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Abstract

An explicit expression of the derivative of the square root of a tensor is provided, by using the expressions of the derivatives of the eigenvalues and eigenvectors of a symmetric tensor. Starting from this result, the derivatives of the right and left stretch tensor $\mathbf{U}$, $\mathbf{V}$ and of the rotation $\mathbf{R}$ with respect to the deformation gradient $\mathbf{F}$, are calculated. Expressions for the material time derivatives of $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{R}$ are also given.

1 Introduction

The right and left stretch tensor $\mathbf{U}$ and $\mathbf{V}$, defined as the square roots of the right and left Cauchy-Green strain tensors $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ and $\mathbf{B} = \mathbf{F}\mathbf{F}^T$, with $\mathbf{F}$ deformation gradient, are widely used in continuum mechanics. Often, knowing their material time derivatives $\dot{\mathbf{U}}$ and $\dot{\mathbf{V}}$ proves to be useful.

In [1] the explicit expressions for $\dot{\mathbf{U}}$ and $\dot{\mathbf{V}}$ are calculated by differentiating the polar decompositions of $\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}$ with respect to the time and solving an equation for $\dot{\mathbf{R}}$. In particular, Guo has used two lemmata, the former regarding the solution of the homogeneous tensor equation $\mathbf{S}\mathbf{X} + \mathbf{X}\mathbf{S} = \mathbf{0}$, with $\mathbf{S}$ a symmetric positive definite tensor, and the latter dealing with solving the equation $\mathbf{S}\mathbf{X} + \mathbf{X}\mathbf{S} = \mathbf{A}$, in which $\mathbf{A}$ is a skew tensor.

In [3] Hoger and Carlson obtain the expression for $\dot{\mathbf{U}}$ and $\dot{\mathbf{V}}$ by applying the chain rule to $\mathbf{U} = \sqrt{\mathbf{C}}$, $\mathbf{V} = \sqrt{\mathbf{B}}$. This requires knowing an expression for the derivative of the square root of a second-order tensor. The main goal of their paper is to provide such an expression, which they accomplish by solving the tensor equation $\mathbf{S}\mathbf{X} + \mathbf{X}\mathbf{S} = \mathbf{T}$, for given $\mathbf{T}$, symmetric, and $\mathbf{S}$, symmetric positive definite. They have provided several expressions for the derivative of the square root, that has a polynomial expression whose coefficients are function of the principal invariants of the square root itself. Finally, they have used their results to arrive at formulas for $\dot{\mathbf{U}}$ and $\dot{\mathbf{V}}$, that are distinct from those given in [1].

In [6] Wheeler considers the right polar decomposition of $\mathbf{F}$ and presents some results concerning the derivative of $\mathbf{R}$ and $\mathbf{U}$ with respect to $\mathbf{F}$. Finally, in [7] the explicit expressions for the derivatives of the stretch and rotation tensors with respect to the deformation gradient are derived. Chen and Wheeler have used the definition of derivatives and found the restrictions of the desired derivatives to the subspaces associated with the tangent space of the stretch and rotation tensors and proved that these subspaces span the entire tensor space.

In this paper, an explicit expression is provided for the derivative of the square root of a tensor, from which the material time derivatives of stretch tensors $\mathbf{U}$ and $\mathbf{V}$ are obtained. Subsequently, the derivative of $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{R}$ with respect to the deformation gradient $\mathbf{F}$ are calculated. The proof follows a wholly different approach from those in the works cited above. In fact, the derivatives of the square root of a symmetric tensor, as well as those of the stretch and rotation tensor are explicitly calculated by using the expressions for the derivatives of the eigenvalues and eigenvectors of a symmetric tensor. These derivatives are briefly recalled in Section 2 (for further details we refer to [4], [10], [11] and [8]).

In particular, [8] deals with the problem of the differentiating the eigenvalues of the stretch tensors with respect to the deformation gradient, as well as differentiating the square root of a tensor. The expressions for the derivatives of the square root and of polar factors of the deformation gradient are given in terms of the exponential tensors.

In Section 3, the derivative of the square root is calculated. We shall first consider the subset of all symmetric tensors constituted by tensors having distinct eigenvalues and calculate the derivative in this set. Subsequently, it is observed that the expression found for the derivative is also valid in the case of repeated eigenvalues, and this leads to the conclusion that the resulting expression is in fact the derivative of the square root of a positive definite symmetric tensor.

In Section 4, the derivatives of the right and left stretch tensors and rotation tensor with respect to the deformation gradient are calculated by using the results obtained in Section 2, and the expressions for $\dot{\mathbf{U}}$, $\dot{\mathbf{V}}$ and $\dot{\mathbf{R}}$ are found.

Finally, in Section 5 a generalization of the isotropic function $\mathbf{F}(\mathbf{X}) = \sum_{i=1}^{3} f(x_i)\ \mathbf{e}_i(\mathbf{X}) \otimes \mathbf{e}_i(\mathbf{X})$, with $\mathbf{X}$ symmetric tensor, studied in [4] is introduced and its derivative is explicitly calculated. The procedures and results presented in this paper deal with the three-dimensional case; treatment of the two-dimensional case is provided in the Appendix.

2 The derivatives of the eigenvalues and eigenvectors of a symmetric tensor

Let $\mathcal{V}$ be a three-dimensional linear space, *Lin* the space of all linear transformations on $\mathcal{V}$ (second-order tensors), with the inner product $\mathbf{A} \cdot \mathbf{B} = tr(\mathbf{A}\mathbf{B}^T)$, where $\mathbf{A},\mathbf{B} \in Lin$ and $\mathbf{B}^T$ denotes the transpose of $\mathbf{B}$. The case of a two-dimensional space will be treated in the Appendix. Sym and Skw are the subspaces of Lin made up of all symmetric tensors and skew tensors, respectively. $Psym$ is the cone of Sym of all positive definite symmetric tensors, and Lin^+ is the subset of Lin of tensors with positive determinant,

$$Sym = \{\mathbf{A} \in Lin \mid \mathbf{A} = \mathbf{A}^T\}, \tag{1}$$

$$Skw = \{\mathbf{A} \in Lin \mid \mathbf{A} = -\mathbf{A}^T\}, \tag{2}$$

$$Psym = \{\mathbf{A} \in Sym \mid \mathbf{v} \cdot \mathbf{A}\mathbf{v} > 0, \quad \forall \mathbf{v} \in \mathcal{V}, \mathbf{v} \neq \mathbf{0}\}, \tag{3}$$

$$Lin^+ = \{\mathbf{A} \in Lin \mid \det \mathbf{A} > 0\}, \tag{4}$$

For $\mathbf{a}$, $\mathbf{b}, \mathbf{c} \in \mathcal{V}$, $\mathbf{a} \otimes \mathbf{b}$ is the second-order tensor defined by $(\mathbf{a} \otimes \mathbf{b})\mathbf{v} = (\mathbf{b} \cdot \mathbf{v})\mathbf{a}, \forall \mathbf{v} \in \mathcal{V}$; $\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c}$ is the linear application from Lin into $\mathcal{V}$ (third-order tensor) such that $\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c}\mathbf{H} = ((\mathbf{b} \otimes \mathbf{c}) \cdot \mathbf{H})\mathbf{a}, \forall \mathbf{H} \in Lin$.

