

# Implementing Poincaré Sections for a Chaotic Relaxation Oscillator

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**Abstract**—We propose an electronic implementation to record Poincaré sections of dynamical systems exhibiting chaos. Poincaré sections are obtained by sampling and holding the maxima of a sequence of pulses of a chaotic relaxation oscillator versus the same temporal sequence shifted by one unit. By using these sections we are able to detail the transition to chaos via torus breakdown.

**Index Terms**—Oscillators, nonlinear systems, nonlinear dynamical systems, electronic circuits, chaos.

## I. INTRODUCTION

THE ONSET of chaos is linked to the loss of stability with consequent bifurcation of periodic solutions. Unfortunately this analysis is difficult to deal with except for particular cases. An elegant powerful method to overcome these difficulties has been proposed by Henri Poincaré more than a century ago, first in his famous memoir “On the Curves defined by differential equations and then in his New Methods of Celestial Mechanics [1], [2]. It was not until 1927 that this method, called Poincaré section, was used for studying dynamical systems by George David Birkhoff. According to Barrow-Green [3]: “Birkhoff first began to generalize Poincaré’s idea of a transverse section and formally developed a theory attached to it. Poincaré had used the idea specifically to reduce the restricted three body problem to the transformation of a ring to itself, but if the method was to have a general validity it was important to establish under what circumstances transverse sections exist. Birkhoff was able to show that not only do they exist in a wide variety of cases but also that they can be of varying genus and have different numbers of boundaries. Concerning the transformation of transverse section, Birkhoff emphasized the fact that it possesses an invariant integral area. This is important because it means that the transformation only involves one arbitrary function of two variables as opposed to the normal form of the differential equations, which involves two arbitrary functions.

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In other words, reducing a dynamical problem to a transformation of transverse section into itself is both a qualitative and an analytic reduction. By considering the invariant points of these transformations Birkhoff derived two important results about the periodic orbits. In the first place he found that the difference between the number of unstable and stable periodic orbits is a constant.” According to Ginoux [4]: “At the time, there were few French scientists who took an interest in Birkhoff’s work. Thus, in the early 1930s, a young French woman, Marie Charpentier (1903–1994) demonstrated Birkhoff’s result (1932) obtained two years before (probably during her stay at Harvard). Indeed, in an essay also published in the *Bulletin de la Société Mathématique de France*, Birkhoff (1932) had introduced the concept of “remarkable closed curves”. In fact, the trajectory curve, integral of a dynamical system, can intersect a Poincaré section (a plane in dimension three, a hyperplane in higher dimensions) at each of its passages. To each intersection corresponds to a point. Then, there exist a transformation leading from a point to the following point. Such a transformation has been called *Poincaré map* or *first recurrence map* or *first return map*. According to Ginoux [4]: “We then observe that the “impacts” of the trajectory on this plane which is where the Poincaré map appears randomly, in such a way that it seems impossible to predict where the next impact will be. Gradually, as the trajectory intersects the plane, a “closed curve” is formed. This curve is Birkhoff’s “remarkable closed curve”. Although, Pianigiani [5] recalled that “the first return map was introduced in 1943 by Kakutani”, it has been used before by Ambrose [6]. Moreover, as highlighted by Ginoux [4], in 1925, the French mathematician Elie Cartan and his son Henri already used a Poincaré map to prove the existence of a periodic solution for the differential equation of the series-dynamo machine’s oscillations and so, of the triode. Thus, after Second World War, it appears in the literature that both concepts of Poincaré section and Poincaré map were widely used for studying dynamical systems. One of their most famous applications has been made by Steve Smale in the end of the sixties who has established the existence of a horseshoe map which is typical of chaotic dynamical systems [7].

This brief is organized as follows. After a short introduction on the Poincaré map and the quasi-periodicity route to chaos (a well known topic from the theoretical point of view but not sufficiently explored from the experimental side), we introduce our case study on a driven relaxation oscillator based on a unipolar junction transistor UJT.

In Section III the schematic circuit of the oscillator together with an hardware implementation of the Poincaré section are presented. The experimental results are also here presented and discussed.

In the Conclusions future investigation is proposed considering that the system behaves like a leaky-integrated-and-fire system.

