THE 0-FRACTIONAL PERIMETER BETWEEN FRACTIONAL PERIMETERS AND RIESZ POTENTIALS

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Abstract. This paper provides a unified point of view on fractional perimeters and Riesz potentials. Denoting by H^σ - for $\sigma \in (0,1)$ - the σ-fractional perimeter and by J^σ - for $\sigma \in (-d,0)$ - the σ-Riesz energies acting on characteristic functions, we prove that both functionals can be seen as limits of renormalized self-attractive energies as well as limits of repulsive interactions between a set and its complement.

We also show that the functionals H^σ and J^σ , up to a suitable additive renormalization diverging when $\sigma \to 0$, belong to a continuous one-parameter family of functionals, which for $\sigma = 0$ gives back a new object we refer to as 0-*fractional perimeter*. All the convergence results with respect to the parameter σ and to the renormalization procedures are obtained in the framework of Γ-convergence. As a byproduct of our analysis, we obtain the isoperimetric inequality for the 0-fractional perimeter.

Keywords. Fractional perimeters, Riesz kernels, Γ-convergence, isoperimetric inequality.

AMS subject classifications. 49J45, 49J10, 49Q10, 49J99.

Contents

Introduction

Given $\sigma \in (0,1)$, the σ-fractional perimeter [7] of a measurable set $E \subseteq \mathbb{R}^d$ is defined as

$$H^\sigma(E) := \int_E \int_{\mathbb{R}^d \setminus E} \frac{1}{|x-y|^{d+\sigma}} \, \mathrm{d}y \, \mathrm{d}x \,. \tag{0.1}$$

Moreover, given $\sigma \in (-d, 0)$ one can define σ-Riesz energy functionals as

$$J^\sigma(E) := -\int_E \int_E \frac{1}{|x-y|^{d+\sigma}} \, \mathrm{d}y \, \mathrm{d}x \,. \tag{0.2}$$

The main purpose of this paper is to introduce meaningful extensions of both functionals for all $\sigma \in (-d, 1)$. Clearly, plugging $\sigma \in (-d, 0]$ in (0.1), as well as $\sigma \in [0, 1)$ in (0.2), would give back infinite tail and core energies, respectively. Moreover, for every set E with $0 < |E| < +\infty$ we have

$$\lim_{\sigma \to 0^+} H^\sigma(E) = +\infty, \qquad \lim_{\sigma \to 0^-} J^\sigma(E) = -\infty \,.$$

A natural question is then to understand the blow up scaling of these energies as $\sigma \to 0$. In [17, 29, 4], the asymptotics of the σ-fractional perimeter and of the corresponding σ-fractional curvature inside a regular domain Ω, as $\sigma \to 0^+$, have been studied. In [17], the authors have proven that, under suitable conditions on the set E, the σ-fractional perimeter of E scaled by σ, converge to $d\omega_d|E|$ as $\sigma \to 0^+$, where ω_d denotes the measure of the unit ball in $\mathbb{R}^d$.

In this paper we provide a unified point of view on fractional perimeters and Riesz potentials and, in particular, we develop a "first order" Γ-convergence analysis (see [3]) of the functionals H^σ and J^σ as $\sigma \to 0$. We first introduce suitable regularization procedures, usually referred to as the *core radius approach*, to cut off the tail and the core energy from H^σ and J^σ respectively, extending these functionals to all $\sigma \in (-d, 1)$, including $\sigma = 0$, by setting

$$\begin{aligned} H^\sigma_\rho(E) &:= \int_E \int_{B_\rho(x)\setminus E} \frac{1}{|x-y|^{d+\sigma}}\, \mathrm{d}y\, \mathrm{d}x\,, \\ J^\sigma_\rho(E) &:= -\int_E \int_{E\setminus B_\rho(x)} \frac{1}{|x-y|^{d+\sigma}}\, \mathrm{d}y\, \mathrm{d}x\,. \end{aligned} \tag{0.3}$$

For $\sigma \in (0, 1)$, the functionals H^σ_ρ converge to the σ-fractional perimeters as $\rho \to +\infty$, while for $\sigma \in (-d, 0)$, the functionals J^σ_ρ converge to σ-Riesz potentials as $\rho \to 0^+$. Clearly, in the remaining range of parameters these functionals still diverge.

Then, we introduce suitable renormalized functionals, removing a tail energy from H^σ_ρ, and adding a core energy to J^σ_ρ. More precisely, for every $\sigma \in (-d, 1)$, for every $\rho > 0$, and for every set $E \subset \mathbb{R}^d$ with finite measure we set

$$\hat H^\sigma_\rho(E) := H^\sigma_\rho(E) - \gamma^\sigma_\rho|E|\,, \qquad \hat J^\sigma_\rho(E) := J^\sigma_\rho(E) - \gamma^\sigma_\rho|E|\,,$$

where the constant γ^σ_ρ is defined in (1.3). These renormalized functionals $\hat H^\sigma_\rho$ and $\hat J^\sigma_\rho$ converge, for all $\sigma \in (-d, 1) \setminus \{0\}$ to fractional and Riesz type functionals. Setting $\gamma^\sigma := \frac{d\omega_d}{\sigma}$, for $\sigma \in (-d, 1) \setminus \{0\}$, and

$$\hat H^\sigma(E) := H^\sigma(E) - \gamma^\sigma|E|\,, \text{ for } \sigma \in (0, 1)\,, \qquad \hat J^\sigma(E) := J^\sigma(E) - \gamma^\sigma|E|\,, \text{ for } \sigma \in (-d, 0)\,,$$

we have that for $\sigma \in (-d, 0)$, $\hat H^\sigma_\rho \to \hat J^\sigma$ as $\rho \to +\infty$ (and clearly $\hat H^\sigma_\rho \to \hat H^\sigma$ for $\sigma \in (0, 1)$), while for $\sigma \in (0, 1)$, $\hat J^\sigma_\rho \to \hat H^\sigma$ as $\rho \to 0^+$ (and clearly $\hat J^\sigma_\rho \to \hat J^\sigma$ for $\sigma \in (-d, 0)$).

These two families $\hat H^\sigma$ and $\hat J^\sigma$ of renormalized energies are separated by the limit case $\sigma = 0$. Indeed, also for $\sigma = 0$, the following limits exist

$$\hat H^0(E) := \lim_{\rho\to+\infty} \hat H^0_\rho(E) = \lim_{\rho\to0^+} \hat J^0_\rho(E) = H^0_1(E) + J^0_1(E)\,,$$

where $H^0_1(E)$ and $J^0_1(E)$ correspond to the functionals defined in (0.3) with $\rho = 1$ and $\sigma = 0$. We refer to the functional $\hat H^0$ as 0-*fractional perimeter* since this is formally the limit of $\hat H^\sigma$ as $\sigma \to 0^+$. Indeed, we shall show that

$$\lim_{\sigma\to0^+} \hat H^\sigma = \lim_{\sigma\to0^-} \hat J^\sigma = \hat H^0\,,$$

where the limits are understood in the sense of Γ-convergence; therefore, we can set $\hat J^0 := \hat H^0$ and understand the 0-fractional perimeter also as a 0-*Riesz functional*.

This functional is closely related to the notion of logarithmic laplacian L_Δ introduced in [12]. There, the authors prove that L_Δ, computed on regular enough functions, is the pointwise limit, as $\sigma \to 0^+$, of a suitable renormalization of the fractional laplacian $(-\Delta)^{\frac{\sigma}{2}}$. While [12] deals with the functional analytic framework of the operator L_Δ, our paper focuses on the geometric framework of characteristic functions, and specifically on the variational analysis of the σ-fractional perimeter as $\sigma \to 0$. In fact, our analysis consists in a Γ-convergence approach to the two-parameter families of functionals introduced above, showing that $\hat H^\sigma_R$ and $\hat J^\sigma_r$ are continuous, in the sense of Γ-convergence, with respect to variations of all the parameters $\sigma \in (-d, 1)$, $r \in [0, +\infty)$ and $R \in (0, +\infty]$ (see Theorems 6.3 and 6.5).

Our Γ-convergence results are completed with compactness properties for sequences with equi-bounded energy. It is well known that families of equi-bounded sets of equi-bounded perimeter are pre-compact in L^1, and such property extends to fractional perimeters. Here we show the same compactness property also for the new 0-fractional perimeter. In fact, we can deal also with the case of varying parameters $\sigma \in [0, 1)$, and $R \in (0, +\infty]$ (see Theorem 5.4). Analogous compactness results hold for the functionals $\hat J^\sigma_r$ for $\sigma \in [0, 1)$ when $r \to 0^+$, for $\hat H^\sigma_R$ when $\sigma \to 0^-$, and for $\hat J^\sigma_r$ when $r \to 0^+$ and $\sigma \to 0^-$ simultaneously.

In all the other cases we expect only weak$*$ compactness in the family of L^1 densities. We address the interested reader to [24, 14], where the authors provide compactness results for nonlocal Sobolev spaces, for a large class of non-integrable kernels.

Summarizing, the main novelty of our approach consists in casting fractional perimeters into the framework of self-attractive Riesz potentials, and viceversa. The underlying idea is that fractional perimeters, defined through interaction potentials of the set with its complement, can be formally seen as the opposite of the self-interaction of the set with itself, but with an infinite core energy. This heuristic point of view is formalized by our analysis through rigorous renormalization procedures. The advantage of this approach is that one can exploit classical techniques for self-attracting energies to the framework of fractional perimeters and to the new 0-fractional perimeter. A clarifying example of this fact is given by the isoperimetric inequality. Indeed, for self-attractive interaction potentials the celebrated Riesz inequality states that the energy is maximized on radially symmetric functions and, under L^∞ constraint, by characteristic functions of balls. In the terminology of fractional perimeters, this is nothing but the fractional isoperimetric inequality, proven in [21, 22, 18, 19]. Here we provide a self-contained proof based on Riesz inequality, which provides the σ-fractional isoperimetric inequality and its stability also in the limit case $\sigma = 0$. We refer to [9, Proposition 3.1] for a similar result in the case of nonlocal perimeters with a general radially symmetric interaction kernel.

Finally, we point out that the 0-fractional perimeter fits into the class of nonlocal perimeters introduced in [11], up to the fact that it is, in general, non-positive. Therefore, it would be interesting to study the corresponding 0-fractional mean curvature flow. We notice that σ-fractional mean curvature flows (for $\sigma \in (0,1)$) are nowadays relatively well understood (see [23, 11]), and that their limit as $\sigma \to 1$ gives back the classical mean curvature flow [23]. This is consistent with the fact that the σ-fractional perimeter, rescaled by $(1-\sigma)$, converges to $d\omega_d$ times the Euclidean perimeter (see [1], for a Γ-convergence result, and the references therein). A natural problem is to study the limit of (suitably rescaled) σ-fractional mean curvature flows as $\sigma \to 0^+$. On the one hand, one could expect that such flows, suitably reparametrized in time, converge to evolutions of sets with constant normal velocity; on the other hand, gradient flows of renormalized σ-fractional perimeters, as well as σ-fractional mean curvature flows with a volume constraint, could converge to the 0-fractional mean curvature flow, as $\sigma \to 0^+$.

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1. Renormalized fractional perimeters and renormalized Riesz energies

Let $\mathcal{M}(\mathbb{R}^d)$ be the family of measurable sets in $\mathbb{R}^d$ and let

$$\mathcal{M}_{\mathrm{f}}(\mathbb{R}^d) := \{E \in \mathcal{M}(\mathbb{R}^d) \, : \, |E| < +\infty\} .$$

For $\sigma \in (0,1)$, the *σ-fractional perimeter* $H^\sigma \colon \mathcal{M}(\mathbb{R}^d) \to [0,+\infty]$ is defined in (0.1), while for every $\sigma \in (-d,0)$, the *σ-Riesz energy* $J^\sigma \colon \mathcal{M}(\mathbb{R}^d) \to [-\infty, 0]$ is defined in (0.2).

For every $\sigma \neq 0$ we define $\gamma^\sigma := \frac{d\omega_d}{\sigma}$. Letting $E \in \mathcal{M}_{\mathrm{f}}(\mathbb{R}^d)$, we recall that

$$\hat{H}^\sigma(E) := H^\sigma(E) - \gamma^\sigma |E| , \qquad \hat{J}^\sigma(E) := J^\sigma(E) - \gamma^\sigma |E| . \tag{1.1}$$

Remark 1.1. By definition, $\hat{H}^\sigma \colon \mathcal{M}_{\mathrm{f}}(\mathbb{R}^d) \to (-\infty, +\infty]$. Moreover, by Riesz inequality (see Theorem A.1), we have

$$J^\sigma(E) \geq - \int_{B^{|E|}} \int_{B^{|E|}} \frac{1}{|x-y|^{d+\sigma}} \, \mathrm{d}y \, \mathrm{d}x > -\infty , \tag{1.2}$$

where $B^{|E|}$ denotes the ball with center at 0 and volume equal to $|E|$. It follows that $\hat{J}^\sigma \colon \mathcal{M}_{\mathrm{f}}(\mathbb{R}^d) \to \mathbb{R}$.

We introduce two types of approximations of the functionals $\hat{H}^\sigma$ and $\hat{J}^\sigma$ above.

Let $\sigma \in (-d, 1)$. For every $R > 0$ we define the functionals $H^\sigma_R \colon \mathcal{M}(\mathbb{R}^d) \to [0, +\infty]$ as

$$H^\sigma_R(E) := \int_E \int_{B_R(x) \setminus E} \frac{1}{|x-y|^{d+\sigma}} \, \mathrm{d}y \, \mathrm{d}x .$$

Moreover, for every $\rho>0$ we set

$$\gamma^\sigma_\rho:=\begin{cases} d\omega_d\dfrac{1-\rho^{-\sigma}}{\sigma} & \text{if } \sigma\neq 0\,,\\ d\omega_d\log\rho & \text{if } \sigma=0\,,\end{cases} \tag{1.3}$$

and for every $R>0$ we introduce the functionals $\hat H^\sigma_R$ defined by

$$\hat H^\sigma_R(E):=H^\sigma_R(E)-\gamma^\sigma_R|E|\,.$$

Notice that if $\sigma\in[0,1)$, then $\hat H^\sigma_R:\mathcal{M}_{\rm f}(\mathbb{R}^d)\to(-\infty,+\infty]$, whereas if $\sigma\in(-d,0)$ then $\hat H^\sigma_R:\mathcal{M}_{\rm f}(\mathbb{R}^d)\to\mathbb{R}$.

Furthermore, for every $r>0$ we define the functionals $J^\sigma_r:\mathcal{M}(\mathbb{R}^d)\to[-\infty,0]$ as

$$J^\sigma_r(E):=-\int_E\int_{E\setminus B_r(x)}\frac{1}{|x-y|^{d+\sigma}}\,\mathrm{d}y\,\mathrm{d}x \tag{1.4}$$

and the renormalized functionals $\hat J^\sigma_r:\mathcal{M}_{\rm f}(\mathbb{R}^d)\to\mathbb{R}$ as

$$\hat J^\sigma_r(E):=J^\sigma_r(E)-\gamma^\sigma_r|E|,\quad \text{for all } E\in\mathcal{M}_{\rm f}(\mathbb{R}^d).$$

Remark 1.2. Let $R>0$. By Fatou Lemma, it immediately follows that the functionals H^σ, $\hat H^\sigma$ (for $\sigma\in(0,1)$) and H^σ_R, $\hat H^\sigma_R$ (for $\sigma\in(-d,1)$) are lower semicontinuous with respect to the strong L^1 convergence of characteristic functions.

