < +\infty,$$

then for all  $s < r$ ,

$$d_s(f, g) \leq \liminf d_s(f_n, g_n).$$

Thanks to the interpolation formula (29) with  $s = 3$  and  $p = 2$ , we obtain from (77) the bound

$$d_2(f_\varepsilon(t), g_\varepsilon(t)) \leq \left[ d_3(\tilde{f}_0, \tilde{g}_0) \right]^{2/3} e^{-2(1-\delta)(\lambda-\sigma)t}. \quad (78)$$

Thus, by Lemma 3.6, letting  $\varepsilon \rightarrow 0$ , we conclude that, if  $f(t)$  and  $g(t)$  are solutions of the Fokker-Planck equation (1), corresponding to initial values  $\tilde{f}_0$  and  $\tilde{g}_0$  such that  $d_3(\tilde{f}_0, \tilde{g}_0)$  is finite, the distance  $d_2(f(t), g(t))$  decays to zero with the explicit exponential rate  $2(1-\delta)(\lambda-\sigma)$ . We can resume the previous result in the following.

**Theorem 3.7.** *Let  $f(v, t)$  be a solution of the initial-boundary value problem for the Fokker-Planck equation (1), corresponding to an initial density  $f_0(v)$  such that, as in Theorem 2.1  $f_0(v) \in L_1(\mathbb{R}_+)$  and  $v^3 f_0(v) \in L_1(\mathbb{R}_+)$ . Then, provided  $d_3(f_0, f_\infty)$  is finite, the solution  $f(v, t)$  converges towards the equilibrium density in the Fourier distance  $d_2$ , and for each constant  $\delta$  such that  $0 < \delta < 1$ , the following bound holds*

$$d_2(f(t), f_\infty) \leq [d_3(f_0, f_\infty)]^{2/3} e^{-2(1-\delta)(\lambda-\sigma)t}.$$

**4. Conclusions.** In this paper, we studied existence, uniqueness and asymptotic behavior of a Fokker-Planck equation for wealth distribution first derived in [2]. In particular, we investigated the connections between a bilinear kinetic model for wealth distribution introduced in [9] and the Fokker-Planck equation (1). In various cases, these connections allow to pass results which have been found for the kinetic model to the Fokker-Planck equation.

Some problems, however, remain open. In particular, the invariant trade limit which allows to obtain the Fokker-Planck equation has been proven to hold only when the initial values in the kinetic equation possess moments bounded up to the order three. This condition reflects also on the limit Fokker-Planck equation, where, for example, the exponential decay towards equilibrium obtained in Theorem 3.7 requires the boundedness of the  $d_3$ -distance between the initial datum and the corresponding equilibrium density. Thus, convergence results towards equilibrium in absence of a sufficiently high number of moments initially bounded is unknown.

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Received xxxx 20xx; revised xxxx 20xx.

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