## II. THEORY

Let us consider a dynamical system, that for sake of simplicity we assume to be tridimensional, and the corresponding phase-space trajectory originated from assigned initial conditions and imaging to transversally intersect it with a surface  $S$ . The trajectory will intercept  $S$  at successive positions  $x_0, x_1, \dots, x_n$ . The related dynamics induced on  $S$  will be of the form  $x_{(n+1)} = G(x_n)$ . The discrete dynamics is referred to as the Poincaré map where  $S$  is the Poincaré surface of section (shortly Poincaré section). For a tridimensional system, introducing a suitable coordinate system in  $S$  the Poincaré map can be written as:

$$x_{(n+1)} = f_1(x_n, y_n) \quad (1)$$

$$y_{(n+1)} = f_2(x_n, y_n) \quad (2)$$

The most important advantage of the Poincaré map is reduction of dimensionality, in our particular case from a tridimensional case to a two dimensional one. Let us consider some particular cases of Poincaré maps. In the case of a limit cycle the successive intersection with a Poincaré surface will coincide. The dynamics on  $S$  will be described by:

$$\bar{x} = f_1(\bar{x}, \bar{y}) \quad (3)$$

$$\bar{y} = f_2(\bar{x}, \bar{y}) \quad (4)$$

where  $\bar{x}$  and  $\bar{y}$  are the coordinates of the intersection point  $P$ . Of particular interest is the case where the intersection points of the phase-space trajectory are localized on a closed curve in the Poincaré section. Such a condition occurs when the attractor is a two dimensional torus, that is, a quasi-periodic motion with two incommensurate frequencies. In this particular case the corresponding dissipative map can be written as:

$$\theta_{(i+1)} = P_1(\theta_i, r_i) \text{ mod } 2\pi \quad (5)$$

$$r_{(i+1)} = P_2(\theta_i, r_i) \quad (6)$$

where  $\theta$  and  $r$  are angular and radial coordinates respectively.  $\theta$  is restricted to the interval  $[0, 2\pi]$  by using the function  $\text{mod}(2\pi)$ . Assuming a strong contraction in the radial direction the Poincaré section approaches a closed curve. The two dimensional map reduces to a one dimensional one as:

$$\theta_{(i+1)} = P_1(\theta_i, [g(\theta)]_i) = F(\theta_i) \text{ where } r_i = [g(\theta)]_i.$$

Such a motion fills up an invariant torus in the phase-space. Different transitions from a quasi-periodic motion on a two dimensional torus exist leading to chaos (torus breakdown) or locking regimes. Most of contributions about torus breakdown consider discrete maps due to the fact that continuous models described by differential equations are much more difficult to study and demonstrate. The appearance of chaos following the breakdown of a 2-D torus has been largely investigated in the literature [8], [9]. From a theoretical point of view, Ruelle and Takens [10] and later, Newhouse, Ruelle and Takens (NRT) [11] demonstrated general theorems on the appearance of chaos starting from a 4-D and a 3-D torus. It has been demonstrated that an arbitrarily small perturbation can lead to the destabilization of a 3-D torus. Moreover, chaos can also occur directly through a destabilization of a two-frequency torus as proposed by Curry and Yorke [12]. Folds and wrinkles in the Poincaré section characterize this general route to chaos. The dynamics of a two frequency torus breakdown has been numerically investigated in a driven double scroll circuit by Baptista and Caldas [13]. Such a dynamics has been also

investigated by Matsumoto *et al.* [14] in a kind of Chua circuit. By using this autonomous system Anishchenko *et al.* [15] gave a confirmation of the Afraimovich-Shilnikov theorem [16] on torus breakdown. Beyond the theoretical interest on this kind of transition only few experimental confirmation exist [17], [18], [19]. From an experimental point of view the Poincaré section is a clear indicator for the torus breakdown as the control parameter is changed. Its real time knowledge provides a valuable and quick tool to characterize the phenomenon. For this reason we have proposed to analyze this dynamics by using a driven electronic relaxation oscillator. Such a kind of oscillator was introduced by van der Pol [20] (for historical considerations see Ginoux [4], [21], [22]) and recently reconsidered for studying a glow discharge tube [23], [24]. In the glow discharge tube one the two competing frequencies, the plasma eigenfrequency, is strictly tied to the physical characteristics of the tube. So, to have a better control on both frequencies, we have decided to investigate an electronic relaxation oscillator using an UJT externally driven by a sinusoidal forcing. A model, based on the negative resistance characteristic of the UJT, has been recently proposed by Ginoux *et al.* [25].