Lemma 1.3. *Let $r>0$. The functionals J^σ, $\hat J^\sigma$ (for $\sigma\in(-d,0)$) and J^σ_r, $\hat J^\sigma_r$ (for $\sigma\in(-d,1)$) are continuous with respect to the strong L^1 convergence of characteristic functions.*

Proof. We first prove the continuity of the functionals J^σ_r and $\hat J^\sigma_r$ for every $\sigma\in(-d,1)$. Moreover we notice that the functionals $\hat J^\sigma_r$ are nothing but a continuous perturbation of the functionals J^σ_r so that it is enough to prove only the continuity of J^σ_r . For all $\eta_1,\,\eta_2\in L^1(\mathbb{R}^d;[0,1])$, we set

$$\mathcal{J}^\sigma_r(\eta_1,\eta_2):=-\int_{\mathbb{R}^d}\eta_1(x)\left[\int_{\mathbb{R}^d\setminus B_r(x)}\frac{\eta_2(y)}{|x-y|^{d+\sigma}}\,\mathrm{d}y\right]\,\mathrm{d}x\,. \tag{1.5}$$

Clearly, $\mathcal{J}^\sigma_r$ is bilinear and continuous, i.e.,

$$|\mathcal{J}^\sigma_r(\eta_1,\eta_2)|\le r^{-(d+\sigma)}\|\eta_1\|_{L^1}\,\|\eta_2\|_{L^1}\qquad\text{for all } \eta_1,\,\eta_2\in L^1(\mathbb{R}^d;[0,1])\,. \tag{1.6}$$

It follows that $\mathcal{J}^\sigma_r(\eta_n,\eta_n)\to\mathcal{J}^\sigma_r(\eta,\eta)$ as $n\to+\infty$ for every $\{\eta_n\}_{n\in\mathbb{N}}\subset L^1(\mathbb{R}^d;[0,1])$ converging to some $\eta\in L^1(\mathbb{R}^d;[0,1])$. Since

$$J^\sigma_r(E)=\mathcal{J}^\sigma_r(\chi_E,\chi_E)\qquad\text{for all } E\in\mathcal{M}_{\rm f}(\mathbb{R}^d)\,,$$

we get the claim.

Now we prove the continuity of the functionals J^σ for $\sigma\in(-d,0)$, which trivially implies also the continuity of $\hat J^\sigma$. In such a case, for every $\eta_1,\,\eta_2\in L^1(\mathbb{R}^d;[0,1])$, we set $R(\eta_2):=(\frac{\|\eta_2\|_{L^1}}{\omega_d})^{\frac1d}$; by an easy rearrangement argument we have

$$\begin{aligned}\int_{\mathbb{R}^d}\eta_1(x)\left[\int_{\mathbb{R}^d}\frac{\eta_2(y)}{|x-y|^{d+\sigma}}\,\mathrm{d}y\right]\,\mathrm{d}x\le&\int_{\mathbb{R}^d}\eta_1(x)\left[\int_{\mathbb{R}^d}\frac{\chi_{B_{R(\eta_2)}(x)}(y)}{|x-y|^{d+\sigma}}\,\mathrm{d}y\right]\,\mathrm{d}x\\ =&\|\eta_1\|_{L^1}d\omega d(-\sigma)[R(\eta_2)]^{-\sigma}=d\omega_d^{1+\frac{\sigma}{d}}(-\sigma)\|\eta_1\|_{L^1}\|\eta_2\|_{L^1}^{-\frac{\sigma}{d}};\end{aligned}$$

again such an estimate yields the claim. □

Remark 1.4. We notice that, for every $\sigma\in(-d,1)$, the renormalization constants introduced above can be seen either as core or tail energy terms; in fact, they are nothing but

$$\gamma^\sigma=\int_{\mathbb{R}^d\setminus B_1}\frac{1}{|z|^{d+\sigma}}\,\mathrm{d}z\qquad\text{if }\sigma\in(0,1),\qquad \gamma^\sigma=-\int_{B_1}\frac{1}{|z|^{d+\sigma}}\,\mathrm{d}z\qquad\text{if }\sigma\in(-d,0),$$

$$\gamma^\sigma_R=\int_{B_R\setminus B_1}\frac{1}{|z|^{d+\sigma}}\,\mathrm{d}z\qquad\text{if } R\ge 1,\qquad \gamma^\sigma_r=-\int_{B_1\setminus B_r}\frac{1}{|z|^{d+\sigma}}\,\mathrm{d}z\qquad\text{if } r\le 1\,,$$

where here and below $B_\rho := B_\rho(0)$ for every $\rho > 0$. It follows that for every $0 < \rho_1 < \rho_2 < +\infty$

$$\gamma^\sigma_{\rho_2} - \gamma^\sigma_{\rho_1} = \int_{B_{\rho_2}\setminus B_{\rho_1}} \frac{1}{|z|^{d+\sigma}}\,\mathrm{d}z\,. \tag{1.7}$$

Lemma 1.5. *For all $\sigma \in (-d, 1)$ and for all $E \in \mathcal{M}_{\mathrm{f}}(\mathbb{R}^d)$, the quantity*

$$H^\sigma_\rho(E) + J^\sigma_\rho(E) - \gamma^\sigma_\rho |E| \tag{1.8}$$

is independent of ρ. Moreover,

$$\hat{H}^\sigma = H^\sigma_1 + J^\sigma_1 \quad \textit{for } \sigma \in (0,1)\,, \qquad \hat{J}^\sigma = H^\sigma_1 + J^\sigma_1 \quad \textit{for } \sigma \in (-d,0)\,. \tag{1.9}$$

Proof. For every $0 < r < R < +\infty$, we have

$$\begin{aligned}
H^\sigma_R(E) =& H^\sigma_r(E) + \int_E \int_{(B_R(x)\setminus B_r(x))\setminus E} \frac{1}{|x-y|^{d+\sigma}}\,\mathrm{d}y\,\mathrm{d}x\\
=& H^\sigma_r(E) + \gamma^\sigma_R|E| - \gamma^\sigma_r|E| - \int_E\int_{E\setminus B_r(x)} \frac{1}{|x-y|^{d+\sigma}}\,\mathrm{d}y\,\mathrm{d}x + \int_E\int_{E\setminus B_R(x)} \frac{1}{|x-y|^{d+\sigma}}\,\mathrm{d}y\,\mathrm{d}x\\
=& H^\sigma_r(E) + \gamma^\sigma_R|E| - \gamma^\sigma_r|E| + J^\sigma_r(E) - J^\sigma_R(E)\,,
\end{aligned}$$

whence we deduce (1.8).

Let us pass to the proof of (1.9). First, we notice that

$$\lim_{R\to+\infty} J^\sigma_R(E) = 0 \quad \text{for all } \sigma \in [0,1)\,, \qquad \lim_{r\to 0} H^\sigma_r(E) = 0 \quad \text{for all } \sigma \in (-d,0)\,. \tag{1.10}$$

Then, recalling also Remark 1.4, (1.9) can be easily deduced by taking the limits as $\rho \to 0$ and $\rho \to +\infty$ in (1.8). □

2. Convergence of $\hat{H}^\sigma_R$ as $R \to +\infty$

In this section, we establish the convergence of $\hat{H}^\sigma_R$ as $R \to +\infty$. We distinguish among three cases: $\sigma \in (-d, 0)$, $\sigma \in (0, 1)$, $\sigma = 0$.

We start by discussing the case $\sigma \in (-d, 0)$.

Proposition 2.1. *Let $\sigma \in (-d, 0)$. For every $E \in \mathcal{M}_{\mathrm{f}}(\mathbb{R}^d)$, $H^\sigma_R(E)$ and γ^σ_R are monotonically increasing with respect to R and*

$$\lim_{R\to+\infty} H^\sigma_R(E) = \lim_{R\to+\infty} \gamma^\sigma_R = +\infty\,. \tag{2.1}$$

Moreover, $\hat{H}^\sigma_R(E)$ is monotonically non-increasing with respect to R and

$$\hat{J}^\sigma(E) = \lim_{R\to+\infty} \hat{H}^\sigma_R(E) \qquad \textit{for all } E \in \mathcal{M}_{\mathrm{f}}(\mathbb{R}^d)\,. \tag{2.2}$$

Furthermore, for all $\varepsilon > 0$ the convergence in (2.2) is uniform with respect to $\sigma \in [-d+\varepsilon, 0)$; precisely, for every $R > 0$

$$0 \le \hat{H}^\sigma_R(E) - \hat{J}^\sigma(E) \le \frac{|E|^2}{R^{d+\sigma}} \qquad \textit{for all } E \in \mathcal{M}_{\mathrm{f}}(\mathbb{R}^d)\,. \tag{2.3}$$

Finally, $\hat{J}^\sigma$ is the Γ-limit of the functionals $\hat{H}^\sigma_R$ as $R \to +\infty$, with respect to the strong L^1 topology.

Proof. The limits in (2.1) are trivial consequences of the definitions of $H^\sigma_R(E)$ and γ^σ_R for $\sigma \in (-d, 0)$. Let now $R > 0$. By Remark 1.4, we have

$$\gamma^\sigma_R - \gamma^\sigma = \int_{B_R} \frac{1}{|z|^{d+\sigma}}\,\mathrm{d}z,$$

whence we deduce

$$\begin{aligned}
\hat{H}^\sigma_R(E) =& -\int_E\int_{B_R(x)\cap E} \frac{1}{|x-y|^{d+\sigma}}\,\mathrm{d}y\,\mathrm{d}x - \gamma^\sigma|E|\\
=& \hat{J}^\sigma(E) + \int_E\int_{E\setminus B_R(x)} \frac{1}{|x-y|^{d+\sigma}}\,\mathrm{d}y\,\mathrm{d}x\,;
\end{aligned} \tag{2.4}$$

since the last integral is monotonically non-increasing with respect to R, the same holds for $\hat{H}^\sigma_R(E)$; moreover, by the monotone convergence Theorem, we deduce that $\hat{H}^\sigma_R(E)$ converge to $\hat{J}^\sigma(E)$. By (2.4), it follows that

$$\hat{H}^\sigma_R(E) - \hat{J}^\sigma(E) = \int_E \int_{E\setminus B_R(x)} \frac{1}{|x-y|^{d+\sigma}}\,\mathrm{d}y\,\mathrm{d}x \le \frac{|E|^2}{R^{d+\sigma}}\,,$$

i.e., (2.3) holds. Finally, the Γ-convergence of $\hat{H}^\sigma_R$ to $\hat{J}^\sigma$ is an obvious consequence of (2.3). □

We now consider the case $\sigma \in (0,1)$. Both the result and its proof are fully analogous to those of Proposition 2.1; we only provide the corresponding statement.

Proposition 2.2. *Let $\sigma \in (0,1)$. For every $E \in \mathcal{M}(\mathbb{R}^d)$, $H^\sigma_R(E)$ is monotonically non-decreasing with respect to R and tends to $H^\sigma(E)$ as $R \to +\infty$.*

Moreover, γ^σ_R is monotonically increasing with respect to R and tends to γ^σ as $R \to +\infty$. As a consequence,

$$\hat{H}^\sigma(E) = \lim_{R\to+\infty} \hat{H}^\sigma_R(E) \qquad \text{for all } E \in \mathcal{M}_{\mathrm{f}}(\mathbb{R}^d). \tag{2.5}$$

Furthermore, $\hat{H}^\sigma_R(E)$ is monotonically non-increasing with respect to R, and the convergence in (2.5) is uniform with respect to σ; precisely, for every $R > 1$

$$0 \le \hat{H}^\sigma_R(E) - \hat{H}^\sigma(E) \le \frac{|E|^2}{R^{d+\sigma}} \qquad \text{for all } E \in \mathcal{M}_{\mathrm{f}}(\mathbb{R}^d)\,.$$

Finally, $\hat{H}^\sigma$ is the Γ-limit of the functionals $\hat{H}^\sigma_R$ as $R \to +\infty$, with respect to the strong L^1 convergence of characteristic functions.

We finally introduce the 0-fractional perimeter as the limit of the functionals $\hat{H}^0_R$ as $R \to +\infty$.

Proposition 2.3. *For every $E \in \mathcal{M}_{\mathrm{f}}(\mathbb{R}^d)$ the functionals $\hat{H}^0_R(E)$ are monotonically non-increasing with respect to R and*

$$\lim_{R\to+\infty} \hat{H}^0_R(E) = H^0_1(E) + J^0_1(E) =: \hat{H}^0(E)\,. \tag{2.6}$$

Finally, $\hat{H}^0$ is the Γ-limit of the functionals $\hat{H}^0_R$ as $R \to +\infty$, with respect to the strong L^1 convergence of characteristic functions.

Proof. Let $0 < R_1 < R_2 < +\infty$; then, recalling (1.7), we have

$$\begin{aligned}
\hat{H}^0_{R_2}(E) &= \hat{H}^0_{R_1}(E) + \int_E \left[\int_{(B_{R_2}(x)\setminus B_{R_1}(x))\setminus E} \frac{1}{|x-y|^d}\,\mathrm{d}y + \gamma^0_{R_1} - \gamma^0_{R_2}\right]\mathrm{d}x \\
&\le \hat{H}^0_{R_1}(E) + \int_E \left[\int_{B_{R_2}(x)\setminus B_{R_1}(x)} \frac{1}{|x-y|^d}\,\mathrm{d}y - d\omega_d \log\frac{R_2}{R_1}\right]\mathrm{d}x \\
&= \hat{H}^0_{R_1}(E)\,.
\end{aligned}$$

Therefore, $\hat{H}^0_R(E)$ are monotonically non-increasing with respect to R. Moreover, by Lemma 1.5, for every $R > 0$, we have

$$H^0_R(E) + J^0_R(E) - \gamma^0_R|E| = H^0_1(E) + J^0_1(E)\,,$$

whence, sending $R \to +\infty$ and recalling (1.10) we deduce (2.6). Finally, by Remark 1.2 and by Lemma 1.3, we have that the functional $\hat{H}^0$ is lower semicontinuous with respect to the strong L^1 convergence of the characteristic functions; therefore, by the monotonicity of $\hat{H}^0_R$ with respect to R and by [15, Proposition 5.7], we deduce the Γ-convergence result. □

Lemma 2.4. *For every $E \in \mathcal{M}_{\mathrm{f}}(\mathbb{R}^d)$ we have*

$$\hat{H}^0(E) \ge -|E|\omega_d \log\Big(\frac{|E|}{\omega_d}\Big).$$

Proof. For every $R>0$ we set $r_{R,E}(x) := (\frac{|E\cap B_R(x)|}{\omega_d})^{\frac{1}{d}}$. By Lemma A.5, we have

$$\begin{aligned} H^0_R(E) &= \int_E \int_{B_R(x)\setminus E} \frac{1}{|x-y|^d}\, \mathrm{d}y\, \mathrm{d}x \\ &\geq \int_E \int_{B_R(x)\setminus B_{r_{R,E}(x)}(x)} \frac{1}{|x-y|^d}\, \mathrm{d}y\, \mathrm{d}x = \int_E d\omega_d \log \frac{R}{r_{R,E}(x)}\, \mathrm{d}x \\ &= \gamma^0_R |E| - \omega_d \int_E \log \frac{|E\cap B_R(x)|}{\omega_d}\, \mathrm{d}x\,, \end{aligned}$$

whence the claim follows by sending $R\to+\infty$ and using Proposition 2.3 and the monotone convergence Theorem. □

Definition 2.5. We refer to the functional $\hat{H}^0 : \mathcal{M}_{\mathrm{f}}(\mathbb{R}^d) \to (-\infty,+\infty]$ as 0-*fractional perimeter*.

We observe that, as a consequence of the definition above and of [12, formula (1.4)], we can write the 0-fractional perimeter of E as

$$\hat{H}^0(E) = \frac{1}{c_d}\int_E L_\Delta \chi_E(x)\, \mathrm{d}x - \frac{\rho_d}{c_d}|E|;$$

here L_Δ denotes the logarithmic laplacian introduced in [12], $c_d := \frac{2}{d\omega_d}$ and $\rho_d := 2\log 2 + \psi(\frac{d}{2}) + \Gamma'(1)$, where Γ is the Euler Gamma function and $\psi := \frac{\Gamma'}{\Gamma}$ is the Digamma function (see [12] for further details).