Let us denote by $\mathbb{L}in$ the space of all linear transformations on Lin (fourth-order tensors) A fourth-order tensor $\mathbb{C}$, for which $\mathbb{C}[\mathbf{A}] \in Sym$ and $\mathbf{A} \cdot \mathbb{C}[\mathbf{B}] = \mathbf{B} \cdot \mathbb{C}[\mathbf{A}]$, $\forall \mathbf{A}$, $\mathbf{B} \in Lin$, is said to be symmetric. For $\mathbf{A}$, $\mathbf{B} \in Lin$, $\mathbf{A} \otimes \mathbf{B}$ is the fourth-order tensor defined by $\mathbf{A} \otimes \mathbf{B}[\mathbf{H}] = (\mathbf{H} \cdot \mathbf{B})\mathbf{A}$, $\forall \mathbf{H} \in Lin$.

For $\mathbf{E} \in Sym$, let $e_1 \leq e_2 \leq e_3$ be its eigenvalues and

$$I_1(\mathbf{E}) = tr\mathbf{E} = e_1 + e_2 + e_3, \tag{5}$$

$$I_2(\mathbf{E}) = \frac{1}{2}[(tr\mathbf{E})^2 - tr(\mathbf{E}^2)], \tag{6}$$

$$I_3(\mathbf{E}) = \det \mathbf{E}, \tag{7}$$

its principal invariants. It is possible to express the eigenvalues of $\mathbf{E}$ as functions of invariants I_1, I_2 and I_3 of $\mathbf{E}$. In fact, if $\mathbf{E}$ is a spherical tensor, then

$$e_1 = e_2 = e_3 = \frac{1}{3}I_1(\mathbf{E}). \tag{8}$$

Now, let us suppose that $\mathbf{E}$ has at least two distinct eigenvalues. Since e_1, e_2 and e_3 are the roots of the characteristic polynomial

$$p(\lambda) = \lambda^3 - I_1(\mathbf{E})\lambda^2 + I_2(\mathbf{E})\lambda - I_3(\mathbf{E}), \tag{9}$$

we can write [9]

$$e_1 = -\frac{2}{\sqrt{3}}\chi \cos\omega + \frac{1}{3}I_1, \tag{10}$$

$$e_2 = \frac{2}{\sqrt{3}}\chi\cos(\omega + \frac{\pi}{3}) + \frac{1}{3}I_1, \tag{11}$$

$$e_3 = \frac{2}{\sqrt{3}}\chi\cos(\omega - \frac{\pi}{3}) + \frac{1}{3}I_1, \tag{12}$$

where

$$\chi = \sqrt{\frac{I_1^2 - 3I_2}{3}}, \tag{13}$$

$$\cos 3\omega = -\frac{3\sqrt{3}\gamma}{2\chi^3}, \tag{14}$$

$$\gamma = I_3 - \frac{1}{2}I_1I_2 + \frac{2}{27}I_1^3. \tag{15}$$

Notice that when $\mathbf{E}$ is not a spherical tensor, χ is different from 0. In this way the function $\widehat{\mathbf{e}}(\mathbf{E}) = (e_1, e_2, e_3)$ which associate to each symmetric tensor $\mathbf{E}$ its ordered eigenvalues, is well defined. Moreover, functions $\widehat{e}_i$ defined from Sym with values in $\mathbb{R}$, such that $\widehat{e}_i(\mathbf{E}) = e_i$, are Lipschitz continuous [8].

Let Sym^* be the subset of Sym constituted by all symmetric tensors having distinct eigenvalues. Sym^* is an open dense subset of Sym [8].

From the spectral theorem, it follows that, if $\mathbf{E} \in Sym^*$, then the function associating the eigenvectors to each symmetric tensor is well defined. In fact, an eigenvector $\mathbf{g}_i$ is determined uniquely to within the change of sign by the equation

$$\mathbf{E}\mathbf{g}_i = e_i\mathbf{g}_i. \tag{16}$$

Given $\mathbf{E} \in Sym^*$, let $\mathbf{g}_1, \mathbf{g}_2$ and $\mathbf{g}_3$ be its eigenvectors corresponding to e_1, e_2 and e_3, respectively, and let us consider the orthonormal basis of Sym,

$$\mathbf{G}_1 = \mathbf{g}_1 \otimes \mathbf{g}_1, \tag{17}$$

$$\mathbf{G}_2 = \mathbf{g}_2 \otimes \mathbf{g}_2, \tag{18}$$

$$\mathbf{G}_3 = \mathbf{g}_3 \otimes \mathbf{g}_3, \tag{19}$$

$$\mathbf{G}_4 = \frac{1}{\sqrt{2}}(\mathbf{g}_1 \otimes \mathbf{g}_2 + \mathbf{g}_2 \otimes \mathbf{g}_1), \tag{20}$$

$$\mathbf{G}_5 = \frac{1}{\sqrt{2}}(\mathbf{g}_1 \otimes \mathbf{g}_3 + \mathbf{g}_3 \otimes \mathbf{g}_1), \tag{21}$$

$$\mathbf{G}_6 = \frac{1}{\sqrt{2}}(\mathbf{g}_2 \otimes \mathbf{g}_3 + \mathbf{g}_3 \otimes \mathbf{g}_2). \tag{22}$$

The following proposition summarizes the derivatives of eigenvalues and eigenvectors of a symmetric tensor [8].

Proposition 1 *For* $\mathbf{E} \in Sym^*$*, the derivatives of eigenvalues and eigenvectors of* $\mathbf{E}$ *with respect to* $\mathbf{E}$ *are respectively*

$$D_E e_1 = \mathbf{G}_1, \tag{23}$$

$$D_E e_2 = \mathbf{G}_2, \tag{24}$$

$$D_E e_3 = \mathbf{G}_3, \tag{25}$$

$$D_E \mathbf{g}_1 = \frac{1}{2(e_1 - e_2)}(\mathbf{g}_2 \otimes \mathbf{g}_1 \otimes \mathbf{g}_2 + \mathbf{g}_2 \otimes \mathbf{g}_2 \otimes \mathbf{g}_1) + \frac{1}{2(e_1 - e_3)}(\mathbf{g}_3 \otimes \mathbf{g}_1 \otimes \mathbf{g}_3 + \mathbf{g}_3 \otimes \mathbf{g}_3 \otimes \mathbf{g}_1), \tag{26}$$

$$D_E \mathbf{g}_2 = \frac{1}{2(e_2 - e_1)}(\mathbf{g}_1 \otimes \mathbf{g}_1 \otimes \mathbf{g}_2 + \mathbf{g}_1 \otimes \mathbf{g}_2 \otimes \mathbf{g}_1) + \frac{1}{2(e_2 - e_3)}(\mathbf{g}_3 \otimes \mathbf{g}_2 \otimes \mathbf{g}_3 + \mathbf{g}_3 \otimes \mathbf{g}_3 \otimes \mathbf{g}_2), \tag{27}$$

$$D_E \mathbf{g}_3 = \frac{1}{2(e_3 - e_1)}(\mathbf{g}_1 \otimes \mathbf{g}_1 \otimes \mathbf{g}_3 + \mathbf{g}_1 \otimes \mathbf{g}_3 \otimes \mathbf{g}_1) + \frac{1}{2(e_3 - e_2)}(\mathbf{g}_2 \otimes \mathbf{g}_2 \otimes \mathbf{g}_3 + \mathbf{g}_2 \otimes \mathbf{g}_3 \otimes \mathbf{g}_2). \tag{28}$$

Moreover, the relations

$$D_E \mathbf{G}_1 = \frac{1}{e_1 - e_2}\mathbf{G}_4 \otimes \mathbf{G}_4 + \frac{1}{e_1 - e_3}\mathbf{G}_5 \otimes \mathbf{G}_5, \tag{29}$$

$$D_E \mathbf{G}_2 = \frac{1}{e_2 - e_1}\mathbf{G}_4 \otimes \mathbf{G}_4 + \frac{1}{e_2 - e_3}\mathbf{G}_6 \otimes \mathbf{G}_6, \tag{30}$$

$$D_E \mathbf{G}_3 = \frac{1}{e_3 - e_1}\mathbf{G}_5 \otimes \mathbf{G}_5 + \frac{1}{e_3 - e_2}\mathbf{G}_6 \otimes \mathbf{G}_6. \tag{31}$$

hold.