## III. EXPERIMENT AND RESULTS

We consider here the dynamics of an UJT relaxation oscillator driven by a sinusoidal forcing as shown in Fig. 1. In the absence of the sinusoidal forcing the relaxation oscillator yields an intrinsic oscillation at a frequency  $f_1 = 4770\text{Hz}$ , imposed by the RC external components. The forcing by the waveform generator  $G$  provides the additional degree of freedom that yields quasi-periodic dynamics and torus breaking [25]. Therefore, it is particularly interesting when Poincaré sections are considered and experimentally implemented. An immediate visualization of the Poincaré section is of considerable interest because it allows following the transition to chaos. As introduced before it is necessary to sample a spiking signal as the current output signal of the UJT relaxation oscillator (the current output signal  $I_{RL}$ , proportional to the voltage output signal  $V_{RL}$ , will be equivalently used in the following). Clearly the essential features are kept when the signal is sampled on the peaks and the corresponding temporal sequences  $I(n)$  are memorized and shifted by one unit by means of the circuit shown in Fig. 2 comprising two sample/hold devices two monostable multivibrators and one differentiator.

In Fig. 3, the upper oscilloscope trace a) shows  $I_{RL}$  while trace b) shows  $V_C$ . Traces c) and d) represent respectively  $I_{pk}(n-1)$  and  $I_{pk}(n)$  that are the peaks memorized by the relative sample/hold device shown in Fig. 2. In Fig. 4, the lower trace d) is a temporal expansion of the current pulse  $I_{RL}$ , traces b) and c) are active-low digital pulses controlling the Sample/Hold devices which retains data of the  $(n-1)$  and  $(n)$  sequences. The upper trace a) is the detection of the current peak in the  $I_{RL}$  by means of the derivative circuit  $D$ .

Let's assume that at least a first cycle of sampling has just passed, we will have a value  $I_{pk}(n)$  held by sampling device number two ( $S/H_2$ ).

When a peak in  $I(t)$  is detected by the derivative circuit  $D$  we will obtain a fast peak marking the top of the peak in  $I(t)$ .

Since the  $D$  circuit is of an inverting kind we obtain a negative fast peak that is proportional to the height of the  $I_{ujt}$  waveform. That's the reason for the consecutive Pulse

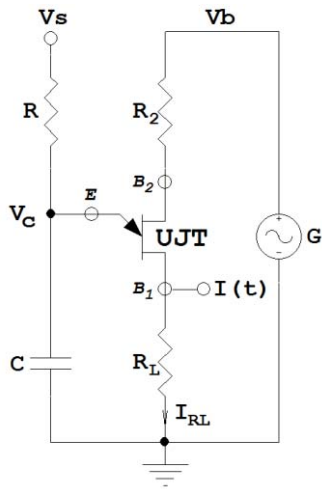


Fig. 1. Schematic circuit of the driven UJT relaxation oscillator; UJT = 2N2646,  $R = 12700\Omega$ ,  $C = 50\text{nF}$ ,  $R_2 = 677\Omega$ ,  $R_L = 56\Omega$ .  $V_s = 7\text{V}$  (supply voltage).  $G = \text{sinewave function generator}$ .  $V_b = 4.85V_{dc} + m \sin(2\pi f_2 t)$  where  $m$  is the amplitude of the driving frequency  $f_2$ .

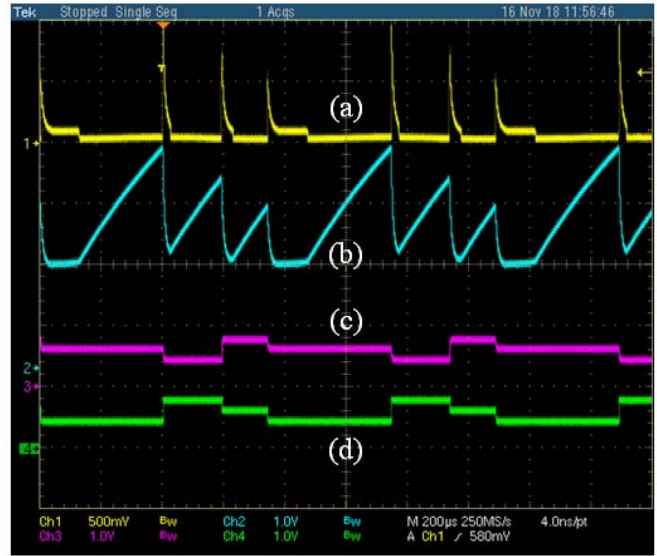


Fig. 3. Trace a) depicts the temporal evolution of the current  $I_{RL}$ , trace b) shows the capacitor voltage  $V_C$ . Trace d) shows  $I_{pk}(n)$  while trace c) shows  $I_{pk}(n - 1)$  obtained by sampling  $I_{RL}$  signals on the maxima values and held for one cycle.