Remark 2.6. By (1.9) and (2.6) it immediately follows that, for every $E\in\mathcal{M}_{\mathrm{f}}(\mathbb{R}^d)$ with positive measure, $\hat{H}^\sigma(E)$ is increasing with respect to $\sigma\in[0,1)$.

Remark 2.7. If E is a bounded set, by arguing as in the proof of Proposition 2.3, one immediately has

$$\hat{H}^0_{R_1}(E) = \hat{H}^0_{R_2}(E) \qquad \text{for every } R_2 > R_1 > \mathrm{diam}\,(E)\,,$$

whence

$$\hat{H}^0(E) = \hat{H}^0_R(E) \qquad \text{for every } R > \mathrm{diam}\,(E)\,.$$

Analogously, one can show that if E is a bounded set, for every $R > \mathrm{diam}\,(E)$ it holds

$$\hat{J}^\sigma(E) = \hat{H}^\sigma_R(E) \quad (\text{for } \sigma\in(-d,0)) \qquad \text{and} \qquad \hat{H}^\sigma(E) = \hat{H}^\sigma_R(E) \quad (\text{for } \sigma\in(0,1))\,.$$

Lemma 2.8. *Let $\sigma\in(-d,1)$. For every $R>0$ the functionals H^σ_R are submodular, i.e., for every $E_1,E_2\in\mathcal{M}_{\mathrm{f}}(\mathbb{R}^d)$*

$$H^\sigma_R(E_1\cup E_2) + H^\sigma_R(E_1\cap E_2) \leq H^\sigma_R(E_1) + H^\sigma_R(E_2)\,. \tag{2.7}$$

As a consequence, also the functionals $\hat{H}^\sigma_R$ are submodular.

Moreover, the functionals $\hat{H}^\sigma$ are submodular for $\sigma\in[0,1)$ and the functionals $\hat{J}^\sigma$ are submodular for $\sigma\in(-d,0)$.

Finally, for every $\sigma\in(-d,1)$ and for every $r>0$, also the functionals J^σ_r and the functionals $\hat{J}^\sigma_r$ are submodular.

Proof. Fix $R>0$ and let $E_1,E_2\in\mathcal{M}_{\mathrm{f}}(\mathbb{R}^d)$. Trivially,

$$|E_1\cup E_2| + |E_1\cap E_2| = |E_1| + |E_2|. \tag{2.8}$$

Therefore, once (2.7) is proven, the submodularity of $\hat{H}^\sigma_R$ follows, and in turn the submodularity of $\hat{H}^\sigma$ and $\hat{J}^\sigma$, by sending $R\to+\infty$ and using Propositions 2.1, 2.2 and 2.3.

To prove (2.7) we preliminarily notice that for every disjoint sets $A_1,A_2\in\mathcal{M}_{\mathrm{f}}(\mathbb{R}^d)$ and for every $x\in\mathbb{R}^d$ we have

$$B_R(x)\cap A_2 \subseteq B_R(x)\setminus A_1, \quad B_R(x)\setminus(A_1\dot{\cup}A_2) = (B_R(x)\setminus A_1)\setminus(B_R(x)\cap A_2)$$

(where $\dot\cup$ denotes the disjoint union), so that

$$
\begin{aligned}
H^\sigma_R(A_1\dot\cup A_2) =&H^\sigma_R(A_1)-\int_{A_1}\int_{B_R(x)\cap A_2}\frac{1}{|x-y|^{d+\sigma}}\,\mathrm{d}y\,\mathrm{d}x\\
&+H^\sigma_R(A_2)-\int_{A_2}\int_{B_R(x)\cap A_1}\frac{1}{|x-y|^{d+\sigma}}\,\mathrm{d}y\,\mathrm{d}x\,.
\end{aligned}\tag{2.9}
$$

Since

$$E_1\cup E_2=(E_2\setminus E_1)\dot\cup E_1\,,$$

by (2.9), we deduce

$$
\begin{aligned}
H^\sigma_R(E_1\cup E_2) =&H^\sigma_R(E_2\setminus E_1)-\int_{E_2\setminus E_1}\int_{B_R(x)\cap E_1}\frac{1}{|x-y|^{d+\sigma}}\,\mathrm{d}y\,\mathrm{d}x\\
&+H^\sigma_R(E_1)-\int_{E_1}\int_{B_R(x)\cap(E_2\setminus E_1)}\frac{1}{|x-y|^{d+\sigma}}\,\mathrm{d}y\,\mathrm{d}x\,.
\end{aligned}\tag{2.10}
$$

Analogously, since

$$E_2=(E_2\setminus E_1)\dot\cup(E_1\cap E_2)\,,$$

then, again by (2.9), we get

$$
\begin{aligned}
H^\sigma_R(E_2) =&H^\sigma_R(E_2\setminus E_1)-\int_{E_2\setminus E_1}\int_{B_R(x)\cap(E_1\cap E_2)}\frac{1}{|x-y|^{d+\sigma}}\,\mathrm{d}y\,\mathrm{d}x\\
&+H^\sigma_R(E_1\cap E_2)-\int_{E_1\cap E_2}\int_{B_R(x)\cap(E_2\setminus E_1)}\frac{1}{|x-y|^{d+\sigma}}\,\mathrm{d}y\,\mathrm{d}x\,.
\end{aligned}\tag{2.11}
$$

In conclusion, by (2.10) and (2.11), we obtain

$$
\begin{aligned}
&H^\sigma_R(E_1\cup E_2)+H^\sigma_R(E_1\cap E_2)=H^\sigma_R(E_1)+H^\sigma_R(E_2)\\
&+\int_{E_2\setminus E_1}\int_{B_R(x)\cap(E_1\cap E_2)}\frac{1}{|x-y|^{d+\sigma}}\,\mathrm{d}y\,\mathrm{d}x+\int_{E_1\cap E_2}\int_{B_R(x)\cap(E_2\setminus E_1)}\frac{1}{|x-y|^{d+\sigma}}\,\mathrm{d}y\,\mathrm{d}x\\
&-\int_{E_2\setminus E_1}\int_{B_R(x)\cap E_1}\frac{1}{|x-y|^{d+\sigma}}\,\mathrm{d}y\,\mathrm{d}x-\int_{E_1}\int_{B_R(x)\cap(E_2\setminus E_1)}\frac{1}{|x-y|^{d+\sigma}}\,\mathrm{d}y\,\mathrm{d}x\\
\le&H^\sigma_R(E_1)+H^\sigma_R(E_2)\,,
\end{aligned}
$$

i.e., (2.7) holds.

The proof of the submodularity of J^σ_r and of $\hat J^\sigma_r$ works analogously. We just sketch it. In view of (2.8), it is enough to prove only the submodularity of J^σ_r. By arguing as in (2.10) and (2.11), we obtain

$$
\begin{aligned}
J^\sigma_r(E_1\cup E_2) =&J^\sigma_r(E_2\setminus E_1)-\int_{E_2\setminus E_1}\int_{E_1\setminus B_r(x)}\frac{1}{|x-y|^{d+\sigma}}\,\mathrm{d}y\,\mathrm{d}x\\
&+J^\sigma_r(E_1)-\int_{E_1}\int_{(E_2\setminus E_1)\setminus B_r(x)}\frac{1}{|x-y|^{d+\sigma}}\,\mathrm{d}y\,\mathrm{d}x,\\
J^\sigma_r(E_2) =&J^\sigma_r(E_2\setminus E_1)-\int_{E_2\setminus E_1}\int_{(E_1\cap E_2)\setminus B_r(x)}\frac{1}{|x-y|^{d+\sigma}}\,\mathrm{d}y\,\mathrm{d}x\\
&+J^\sigma_r(E_1\cap E_2)-\int_{E_1\cap E_2}\int_{(E_2\setminus E_1)\setminus B_r(x)}\frac{1}{|x-y|^{d+\sigma}}\,\mathrm{d}y\,\mathrm{d}x,
\end{aligned}
$$

whence we get

$$
\begin{aligned}
&J^\sigma_r(E_1\cup E_2)+J^\sigma_r(E_1\cap E_2)=J^\sigma_r(E_1)+J^\sigma_r(E_2)\\
&+\int_{E_2\setminus E_1}\int_{(E_1\cap E_2)\setminus B_r(x)}\frac{1}{|x-y|^{d+\sigma}}\,\mathrm{d}y\,\mathrm{d}x+\int_{E_1\cap E_2}\int_{(E_2\setminus E_1)\setminus B_r(x)}\frac{1}{|x-y|^{d+\sigma}}\,\mathrm{d}y\,\mathrm{d}x\\
&-\int_{E_2\setminus E_1}\int_{E_1\setminus B_r(x)}\frac{1}{|x-y|^{d+\sigma}}\,\mathrm{d}y\,\mathrm{d}x-\int_{E_1}\int_{(E_2\setminus E_1)\setminus B_r(x)}\frac{1}{|x-y|^{d+\sigma}}\,\mathrm{d}y\,\mathrm{d}x\\
&\le J^\sigma_r(E_1)+J^\sigma_r(E_2),
\end{aligned}
$$

i.e., the submodularity of J_r^σ.

□

For every $\sigma \in (-d,1)$ we extend the functional H_R^σ to L^1 functions, obtaining a *generalized total variation functional* $TV_{H_R^\sigma} : L^1(\mathbb{R}^d) \to [0,+\infty]$ (see [31, 32]) defined by

$$TV_{H_R^\sigma}(u) := \int_{-\infty}^{+\infty} H_R^\sigma(\{u>t\})\,\mathrm{d}t \qquad \text{for all } u \in L^1(\mathbb{R}^d). \tag{2.12}$$

By Lemma 2.8 and by [10, Proposition 3.4], $TV_{H_R^\sigma}$ is convex. Analogously, one can consider for all $\sigma \in (-d,1)$ the functionals $TV_{\hat H_R^\sigma} : L^1_{\mathrm{c}}(\mathbb{R}^d) \to (-\infty,+\infty]$ defined by

$$TV_{\hat H_R^\sigma}(u) := \int_{-\infty}^{+\infty} \hat H_R^\sigma(\{u>t\})\,\mathrm{d}t \qquad \text{for all } u \in L^1_{\mathrm{c}}(\mathbb{R}^d)\,,$$

where $L^1_{\mathrm{c}}(\mathbb{R}^d)$ denotes the set of L^1 functions compactly supported in $\mathbb{R}^d$.

3. Convergence of $\hat J_r^\sigma$ as $r \to 0^+$

In this section, we study the convergence of the functionals $\hat J_r^\sigma$ as $r \to 0^+$. We preliminarily notice that for every $\sigma \in (-d,1)$ and for every $r>0$

$$\hat J_r^\sigma(E) = \int_E j_r^\sigma(x,E)\,\mathrm{d}x, \qquad \text{where } j_r^\sigma(x,E) := -\int_{E\setminus B_r(x)} \frac{1}{|x-y|^{d+\sigma}}\,\mathrm{d}y - \gamma_r^\sigma\,. \tag{3.1}$$

Lemma 3.1. *Let $\sigma \in (-d,1)$. For every $E \in \mathcal{M}_{\mathrm{f}}(\mathbb{R}^d)$, the functions $j_r^\sigma(\cdot,E) : \mathbb{R}^d \to \mathbb{R}$, as well as the functionals $\hat J_r^\sigma(E)$, are monotonically non-increasing with respect to r. In particular, for every $x \in \mathbb{R}^d$ there exists $j^\sigma(x,E) := \lim_{r\to 0^+} j_r^\sigma(x,E)$ and there exists the limit*

$$\lim_{r\to 0^+} \hat J_r^\sigma(E) = \int_E j^\sigma(x,E)\,\mathrm{d}x\,. \tag{3.2}$$

Moreover, if $\sigma \in (-d,0)$, then

$$j^\sigma(x,E) = -\int_E \frac{1}{|x-y|^{d+\sigma}}\,\mathrm{d}y - \gamma^\sigma\,, \tag{3.3}$$

and

$$\hat J^\sigma(E) = \lim_{r\to 0^+} \hat J_r^\sigma(E)\,,$$

where $\hat J^\sigma$ is the functional defined in (1.1).

Proof. Let $0 < r_1 < r_2 < +\infty$. By the very definition of $j_r^\sigma(\cdot,E)$ and by (1.7), for every $x \in \mathbb{R}^d$ we have

$$\begin{aligned} j_{r_1}^\sigma(x,E) &= j_{r_2}^\sigma(x,E) - \int_{(B_{r_2}(x)\setminus B_{r_1}(x))\cap E} \frac{1}{|x-y|^{d+\sigma}}\,\mathrm{d}y + \gamma_{r_2}^\sigma - \gamma_{r_1}^\sigma \\ &\geq j_{r_2}^\sigma(x,E) - \int_{B_{r_2}(x)\setminus B_{r_1}(x)} \frac{1}{|x-y|^{d+\sigma}}\,\mathrm{d}y + \gamma_{r_2}^\sigma - \gamma_{r_1}^\sigma \\ &= j_{r_2}^\sigma(x,E)\,. \end{aligned}$$

Therefore, $j_r^\sigma(x,E)$ monotonically converge to some $j^\sigma(x,E)$ for every $x \in \mathbb{R}^d$. Moreover, by (3.1) and by the monotone convergence Theorem, we deduce (3.2). Finally, (3.3) is again a consequence of the monotone convergence Theorem and of the very definition of $\hat J^\sigma$. □

Definition 3.2. Thanks to Lemma 3.1 we can extend the definition of $\hat J^\sigma(E)$ also to the case $\sigma \in [0,1)$, by setting $\hat J^\sigma(E) := \lim_{r\to 0^+} \hat J_r^\sigma(E)$ for all $E \in \mathcal{M}_{\mathrm{f}}(\mathbb{R}^d)$, as in (3.2). In Remark 3.3 below we show that actually $\hat J^\sigma : \mathcal{M}_{\mathrm{f}}(\mathbb{R}^d) \to (-\infty,+\infty]$.

Remark 3.3. Let $\sigma \in (-d,1)$, $E \in \mathcal{M}_{\mathrm{f}}(\mathbb{R}^d)$, and $r \in (0,1]$; by the very definition of J^σ_r in (1.4), we have

$$|J^\sigma_r(E)| = -J^\sigma_r(E) \le \frac{|E|^2}{r^{d+\sigma}}\,, \tag{3.4}$$

and hence

$$\hat{J}^\sigma_r(E) \ge -\frac{|E|^2}{r^{d+\sigma}} - \gamma^\sigma_r |E|\,. \tag{3.5}$$

Therefore, by (3.5) and by Lemma 3.1, we deduce

$$\hat{J}^\sigma(E) \ge \hat{J}^\sigma_1(E) = J^\sigma_1(E) \ge -|E|^2\,. \tag{3.6}$$

Notice that the lower bound in (3.6) is worse than the one obtained in (1.2) for $\sigma \in (-d,0)$. Nevertheless, such a lower bound is enough to guarantee that $\hat{J}^\sigma : \mathcal{M}_{\mathrm{f}}(\mathbb{R}^d) \to (-\infty,+\infty]$ for every $\sigma \in [0,1)$.