If E *has two coincident eigenvalues, for example* $e_1 < e_2 = e_3$*, relations*

$$D_E e_1 = \mathbf{G}_1, \tag{32}$$

$$D_E e_2 = \frac{1}{2}(\mathbf{I} - \mathbf{G}_1), \tag{33}$$

$$D_E \mathbf{G}_1 = \frac{1}{e_1 - e_2}(\mathbf{G}_4 \otimes \mathbf{G}_4 + \mathbf{G}_5 \otimes \mathbf{G}_5), \tag{34}$$

$$D_E(\mathbf{I} - \mathbf{G}_1) = \frac{1}{e_2 - e_1}(\mathbf{G}_4 \otimes \mathbf{G}_4 + \mathbf{G}_5 \otimes \mathbf{G}_5), \tag{35}$$

take the place of (23)-(25) and (29)-(31).

Finally, if $\boldsymbol{E}$ *is a spherical tensor,* $\mathbf{E} = e\mathbf{I}$*, then (23)-(25) and (29)-(31) reduce to relations*

$$D_E e = \frac{1}{3}\mathbf{I}, \tag{36}$$

$$D_E \mathbf{I} = \mathbb{O}. \tag{37}$$

The derivatives of the eigenvalues and eigenvectors of a symmetric tensor have been already provided in [4] (*cfr.* relations (3.1.4) and (3.1.5)) and [8] (*cfr.* relations (1.2.4) and (1.2.5)). The results for the two-dimensional case are summarized in the Appendix.

3 The derivative of the square root

Let us consider the function Π, defined from *Psym* onto *Psym* which to every tensor $\mathbf{A}$ associated its square root

$$\Pi(\mathbf{A}) = \sqrt{\mathbf{A}}. \tag{38}$$

Π is infinitely differentiable [8], and its derivative has been explicitly calculated by many authors [3], [8]. Here we wish to obtain the derivative of Π by using the results set forth in Section 2. Let $\mathbf{A} = \sum_{i=1}^{3} a_i \, \mathbf{g}_i \otimes \mathbf{g}_i$ be the spectral representation of $\mathbf{A}$, it then holds that

$$\Pi(\mathbf{A}) = \sum_{i=1}^{3} \sqrt{a_i} \, \mathbf{g}_i \otimes \mathbf{g}_i. \tag{39}$$

Firstly, we suppose that the eigenvalues of $\mathbf{A}$ are distinct, namely $\mathbf{A} \in Psym \cap Sym^*$. By differentiating (39) with respect to $\mathbf{A}$, and accounting for relations (23)-(25) and (29)-(31), we get

$$D_A \Pi(\mathbf{A}) = \frac{1}{2\sqrt{a_1}}\mathbf{G}_1 \otimes \mathbf{G}_1 + \frac{1}{2\sqrt{a_2}}\mathbf{G}_2 \otimes \mathbf{G}_2 + \frac{1}{2\sqrt{a_3}}\mathbf{G}_3 \otimes \mathbf{G}_3 +$$

$$+\sqrt{a_1}\left(\frac{1}{a_1-a_2}\mathbf{G}_4\otimes\mathbf{G}_4+\frac{1}{a_1-a_3}\mathbf{G}_5\otimes\mathbf{G}_5\right)+$$

$$+\sqrt{a_2}\left(\frac{1}{a_2-a_1}\mathbf{G}_4\otimes\mathbf{G}_4+\frac{1}{a_2-a_3}\mathbf{G}_6\otimes\mathbf{G}_6\right)+$$

$$+\sqrt{a_3}\left(\frac{1}{a_3-a_1}\mathbf{G}_5\otimes\mathbf{G}_5+\frac{1}{a_3-a_2}\mathbf{G}_6\otimes\mathbf{G}_6\right). \tag{40}$$

From (40), simple calculations yield

$$D_A\Pi(\mathbf{A})=\frac{1}{2\sqrt{a_1}}\mathbf{G}_1\otimes\mathbf{G}_1+\frac{1}{2\sqrt{a_2}}\mathbf{G}_2\otimes\mathbf{G}_2+\frac{1}{2\sqrt{a_3}}\mathbf{G}_3\otimes\mathbf{G}_3+$$

$$+\frac{1}{\sqrt{a_1}+\sqrt{a_2}}\mathbf{G}_4\otimes\mathbf{G}_4+\frac{1}{\sqrt{a_1}+\sqrt{a_3}}\mathbf{G}_5\otimes\mathbf{G}_5++\frac{1}{\sqrt{a_2}+\sqrt{a_3}}\mathbf{G}_6\otimes\mathbf{G}_6. \tag{41}$$

Relation (41) also holds when $\mathbf{A}$ has either two or three coincident eigenvalues. (41) is the spectral representation of the symmetric fourth-order tensor $D_A\Pi(\mathbf{A})$; in particular, $D_A\Pi(\mathbf{A})$ is positive definite with eigenvalues $\frac{1}{2\sqrt{a_1}}$, $\frac{1}{2\sqrt{a_2}}$, $\frac{1}{2\sqrt{a_3}}$, $\frac{1}{\sqrt{a_1}+\sqrt{a_2}}$, $\frac{1}{\sqrt{a_1}+\sqrt{a_3}}$ and $\frac{1}{\sqrt{a_2}+\sqrt{a_3}}$ and eigenvectors $\mathbf{G}_1$, $\mathbf{G}_2$, $\mathbf{G}_3$, $\mathbf{G}_4$, $\mathbf{G}_5$ and $\mathbf{G}_6$.

Alternative representations of (41) have been given in [3] and [8] (*cfr.* (1.2.11) and (1.2.12)). Specifically, formulae (3.1) and (3.2) in [3] have been obtained by using the definition of derivative and resolving a tensorial equation of the type $\mathbf{XS}+\mathbf{SX}=\mathbf{T}$, with $\mathbf{S}\in Psym$, $\mathbf{T}\in Sym$.

Equation (41) allows calculating the material time derivative of the right stretch tensor

$$\mathbf{U}=\Pi(\mathbf{C})=\sqrt{\mathbf{C}}, \tag{42}$$

where

$$\mathbf{C}=\mathbf{F}^T\mathbf{F}, \tag{43}$$

is the right Cauchy-Green strain tensor, with $\mathbf{F}$ the deformation gradient. The expression of $\dot{\mathbf{U}}$ has already been provided in [1] and [3]. Let us now designate $c_1\le c_2\le c_3$ as the eigenvalues of $\mathbf{C}$, and $\mathbf{g}_1$, $\mathbf{g}_2$, $\mathbf{g}_3$ the corresponding eigenvectors; moreover, let $\mathbf{G}_i$, $i=1,...,6$, be the tensors defined in (17)-(22) by using $\mathbf{g}_1$, $\mathbf{g}_2$, $\mathbf{g}_3$. Applying the chain rule to (42) yields

$$\dot{\mathbf{U}}=D_C\Pi(\mathbf{C})[\dot{\mathbf{C}}], \tag{44}$$

from which, by using (41), we get

$$\dot{\mathbf{U}}=\frac{1}{2\sqrt{c_1}}(\mathbf{G}_1\cdot\dot{\mathbf{C}})\mathbf{G}_1+\frac{1}{2\sqrt{c_2}}(\mathbf{G}_2\cdot\dot{\mathbf{C}})\mathbf{G}_2+\frac{1}{2\sqrt{c_3}}(\mathbf{G}_2\cdot\dot{\mathbf{C}})\mathbf{G}_3+$$

$$+\frac{1}{\sqrt{c_1}+\sqrt{c_2}}(\mathbf{G}_4\cdot\dot{\mathbf{C}})\mathbf{G}_4+\frac{1}{\sqrt{c_1}+\sqrt{c_3}}(\mathbf{G}_5\cdot\dot{\mathbf{C}})\mathbf{G}_5+$$

$$+\frac{1}{\sqrt{c_2}+\sqrt{c_3}}(\mathbf{G}_6\cdot\dot{\mathbf{C}})\mathbf{G}_6. \tag{45}$$