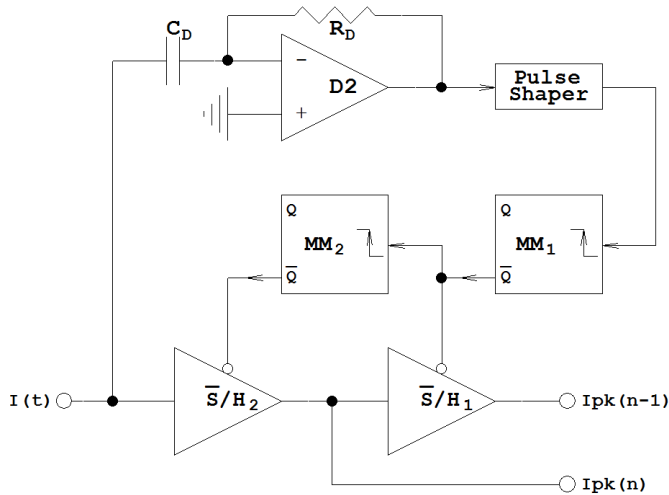


Fig. 2. Schematic circuit implementing the Poincaré section.  $S/H_1$  and  $S/H_2$  are sample/hold devices SMP04 from Analog Devices,  $MM_1$  and  $MM_2$  are monostable multivibrators SN74LS123 from Texas Instruments.  $D$  is a derivative circuit implemented by Operational amplifier LT1114 from Linear Technology,  $R_D = 4400\Omega$ ,  $C_D = 120\text{pF}$ . The Pulse Shaper block consists of an inverting and a logarithmic amplifier.

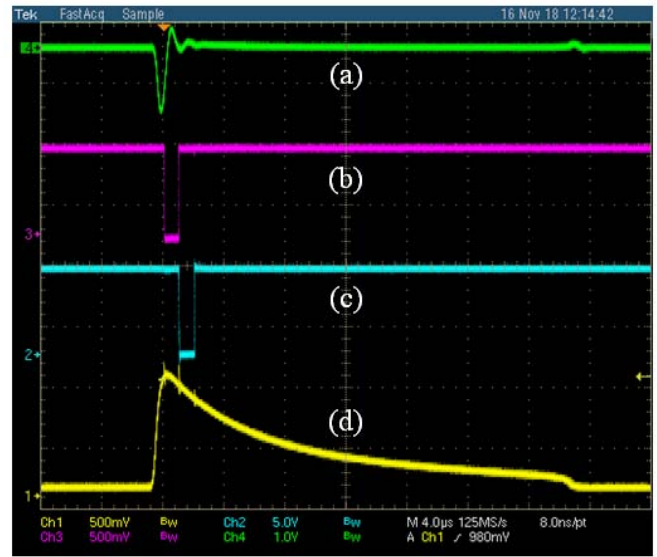
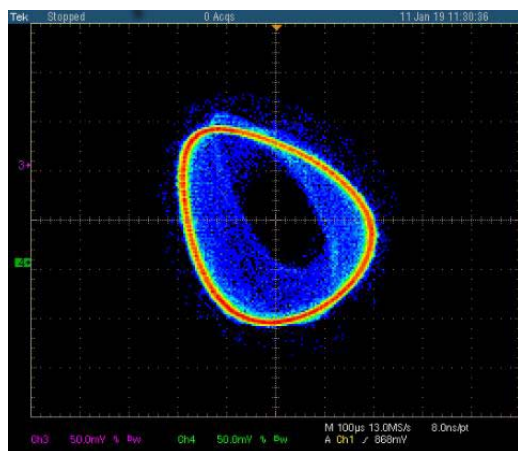


Fig. 4. Trace d) is an expansion of the current  $I_{RL}$  while trace a) is its derivative. b) and c) traces are sample logic commands for the sample/hold devices.