Now we extend the functionals $\hat{J}^\sigma_r$ and $\hat{J}^\sigma$ to L^1 densities by setting, for all $\rho \in L^1(\mathbb{R}^d;[0,1])$,

$$\begin{aligned}
&\hat{J}^\sigma_r(\rho) = \int_{\mathbb{R}^d} \rho(x) j^\sigma_r(x,\rho)\,\mathrm{d}x,\\
&\text{where } j^\sigma_r(x,\rho) := -\int_{\mathbb{R}^d\setminus B_r(x)} \frac{\rho(y)}{|x-y|^{d+\sigma}}\,\mathrm{d}y - \gamma^\sigma_r\,.
\end{aligned} \tag{3.7}$$

Arguing as in the proof of Lemma 3.1, one can prove the following result.

Lemma 3.4. *Let $\sigma \in (-d,1)$; for every $\rho \in L^1(\mathbb{R}^d;[0,1])$, the functions $j^\sigma_r(\cdot,\rho) : \mathbb{R}^d \to \mathbb{R}$, as well as the functionals $\hat{J}^\sigma_r(\rho)$, are monotonically non-increasing with respect to r . In particular, for every $x \in \mathbb{R}^d$ there exists $j^\sigma(x,\rho) := \lim_{r\to 0^+} j^\sigma_r(x,\rho)$ and there exists*

$$\hat{J}^\sigma(\rho) := \lim_{r\to 0^+} \hat{J}^\sigma_r(\rho) = \int_{\mathbb{R}^d} \rho(x) j^\sigma(x,\rho)\,\mathrm{d}x\,.$$

Moreover, if $\sigma \in (-d,0)$, then

$$j^\sigma(x,\rho) = -\int_{\mathbb{R}^d} \frac{\rho(y)}{|x-y|^{d+\sigma}}\,\mathrm{d}y - \gamma^\sigma\,.$$

By arguing as in Remark 3.3 we have that $\hat{J}^\sigma_r(\rho)$ and $\hat{J}^\sigma(\rho)$ are bounded from below by $-\|\rho\|^2_{L^1}$. Moreover, in Remark 5.2 we will see that if $\sigma \in [0,1)$ then $\hat{J}^\sigma(\rho) = +\infty$ whenever ρ is not the characteristic function of a set with finite measure.

Proposition 3.5. *Let $\sigma \in (-d,1)$; for every $r > 0$, the functionals $\hat{J}^\sigma_r : L^1(\mathbb{R}^d;[0,1]) \to (-\infty,+\infty]$ are continuous with respect to the strong L^1 convergence. As a consequence, their monotone limit $\hat{J}^\sigma$ is lower semicontinuous; more precisely, $\hat{J}^\sigma$ is the Γ-limit of $\{\hat{J}^\sigma_r\}_{r>0}$ with respect to the strong L^1 topology, as $r \to 0^+$. The same Γ-convergence result holds true for the functionals $\hat{J}^\sigma_r$, $\hat{J}^\sigma$ defined on $\mathcal{M}_{\mathrm{f}}(\mathbb{R}^d)$.*

Proof. By arguing verbatim as in the proof of Lemma 1.3 (see formulas (1.5) and (1.6)), one can prove the continuity of the functionals $\hat{J}^\sigma_r$ with respect to the strong L^1 convergence. Moreover, it is well known that monotone convergence of continuous functionals implies Γ-convergence to the pointwise (lower semicontinuous) limit [15]. □

Remark 3.6. Notice that for every $r \in (0,1)$ the functionals $\hat{J}^\sigma_r$ are monotonically non-decreasing with respect to $\sigma \in (-d,1)$. Indeed, recalling (3.1), it is easy to check that

$$j^\sigma_r(x,E) = \int_{(B_1(x)\setminus B_r(x))\setminus E} \frac{1}{|x-y|^{d+\sigma}}\,\mathrm{d}y + \int_{E\setminus B_1(x)} -\frac{1}{|x-y|^{d+\sigma}}\,\mathrm{d}y,$$

and that both the addenda are monotonically non-decreasing with respect to σ.

4. $\hat H^\sigma = \hat J^\sigma$

In view of Proposition 2.1 and Lemma 3.1 we have that for every $\sigma \in (-d,0)$ and for every $E \in \mathcal{M}_{\rm f}(\mathbb{R}^d)$

$$\lim_{R\to+\infty} \hat H^\sigma_R(E) = \hat J^\sigma(E) = \lim_{r\to 0^+} \hat J^\sigma_r(E)\,,$$

where $\hat J^\sigma(E)$ is finite for every $E \in \mathcal{M}_{\rm f}(\mathbb{R}^d)$. By definition we set $\hat H^\sigma := \hat J^\sigma$ for every $\sigma \in (-d,0)$. Moreover, we recall that $\hat H^\sigma := \hat J^\sigma = H^\sigma_1 + J^\sigma_1$ for every $\sigma \in (-d,0)$. In this section we show that the identities above extends also to the functionals $\hat H^\sigma$ and $\hat J^\sigma$ for $\sigma \in [0,1)$. More precisely, we prove that for every $\sigma \in [0,1)$ the functionals $\hat H^\sigma$ and $\hat J^\sigma$ coincide on all the measurable sets with finite measure and are finite on smooth sets.

Theorem 4.1. *Let $\sigma \in [0,1)$. For every $E \in \mathcal{M}_{\rm f}(\mathbb{R}^d)$, it holds*

$$\hat J^\sigma(E) = \hat H^\sigma(E)\,.$$

Moreover, $\hat J^\sigma(E)$ and $\hat H^\sigma(E)$ are finite if and only if $H^\sigma_1(E) < +\infty$.

Proof. We distinguish among two cases.

Case 1: $\hat H^\sigma(E) = +\infty$. By Proposition 2.2 and by Proposition 2.3, we have that $\hat H^\sigma_1(E) = +\infty$ which, by the monotone convergence Theorem, implies

$$\lim_{r\to0^+} \int_E \int_{(B_1(x)\setminus B_r(x))\setminus E} \frac{1}{|x-y|^{d+\sigma}}\,\mathrm{d}y\,\mathrm{d}x = H^\sigma_1(E) = \hat H^\sigma_1(E) = +\infty\,. \tag{4.1}$$

Moreover, by Remark 1.4 for every $r \in (0,1]$ we have

$$\begin{aligned} &\int_E \int_{(B_1(x)\setminus B_r(x))\setminus E} \frac{1}{|x-y|^{d+\sigma}}\,\mathrm{d}y\,\mathrm{d}x \\ =& -\gamma^\sigma_r|E| - \int_E\int_{E\cap(B_1(x)\setminus B_r(x))} \frac{1}{|x-y|^{d+\sigma}}\,\mathrm{d}y\,\mathrm{d}x \\ =& -\gamma^\sigma_r|E| - \int_E\int_{E\setminus B_r(x)} \frac{1}{|x-y|^{d+\sigma}}\,\mathrm{d}y\,\mathrm{d}x + \int_E\int_{E\setminus B_1(x)} \frac{1}{|x-y|^{d+\sigma}}\,\mathrm{d}y\,\mathrm{d}x \\ =& \hat J^\sigma_r(E) - \hat J^\sigma_1(E)\,. \end{aligned} \tag{4.2}$$

Therefore, by taking the limit as $r \to 0^+$ in (4.2), using (4.1), Lemma 3.1 and (3.4), we deduce that $\hat J^\sigma(E) = +\infty$.

Case 2: $\hat H^\sigma(E) < +\infty$. By Proposition 2.2 and by Proposition 2.3, we have that there exists $R_1 \ge 1$ such that

$$\hat H^\sigma_R(E) \le \hat H^\sigma(E) + 1 \qquad \text{for } R \ge R_1\,. \tag{4.3}$$

Let now $r \le 1$ and $R \ge R_1$; then, by (4.3) and Lemma 1.5, we get

$$\begin{aligned} \hat H^\sigma(E) + 1 \ge& \hat H^\sigma_R(E) = \hat J^\sigma_r(E) + H^\sigma_r(E) - J^\sigma_R(E) \ge \hat J^\sigma_1(E) + H^\sigma_r(E) \\ \ge& -|E|^2 + H^\sigma_r(E)\,, \end{aligned} \tag{4.4}$$

where we have used also that $-J^\sigma_R(E) \ge 0$, Lemma 3.1 and (3.6); it follows that

$$H^\sigma_r(E) \le |E|^2 + \hat H^\sigma(E) + 1 < +\infty \qquad \text{for every } 0 < r \le 1\,. \tag{4.5}$$

By the non-negativity and the monotonicity of $H^\sigma_r(E)$ with respect to r, we deduce that there exists

$$\lim_{r\to0^+} H^\sigma_r(E) \in [0,+\infty)\,.$$

Let now $0 < \bar r \le 1$; by the monotone convergence Theorem, we have

$$\begin{aligned} H^\sigma_{\bar r}(E) - \lim_{r\to0^+} H^\sigma_r(E) &= \lim_{r\to0^+} (H^\sigma_{\bar r}(E) - H^\sigma_r(E)) \\ &= \lim_{r\to0^+} \int_E\int_{(B_{\bar r}(x)\setminus B_r(x))\setminus E} \frac{1}{|x-y|^{d+\sigma}}\,\mathrm{d}y\,\mathrm{d}x \\ &= H^\sigma_{\bar r}(E)\,, \end{aligned}$$

i.e.,

$$\lim_{r\to 0^+} H^\sigma_r(E) = 0\,. \tag{4.6}$$

In conclusion, by taking first the limit as $r\to 0^+$ and then the limit as $R\to+\infty$ in the equality in (4.4), by Lemma 3.1, (4.6), Proposition 2.3 and (1.10), we deduce that $\hat H^\sigma(E)=\hat J^\sigma(E)$.

Finally, we prove the last sentence in the statement. If $\hat H^\sigma(E)=+\infty$, then $H^\sigma_1(E)=\hat H^\sigma_1(E)\ge \hat H^\sigma(E)=+\infty$, whereas, if $\hat H^\sigma(E)<+\infty$, then (4.5) with $r=1$ yields $H^\sigma_1(E)<+\infty$. □

It is well-known that fractional perimeters are finite on smooth sets. Next, we extend this property to the 0-fractional perimeter. This fact is not completely obvious, since $H^\sigma(E)$, together with its renormalization $\gamma^\sigma|E|$, diverges as $\sigma\to 0^+$; clearly, we expect that the C^2 regularity condition assumed below is far from being sharp in order to have finite $\hat H^0$-perimeter.

Proposition 4.2. *Let $\sigma\in[0,1)$. If E is an open bounded set with boundary of class C^2, then $\hat H^\sigma(E)=\hat J^\sigma(E)<+\infty$.*

Proof. In view of Theorem 4.1 it is enough to show that $H^\sigma_1(E)<+\infty$. Let $T:=\frac12\|\mathscr{H}\|^{-1}_{L^\infty(\partial E)}$, where $\mathscr{H}$ is the second fundamental form. Moreover, for all $t>0$ we set

$$E_t:=\{x\in E:\,\mathrm{dist}(x,\partial E)>t\}. \tag{4.7}$$

Since E has boundary of class C^2, we have that $T<+\infty$; moreover, there exists $C>0$ such that $\mathcal{H}^{d-1}(\partial E_t)\le C$ and E_t has boundary of class C^2 for all $t\in(0,T)$. Then, for $0<r<\min\{1,T\}$ by coarea formula we have

$$\begin{aligned}
H^\sigma_r(E)&=\int_E\int_{B_r(x)\setminus E}\frac{1}{|x-y|^{d+\sigma}}\,\mathrm{d}y\,\mathrm{d}x=\int_{E\setminus E_r}\int_{B_r(x)\setminus E}\frac{1}{|x-y|^{d+\sigma}}\,\mathrm{d}y\,\mathrm{d}x\\
&=\int_0^r\int_{\partial E_t}\int_{B_r(x)\setminus E}\frac{1}{|x-y|^{d+\sigma}}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t\\
&\le\int_0^r\int_{\partial E_t}\int_{B_r(x)\setminus B_t(x)}\frac{1}{|x-y|^{d+\sigma}}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t\\
&\le C\int_0^r(G^\sigma(r)-G^\sigma(t))\,\mathrm{d}t<+\infty,
\end{aligned}$$

where $G^\sigma(\tau)$ is the primitive of $\tau^{-1-\sigma}$ and in the last inequality we have used that it is integrable around the origin. By Propositions 2.2 and 2.3, we conclude that

$$H^\sigma_1(E)=\hat H^\sigma_1(E)\le\hat H^\sigma_r(E)=H^\sigma_r(E)-\gamma^\sigma_r|E|<+\infty.$$

□

The following lemma clarifies the scaling property of $\hat H^\sigma$ for $\sigma\in(-d,1)$.

Lemma 4.3. *Let $\sigma\in(-d,1)$. For every $E\in\mathcal{M}_{\mathrm f}(\mathbb{R}^d)$ and for every $\lambda>0$ it holds*

$$\hat H^\sigma(\lambda E)=\lambda^{d-\sigma}\hat H^\sigma(E)-\lambda^d\gamma^\sigma_\lambda|E|, \tag{4.8}$$

with γ^σ_λ defined in (1.3).

Proof. Let $E\in\mathcal{M}_{\mathrm f}(\mathbb{R}^d)$ and $\lambda>0$. We preliminary notice that, by the very definition of γ^σ_ρ in (1.3) and by a change of variable in the functional H^σ_ρ defined in (0.3), for every $R>0$ it holds

$$\gamma^\sigma_{\lambda R}=\lambda^{-\sigma}\gamma^\sigma_R+\gamma^\sigma_\lambda,\qquad\qquad H^\sigma_{\lambda R}(\lambda E)=\lambda^{d-\sigma}H^\sigma_R(E).$$

These two facts imply that, for every $R>0$,

$$\begin{aligned}
\hat H^\sigma_{\lambda R}(\lambda E)&=H^\sigma_{\lambda R}(\lambda E)-\gamma^\sigma_{\lambda R}|\lambda E|\\
&=\lambda^{d-\sigma}H^\sigma_R(E)-(\lambda^{-\sigma}\gamma^\sigma_R+\gamma^\sigma_\lambda)\lambda^d|E|\\
&=\lambda^{d-\sigma}H^\sigma_R(E)-\lambda^{d-\sigma}\gamma^\sigma_R|E|+\lambda^d\gamma^\sigma_\lambda|E|\\
&=\lambda^{d-\sigma}\hat H^\sigma_R(E)+\lambda^d\gamma^\sigma_\lambda|E|,
\end{aligned}$$

whence (4.8) follows by sending $R\to+\infty$ and using Propositions 2.1, 2.2, and 2.3. □

5. Compactness

This section is devoted to the proof of compactness results for the functionals $\hat{J}^\sigma$, $\hat{J}^\sigma_r$, $\hat{H}^\sigma_R$. First we prove compactness properties for the functionals $\hat{J}^\sigma_r$ and $\hat{J}^\sigma$ for $\sigma \in [0,1)$.

Theorem 5.1 (Compactness)**.** *Let $\{\sigma_n\}_{n\in\mathbb{N}} \subset [0,1)$ and let $r_n \to 0^+$. Let $U \subset \mathbb{R}^d$ be an open bounded set and let $\{E_n\}_{n\in\mathbb{N}} \subset \mathcal{M}_{\mathrm{f}}(\mathbb{R}^d)$ be such that $E_n \subset U$ for all $n \in \mathbb{N}$. Finally, let $C > 0$.*

If $\hat{J}^{\sigma_n}_{r_n}(E_n) \le C$ for all $n \in \mathbb{N}$, then, up to a subsequence, $\chi_{E_n} \to \chi_E$ in $L^1(\mathbb{R}^d)$ for some $E \in \mathcal{M}_{\mathrm{f}}(\mathbb{R}^d)$.

In particular, if $\hat{J}^{\sigma_n}(E_n) \le C$ for all $n \in \mathbb{N}$, then, up to a subsequence, $\chi_{E_n} \to \chi_E$ in $L^1(\mathbb{R}^d)$ for some $E \in \mathcal{M}_{\mathrm{f}}(\mathbb{R}^d)$.