From the expressions

$$\dot{\mathbf{C}}=\sum_{i=1}^{3}(\dot{c}_i\mathbf{G}_i+c_i\dot{\mathbf{G}}_i), \tag{46}$$

and

$$\dot{\mathbf{G}}_i=\dot{\mathbf{g}}_i\otimes\mathbf{g}_i+\mathbf{g}_i\otimes\dot{\mathbf{g}}_i,\quad i=1,2,3, \tag{47}$$

we can state that for $k=1,2,3$, it holds that

$$\dot{\mathbf{C}}\cdot\mathbf{G}_k=\dot{c}_k+\sum_{i=1}^{3}2c_i(\dot{\mathbf{g}}_i\cdot\mathbf{g}_k)(\mathbf{g}_i\cdot\mathbf{g}_k)=\dot{c}_k, \tag{48}$$

because $(\dot{\mathbf{g}}_i\cdot\mathbf{g}_k)=0$, if $i=k$, and $(\mathbf{g}_i\cdot\mathbf{g}_k)=0$, if $i\neq k$. Moreover, recalling (20)-(22) we have

$$\dot{\mathbf{C}}\cdot\mathbf{G}_4=\sqrt{2}(c_1-c_2)\dot{\mathbf{g}}_1\cdot\mathbf{g}_2, \tag{49}$$

$$\dot{\mathbf{C}}\cdot\mathbf{G}_5=\sqrt{2}(c_1-c_3)\dot{\mathbf{g}}_1\cdot\mathbf{g}_3, \tag{50}$$

$$\dot{\mathbf{C}}\cdot\mathbf{G}_6=\sqrt{2}(c_2-c_3)\dot{\mathbf{g}}_2\cdot\mathbf{g}_3. \tag{51}$$

Finally, by taking the foregoing into account, (45) becomes

$$\dot{\mathbf{U}}=\frac{\dot{c}_1}{2\sqrt{c_1}}\mathbf{G}_1+\frac{\dot{c}_2}{2\sqrt{c_2}}\mathbf{G}_2+\frac{\dot{c}_3}{2\sqrt{c_3}}\mathbf{G}_3+$$

$$\sqrt{2}(\sqrt{c_1}-\sqrt{c_2})(\dot{\mathbf{g}}_1\cdot\mathbf{g}_2)\mathbf{G}_4+\sqrt{2}(\sqrt{c_1}-\sqrt{c_3})(\dot{\mathbf{g}}_1\cdot\mathbf{g}_3)\mathbf{G}_5+$$

$$+\sqrt{2}(\sqrt{c_2}-\sqrt{c_3})(\dot{\mathbf{g}}_2\cdot\mathbf{g}_3)\mathbf{G}_6. \tag{52}$$

(52) represents an alternative to the expressions (4.1) and (4.2) in [3] and (41) in [1] for the time derivative of **U**. In fact, (52) expresses $\dot{\mathbf{U}}$ in terms of eigenvalues and eigenvectors of **C** and their time derivatives, whereas [3] and [1] state it as a function of **R**, **C**, the spatial gradient of velocity $\dot{\mathbf{F}}\mathbf{F}^{-1}$, the left stretch tensor $\mathbf{V}=\mathbf{R}\mathbf{U}\mathbf{R}^T$ and the principal invariants of **U**.

Let us now consider the left stretch tensor

$$\mathbf{V} = \Pi(\mathbf{B}) = \sqrt{\mathbf{B}}, \tag{53}$$

where

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T \tag{54}$$

is the left Cauchy-Green strain tensor corresponding to $\mathbf{F}$. For the time derivative of $\mathbf{V}$ we have

$$\dot{\mathbf{V}} = D_B\Pi(\mathbf{B})[\dot{\mathbf{B}}] = \frac{\dot{c}_1}{2\sqrt{c_1}}\mathbf{Q}_1 + \frac{\dot{c}_2}{2\sqrt{c_2}}\mathbf{Q}_2 + \frac{\dot{c}_3}{2\sqrt{c_3}}\mathbf{Q}_3 +$$

$$\sqrt{2}(\sqrt{c_1} - \sqrt{c_2})(\dot{\mathbf{q}}_1 \cdot \mathbf{q}_2)\mathbf{Q}_4 + \sqrt{2}(\sqrt{c_1} - \sqrt{c_3})(\dot{\mathbf{q}}_1 \cdot \mathbf{q}_3)\mathbf{Q}_5 +$$

$$+\sqrt{2}(\sqrt{c_2} - \sqrt{c_3})(\dot{\mathbf{q}}_2 \cdot \mathbf{q}_3)\mathbf{Q}_6, \tag{55}$$

where tensors $\mathbf{Q}_i$ are defined as $\mathbf{G}_i$ in (17)-(22), the eigenvectors $\mathbf{q}_i$ of $\mathbf{B}$ replace the eigenvectors $\mathbf{g}_i$ of $\mathbf{C}$, and c_i are the common eigenvalues of $\mathbf{C}$ and $\mathbf{B}$.

4 The derivative of the right and left stretch tensors and rotation tensor

Let us consider the functions $\widehat{\mathbf{U}}$ and $\widehat{\mathbf{V}}$, defined on Lin^+ with values in $Psym$, as per

$$\mathbf{U} = \widehat{\mathbf{U}}(\mathbf{F}) = \sqrt{\mathbf{F}^T\mathbf{F}}, \tag{56}$$

$$\mathbf{V} = \widehat{\mathbf{V}}(\mathbf{F}) = \sqrt{\mathbf{F}\mathbf{F}^T}, \tag{57}$$

which deliver the tensor $\mathbf{U}$ and $\mathbf{V}$ of the right and left polar decomposition of $\mathbf{F}$, respectively. In [8] it has been proved that $\widehat{\mathbf{U}}$ and $\widehat{\mathbf{V}}$ are infinitely differentiable. The aim here is to obtain explicit expressions for the derivatives of $\widehat{\mathbf{U}}$ and $\widehat{\mathbf{V}}$ with respect to $\mathbf{F}$, distinct from those calculated in [7] and [8].