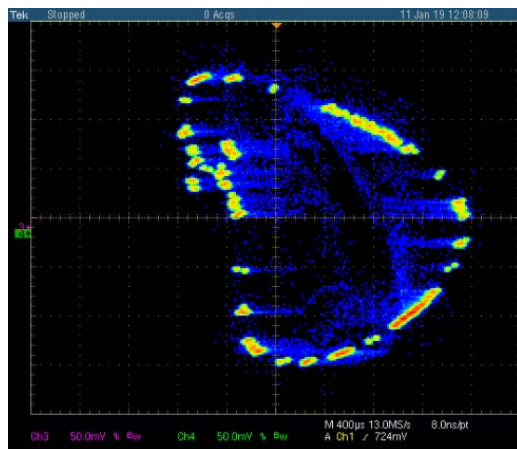
Shaper (PS) block to exist: inverting the peak and to render it's height in a logarithmic way so almost no difference in the peak itself can be observed; after this stage some amplification is applied to make the signal suitable to the digital input of two monostable multivibrators (it becomes a big positive peak). When the first monostable device  $MM_1$  is triggered by the former shaped pulse it shoots out a negative digital pulse on the output  $\bar{Q}$ . This pulse has an appropriate duration to let the capacitor that is embedded in the sampling device charge. The voltage stored in  $S/H_2$  is passed to  $S/H_1$ , meaning that  $I_{pk}(n)$  becomes  $I_{pk}(n - 1)$  since a new cycle has begun. When  $MM_1$  ends its function,  $\bar{Q}$  returns up to its high value and this Low to High transition triggers the second monostable  $MM_2$  that, by mean of its  $\bar{Q}$  output pulse, starts sampling in  $S/H_2$  getting the new value for  $I_{pk}(n)$ .

$I_{pk}(n)$  is then plotted vs  $I_{pk}(n - 1)$  obtaining an online Poincaré-section of the system that let us evaluate its temporal evolution thus avoiding us from acquiring many and many data strings to be analyzed then offline with appropriate software.

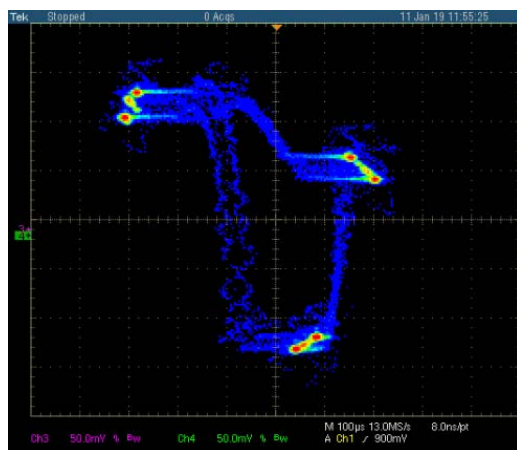
There are other real time software implementations using routines to calculate Poincaré sections demonstrated on chaotic systems like a modified Chua circuit [26] and on a mechanical system like the double pendulum [27]. In these cases the intrinsic frequencies are one or two orders of magnitude lower than our system. In addition our spiking system requires a high sampling rate in order to detect the rising edge of the current signal thus entailing an advantage of the analog implementation over the software one. Other software implementations of



(a)



(b)

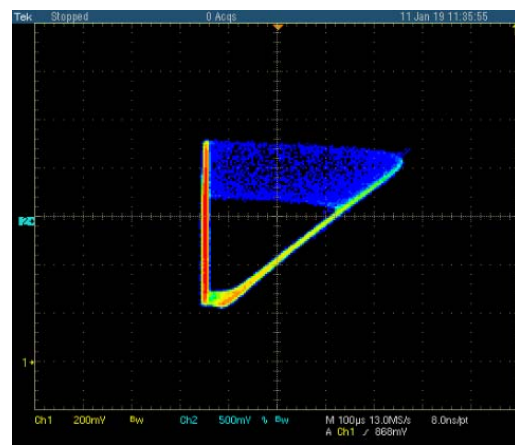


(c)

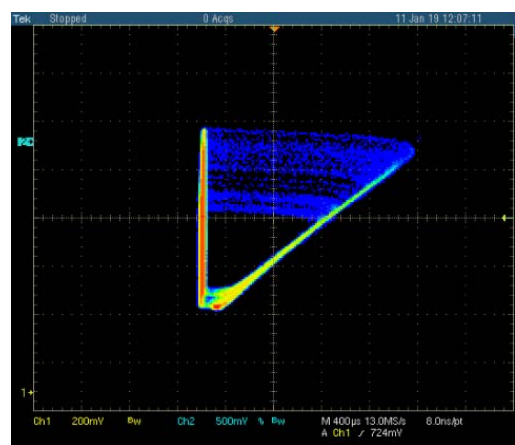
Fig. 5. Poincaré sections  $I_{pk}(n)$  vs  $I_{pk}(n-1)$  obtained by the sample/hold circuit shown in Fig. 2. a) torus,  $m = 250\text{mV}$ . b) first torus breakdown with wrinkles,  $m = 371\text{mV}$ . c) period three chaotic phase locking,  $m = 390\text{mV}$ .