Proof. Recalling the definition of j^σ_r in (3.7), we claim the following two properties satisfied by all $\eta \in L^1(\mathbb{R}^d;[0,1])$:

(1) For every $x \in \mathbb{R}^d$, $r \in (0,1)$ it holds $j^0_r(x,\eta) \ge -\|\eta\|_{L^1}$.

(2) For every Lebesgue point $x \in \mathbb{R}^d$ with Lebesgue value $\lambda \in (0,1)$ it holds

$$\lim_{r\to 0^+} j^0_r(x,\eta) = +\infty.$$

Proof of (1). For every $r \in (0,1]$ we write

$$j^0_r(x,\eta) = -\int_{\mathbb{R}^d\setminus B_1(x)} \frac{\eta(y)}{|x-y|^d}\,\mathrm{d}y - \left[\int_{B_1(x)\setminus B_r(x)} \frac{\eta(y)}{|x-y|^d}\,\mathrm{d}y + \gamma^0_r\right]. \tag{5.1}$$

By Remark 1.4 the last term in square brackets is always non-positive, whence property (1) easily follows.

Proof of (2). We have to show that, whenever the Lebesgue value λ of η at x is in $(0,1)$, the last term in square brackets in (5.1) in fact tends to $-\infty$ as $r \to 0^+$. To this purpose, in order to short notation we assume $x = 0$, we let $\theta \in (0,1)$ be defined by $\theta^d = \frac{1-\lambda}{2}$, and for all $k \ge 1$ we set $A^k := B_{\theta^{k-1}} \setminus B_{\theta^k}$. Since λ is the Lebesgue value of η at 0, there exists $\bar{k} \in \mathbb{N}$ such that, for all $k > \bar{k}$ we have

$$\frac{1}{\omega_d \theta^{(k-1)d}} \int_{B_{\theta^{k-1}}} \eta(y)\,\mathrm{d}y \le \lambda + \frac{1-\lambda}{4} = \frac{1+3\lambda}{4}.$$

It follows that, for all $k > \bar{k}$,

$$\int_{A^k} \eta(y)\,\mathrm{d}y \le \int_{B_{\theta^{k-1}}} \eta(y)\,\mathrm{d}y \le \omega_d \frac{1+3\lambda}{4}\theta^{(k-1)d} =: m_k.$$

Now, we apply Lemma A.6 with m replaced by m_k, s replaced by θ^k and in turn $R(m,s)$ replaced by $R(m_k,\theta^k)$. Therefore, setting $\hat{A}^k := B_{R(m_k,\theta^k)} \setminus B_{\theta^k}$, for all $k > \bar{k}$ we have

$$\int_{A^k} \frac{\eta(y)}{|y|^d}\,\mathrm{d}y \le \int_{\hat{A}^k} \frac{1}{|y|^d}\,\mathrm{d}y. \tag{5.2}$$

Now we prove that there exists $\delta_{d,\lambda} > 0$ (independent of k) such that

$$\int_{\hat{A}^k} \frac{1}{|y|^d}\,\mathrm{d}y + \gamma^0_{\theta^k} - \gamma^0_{\theta^{k-1}} \le -\delta_{d,\lambda}. \tag{5.3}$$

By the very definition of $R(m_k,\theta^k)$ in Lemma A.6, we have that $|\hat{A}_k| = m_k$ so that

$$R(m_k,\theta^k) = \theta^{k-1}\left(\theta^d + \frac{1+3\lambda}{4}\right)^{\frac{1}{d}} = \theta^{k-1}\left(\frac{\lambda+3}{4}\right)^{\frac{1}{d}}. \tag{5.4}$$

By using (5.4), we deduce (5.3) as follows:

$$\begin{aligned}\int_{\hat{A}^k} \frac{1}{|y|^d}\,\mathrm{d}y + \gamma^0_{\theta^k} - \gamma^0_{\theta^{k-1}} &= d\omega_d \log\frac{R(m_k,\theta^k)}{\theta^k} + d\omega_d \log\theta \\ &= \omega_d \log\left(\frac{\lambda+3}{4}\right) =: -\delta_{d,\lambda},\end{aligned}$$

Therefore, by (5.2), (5.3) and by the fact that $\gamma^0_1=0$, for all $K>\bar k$ we get

$$\begin{aligned}&\int_{B_1\setminus B_{\theta K}}\frac{\eta(y)}{|y|^d}\,\mathrm{d}y+\gamma^0_{\theta^K}\\&=\sum_{k=1}^{\bar k}\Big(\int_{A^k}\frac{\eta(y)}{|y|^d}\,\mathrm{d}y+\gamma^0_{\theta^k}-\gamma^0_{\theta^{k-1}}\Big)+\sum_{k=\bar k+1}^{K}\Big(\int_{A^k}\frac{\eta(y)}{|y|^d}\,\mathrm{d}y+\gamma^0_{\theta^k}-\gamma^0_{\theta^{k-1}}\Big)\\&\le\sum_{k=\bar k+1}^{K}\Big(\int_{A^k}\frac{\eta(y)}{|y|^d}\,\mathrm{d}y+\gamma^0_{\theta^k}-\gamma^0_{\theta^{k-1}}\Big)\le\sum_{k=\bar k+1}^{K}\Big(\int_{\hat A^k}\frac{\eta(y)}{|y|^d}\,\mathrm{d}y+\gamma^0_{\theta^k}-\gamma^0_{\theta^{k-1}}\Big)\\&\le-(K-\bar k)\delta_{d,\lambda}.\end{aligned}$$

Letting $K\to+\infty$, we deduce property (2).

Conclusion. Up to a subsequence, $\rho_n:=\chi_{E_n}\overset{*}{\rightharpoonup}\rho$ for some ρ in $L^1(\mathbb{R}^d;[0,1])$. For every $\bar r\in(0,1)$, by Remark 3.6 and Lemma 3.4, we have

$$\begin{aligned}\liminf_{n\to+\infty}\hat J^{\sigma_n}_{r_n}(E_n)&\ge\liminf_{n\to+\infty}\hat J^0_{r_n}(E_n)\\&\ge\liminf_{n\to+\infty}\int_U\rho_n(x)j^0_{\bar r}(x,\rho_n)\,\mathrm{d}x\\&=\int_U\rho(x)j^0_{\bar r}(x,\rho)\,\mathrm{d}x,\end{aligned}$$

where in the last equality we have used that $\rho_n\overset{*}{\rightharpoonup}\rho$ in $L^\infty(U)$ and that, by the dominated convergence Theorem, $j^0_{\bar r}(\cdot,\rho_n)\to j^0_{\bar r}(\cdot,\rho)$ in $L^1(U)$ as $n\to+\infty$.

Setting $\mathcal{N}:=\{x\in U:\ \rho(x)\in(0,1)\}$ and using the claims (1) and (2) we deduce that

$$C\ge\liminf_{n\to+\infty}\hat J^{\sigma_n}_{r_n}(E_n)\ge\lim_{\bar r\to0}\int_U\rho(x)j^0_{\bar r}(x,\rho)\,\mathrm{d}x\ge+\infty|\mathcal{N}|-\|\rho\|^2_{L^1}.$$

As a consequence $\mathcal{N}$ is a negligible set, hence ρ is the characteristic function of some set $E\subset U$. It follows that $\chi_{E_n}\to\chi_E$ strongly in $L^1(\mathbb{R}^d)$.

The last part of the theorem is a trivial consequence of the monotonicity of $\hat J^\sigma_r$ established in Lemma 3.4. □

Remark 5.2. By the proof of Theorem 5.1, and in particular by claims (1) and (2), it immediately follows that, for every $\sigma\in[0,1)$, $\hat J^\sigma(\rho)=+\infty$ whenever ρ is not the characteristic function of a set with finite measure.

We notice that the compactness property stated in Theorem 5.1 is not satisfied by the functionals $\hat J^\sigma$ for $\sigma\in(-d,0)$. In this case, indeed, $\hat J^\sigma(\eta)$ is finite for every density $\eta\in L^1(\mathbb{R}^d;[0,1])$. Nevertheless we have the following compactness result for the functionals $\hat J^\sigma$ as $\sigma\to0^-$.

Theorem 5.3. *Let $\{\sigma_n\}_{n\in\mathbb{N}}\subset(-d,0)$ and $\{r_n\}_{n\in\mathbb{N}}\subset\mathbb{R}^+$ be such that $\sigma_n\to0^-$ and $r_n\to0^+$. Let $U\subset\mathbb{R}^d$ be an open bounded set and let $\{E_n\}_{n\in\mathbb{N}}\subset\mathcal{M}_{\mathrm{f}}(\mathbb{R}^d)$ be such that $E_n\subset U$ for all $n\in\mathbb{N}$. Finally, let $C>0$.*

If $\hat J^{\sigma_n}_{r_n}(E_n)\le C$ for all $n\in\mathbb{N}$, then, up to a subsequence, $\chi_{E_n}\to\chi_E$ in $L^1(\mathbb{R}^d)$ for some $E\in\mathcal{M}_{\mathrm{f}}(\mathbb{R}^d)$.

In particular, if $\hat J^{\sigma_n}(E_n)\le C$ for all $n\in\mathbb{N}$, then, up to a subsequence, $\chi_{E_n}\to\chi_E$ in $L^1(\mathbb{R}^d)$ for some $E\in\mathcal{M}_{\mathrm{f}}(\mathbb{R}^d)$.

Proof. Up to a subsequence, $\rho_n:=\chi_{E_n}\overset{*}{\rightharpoonup}\rho$ for some ρ in $L^1(\mathbb{R}^d;[0,1])$. For every $\bar r\in(0,1)$, by Lemma 3.4, we have

$$\begin{aligned}\liminf_{n\to+\infty}\hat J^{\sigma_n}_{r_n}(E_n)&\ge\liminf_{n\to+\infty}\int_U\rho_n(x)j^{\sigma_n}_{\bar r}(x,\rho_n)\,\mathrm{d}x\\&=\int_U\rho(x)j^0_{\bar r}(x,\rho)\,\mathrm{d}x,\end{aligned}$$

where in the last equality we have used that $\rho_n \stackrel{*}{\rightharpoonup} \rho$ in $L^\infty(U)$ and that, by the dominated convergence Theorem, $j_{\bar r}^{\sigma_n}(\cdot,\rho_n) \to j_{\bar r}^{0}(\cdot,\rho)$ in $L^1(U)$ as $n \to +\infty$. By using claim (2) in the proof of Theorem 5.1 and arguing as in the conclusion therein we get the statements. $\square$

Finally, we prove the following compactness result also for the functionals $\hat H^\sigma_R$.

Theorem 5.4. *Let $\{\sigma_n\}_{n\in\mathbb{N}} \subset (-d,1)$ and $\{R_n\}_{n\in\mathbb{N}} \subset (0,+\infty)$. Let $U \subset \mathbb{R}^d$ be an open bounded set and let $\{E_n\}_{n\in\mathbb{N}} \subset \mathcal{M}_{\mathrm{f}}(\mathbb{R}^d)$ be such that $E_n \subset U$ for all $n \in \mathbb{N}$. Finally, let $C > 0$.*

If $\hat H^{\sigma_n}_{R_n}(E_n) \le C$ for all $n \in \mathbb{N}$, we have:

(a) *if $\{\sigma_n\}_{n\in\mathbb{N}} \subset [0,1)$, then, up to a subsequence, $\chi_{E_n} \to \chi_E$ in $L^1(\mathbb{R}^d)$ for some $E \in \mathcal{M}_{\mathrm{f}}(\mathbb{R}^d)$,*
(b) *if $\sigma_n \to 0$, then, up to a subsequence, $\chi_{E_n} \to \chi_E$ in $L^1(\mathbb{R}^d)$ for some $E \in \mathcal{M}_{\mathrm{f}}(\mathbb{R}^d)$.*

Proof. By Proposition 2.2 we can assume without loss of generality that $R_n > \mathrm{diam}\,(U)$. By Remark 2.7, we have that $\hat H^{\sigma_n}_{R_n}(E_n) = \hat H^{\sigma_n}(E_n) = \hat J^{\sigma_n}(E_n)$. By Theorems 5.1 and 5.3 we deduce (a) and (b). $\square$

6. Γ-convergence

This section is devoted to the Γ-convergence analysis of the functionals $\hat J^\sigma$, $\hat J^\sigma_r$, $\hat H^\sigma_R$ as $\sigma \to \bar\sigma$, $r \to \bar r$, $R \to \bar R$ for some $\bar\sigma \in (-d,1)$, $\bar r \in [0,+\infty)$, $\bar R \in (0,+\infty]$.

Next, we shall prove the Γ-convergence of the functionals $\hat H^\sigma_R$ as $\sigma \to \bar\sigma$. Firstly, for smooth sets E, we show the pointwise convergence of $\hat H^\sigma_R(E)$ to $\hat H^{\bar\sigma}_{\bar R}(E)$ as $\sigma \to \bar\sigma$ and $R \to \bar R$ for some $\bar\sigma \in (-d,1)$ and $\bar R \in (0,+\infty]$. From now on, it is convenient to adopt the notation $\hat H^\sigma_\infty := \hat H^\sigma$.

Proposition 6.1. *Let $\bar\sigma \in (-d,1)$ and $\bar R \in (0,+\infty]$. Let moreover $\{\sigma_n\}_{n\in\mathbb{N}} \subset (-d,1)$ and $\{R_n\}_{n\in\mathbb{N}} \subset (0,+\infty]$ be such that $\sigma_n \to \bar\sigma$ and $R_n \to \bar R$ as $n \to +\infty$. If $E \in \mathcal{M}_{\mathrm{f}}(\mathbb{R}^d)$ is an open bounded set with boundary of class C^2, then*

$$\lim_{n\to+\infty} \hat H^{\sigma_n}_{R_n}(E) = \hat H^{\bar\sigma}_{\bar R}(E). \tag{6.1}$$

Proof. We claim that

$$\lim_{\sigma\to\bar\sigma} \hat H^{\sigma}_{R}(E) = \hat H^{\bar\sigma}_{R}(E) \qquad \text{for every } R \in (0,+\infty]. \tag{6.2}$$

Now we prove that (6.2) implies (6.1). If $\bar R \in (0,+\infty)$, in view of (1.7), we have

$$\begin{aligned} |\hat H^{\sigma_n}_{R_n}(E) - \hat H^{\sigma_n}_{\bar R}(E)| &\le |H^{\sigma_n}_{R_n}(E) - H^{\sigma_n}_{\bar R}(E)| + |\gamma^{\sigma_n}_{R_n} - \gamma^{\sigma_n}_{\bar R}||E| \\ &\le \Big| \int_E \int_{A_{R_n,\bar R}(x)} \frac{1}{|x-y|^{d+\sigma_n}} \,\mathrm{d}y\,\mathrm{d}x \Big| + |\gamma^{\sigma_n}_{R_n} - \gamma^{\sigma_n}_{\bar R}||E| \\ &= 2|\gamma^{\sigma_n}_{R_n} - \gamma^{\sigma_n}_{\bar R}||E| \to 0 \qquad \text{as } n \to +\infty, \end{aligned} \tag{6.3}$$

where $A_{R_n,\bar R}(x)$ denotes the annular ring centered at x having as inner radius $\min\{R_n, \bar R\}$ and as outer radius $\max\{R_n,\bar R\}$. Moreover, if $\bar R = +\infty$, then for n large enough we have that $R_n \ge 1$ and $\sigma_n \ge \hat\sigma$ for some $\hat\sigma \in (-d,1)$. Therefore, by Proposition 2.1, Proposition 2.2 and Proposition 2.3, for every $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that, for every $\sigma \in (-d,1)$ and $R_n \ge R_\varepsilon$ we have

$$|\hat H^\sigma(E) - \hat H^\sigma_{R_n}(E)| < \varepsilon. \tag{6.4}$$

By (6.3), (6.4), and (6.2), we get

$$\lim_{n\to+\infty} \hat H^{\sigma_n}_{R_n}(E) = \lim_{n\to+\infty} (\hat H^{\sigma_n}_{R_n}(E) - \hat H^{\sigma_n}_{\bar R}(E)) + \lim_{n\to+\infty} \hat H^{\sigma_n}_{\bar R}(E) = \hat H^{\bar\sigma}_{\bar R}(E),$$

i.e., (6.1) holds.