$\widehat{\mathbf{U}}$ is the composition of functions Π in (39) and Ψ, defined from Lin^+ into $Psym$, $\Psi(\mathbf{F}) = \mathbf{F}^T\mathbf{F} = \mathbf{C}$,

$$\widehat{\mathbf{U}}(\mathbf{F}) = \Pi(\Psi(\mathbf{F})); \tag{58}$$

therefore,

$$D_F\widehat{\mathbf{U}}(\mathbf{F})[\mathbf{H}] = D_{\Psi(F)}\Pi(\Psi(\mathbf{F}))[D_F\Psi(\mathbf{F})[\mathbf{H}]], \quad \mathbf{H} \in Lin. \tag{59}$$

As before, let $0 < c_1 \leq c_2 \leq c_3$ denote the eigenvalues of the symmetric definite positive tensor $\mathbf{C} = \Psi(\mathbf{F})$, and $\mathbf{g}_1, \mathbf{g}_2$ and $\mathbf{g}_3$ the corresponding eigenvectors. By virtue of (41), it holds that

$$D_{\Psi(F)}\Pi(\Psi(\mathbf{F})) = \frac{1}{2\sqrt{c_1}}\mathbf{G}_1 \otimes \mathbf{G}_1 + \frac{1}{2\sqrt{c_2}}\mathbf{G}_2 \otimes \mathbf{G}_2 + \frac{1}{2\sqrt{c_3}}\mathbf{G}_3 \otimes \mathbf{G}_3 +$$

$$+\frac{1}{\sqrt{c_1}+\sqrt{c_2}}\mathbf{G}_4 \otimes \mathbf{G}_4 + \frac{1}{\sqrt{c_1}+\sqrt{c_3}}\mathbf{G}_5 \otimes \mathbf{G}_5 + \frac{1}{\sqrt{c_2}+\sqrt{c_3}}\mathbf{G}_6 \otimes \mathbf{G}_6. \quad (60)$$

By accounting for (60) and the relation

$$D_F\Psi(\mathbf{F}) = \mathbf{H}^T\mathbf{F} + \mathbf{F}^T\mathbf{H}, \quad \forall \mathbf{H} \in Lin, \quad (61)$$

the chain rule (59) can be rewritten as follows,

$$D_F\widehat{\mathbf{U}}(\mathbf{F})[\mathbf{H}] = \frac{1}{2\sqrt{c_1}}(\mathbf{G}_1 \cdot (\mathbf{H}^T\mathbf{F} + \mathbf{F}^T\mathbf{H}))\mathbf{G}_1 +$$

$$\frac{1}{2\sqrt{c_2}}(\mathbf{G}_2 \cdot (\mathbf{H}^T\mathbf{F} + \mathbf{F}^T\mathbf{H}))\mathbf{G}_2 + \frac{1}{2\sqrt{c_3}}(\mathbf{G}_3 \cdot (\mathbf{H}^T\mathbf{F} + \mathbf{F}^T\mathbf{H}))\mathbf{G}_3 +$$

$$+\frac{1}{\sqrt{c_1}+\sqrt{c_2}}(\mathbf{G}_4 \cdot (\mathbf{H}^T\mathbf{F} + \mathbf{F}^T\mathbf{H}))\mathbf{G}_4 +$$

$$+\frac{1}{\sqrt{c_1}+\sqrt{c_3}}(\mathbf{G}_5 \cdot (\mathbf{H}^T\mathbf{F} + \mathbf{F}^T\mathbf{H}))\mathbf{G}_5 +$$

$$+\frac{1}{\sqrt{c_2}+\sqrt{c_3}}(\mathbf{G}_6 \cdot (\mathbf{H}^T\mathbf{F} + \mathbf{F}^T\mathbf{H}))\mathbf{G}_6. \quad (62)$$

It is easy to verify that for $i = 1, ..., 6$, we have

$$\mathbf{G}_i \cdot (\mathbf{H}^T\mathbf{F} + \mathbf{F}^T\mathbf{H}) = 2\mathbf{F}\mathbf{G}_i \cdot \mathbf{H}, \quad \forall \mathbf{H} \in Lin, \quad (63)$$

and thereby, (62) becomes

$$D_F\widehat{\mathbf{U}}(\mathbf{F})[\mathbf{H}] = \{\frac{1}{\sqrt{c_1}}\mathbf{G}_1 \otimes \mathbf{F}\mathbf{G}_1 + \frac{1}{\sqrt{c_2}}\mathbf{G}_2 \otimes \mathbf{F}\mathbf{G}_2 +$$

$$+\frac{1}{\sqrt{c_3}}\mathbf{G}_3 \otimes \mathbf{F}\mathbf{G}_3 + \frac{2}{\sqrt{c_1}+\sqrt{c_2}}\mathbf{G}_4 \otimes \mathbf{F}\mathbf{G}_4 +$$

$$+\frac{2}{\sqrt{c_1}+\sqrt{c_3}}\mathbf{G}_5 \otimes \mathbf{F}\mathbf{G}_5 + \frac{2}{\sqrt{c_2}+\sqrt{c_3}}\mathbf{G}_6 \otimes \mathbf{F}\mathbf{G}_6\}[\mathbf{H}]. \quad (64)$$

From (64) and (70), and accounting for the fact that $(\mathbf{A} \otimes \mathbf{B})^T = \mathbf{B} \otimes \mathbf{A}$ and $(\mathbf{A} \boxtimes \mathbf{B})^T = \mathbf{A} \boxtimes \mathbf{B}$, for $\mathbf{A}, \mathbf{B} \in Sym$, we can easily derive the relations

$$(D_F \widehat{\mathbf{U}}(\mathbf{F}))^T[\mathbf{G}_k] = \mathbf{R}\mathbf{G}_k, \quad k = 1,2,3. \tag{71}$$

$$(D_F \widehat{\mathbf{U}}(\mathbf{F}))^T[\mathbf{U}] = \mathbf{F}, \tag{72}$$

$$(D_F \widehat{\mathbf{U}}(\mathbf{F}))^T[\mathbf{U}^n] = \mathbf{R}\mathbf{U}^n, \quad n \geq 2, \tag{73}$$

$$(D_F \widehat{\mathbf{R}}(\mathbf{F}))^T[\mathbf{R}\mathbf{G}_k] = \mathbf{0}, \quad k = 1,2,3. \tag{74}$$

$$(D_F \widehat{\mathbf{R}}(\mathbf{F}))^T[\mathbf{F}] = \mathbf{0}, \tag{75}$$

$$(D_F \widehat{\mathbf{R}}(\mathbf{F}))^T[\mathbf{R}\mathbf{U}^n] = \mathbf{0}, \quad n \geq 2, \tag{76}$$

already proven in [6] without explicit calculation of the derivatives of $\widehat{\mathbf{U}}$ and $\widehat{\mathbf{R}}$ with respect to $\mathbf{F}$.

For $\mathbf{S} \in Sym$, $\mathbf{W} \in Skw$, the following relations

$$\mathbf{F}\mathbf{G}_k \cdot \mathbf{R}\mathbf{S} = \sqrt{c_k}\mathbf{S} \cdot \mathbf{G}_k, \quad k = 1,2,3, \tag{77}$$

$$\mathbf{F}\mathbf{G}_4 \cdot \mathbf{R}\mathbf{S} = \frac{\sqrt{c_1} + \sqrt{c_2}}{2}\mathbf{S} \cdot \mathbf{G}_4, \tag{78}$$

$$\mathbf{F}\mathbf{G}_5 \cdot \mathbf{R}\mathbf{S} = \frac{\sqrt{c_1} + \sqrt{c_3}}{2}\mathbf{S} \cdot \mathbf{G}_5, \tag{79}$$

$$\mathbf{F}\mathbf{G}_6 \cdot \mathbf{R}\mathbf{S} = \frac{\sqrt{c_2} + \sqrt{c_3}}{2}\mathbf{S} \cdot \mathbf{G}_6, \tag{80}$$

$$\mathbf{F}\mathbf{G}_k \cdot \mathbf{W}\mathbf{F} = 0, \quad k = 1, ..., 6. \tag{81}$$

hold. From (64), by using (77)-(81) we get the following equalities already proven in [7]

$$D_F \widehat{\mathbf{U}}(\mathbf{F})[\mathbf{R}\mathbf{S}] = \mathbf{S}, \quad \forall \mathbf{S} \in Sym, \tag{82}$$

$$D_F \widehat{\mathbf{U}}(\mathbf{F})[\mathbf{W}\mathbf{F}] = \mathbf{0}, \quad \forall \mathbf{W} \in Skw. \tag{83}$$

