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ON GLOBAL OPTIMALITY OF LOCAL OPTIMA IN
QUADRATIC 0-1 PROGRAMMING

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**On global optimality of local optima in
quadratic 0-1 programming**

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Abstract

Many algorithms for quadratic 0-1 programming find a global solution by searching some or all the local solutions of the problem. In this paper we present some necessary and/or sufficient conditions for global optimality of local optima in quadratic 0-1 programming.

Key Words: Quadratic 0-1 programming, global optimization, local optima .

1. Introduction and notations

Consider the quadratic 0-1 minimization problem

$$(1) \quad \min f(x) \quad \text{s.t.} \quad x \in B^n = \{0,1\}^n,$$

where $f(x) = x^T C x + q^T x$, $C \in \mathbf{R}^{n \times n}$ and $q \in \mathbf{R}^n$. Any unconstrained minimization problem in 0-1 variables, as well as many other problems in combinatorial optimization, can be rather naturally formulated as (1) (see [3,5]). Unfortunately, (1) is an NP-hard problem [1] and therefore it is not easy to solve when n is large. One of the difficulties involved in its solution is due to the fact that in general f may have many local minima over B^n that are not global (see [6]). However, f has only one local minimum, which is obviously global, when it belongs to some special class of functions [4].

Let us mention that from a complexity viewpoint it is not yet clear whether the problem of finding a local minimum for f is polynomial, but in practice a simple greedy algorithm may be employed to compute a local minimum in a quite efficient manner [2,7].

If a local minimum point for f is available, the question arises whether it is also a global minimum. Also for this problem the theoretical complexity is not known. However, in the following section we will present some necessary and/or sufficient conditions which may be employed for testing global minimality of a local minimum point for f over B^n .

Let us now introduce some notations and some preliminary remarks. Let I denote the set of indices $\{1, \dots, n\}$. Given two points $x, y \in B^n$ the Hamming distance $d^H(x, y)$ between them is equal to the number of indices $i \in I$ such that $x_i \neq y_i$. A point $x \in B^n$ is called a k -local minimum point for f if $f(x) \leq f(y)$, for every $y \in B^n$ such that $d^H(x, y) \leq k$. If $k = 1$, then x is called simply a

local minimum point. Note that an n -local minimum point is obviously a global minimum point for f over B^n . With every point $x \in B^n$ we associate the point $x' \in \{-1, 1\}^n$ such that $x'_k = 1 - 2x_k$, for $k = 1, \dots, n$. Furthermore, we define $x^k = x + x'_k e_k$, where e_k is the k^{th} unit vector in \mathbb{R}^n , for $k = 1, \dots, n$. The discrete first order derivatives of f are defined as follows

$$d_k(x) = f(x^k) - f(x), \quad k = 1, \dots, n.$$

It is trivial to see that x is a local minimum for f if and only if $d_k(x) \geq 0$ for every k . Let $x, y \in B^n$ and set $S = \{i \in I : x_i \neq y_i\}$. Then,

$$(2) \quad f(y) - f(x) = \sum_{i \in S} d_i(x) + \sum_{i, j \in S} c_{ij} x'_i x'_j.$$

2. Necessary and sufficient conditions for global optimality

If $x \in B^n$ is a k -local minimum point for f for some $k \leq n$, then clearly x is an h -local minimum point for every $h \leq k$. In particular, if x is a solution of (1), then x is a k -local minimum point for every $k \leq n$. In the next proposition we present a necessary condition for k -local minimality of x .

Proposition 1

If $x \in B^n$ is a k -local minimum point for f , then for every $2 \leq h \leq k$ the following relation holds :

$$(3) \quad \exists S \subset I \text{ such that } |S| = h \text{ and } \sum_{i, j \in S} c_{ij} x'_i x'_j \geq - \frac{(h-1)}{(k-1)} \sum_{i \in S} d_i(x).$$

Proof

Assume on the contrary that for some $h \leq k$ and for every $S \subset I$ with

$|S| = h$ one has

$$\sum_{i,j \in S} c_{ij} x'_i x'_j < - \frac{h-1}{k-1} \sum_{i \in S} d_i(x) .$$