Now we prove (6.2) and we consider only in the case $\bar\sigma = 0$, being the proof in the other cases fully analogous. By (6.4) and triangular inequality, for every $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that, for every $\sigma \in (-d,1)$ and $R \ge R_\varepsilon$ we have

$$|\hat H^\sigma(E) - \hat H^0(E)| < 2\varepsilon + |\hat H^\sigma_R(E) - \hat H^0_R(E)|.$$

Therefore, in order to get (6.2) for $R \in (0,+\infty]$, it is enough to prove it only for $R \in (0,+\infty)$.

Claim: For every $\varepsilon>0$ and for every $R>0$ there exists $\sigma_{\varepsilon,R}$ with $|\sigma_{\varepsilon,R}|>0$ such that

$$|\hat{H}^{\sigma}_R(E)-\hat{H}^0_R(E)|<\varepsilon \qquad \textit{for all } \sigma \textit{ with } |\sigma|<|\sigma_{\varepsilon,R}|\,.$$

In order to prove the claim, we preliminarily notice that

$$|\hat{H}^{\sigma}_R(E)-\hat{H}^0_R(E)|\le|\gamma^\sigma_R-\gamma^0_R||E|+|H^\sigma_R(E)-H^0_R(E)|\,. \tag{6.5}$$

As for the first addendum in (6.5), by the very definition of γ^σ_R and γ^0_R in (1.3) , we get

$$|\gamma^\sigma_R-\gamma^0_R|=d\omega_d|\log R|\left|\frac{1-e^{-\sigma\log R}}{\sigma\log R}-1\right|\le|\sigma|d\omega_d\log^2R\max\{R^\sigma,R^{-\sigma}\}, \tag{6.6}$$

where the inequality follows by applying

$$\left|\frac{1-e^{-t}}{t}-1\right|=\left|\frac{e^{-t}-(1-t)}{t}\right|\le\frac{t^2e^{|t|}}{|t|}=|t|e^{|t|}$$

with $t=\sigma\log R$. In order to estimate the second addendum in (6.5), we notice that

$$\begin{aligned}|H^\sigma_R(E)-H^0_R(E)|\le&\int_E\int_{(B_R(x)\setminus E)\setminus B_1(x)}\left|\frac{1}{|x-y|^{d+\sigma}}-\frac{1}{|x-y|^d}\right|\,\mathrm{d}y\,\mathrm{d}x\\&+\int_E\int_{(B_R(x)\setminus E)\cap B_1(x)}\left|\frac{1}{|x-y|^{d+\sigma}}-\frac{1}{|x-y|^d}\right|\,\mathrm{d}y\,\mathrm{d}x\,.\end{aligned} \tag{6.7}$$

Notice also that, if $R\le1$, the first integral in (6.7) is equal to 0, whereas, if $R>1$, in view of (6.6), we have

$$\begin{aligned}&\int_E\int_{(B_R(x)\setminus E)\setminus B_1(x)}\left|\frac{1}{|x-y|^{d+\sigma}}-\frac{1}{|x-y|^d}\right|\,\mathrm{d}y\,\mathrm{d}x\\\le&\int_E\int_{B_R(x)\setminus B_1(x)}\left|\frac{1}{|x-y|^{d+\sigma}}-\frac{1}{|x-y|^d}\right|\,\mathrm{d}y\,\mathrm{d}x=\quad|E||\gamma^\sigma_R-\gamma^0_R|\\\le&|\sigma|d\omega_d\log^2R\max\{R^\sigma,R^{-\sigma}\}|E|\,,\end{aligned} \tag{6.8}$$

where the last equality follows by the fact that the integrand in the modulus has constant sign in the annulus $B_R(x)\setminus B_1(x)$.

In order to estimate the second integral in (6.7), we first consider the case $\sigma>0$. Setting $\bar{R}:=\min\{1,R\}$ and $r_x:=\mathrm{dist}(x,\partial E)$ for every $x\in E$, we have

$$\begin{aligned}&\int_E\int_{(B_R(x)\setminus E)\cap B_1(x)}\left|\frac{1}{|x-y|^{d+\sigma}}-\frac{1}{|x-y|^d}\right|\,\mathrm{d}y\,\mathrm{d}x\\\le&\int_E\int_{B_{\bar{R}}(x)\setminus B_{r_x}(x)}\left(\frac{1}{|x-y|^{d+\sigma}}-\frac{1}{|x-y|^d}\right)\,\mathrm{d}y\,\mathrm{d}x\\\le&\sigma\int_E\int_{B_{\bar{R}}(x)\setminus B_{r_x}(x)}-\frac{\log|x-y|}{|x-y|^{d+\sigma}}\,\mathrm{d}y\,\mathrm{d}x\le\sigma d\omega_d\int_E\frac{1}{r_x^\sigma}\int_{r_x}^{\bar{R}}-\frac{\log\rho}{\rho}\,\mathrm{d}\rho\,\mathrm{d}x\\\le&\sigma d\omega_d\frac12\int_E\frac{1}{r_x^\sigma}\log^2r_x\,\mathrm{d}x-\sigma d\omega_d\frac12\frac{1}{\bar{R}^\sigma}\log^2\bar{R}|E|\,,\end{aligned} \tag{6.9}$$

where the first inequality follows by applying the bound (valid for $t\ge1$)

$$t^{d+\sigma}-t^d=t^{d+\sigma}(1-t^{-\sigma})\le t^{d+\sigma}\sigma\log t$$

with $t=\frac{1}{|x-y|}$.

As for the case $\sigma<0$, by arguing as in (6.9) one can easily show that

$$\int_E\int_{(B_R(x)\setminus E)\cap B_1(x)}\left|\frac{1}{|x-y|^{d+\sigma}}-\frac{1}{|x-y|^d}\right|\,\mathrm{d}y\,\mathrm{d}x\le|\sigma|d\omega_d\int_E\log^2r_x\,\mathrm{d}x-|\sigma|d\omega_d\log^2\bar{R}|E|\,. \tag{6.10}$$

Therefore, in view of (6.5), (6.6), (6.7), (6.8), (6.9), (6.10), the equality (6.2) is proven once we show that there exists a constant $C(E) > 0$ such that

$$\int_E \frac{1}{r_x^\sigma} \log^2 r_x \,\mathrm{d}x \leq C(E) \qquad \text{for every } \sigma \in [0,1)\,. \tag{6.11}$$

Recalling that E has boundary of class C^2, let $T := \frac{1}{2}\|\mathscr{H}\|^{-1}_{L^\infty(\partial E)}$, where $\mathscr{H}$ is the second fundamental form. Moreover, for all $t > 0$ let

$$E_t := \{x \in E : \operatorname{dist}(x, \partial E) > t\}.$$

Then, for all $t \in (0, T)$ we have that E_t has boundary of class C^2, and $\mathcal{H}^{d-1}(\partial E_t) \leq C$ for some C independent of t. Then, by coarea formula we have

$$\begin{aligned}\int_E \frac{1}{r_x^\sigma} \log^2 r_x \,\mathrm{d}x \leq & \int_0^T \mathcal{H}^{d-1}(\partial E_t) \frac{1}{t^\sigma} \log^2 t \,\mathrm{d}t + \Big(\frac{1}{T^\sigma} \max\{\log^2 \operatorname{diam}(E), \log^2 T\}\Big)|E_T| \\ \leq & c_1 \int_0^T \frac{1}{t^\sigma} \log^2 t \,\mathrm{d}t + c_2 \leq C(E),\end{aligned}$$

i.e., (6.11) holds. □

In the following proposition we show that a set $E \in \mathcal{M}_{\mathrm{f}}(\mathbb{R}^d)$ with $\hat{H}^\sigma(E) < +\infty$ can be approximated by a sequence of smooth sets. The same property related to the σ-fractional perimeters, with $\sigma > 0$, has been proved in [28].

Proposition 6.2 (Density of smooth sets). *Let $\sigma \in (-d, 1)$ and let $R \in (0, +\infty]$. Let $E \in \mathcal{M}_{\mathrm{f}}(\mathbb{R}^d)$. If $\hat{H}^\sigma_R(E) < +\infty$, then, there exists a sequence $\{E_n\}_{n\in\mathbb{N}}$ of bounded sets with smooth boundary such that $\chi_{E_n} \to \chi_E$ strongly in $L^1(\mathbb{R}^d)$ and $\hat{H}^\sigma_R(E_n) \to \hat{H}^\sigma_R(E)$ as $n \to +\infty$.*

Proof. First, we recall that for sets of finite perimeter, this result is classical, and its proof is based on the lower semicontinuity of the perimeter, on the convexity of the total variation functional, and on the coarea formula (see [2, Theorem 3.42]). Recalling that H^σ_R are lower semicontinuous and that the functionals $TV_{H^\sigma_R}$ introduced in (2.12) are convex, the same proof shows that, for every $R \in (0, +\infty)$, there exists a sequence $\{E_{R,m}\}_{m\in\mathbb{N}}$ of bounded sets with smooth boundary such that $\chi_{E_{R,m}} \to \chi_E$ strongly in $L^1(\mathbb{R}^d)$ and $H^\sigma_R(E_{R,m}) \to H^\sigma_R(E)$ as $m \to +\infty$. As a consequence, $\hat{H}^\sigma_R(E_{R,m}) \to \hat{H}^\sigma_R(E)$ as $m \to +\infty$. We now prove the statement for $R = +\infty$. By Propositions 2.1 and 2.2, the functionals $\hat{H}^\sigma$ are lower semicontinuous; this fact, together with Propositions 2.1, 2.2, and 2.3, implies

$$\hat{H}^\sigma(E) \leq \liminf_{m\to+\infty} \hat{H}^\sigma(E_{R,m}) \leq \liminf_{m\to+\infty} \hat{H}^\sigma_R(E_{R,m}) = \hat{H}^\sigma_R(E).$$

Since $\hat{H}^\sigma_R(E) \to \hat{H}^\sigma(E)$ as $R \to +\infty$, a standard diagonal argument provides a sequence $\{E_n\}_{n\in\mathbb{N}}$ with $E_n = E_{R_n,m_n}$ satisfying all the claimed properties. □

We are now in a position to prove the Γ-convergence result for the functionals $\hat{H}^\sigma_R$ as $\sigma \to \bar{\sigma}$ for some $\bar{\sigma} \in (-d, 1)$, and $R \to \bar{R}$ for some $\bar{R} \in (0, +\infty]$.

Theorem 6.3. *Let $\bar{R} \in (0, +\infty]$ and $\bar{\sigma} \in (-d, 1)$. Let moreover $\{R_n\}_{n\in\mathbb{N}} \subset (0, +\infty]$ and $\{\sigma_n\}_{n\in\mathbb{N}} \subset (-d, 1)$ be such that $R_n \to \bar{R}$ and $\sigma_n \to \bar{\sigma}$ as $n \to +\infty$. The following Γ-convergence result holds true.*

(i) *(Γ-liminf inequality) For every $E \in \mathcal{M}_{\mathrm{f}}(\mathbb{R}^d)$ and for every sequence $\{E_n\}_{n\in\mathbb{N}}$ with $\chi_{E_n} \to \chi_E$ strongly in $L^1(\mathbb{R}^d)$ it holds*

$$\hat{H}^{\bar{\sigma}}_{\bar{R}}(E) \leq \liminf_{n\to+\infty} \hat{H}^{\sigma_n}_{R_n}(E_n)\,.$$

(ii) *(Γ-limsup inequality) For every $E \in \mathcal{M}_{\mathrm{f}}(\mathbb{R}^d)$, there exists a sequence $\{E_n\}_{n\in\mathbb{N}}$ such that $\chi_{E_n} \to \chi_E$ strongly in $L^1(\mathbb{R}^d)$ and*

$$\hat{H}^{\bar{\sigma}}_{\bar{R}}(E) \geq \limsup_{n\to+\infty} \hat{H}^{\sigma_n}_{R_n}(E_n)\,.$$

Proof. We first prove (i). We distinguish among two cases.

Case 1: $\bar{R} \in (0, +\infty)$. Trivially, we have

$$\begin{aligned}\liminf_{n\to+\infty} \hat{H}^{\sigma_n}_{R_n}(E_n) - \hat{H}^{\bar{\sigma}}_{\bar{R}}(E) \geq &\liminf_{n\to+\infty} H^{\sigma_n}_{R_n}(E_n) - H^{\bar{\sigma}}_{\bar{R}}(E) \\ &+ \lim_{n\to+\infty} \gamma^{\sigma_n}_{R_n}|E_n| - \gamma^{\bar{\sigma}}_{\bar{R}}|E| \,.\end{aligned} \tag{6.12}$$

Moreover, by Fatou Lemma

$$\liminf_{n\to+\infty} H^{\sigma_n}_{R_n}(E_n) \geq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \liminf_{n\to+\infty} \chi_{B_{R_n}(x)}(y) \frac{\chi_{E_n}(x)(1-\chi_{E_n}(y))}{|x-y|^{d+\sigma_n}} \,\mathrm{d}y\,\mathrm{d}x = H^{\bar{\sigma}}_{\bar{R}}(E)\,,$$

which, together with (6.12) and (6.6) implies (i).

Case 2: $\bar{R} = +\infty$. By Theorem 4.1, Lemma 3.1, Lemma 6.4 below, and Propositions 2.1, 2.2, 2.3, we have

$$\begin{aligned}\hat{H}^{\bar{\sigma}}(E) = \hat{J}^{\bar{\sigma}}(E) = \lim_{r\to 0^+} \hat{J}^{\bar{\sigma}}_r(E) = \lim_{r\to 0^+}\lim_{n\to+\infty} \hat{J}^{\sigma_n}_r(E_n) \leq \liminf_{n\to+\infty} \hat{J}^{\sigma_n}(E_n) \\ = \liminf_{n\to+\infty} \hat{H}^{\sigma_n}(E_n) \leq \liminf_{n\to+\infty} \hat{H}^{\sigma_n}_{R_n}(E_n)\,,\end{aligned}$$

i.e., (i) holds.