Moreover, we have ([7])

$$D_F\widehat{\mathbf{R}}(\mathbf{F})[\mathbf{RS}] = D_F\widehat{\mathbf{R}}(\mathbf{F})[\mathbf{SR}] = \mathbf{0}, \quad \forall \mathbf{S} \in Sym, \tag{84}$$

$$D_F\widehat{\mathbf{R}}(\mathbf{F})[\mathbf{WF}] = \mathbf{WR}, \quad D_F\widehat{\mathbf{R}}(\mathbf{F})[\mathbf{FW}] = \mathbf{RW}, \quad \forall \mathbf{W} \in Skw. \tag{85}$$

By following the procedure used for the derivative of $\widehat{\mathbf{U}}$ with respect to $\mathbf{F}$, we can calculate the derivative of the function defined in (57)

$$\mathbf{V} = \widehat{\mathbf{V}}(\mathbf{F}) = \Pi(\Phi(\mathbf{F})), \tag{86}$$

in which $\Phi(\mathbf{F}) = \mathbf{F}\mathbf{F}^T$.

The symmetric definite positive tensor $\Phi(\mathbf{F})$ has the same eigenvalues $0 < c_1 \leq c_2 \leq c_3$ of $\Psi(\mathbf{F})$. Vectors

$$\mathbf{q}_1 = \mathbf{R}\mathbf{g}_1, \quad \mathbf{q}_2 = \mathbf{R}\mathbf{g}_2, \quad \mathbf{q}_3 = \mathbf{R}\mathbf{g}_3 \tag{87}$$

are the eigenvectors of $\Phi(\mathbf{F})$ corresponding to c_1, c_2, c_3. Let us consider tensors $\mathbf{Q}_i$, $i = 1, 6$, defined as $\mathbf{G}_i$, where vectors $\mathbf{q}_i$ take the place of $\mathbf{g}_i$; by virtue of (41), it holds that

$$D_{\Phi(F)}\Pi(\Phi(\mathbf{F})) = \frac{1}{2\sqrt{c_1}}\mathbf{Q}_1 \otimes \mathbf{Q}_1 + \frac{1}{2\sqrt{c_2}}\mathbf{Q}_2 \otimes \mathbf{Q}_2 + \frac{1}{2\sqrt{c_3}}\mathbf{Q}_3 \otimes \mathbf{Q}_3 +$$

$$+\frac{1}{\sqrt{c_1}+\sqrt{c_2}}\mathbf{Q}_4 \otimes \mathbf{Q}_4 + \frac{1}{\sqrt{c_1}+\sqrt{c_3}}\mathbf{Q}_5 \otimes \mathbf{Q}_5 + \frac{1}{\sqrt{c_2}+\sqrt{c_3}}\mathbf{Q}_6 \otimes \mathbf{Q}_6. \tag{88}$$

By accounting for (88) and the relation

$$D_F\Phi(\mathbf{F}) = \mathbf{F}\mathbf{H}^T + \mathbf{H}\mathbf{F}^T, \quad \forall \mathbf{H} \in Lin, \tag{89}$$

from (86) we get

$$D_F\widehat{\mathbf{V}}(\mathbf{F})[\mathbf{H}] = \frac{1}{2\sqrt{c_1}}(\mathbf{Q}_1 \cdot (\mathbf{F}\mathbf{H}^T + \mathbf{H}\mathbf{F}^T))\mathbf{Q}_1 +$$

$$\frac{1}{2\sqrt{c_2}}(\mathbf{Q}_2 \cdot (\mathbf{F}\mathbf{H}^T + \mathbf{H}\mathbf{F}^T))\mathbf{Q}_2 + \frac{1}{2\sqrt{c_3}}(\mathbf{Q}_3 \cdot (\mathbf{F}\mathbf{H}^T + \mathbf{H}\mathbf{F}^T))\mathbf{Q}_3 +$$

$$+\frac{1}{\sqrt{c_1}+\sqrt{c_2}}(\mathbf{Q}_4 \cdot (\mathbf{F}\mathbf{H}^T + \mathbf{H}\mathbf{F}^T))\mathbf{Q}_4 +$$

$$+\frac{1}{\sqrt{c_1}+\sqrt{c_3}}(\mathbf{Q}_5 \cdot (\mathbf{H}^T\mathbf{F} + \mathbf{F}^T\mathbf{H}))\mathbf{Q}_5 +$$

$$+\frac{1}{\sqrt{c_2}+\sqrt{c_3}}(\mathbf{Q}_6\cdot(\mathbf{F}\mathbf{H}^T+\mathbf{H}\mathbf{F}^T))\mathbf{Q}_6. \tag{90}$$

It is easily verified that for $i = 1, ..., 6$, we have

$$\mathbf{Q}_i\cdot(\mathbf{F}\mathbf{H}^T+\mathbf{H}\mathbf{F}^T) = 2\mathbf{Q}_i\mathbf{F}\cdot\mathbf{H}, \quad \forall\mathbf{H}\in Lin, \tag{91}$$

and thereby, (90) becomes

$$D_F\widehat{\mathbf{V}}(\mathbf{F})[\mathbf{H}] = \{\frac{1}{\sqrt{c_1}}\mathbf{Q}_1\otimes\mathbf{Q}_1\mathbf{F}+\frac{1}{\sqrt{c_2}}\mathbf{Q}_2\otimes\mathbf{Q}_2\mathbf{F}+$$

$$+\frac{1}{\sqrt{c_3}}\mathbf{Q}_3\otimes\mathbf{Q}_3\mathbf{F}+\frac{2}{\sqrt{c_1}+\sqrt{c_2}}\mathbf{Q}_4\otimes\mathbf{Q}_4\mathbf{F}+$$

$$+\frac{2}{\sqrt{c_1}+\sqrt{c_3}}\mathbf{Q}_5\otimes\mathbf{Q}_5\mathbf{F}+\frac{2}{\sqrt{c_2}+\sqrt{c_3}}\mathbf{Q}_6\otimes\mathbf{Q}_6\mathbf{F}\}[\mathbf{H}]. \tag{92}$$

Since for $k = 1, 2, 3$, we have

$$\mathbf{Q}_k\mathbf{F} = \sqrt{c_k}\mathbf{Q}_k\mathbf{R}, \tag{93}$$

(92) becomes

$$D_F\widehat{\mathbf{V}}(\mathbf{F})[\mathbf{H}] = \{\mathbf{Q}_1\otimes\mathbf{Q}_1\mathbf{R}+\mathbf{Q}_2\otimes\mathbf{Q}_2\mathbf{R}+\mathbf{Q}_3\otimes\mathbf{Q}_3\mathbf{R}+$$

$$\frac{2}{\sqrt{c_1}+\sqrt{c_2}}\mathbf{Q}_4\otimes\mathbf{Q}_4\mathbf{F}+\frac{2}{\sqrt{c_1}+\sqrt{c_3}}\mathbf{Q}_5\otimes\mathbf{Q}_5\mathbf{F}+$$

$$+\frac{2}{\sqrt{c_2}+\sqrt{c_3}}\mathbf{Q}_6\otimes\mathbf{Q}_6\mathbf{F}\}[\mathbf{H}]. \tag{94}$$