Take any set T such that $T \subset I$ and $|T| = k$. Then

$$\begin{aligned} \sum_{i,j \in T} c_{ij} x'_i x'_j &= \binom{k-2}{h-2}^{-1} \sum_{\substack{S \subset T \\ |S|=h}} \sum_{i,j \in S} c_{ij} x'_i x'_j < \\ < - \frac{h-1}{k-1} \binom{k-2}{h-2}^{-1} \sum_{\substack{S \subset I \\ |S|=h}} \sum_{i \in S} d_i(x) = - \frac{h-1}{k-1} \binom{k-2}{h-2}^{-1} \binom{k-1}{h-1} \sum_{i \in T} d_i(x) = - \sum_{i \in T} d_i(x) . \end{aligned}$$

If we set $y = x + \sum_{k \in T} x'_k e^k$, then the above inequality implies $f(y) < f(x)$, contradicting the k -local minimality of x . \square

Corollary 2

A point $x \in B^n$ is a global minimum point for f over B^n if and only if x is a local minimum point and satisfies (3) for every $2 \leq h \leq k \leq n$.

Proof

The necessity of (3) is a trivial consequence of proposition 1. On the other hand, if (3) holds for every $2 \leq h = k \leq n$, then (2) implies that x is a global minimum point for f . \square

We now present a condition that guarantees k -local minimality of an h -local minimum point with $h < k$.

Proposition 3

Let x be an $(h-1)$ -local minimum point for f over B^n for some $h \geq 2$ and take $k \geq h$. If

$$(4) \quad \sum_{i,j \in S} c_{ij} x'_i x'_j \geq - \frac{h-1}{k-1} \sum_{i \in S} d_i(x) , \quad \forall S \subset I \text{ such that } |S| = h ,$$

then x is a k -local minimum point for f on B^n .

Proof

Let y be any point in B^n such that $d^H(x,y)=m \leq k$ and set $T = \{i \in I : x_i \neq y_i\}$.

If $m \leq h-1$, then $f(y) \geq f(x)$ because x is an $(h-1)$ -local minimum. Hence we assume that $m \geq h$. Taking into account (4), the following relations hold

$$\begin{aligned} \sum_{i,j \in T} c_{ij} x'_i x'_j &= \binom{m-2}{h-2}^{-1} \sum_{\substack{S \subset T \\ |S|=h}} \sum_{i,j \in S} c_{ij} x'_i x'_j \geq \\ &\geq - \frac{h-1}{k-1} \binom{m-2}{h-2}^{-1} \sum_{\substack{S \subset T \\ |S|=h}} \sum_{i \in S} d_i(x) = - \frac{h-1}{k-1} \binom{m-2}{h-2}^{-1} \binom{m-1}{h-1} \sum_{i \in T} d_i(x) = \\ &= - \frac{m-1}{k-1} \sum_{i \in T} d_i(x) \geq - \sum_{i \in T} d_i(x) . \end{aligned}$$

Hence, by (2)

$$f(y) \geq f(x) . \quad \square$$

Corollary 4

If x is an $(h-1)$ -local minimum for f over B^n for some $h \geq 2$ and (3) holds with $k = n$, then x is a global minimum point for f over B^n .

Corollary 5

Under the assumptions of proposition 3, if $\sum_{i \in I} x_i = p \leq m$, then x is a global minimum point for f over the set $K_m = \{x \in B^n : \sum_{i \in I} x_i \leq m\}$ for every m such that $\min\{m+p, n\} \leq k$.

Proof

It is sufficient to observe that the assumption on m implies $d^H(x,y) \leq k$ for every $y \in K_m$. \square

Note that if x is an $(h-1)$ -local minimum point, then (4) holds with $k = h$ if and only if x is an h -local minimum. This trivially implies that x is a k -local minimum for f over B^n if and only if the hypothesis of proposition 3 holds for some $h < k$. However, when h is fixed and $k > h$, condition (4) is only sufficient for the k -local minimality of x .

It is easy to check that, even if $h=n$, conditions (3) and (4) can be tested in polynomial time for any fixed k . However, the number of operations required increases exponentially with h . Hence, these conditions may be practically useful only for small values of h .

Note also that if (4) does not hold for every set $S \subset I$, it may still be valid for all $S \subset J$, where $J \subset I$. In such a case, if x is an $(h-1)$ -local minimum on the subcube $B_J^n(x) = \{y \in B^n : y_i = x_i, \forall i \in J\}$, we can clearly conclude that x is a k -local minimum on the same subcube. This remark may be employed in a branch-and-bound algorithm for solving (1).

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