Now we prove (ii). We can assume without loss of generality that $\hat{H}^{\bar{\sigma}}_{\bar{R}}(E) < +\infty$. If E is smooth, in view of Proposition 6.1, in particular by (6.1), the constant sequence $E_n \equiv E$ satisfies the Γ-limsup inequality. The Γ-limsup inequality in the general case is an easy consequence of Proposition 6.2 and of a standard diagunal argument, usually referred to as *density argument in* Γ*-convergence.* The details are left to the reader. □

Lemma 6.4. *Let* $\bar{\sigma} \in (-d, 1)$ *and let* $\bar{r} > 0$*. Let* $\{\sigma_n\}_{n\in\mathbb{N}} \subset (-d, 1)$ *and* $\{r_n\}_{n\in\mathbb{N}} \subset (0, +\infty)$ *be such that* $\sigma_n \to \bar{\sigma}$ *and* $r_n \to \bar{r}$ *as* $n \to +\infty$ *. Let moreover* $E \in \mathcal{M}_{\mathrm{f}}(\mathbb{R}^d)$ *and* $\{E_n\}_{n\in\mathbb{N}} \subset \mathcal{M}_{\mathrm{f}}(\mathbb{R}^d)$ *be such that* $\chi_{E_n} \to \chi_E$ *strongly in* $L^1(\mathbb{R}^d)$ *as* $n \to +\infty$ *. Then,*

$$j^{\bar{\sigma}}_{\bar{r}}(x, E) = \lim_{n\to+\infty} j^{\sigma_n}_{r_n}(x, E_n) \qquad \text{for every } x \in \mathbb{R}^d \tag{6.13}$$

$$\hat{J}^{\bar{\sigma}}_{\bar{r}}(E) = \lim_{n\to+\infty} \hat{J}^{\sigma_n}_{r_n}(E_n)\,. \tag{6.14}$$

Proof. We start by proving (6.13). Let $x \in \mathbb{R}^d$. It is easy to see that

$$\begin{aligned}j^{\sigma_n}_{r_n}(x, E_n) = &\int_{\mathbb{R}^d\setminus B_{r_n}(x)} \frac{\chi_E(y) - \chi_{E_n}(y)}{|x-y|^{d+\sigma_n}}\,\mathrm{d}y \\ &- \int_{\mathbb{R}^d\setminus B_{r_n}(x)} \frac{\chi_E(y)}{|x-y|^{d+\sigma_n}}\,\mathrm{d}y - \gamma^{\sigma_n}_{r_n}\,.\end{aligned} \tag{6.15}$$

As for the first integral in (6.15) we have

$$\int_{\mathbb{R}^d\setminus B_{r_n}(x)} \frac{|\chi_{E_n}(y) - \chi_E(y)|}{|x-y|^{d+\sigma_n}}\,\mathrm{d}y \leq \frac{1}{{r_n}^{d+\sigma_n}}|E_n \Delta E| \to 0\,, \tag{6.16}$$

while for the remaining terms, by the dominated convergence Theorem, we obtain

$$-\int_{\mathbb{R}^d\setminus B_{r_n}(x)} \frac{\chi_E(y)}{|x-y|^{d+\sigma_n}}\,\mathrm{d}y - \gamma^{\sigma_n}_{r_n} \to j^{\bar{\sigma}}_{\bar{r}}(x, E) \qquad \text{as } n \to +\infty\,,$$

which together with (6.15), and (6.16), implies (6.13).

Now we prove (6.14). In view of the strong L^1 convergence of the functions χ_{E_n} we have that there exists a constant $C > 0$ such that

$$\sup_{n\in\mathbb{N}} |E_n| \leq C\,. \tag{6.17}$$

By (3.1), we have

$$\hat{J}^{\sigma_n}_{r_n}(E_n) = \int_{\mathbb{R}^d} (\chi_{E_n}(x) - \chi_E(x)) j^{\sigma_n}_{r_n}(x, E_n)\,\mathrm{d}x + \int_E j^{\sigma_n}_{r_n}(x, E_n)\,\mathrm{d}x\,. \tag{6.18}$$

By (6.17) we have

$$\int_{\mathbb{R}^d} |\chi_{E_n}(x) - \chi_E(x)||j^{\sigma_n}_{r_n}(x, E_n)|\,\mathrm{d}x \leq \frac{C}{{r_n}^{d+\sigma_n}}|E_n\Delta E| \to 0 \text{ as } n \to +\infty\,, \tag{6.19}$$

whereas by (6.13) and by the dominated convergence Theorem we deduce

$$\int_E j^{\sigma_n}_{r_n}(x, E_n)\,\mathrm{d}x \to \hat{J}^{\bar\sigma}_{\bar r}(E) \qquad \text{as } n \to +\infty\,. \tag{6.20}$$

Therefore, (6.14) follows by (6.18), (6.19), and (6.20). □

Finally we prove the Γ-convergence result for the functionals $\hat{J}^\sigma_r$ as $\sigma \to \bar\sigma$ for some $\bar\sigma \in (-d, 1)$ and $r \to \bar r$ for some $\bar r \in [0, +\infty)$. To this purpose, it is convenient to adopt the notation $\hat{J}^\sigma_0 := \hat{J}^\sigma$.

Theorem 6.5. *Let $\bar\sigma \in (-d, 1)$ and let $\bar r \in [0, +\infty)$. Let $\{\sigma_n\}_{n\in\mathbb{N}} \subset (-d, 1)$ and $\{r_n\}_{n\in\mathbb{N}} \subset [0, +\infty)$ be such that $\sigma_n \to \bar\sigma$ and $r_n \to \bar r$ as $n \to +\infty$. The following Γ-convergence result holds true.*

(i) *(Γ-liminf inequality) For every $E \in \mathcal{M}_{\mathrm{f}}(\mathbb{R}^d)$ and for every sequence $\{E_n\}_{n\in\mathbb{N}}$ with $\chi_{E_n} \to \chi_E$ strongly in $L^1(\mathbb{R}^d)$ it holds*

$$\hat{J}^{\bar\sigma}_{\bar r}(E) \leq \liminf_{n\to+\infty} \hat{J}^{\sigma_n}_{\bar r_n}(E_n)\,.$$

(ii) *(Γ-limsup inequality)For every $E \in \mathcal{M}_{\mathrm{f}}(\mathbb{R}^d)$, there exists a sequence $\{E_n\}_{n\in\mathbb{N}}$ such that $\chi_{E_n} \to \chi_E$ strongly in $L^1(\mathbb{R}^d)$ and*

$$\hat{J}^{\bar\sigma}_{\bar r}(E) \geq \limsup_{n\to+\infty} \hat{J}^{\sigma_n}_{r_n}(E_n)\,.$$

Proof. If $\bar r \in (0, +\infty)$, the statement follows immediately by (6.14). We discuss the case $\bar r = 0$. By Lemma 3.1 and by (6.14) we have

$$\hat{J}^{\bar\sigma}(E) = \lim_{r\to0^+} \hat{J}^{\bar\sigma}_r(E) = \lim_{r\to0^+}\lim_{n\to+\infty} \hat{J}^{\sigma_n}_r(E_n) \leq \liminf_{n\to+\infty} \hat{J}^{\sigma_n}_{r_n}(E_n)\,,$$

i.e., (i). We prove (ii) for $\bar r = 0$. By Theorem 6.3, Theorem 4.1, and Lemma 3.1, there exists a sequence $\{E_n\}_{n\in\mathbb{N}} \subset \mathcal{M}_{\mathrm{f}}(\mathbb{R}^d)$ such that $\chi_{E_n} \to \chi_E$ as $n \to +\infty$ and

$$\hat{J}^{\bar\sigma}(E) = \lim_{n\to+\infty} \hat{J}^{\sigma_n}(E_n) \geq \limsup_{n\to+\infty} \hat{J}^{\sigma_n}_{r_n}(E_n)\,.$$

□

7. The fractional isoperimetric inequality

The isoperimetric inequality for the functionals $\hat{J}^\sigma$ for $\sigma \in (-d, 0)$ is nothing but the Riesz inequality (see [30] and Theorem A.1). For $\sigma \in (0, 1)$, one deals with fractional isoperimetric inequalities, that have been proven in [21], while their quantitative counterpart has been established in [22] (see also [18, 16, 19]). Here we prove the isoperimetric inequality and its stability also for the 0-fractional perimeter. In fact, our short proof based on Riesz inequality yields the result for every exponent $\sigma \in [0, 1)$.

Let $\sigma \in [0, 1)$. For every $r > 0$, we set

$$k^\sigma_r(t) := \frac{1}{\max\{t^{d+\sigma}, r^{d+\sigma}\}} + (r - t)^+\,, \tag{7.1}$$

and we define the functionals $\mathscr{J}^\sigma_r : \mathcal{M}_{\mathrm{f}}(\mathbb{R}^d) \to (-\infty, 0]$ as

$$\mathscr{J}^\sigma_r(E) := -\int_E\int_E k^\sigma_r(|x - y|)\,\mathrm{d}y\,\mathrm{d}x \qquad \text{for all } E \in \mathcal{M}_{\mathrm{f}}(\mathbb{R}^d)\,.$$

Notice that k^σ_r is strictly decreasing with respect to t and that, for every $E \in \mathcal{M}_{\mathrm{f}}(\mathbb{R}^d)$,

$$\mathscr{J}^\sigma_r(E) = J^\sigma_r(E) - \int_E\int_{B_r(x)\cap E} \frac{1}{r^{d+\sigma}} + (r - |x - y|)^+\,\mathrm{d}y\,\mathrm{d}x\,, \tag{7.2}$$

We have the following result.

Lemma 7.1. *Let E, $F \in \mathcal{M}_{\mathrm{f}}(\mathbb{R}^d)$ with $|E| = |F|$, and, for $\sigma \neq 0$, assume also that E and F have C^2 compact boundary. Then,*

$$\lim_{r\to 0}(\mathscr{J}_r^\sigma(E) - \mathscr{J}_r^\sigma(F)) = \lim_{r\to 0}(J_r^\sigma(E) - J_r^\sigma(F)) = \hat{J}^\sigma(E) - \hat{J}^\sigma(F). \tag{7.3}$$

Proof. We preliminarily notice that the second equality in (7.3) is a trivial consequence of the very definition of $\hat{J}^\sigma$ and of the fact that $|E| = |F|$.

Moreover, we notice that, for all $G \in \mathcal{M}_{\mathrm{f}}(\mathbb{R}^d)$ we have

$$\int_G \int_{B_r(x)\cap G} (r - |x-y|)^+ \,\mathrm{d}y\,\mathrm{d}x = \delta_r\,,$$

with $\delta_r \to 0$ as $r \to 0^+$. Therefore, by (7.2) we have

$$\begin{aligned} \mathscr{J}_r^\sigma(E) - \mathscr{J}_r^\sigma(F) = {} & J_r^\sigma(E) - J_r^\sigma(F) \\ & + \frac{1}{r^{d+\sigma}}\Big[\int_F |B_r(x)\cap F|\,\mathrm{d}x - \int_E |B_r(x)\cap E|\,\mathrm{d}x\Big] + \delta_r\,, \end{aligned} \tag{7.4}$$

with $\delta_r \to 0$ as $r \to 0^+$.

We first consider the case $\sigma = 0$. By the mean value and dominated convergence Theorems, for all $G \in \mathcal{M}_{\mathrm{f}}(\mathbb{R}^d)$ we have

$$\lim_{r\to 0^+}\int_G \int_{B_r(x)\cap G} \frac{1}{r^d} + (r - |x-y|)^+ \,\mathrm{d}y\,\mathrm{d}x = \omega_d|G|, \tag{7.5}$$

whence, by (7.4) we deduce (7.3).

Let now $\sigma \in (0,1)$. Using the notation in (4.7), by coarea formula, for all $G \in \mathcal{M}_{\mathrm{f}}(\mathbb{R}^d)$ with C^2 compact boundary we have

$$\int_{G\setminus G_r} |B_r(x)\cap G|\,\mathrm{d}x \le \omega_d r^d \int_0^r \mathcal{H}^{d-1}(\partial G_t)\,\mathrm{d}t \le C\omega_d r^{d+1}. \tag{7.6}$$

Moreover, we have

$$\begin{aligned} & \Big|\int_{F_r} |B_r(x)\cap F|\,\mathrm{d}x - \int_{E_r} |B_r(x)\cap E|\,\mathrm{d}x\Big| = \omega_d r^d(|E_r| - |F_r|) \\ = {} & \omega_d r^d\Big||F\setminus F_r| - |E\setminus E_r|\Big| \le \omega_d r^d\Big(|F\setminus F_r| + |E\setminus E_r|\Big) \le C\omega_d r^{d+1}\,, \end{aligned} \tag{7.7}$$

where the last inequality easily follows by the coarea formula and the regularity of ∂E, ∂F.

By (7.4), (7.6) and (7.7), we deduce (7.3) also for $\sigma \in (0,1)$. □

Theorem 7.2 (Isoperimetric inequality)**.** *For every $\sigma \in [0,1)$, the ball B^m of measure equal to $m > 0$ is the unique, up to translations, minimizer of the σ-fractional perimeter $\hat{J}^\sigma$ among all the measurable sets with measure equal to m.*

Moreover, if $\{E_n\}_{n\in\mathbb{N}}$ is a sequence of sets such that $|E_n| \equiv m$ and $\hat{J}^\sigma(E_n) \to \hat{J}^\sigma(B^m)$, then, there exists a sequence of translations $\{\tau_n\}_{n\in\mathbb{N}}$ such that $\chi_{E_n+\tau_n} \to \chi_{B^m}$ strongly in $L^1(\mathbb{R}^d)$.

Proof. We set $\mathrm{Inf} := \inf_{\substack{E\in\mathcal{M}_{\mathrm{f}}(\mathbb{R}^d)\\ |E|=m}} \hat{J}^\sigma(E)$; for all $\varepsilon > 0$ let E_ε (see Proposition 6.2 and [28] for $\sigma > 0$) be a smooth set such that

$$\hat{J}^\sigma(E_\varepsilon) - \mathrm{Inf} \le \varepsilon\,. \tag{7.8}$$

In view of the scaling property (4.8) (and recalling that $\hat{H}^\sigma = \hat{J}^\sigma$) we can assume, without loss of generality, that $|E_\varepsilon| = m$ for every $\varepsilon > 0$. Recalling the definition of k_r^σ in (7.1), for every $0 < r_1 \le r_2 \le 1$, we set

$$k_{r_1,r_2}(t) := k_{r_1}(t) - k_{r_2}(t) \qquad \text{for all } t > 0\,.$$

Noticing that k_{r_1,r_2} is monotonically non-increasing with respect to t, and using Riesz inequality (Theorem A.1) we have

$$\begin{aligned}\mathscr{J}^\sigma_{r_1}(E_\varepsilon)-\mathscr{J}^\sigma_{r_1}(B^m)&=\mathscr{J}^\sigma_{r_2}(E_\varepsilon)-\mathscr{J}^\sigma_{r_1}(B^m)-\int_{E_\varepsilon}\int_{E_\varepsilon}k_{r_1,r_2}(|x-y|)\,\mathrm{d}y\,\mathrm{d}x\\&\geq \mathscr{J}^\sigma_{r_2}(E_\varepsilon)-\mathscr{J}^\sigma_{r_1}(B^m)-\int_{B^m}\int_{B^m}k_{r_1,r_2}(|x-y|)\,\mathrm{d}y\,\mathrm{d}x\\&=\mathscr{J}^\sigma_{r_2}(E_\varepsilon)-\mathscr{J}^\sigma_{r_2}(B^m)=:c_{r_2}(\varepsilon)\geq 0\,,\end{aligned}$$

where the last inequality follows again by Riesz inequality. Chosing $r_2=1$ and letting $r=r_1\to 0^+$, by Lemma 7.1 we deduce that

$$\hat{J}^\sigma(E_\varepsilon)-\hat{J}^\sigma(B^m)\geq c_1(\varepsilon), \tag{7.9}$$

and by (7.8) we conclude that $c_1(\varepsilon)$ (and in fact $c_r(\varepsilon)$ for all positive r) vanishes as $\varepsilon\to 0^+$. Therefore $\mathscr{J}^\sigma_1(E_\varepsilon)\to\mathscr{J}^\sigma_1(B^m)$ as $\varepsilon\to 0^+$. Noticing that k^σ_1 is strictly decreasing, by Theorem A.4 we deduce that, up to translations, $\chi_{E_\varepsilon}\to\chi_{B^m}$ strongly in $L^1(\mathbb{R}^d)$. The minimality of B^m is then a consequence of the lower semicontinuity (together with the translational invariance) of $\hat{J}^\sigma$. □