For $k = 1, 2, 3$, it holds that

$$\mathbf{Q}_k\otimes\mathbf{Q}_k\mathbf{R} = \mathbf{Q}_k\otimes\mathbf{R}\mathbf{G}_k = \mathbf{Q}_k\otimes(\mathbf{q}_k\otimes\mathbf{g}_k) \tag{95}$$

and, moreover, the relations

$$\mathbf{Q}_4\otimes\mathbf{Q}_4\mathbf{F} = \frac{1}{\sqrt{2}}\mathbf{Q}_4\otimes(\sqrt{c_1}\mathbf{q}_2\otimes\mathbf{g}_1+\sqrt{c_2}\mathbf{q}_1\otimes\mathbf{g}_2), \tag{96}$$

$$\mathbf{Q}_5\otimes\mathbf{Q}_5\mathbf{F} = \frac{1}{\sqrt{2}}\mathbf{Q}_5\otimes(\sqrt{c_1}\mathbf{q}_3\otimes\mathbf{g}_1+\sqrt{c_3}\mathbf{q}_1\otimes\mathbf{g}_3), \tag{97}$$

$$\mathbf{Q}_6\otimes\mathbf{Q}_6\mathbf{F} = \frac{1}{\sqrt{2}}\mathbf{Q}_6\otimes(\sqrt{c_2}\mathbf{q}_3\otimes\mathbf{g}_2+\sqrt{c_3}\mathbf{q}_2\otimes\mathbf{g}_3), \tag{98}$$

Now we are in the position to calculate the derivatives of $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{R}$ with respect to time. From the relation

$$\dot{\mathbf{U}} = D_F\widehat{\mathbf{U}}(\mathbf{F})[\dot{\mathbf{F}}], \tag{107}$$

from (64) and equalities

$$\mathbf{FG}_i \cdot \dot{\mathbf{F}} = tr(\mathbf{G}_i\mathbf{F}^T\dot{\mathbf{F}}) = tr(\mathbf{G}_i\mathbf{F}^T\dot{\mathbf{F}}\mathbf{F}^{-1}\mathbf{F}) =$$

$$= tr(\mathbf{LFG}_i\mathbf{F}^T) = \mathbf{D}\cdot\mathbf{FG}_i\mathbf{F}^T, \tag{108}$$

where $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$ is the spatial gradient of velocity and $\mathbf{D}$ its symmetric part, we get

$$\dot{\mathbf{U}} = \{\frac{1}{\sqrt{c_1}}\mathbf{G}_1 \otimes \mathbf{FG}_1\mathbf{F}^T + \frac{1}{\sqrt{c_2}}\mathbf{G}_2 \otimes \mathbf{FG}_2\mathbf{F}^T +$$

$$+\frac{1}{\sqrt{c_3}}\mathbf{G}_3 \otimes \mathbf{FG}_3\mathbf{F}^T + \frac{2}{\sqrt{c_1}+\sqrt{c_2}}\mathbf{G}_4 \otimes \mathbf{FG}_4\mathbf{F}^T +$$

$$+\frac{2}{\sqrt{c_1}+\sqrt{c_3}}\mathbf{G}_5 \otimes \mathbf{FG}_5\mathbf{F}^T + \frac{2}{\sqrt{c_2}+\sqrt{c_3}}\mathbf{G}_6 \otimes \mathbf{FG}_6\mathbf{F}^T\}[\mathbf{D}]. \tag{109}$$

The time derivative of $\mathbf{R}$ can be obtained from

$$\dot{\mathbf{R}}\mathbf{U} + \mathbf{R}\dot{\mathbf{U}} = \dot{\mathbf{F}}, \tag{110}$$

by accounting for (109), which leads to

$$\dot{\mathbf{R}} = \mathbf{LR} - \{\frac{1}{c_1}\mathbf{RG}_1 \otimes \mathbf{FG}_1\mathbf{F}^T + \frac{1}{c_2}\mathbf{RG}_2 \otimes \mathbf{FG}_2\mathbf{F}^T +$$

$$+\frac{1}{c_3}\mathbf{RG}_3 \otimes \mathbf{FG}_3\mathbf{F}^T + \frac{2}{\sqrt{c_1}\sqrt{c_2}(\sqrt{c_1}+\sqrt{c_2})}\mathbf{R}\widetilde{\mathbf{G}}_4 \otimes \mathbf{FG}_4\mathbf{F}^T +$$

$$+\frac{2}{\sqrt{c_1}\sqrt{c_3}(\sqrt{c_1}+\sqrt{c_3})}\mathbf{R}\widetilde{\mathbf{G}}_5 \otimes \mathbf{FG}_5\mathbf{F}^T +$$

$$+\frac{2}{\sqrt{c_2}\sqrt{c_3}(\sqrt{c_2}+\sqrt{c_3})}\mathbf{R}\widetilde{\mathbf{G}}_6 \otimes \mathbf{FG}_6\mathbf{F}^T\}[\mathbf{D}], \tag{111}$$

where

$$\widetilde{\mathbf{G}}_4 = \frac{1}{\sqrt{2}}(\sqrt{c_1}\mathbf{g}_1 \otimes \mathbf{g}_2 + \sqrt{c_2}\mathbf{g}_2 \otimes \mathbf{g}_1), \tag{112}$$

$$D_A b_{12} = \frac{b_{22} - b_{11}}{\sqrt{2}(a_1 - a_2)} \mathbf{G}_4 + \frac{b_{23}}{\sqrt{2}(a_1 - a_3)} \mathbf{G}_5 + \frac{b_{13}}{\sqrt{2}(a_2 - a_3)} \mathbf{G}_6, \tag{135}$$

$$D_A b_{13} = \frac{b_{23}}{\sqrt{2}(a_1 - a_2)} \mathbf{G}_4 + \frac{b_{33} - b_{11}}{\sqrt{2}(a_1 - a_3)} \mathbf{G}_5 + \frac{b_{12}}{\sqrt{2}(a_3 - a_2)} \mathbf{G}_6, \tag{136}$$

$$D_A b_{23} = \frac{b_{13}}{\sqrt{2}(a_2 - a_1)} \mathbf{G}_4 + \frac{b_{12}}{\sqrt{2}(a_3 - a_1)} \mathbf{G}_5 + \frac{b_{33} - b_{22}}{\sqrt{2}(a_2 - a_3)} \mathbf{G}_6. \tag{137}$$

In particular, in view of (130), (132)-(137), (29)-(31) and (124)-(128), we have $D_A \, \widehat{\mathbf{B}}(\mathbf{A}) = \mathbb{O}$.

6 Appendix

The two-dimensional case is considered in the following. For $\mathbf{E} \in Sym$, let $e_1 \leq e_2$ be its ordered eigenvalues and $\mathbf{g}_1, \mathbf{g}_2$ the corresponding eigenvectors. Let us set

$$\mathbf{G}_1 = \mathbf{g}_1 \otimes \mathbf{g}_1, \tag{138}$$

$$\mathbf{G}_2 = \mathbf{g}_2 \otimes \mathbf{g}_2, \tag{139}$$

$$\mathbf{G}_3 = \frac{1}{\sqrt{2}}(\mathbf{g}_1 \otimes \mathbf{g}_2 + \mathbf{g}_2 \otimes \mathbf{g}_1). \tag{140}$$

We have

$$D_E e_1 = \mathbf{G}_1, \tag{141}$$

$$D_E e_2 = \mathbf{G}_2, \tag{142}$$

$$D_E \mathbf{g}_1 = \frac{1}{\sqrt{2}(e_1 - e_2)} \mathbf{g}_2 \otimes \mathbf{G}_3, \tag{143}$$

$$D_E \mathbf{g}_2 = \frac{1}{\sqrt{2}(e_2 - e_1)} \mathbf{g}_1 \otimes \mathbf{G}_3, \tag{144}$$

$$D_E \mathbf{G}_1 = \frac{1}{e_1 - e_2} \mathbf{G}_3 \otimes \mathbf{G}_3, \tag{145}$$

$$D_E \mathbf{G}_2 = \frac{1}{e_2 - e_1} \mathbf{G}_3 \otimes \mathbf{G}_3, \tag{146}$$

for $e_1 < e_2$, and

$$D_E e = \frac{1}{2}\mathrm{I}, \tag{147}$$

$$D_E \mathrm{I} = \mathbb{O}, \tag{148}$$

for $e_1 = e_2 = e$.