Poincaré sections deal with the analysis of EEG signals [28], mobile chaotic robots [29] and biomedical applications [30].

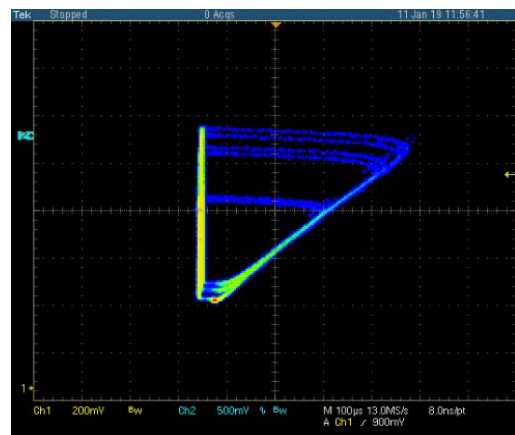
Let us consider the case where the intrinsic oscillation frequency  $f_1$  of the UJT relaxation oscillator and the driving frequency  $f_2$  provided by the sinusoidal waveform generator G are irrational. In our setup the ratio between them has been selected at a value  $0.309 \varphi_{GM}$ , where  $\varphi_{GM}$  is the golden mean number 0.61803. The corresponding Poincaré section of the quasiperiodic motion, obtained by recording on a digital scope



(a)



(b)



(c)

Fig. 6. Phase-Space representation of the attractors. a) torus,  $m = 250\text{mV}$ . b) first torus breakdown with wrinkles,  $m = 371\text{mV}$ . c) period three chaotic phase locking,  $m = 390\text{mV}$ .

in  $x$ - $y$  configuration, the sequence of the maxima  $I_{pk}(n)$  versus  $I_{pk}(n-1)$  is shown in Fig. 5a.

The Poincaré section is a closed line, a clear indication that the associated attractor in the phase space is a two dimensional torus. Using the amplitude  $m$  of the driving frequency as the control parameter it is possible to observe the transition to chaos throughout torus breakdown, that is, the losing of its smoothness as observed in the Poincaré section shown in Fig. 5b. In such a case, the Poincaré section becomes more and



more complex developing wrinkles and discontinuities [9]. In Fig. 6 the corresponding phase-space attractor are reported. The associated colors in both Poincaré sections and Phase-Space attractors provide us an indication of how many times a certain screen pixel is visited. The large blue zone in the attractors displays that the current and voltage pulses are different in their heights. For the Poincaré section of the torus (see Fig. 5a) the uniform red color shows that all the red pixels all almost continuously marked. Wrinkles and discontinuity are clearly shown in Fig. 5b. In correspondence of the wrinkle where three transverse branches appear the corresponding return map is no more invertible and the associated temporal behavior is chaotic. A slight increase of the control parameter  $m$  leads to another chaotic structure displayed in Fig. 5c. The Poincaré section seems to indicate a locking condition with period three, however the transverse structure of the section leads to temporal chaos. In particular we observe with more evidence the presence of a further frequency which is associated with the small oscillations around the steady state of the current pulse.

#### IV. CONCLUSION

The possibility to follow in real time the transition to chaos is of fundamental interest in various fields that range from physics, electronics and biological systems. Poincaré section is a suitable tool for this purpose. The proposed analog method allows us to visualize in real time the Poincaré sections, showing the main features of the breaking of a two-dimensional torus occurring in a driven UJT relaxation oscillator. Although there exist software that calculates Poincaré sections when the intrinsic dynamics becomes fast and characterized by spiking behavior the analog implementation still presents advantages. On the other hand hardware implementations still have room to be improved using faster Sample/Hold components as for example DS1843 by Maxim which is one order of magnitude faster than SMP04. Software implementations of Poincaré sections could be foreseen in the next future directly installed on board of digital oscilloscopes allowing real-time analysis of fast dynamic systems.

Considering the analogy of the spiking behavior of our relaxation oscillator with a single neuron considered as a system collecting information (integrate process) and sending it to the nearby neurons (fire process) when a given threshold is overcome, we retain the proposed method of potential interest in neuroscience [31].

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