Let $\mathbf{A} = \sum_{i=1}^{2} a_i\, \mathbf{g}_i \otimes \mathbf{g}_i$ be the spectral representation of $\mathbf{A} \in Psym$, then the square root of $\mathbf{A}$ is A

$$\Pi(\mathbf{A}) = \sum_{i=1}^{2} \sqrt{a_i}\mathbf{g}_i \otimes \mathbf{g}_i \tag{149}$$

The derivative of Π with respect to $\mathbf{A}$ is

$$D_A\Pi(\mathbf{A}) = \frac{1}{2\sqrt{a_1}}\mathbf{G}_1 \otimes \mathbf{G}_1 + \frac{1}{2\sqrt{a_2}}\mathbf{G}_2 \otimes \mathbf{G}_2 + \frac{1}{\sqrt{a_1}+\sqrt{a_2}}\mathbf{G}_3 \otimes \mathbf{G}_3. \tag{150}$$

In particular, from (150) we get the material time derivatives of the right stretch tensor $\mathbf{U} = \Pi(\mathbf{C})$, with $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ and of the left stretch tensor $\mathbf{V} = \Pi(\mathbf{B})$, with $\mathbf{B} = \mathbf{F}\mathbf{F}^T$,

$$\dot{\mathbf{U}} = \frac{\dot{c}_1}{2\sqrt{c_1}}\mathbf{G}_1 + \frac{\dot{c}_2}{2\sqrt{c_2}}\mathbf{G}_2 + \sqrt{2}(\sqrt{c_1} - \sqrt{c_2})(\dot{\mathbf{g}}_1 \cdot \mathbf{g}_2)\mathbf{G}_3, \tag{151}$$

and

$$\dot{\mathbf{V}} = \frac{\dot{c}_1}{2\sqrt{c_1}}\mathbf{Q}_1 + \frac{\dot{c}_2}{2\sqrt{c_2}}\mathbf{Q}_2 + \sqrt{2}(\sqrt{c_1} - \sqrt{c_2})(\dot{\mathbf{q}}_1 \cdot \mathbf{q}_2)\mathbf{Q}_3, \tag{152}$$

with $c_1 \leq c_2$ the common eigenvalues of $\mathbf{C}$ and $\mathbf{B}$, and $\mathbf{g}_1$, $\mathbf{g}_2$, $\mathbf{q}_1$, $\mathbf{q}_2$, the eigenvalues of $\mathbf{C}$ and $\mathbf{B}$, respectively.

Let us now consider the following functions

$$\mathbf{U} = \widehat{\mathbf{U}}(\mathbf{F}) = \sqrt{\mathbf{F}^T\mathbf{F}}, \quad \mathbf{V} = \widehat{\mathbf{V}}(\mathbf{F}) = \sqrt{\mathbf{F}\mathbf{F}^T}, \quad \mathbf{R} = \widehat{\mathbf{R}}(\mathbf{F}) = \mathbf{F}\widehat{\mathbf{U}}(\mathbf{F})^{-1}, \tag{153}$$

by analogy to the three-dimensional case, it can be shown that

$$D_F\widehat{\mathbf{U}}(\mathbf{F}) = \mathbf{G}_1 \otimes \mathbf{R}\mathbf{G}_1 + \mathbf{G}_2 \otimes \mathbf{R}\mathbf{G}_2 + \frac{2}{\sqrt{c_1}+\sqrt{c_2}}\mathbf{G}_3 \otimes \mathbf{F}\mathbf{G}_3, \tag{154}$$

$$D_F\widehat{\mathbf{V}}(\mathbf{F}) = \mathbf{Q}_1 \otimes \mathbf{Q}_1\mathbf{R} + \mathbf{Q}_2 \otimes \mathbf{Q}_2\mathbf{R} + \frac{2}{\sqrt{c_1}+\sqrt{c_2}}\mathbf{Q}_3 \otimes \mathbf{Q}_3\mathbf{F}, \tag{155}$$

$$D_F\widehat{\mathbf{R}}(\mathbf{F}) = \mathbf{I}\boxtimes\mathbf{U}^{-1} - \frac{1}{\sqrt{c_1}}\mathbf{R}\mathbf{G}_1\mathbf{U}^{-1}\otimes\mathbf{F}\mathbf{G}_1 - \frac{1}{\sqrt{c_2}}\mathbf{R}\mathbf{G}_2\mathbf{U}^{-1}\otimes\mathbf{F}\mathbf{G}_2 +$$

$$-\frac{2}{\sqrt{c_1}+\sqrt{c_2}}\mathbf{R}\mathbf{G}_3\mathbf{U}^{-1}\otimes\mathbf{F}\mathbf{G}_3, \tag{156}$$

Considering $\mathbf{B}\in Sym$, for each tensor $\mathbf{A}\in Sym^*$, with eigenvalues $a_1 < a_2$, and eigenvectors $\mathbf{g}_1, \mathbf{g}_2$, it is possible to write

$$\mathbf{B} = \sum_{i=1}^{2} b_{ij}(\mathbf{A})\mathbf{g}_i\otimes\mathbf{g}_j, \tag{157}$$

where

$$b_{ij}(\mathbf{A}) = \mathbf{g}_i\cdot\mathbf{B}\mathbf{g}_j. \tag{158}$$

Now, we wish to evaluate the derivatives of the components $b_{ij}(\mathbf{A})$ of $\mathbf{B}$ with respect to the basis of eigenvectors $\mathbf{g}_1, \mathbf{g}_2$, when $\mathbf{A}$ varies in Sym^*. In view of the relation

$$D_A b_{ij}(\mathbf{A})[\mathbf{H}] = D_A\mathbf{g}_i(\mathbf{A})[\mathbf{H}]\cdot\mathbf{B}\mathbf{g}_j(\mathbf{A}) + \mathbf{B}\mathbf{g}_i(\mathbf{A})\cdot D_A\mathbf{g}_j(\mathbf{A})[\mathbf{H}], \quad \mathbf{H}\in Sym, \tag{159}$$

from (143) and (144) we get

$$D_A b_{11}(\mathbf{A}) = \frac{\sqrt{2}\,b_{12}}{a_1-a_2}\mathbf{G}_3, \tag{160}$$

$$D_A b_{22}(\mathbf{A}) = \frac{\sqrt{2}\,b_{12}}{a_1-a_2}\mathbf{G}_3, \tag{161}$$

$$D_A b_{12}(\mathbf{A}) = \frac{b_{22}-b_{11}}{\sqrt{2}(a_1-a_2)}\mathbf{G}_3, \tag{162}$$

and from (143) and (144), by virtue of the bilinearity of the tensor product, we have

$$D_A(\mathbf{g}_1\otimes\mathbf{g}_2) = D_a(\mathbf{g}_1\otimes\mathbf{g}_2) = \frac{1}{\sqrt{2}(a_1-a_2)}(\mathbf{G}_2\otimes\mathbf{G}_3 - \mathbf{G}_1\otimes\mathbf{G}_3). \tag{163}$$

In particular, from (157), by accounting of (160)-(162) and (163), we get that $D_A\mathbf{B} = \mathbb{O}$.

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