Remark 7.3. Notice that our proof of Theorem 7.2 relies on elementary rearrangement inequalities which provide uniqueness and stability for the 0-isoperimetric problem. Actually, using the more refined result in [13] (see also [20]) and the quantitative isoperimetric inequality for the Riesz functionals in [19], we can easily show also a quantitative isoperimetric inequality for the functional $\hat{H}^0$. Indeed, writing the last estimate in [19, proof of Theorem 4] in our notation, we have that there exists a constant c_d depending only on the dimension such that for every $\sigma\in(-d,0)$, $m>0$, and for every measurable set $F\subset\mathbb{R}^d$ with $|F|=m$, it holds

$$J^\sigma(F)-J^\sigma(B^m)\geq (d+\sigma)c_d|m|^2A^2[F]\int_I\frac{1}{R^{d+\sigma+1}}\,\mathrm{d}R\,, \tag{7.10}$$

where

$$A[F]:=\frac{1}{2m}\inf_{a\in\mathbb{R}^d}\|\chi_F-\chi_{B^m+a}\|_{L^1(\mathbb{R}^d)}$$

and $I:=\{R>0\,:\,\frac14\leq\frac{|B_R|^{\frac1d}}{2m^{\frac1d}}\leq\frac34\}$; by (7.10) we get

$$J^\sigma(F)-J^\sigma(B^m)\geq c_d2^{d+\sigma}\Big(1-\frac{1}{3^{d+\sigma}}\Big)\omega_d^{1+\frac{\sigma}{d}}m^{1-\frac{\sigma}{d}}A^2[F]. \tag{7.11}$$

Now, let $E\subset\mathbb{R}^d$ with $|E|=m$, and let $\{\sigma_n\}_{n\in\mathbb{N}}\subset(-d,0)$ with $\sigma_n\to 0^-$ as $n\to+\infty$. By Theorem 6.5(ii), there exists a sequence $\{E_n\}_{n\in\mathbb{N}}$ with $\|\chi_{E_n}-\chi_E\|_{L^1(\mathbb{R}^d)}\to 0$ as $n\to+\infty$ such that

$$\hat{J}^0(E)\geq\limsup_{n\to+\infty}\hat{J}^{\sigma_n}(E_n). \tag{7.12}$$

Set moreover $m_n:=|E_n|$. Clearly, $\|\chi_{B^{m_n}}-\chi_{B^m}\|_{L^1(\mathbb{R}^d)}\to 0$ as $n\to+\infty$ which in view of Theorem 6.5(i), yields

$$\hat{J}^0(B^m)\leq\liminf_{n\to+\infty}\hat{J}^{\sigma_n}(B^{m_n}). \tag{7.13}$$

By (7.12) and (7.13), using that $|E_n|=m_n=|B^{m_n}|$ and (7.11), we deduce that

$$\begin{aligned}\hat{J}^0(E)-\hat{J}^0(B^m)&\geq\limsup_{n\to+\infty}(\hat{J}^{\sigma_n}(E_n)-\hat{J}^{\sigma_n}(B^{m_n}))=\limsup_{n\to+\infty}(J^{\sigma_n}(E_n)-J^{\sigma_n}(B^{m_n}))\\&\geq C_d\limsup_{n\to+\infty}A^2[E_n]=C_dA^2[E]\,,\end{aligned}$$

where $C_d>0$ depends only on d.

In view of (1.9) one may wonder whether both the functionals H^σ_1 and J^σ_1 are minimized, under volume constraints, by the ball. We show that this is the case for H^σ_1 (see Proposition 7.5) but, in general, not for J^σ_1 (see Remark 7.4).

Remark 7.4. Let $\sigma \in (-d,1)$ and let $r>0$. We set $m_r := \omega_d(\frac{r}{2})^d$. Clearly, for all $m\in(0,m_r)$ we have that $0=J^\sigma_r(B^m)\ge J^\sigma_r(E)$ for all E with $|E|=m$. Moreover, taking $E:=B^{\frac{m}{2}}\cup B^{\frac{m}{2}}(\xi)$ with $|\xi|=2r$, we have immediately that $|E|=m$ and $J^\sigma_r(E)<0$. Therefore, for all $m\in(0,m_r)$ the ball B^m is a maximizer of J^σ_r; in particular, for general values of m and r the ball is not a solution of the isoperimetric inequality.

Proposition 7.5. *Let $\sigma\in(-d,1)$ and let $R>0$. The ball B^m of measure equal to $m>0$ is the unique, up to translations, minimizer of H^σ_R among all the measurable sets with measure equal to m.*

Moreover, if $\{E_n\}_{n\in\mathbb{N}}$ is a sequence of sets such that $|E_n|\equiv m$ and $H^\sigma_R(E_n)\to H^\sigma_R(B^m)$as $n\to+\infty$, then, there exists a sequence of translations $\{\tau_n\}_{n\in\mathbb{N}}$ such that $\chi_{E_n+\tau_n}\to\chi_{B^m}$ strongly in $L^1(\mathbb{R}^d)$.

Proof. For every $0<r\le R$ we set $k^\sigma_{r,R}(t):=\chi_{[0,R]}(t)k^\sigma_r(t)$ where k^σ_r is defined in (7.1), and we define $\mathcal{K}^\sigma_{r,R}:\mathcal{M}_{\rm f}(\mathbb{R}^d)\to(-\infty,0)$ as

$$\mathcal{K}^\sigma_{r,R}(E):=-\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\chi_E(x)\chi_E(y)k^\sigma_{r,R}(|x-y|)\,\mathrm{d}y\,\mathrm{d}x\,.$$

Let $E\in\mathcal{M}_{\rm f}(\mathbb{R}^d)$ with $|E|=m$ and $H^\sigma_R(E)<+\infty$. For every $0<r<\bar r<\min\{R,1\}$ we have

$$\begin{aligned}
H^\sigma_R(E)-H^\sigma_R(B^m) &= \mathcal{K}^\sigma_{\bar r,R}(E)-\mathcal{K}^\sigma_{\bar r,R}(B^m)\\
&\quad+(\mathcal{K}^\sigma_{r,R}(E)-\mathcal{K}^\sigma_{\bar r,R}(E))-(\mathcal{K}^\sigma_{r,R}(B^m)-\mathcal{K}^\sigma_{\bar r,R}(B^m)) \qquad (7.14)\\
&\quad+H^\sigma_R(E)-\mathcal{K}^\sigma_{r,R}(E)-H^\sigma_R(B^m)+\mathcal{K}^\sigma_{r,R}(B^m)\\
&\ge \mathcal{K}^\sigma_{\bar r,R}(E)-\mathcal{K}^\sigma_{\bar r,R}(B^m) \qquad (7.15)\\
&\quad+H^\sigma_R(E)-\mathcal{K}^\sigma_{r,R}(E)-H^\sigma_R(B^m)+\mathcal{K}^\sigma_{r,R}(B^m)\,, \qquad (7.16)
\end{aligned}$$

where the non-negativity of the quantity in (7.14) follows by the monotonicity of $k^\sigma_r-k^\sigma_{\bar r}$ and by Riesz inequality.

Now we show that the sum in (7.16) tends to 0 as $r\to0^+$. Indeed, by the very definiton of $\mathcal{K}^\sigma_{r,R}$ and by (7.1) we have

$$\begin{aligned}
&H^\sigma_R(E)-\mathcal{K}^\sigma_{r,R}(E)-H^\sigma_R(B^m)+\mathcal{K}^\sigma_{r,R}(B^m)\\
&= \int_E\int_{\mathbb{R}^d\setminus E}\frac{\chi_{B_R(x)}(y)}{|x-y|^{d+\sigma}}\,\mathrm{d}y\,\mathrm{d}x-\int_E\int_{\mathbb{R}^d\setminus E}\chi_{B_R(x)}(y)k^\sigma_r(|x-y|)\,\mathrm{d}y\,\mathrm{d}x \qquad (7.17)\\
&\quad-\int_{B^m}\int_{\mathbb{R}^d\setminus B^m}\frac{\chi_{B_R(x)}(y)}{|x-y|^{d+\sigma}}\,\mathrm{d}y\,\mathrm{d}x+\int_{B^m}\int_{\mathbb{R}^d\setminus B^m}\chi_{B_R(x)}(y)k^\sigma_r(|x-y|)\,\mathrm{d}y\,\mathrm{d}x, \qquad (7.18)
\end{aligned}$$

and $k^\sigma_r(t)$ monotonically increases to $\frac{1}{t^{d+\sigma}}$ as $r\to0^+$. By the monotone convergence Theorem, we have that the expressions in (7.17) and (7.18) tend to zero as $r\to0^+$. Therefore, by taking the limit as $r\to0^+$ in (7.16) and by (7.15), we get

$$H^\sigma_R(E)-H^\sigma_R(B^m)\ge\mathcal{K}^\sigma_{\bar r,R}(E)-\mathcal{K}^\sigma_{\bar r,R}(B^m)\ge0\,, \qquad (7.19)$$

where the last inequality follows by Theorem A.1. Noticing that $k^\sigma_{\bar r,R}$ is strictly decreasing in $(0,R)$ and using Proposition A.2, we get that, up to translations, the ball B^m is the unique minimizer of $\mathcal{K}^\sigma_{\bar r,R}$, and hence of H^σ_R.

Finally, if $\{E_n\}_{n\in\mathbb{N}}$ is a sequence of sets such that $|E_n|\equiv m$ and $H^\sigma_R(E_n)\to H^\sigma_R(\chi_{B^m})$, by (7.19) we have that $\mathcal{K}^\sigma_{\bar r,R}(E_n)\to\mathcal{K}^\sigma_{\bar r,R}(B^m)$ as $n\to+\infty$. By Theorem A.4 we deduce that there exists a sequence of translations $\{\tau_n\}_{n\in\mathbb{N}}$ such that $\chi_{E_n+\tau_n}\to\chi_{B^m}$ strongly in $L^1(\mathbb{R}^d)$. □

Appendix A. Rearrangement inequalities

In this appendix we recall some results on rearrangement inequalities and we provide some cases of uniqueness and stability for the Riesz inequality, in the specific case of a set interacting with itself.

Let $K\in L^1_{\rm loc}(\mathbb{R}^d;[0,+\infty))$ be such that $K(z)=k(|z|)$ for some $k:[0,+\infty)\to[0,+\infty)$ monotonically non-increasing. For every $\eta_1,\eta_2\in L^1(\mathbb{R}^d;[0,+\infty))$ we set

$$I(\eta_1,\eta_2):=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\eta_1(x)\eta_2(y)K(x-y)\,\mathrm{d}y\,\mathrm{d}x\,.$$

First, we recall the classical Riesz inequality [30]. To this purpose, for every $m > 0$ and $x_0 \in \mathbb{R}^d$, we denote by $B^m(x_0)$ the ball centered in x_0 with $|B^m(x_0)| = m$ (B^m if $x_0 = 0$). With a little abuse of notation, for any $x_0 \in \mathbb{R}^d$ and for any $\eta \in L^1(\mathbb{R}^d; [0, +\infty))$, we set $B^\eta(x_0) := B^{\|\eta\|_{L^1}}(x_0)$ ($B^\eta := B^{\|\eta\|_{L^1}}$ if $x_0 = 0$). Moreover, for every function $\eta \in L^1(\mathbb{R}^d; [0, +\infty))$ we denote by η^* the spherical symmetric nonincreasing rearrangement of η, satisfying

$$\{\eta^* > t\} = B^{m_t} \text{ where } m_t := |\{\eta > t\}| \qquad \text{for all } t > 0\,. \tag{A.1}$$

Now, we state the Riesz inequality, restricting our analysis to densities with values in $[0, 1]$. The proof of Theorem A.2 below follows from the Riesz rearrangement inequality [30, 6, 5] and the bathtub principle [26, Theorem 1.14].

Theorem A.1 (Riesz inequality)**.** *Let η_1, $\eta_2 \in L^1(\mathbb{R}^d; [0, 1])$ with $\|\eta_1\|_{L^1}, \|\eta_2\|_{L^1} > 0$. Then,*

$$I(\eta_1, \eta_2) \le I(\eta_1^*, \eta_2^*) \le I(\chi_{B^{\eta_1}}, \chi_{B^{\eta_2}})\,. \tag{A.2}$$

Moreover, if k is strictly decreasing, then the first inequality in (A.2) *is an equality if and only if $\eta_i(\cdot) = \eta_i^*(\cdot - x_0)$ ($i = 1, 2$) for some $x_0 \in \mathbb{R}^d$, whereas the second inequality in* (A.2) *holds with the equality if and only if $\eta_i^* = \chi_{B^{\eta_i}}$.*

Equality cases have been largely studied in the literature (see [25, 5, 8, 9]); here we provide a case of equality specific for a characteristic function interacting with itself.

For every $E \in \mathcal{M}_{\mathrm{f}}(\mathbb{R}^d)$, we set $\mathcal{K}(E) := I(\chi_E, \chi_E)$.

Proposition A.2 (An equality case)**.** *Assume that k is strictly decreasing in a neighborhood of the origin. If $E \in \mathcal{M}_{\mathrm{f}}(\mathbb{R}^d)$ satisfies*

$$\mathcal{K}(E) = \mathcal{K}(B^{|E|})\,, \tag{A.3}$$

Then $E = B^{|E|}(x_0)$ for some $x_0 \in \mathbb{R}^d$.

Proof. By the layer-cake principle, we have

$$\mathcal{K}(E) = \int_0^{+\infty} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \chi_E(x)\chi_E(y)\chi_{\{K>t\}}(x - y)\, \mathrm{d}y\, \mathrm{d}x\, \mathrm{d}t\,.$$

By (A.2) and (A.3) we have that for a.e. $t > 0$

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \chi_E(x)\chi_E(y)\chi_{\{K>t\}}(x - y)\, \mathrm{d}y\, \mathrm{d}x = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \chi_{B^{|E|}}(x)\chi_{B^{|E|}}(y)\chi_{\{K>t\}}(x - y)\, \mathrm{d}y\, \mathrm{d}x\,.$$

Set $\beta(t) := |\{K > t\}|$ for every t . Since K is radially symmetric and k is monotonically decreasing, we clearly have that $\{K > t\} = B^{\beta(t)}$ for all $t > 0$. Moreover, since k is strictly monotone in a neighborhood of the origin, we have that for all $\bar{\beta} > 0$ the set $F_{\bar{\beta}} := \{t > 0 : 0 < \beta(t) < \bar{\beta}\}$ has positive measure. Furthermore, for a.e. $t \in F_{\bar{\beta}}$ we have

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \chi_E(x)\chi_E(y)\chi_{B^{\beta(t)}}(x - y)\, \mathrm{d}x\, \mathrm{d}y = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \chi_{B^{|E|}}(x)\chi_{B^{|E|}}(y)\chi_{B^{\beta(t)}}(x - y)\, \mathrm{d}x\, \mathrm{d}y\,. \tag{A.4}$$

Now fix $\bar{\beta} = 2|E|$ and let $t \in F_{\bar{\beta}}$ be such that (A.4) holds; by [5, Theorem 1] we conclude that, up to a translation, $E = B^{|E|}$. □

We will also need the following result.

Proposition A.3. *Assume that k is strictly decreasing in a neighborhood of the origin. Let $\eta \in L^1(\mathbb{R}^d; [0, 1])$ be such that $I(\eta, \eta) = I(\chi_{B^{\|\eta\|_{L^1}}}, \chi_{B^{\|\eta\|_{L^1}}})$. Then η is a characteristic function.*

Proof. The proof is based on first variation arguments; we briefly sketch it. Let η^* be the spherical symmetric rearrangement of η defined in (A.1). By Riesz inequality (A.2) we have $I(\eta, \eta) \le I(\eta^*, \eta^*)$. Assume by contradiction that there exists a Lebesgue point $\bar{x}$ of η^* with $\eta^*(\bar{x}) \in (0, 1)$. Since η^* is radially non-increasing there exists $\hat{t} > |\bar{x}|$ such that $\eta^*(x) \in (0, 1)$ for a.e. $x \in B_{\hat{t}} \setminus B_{|\bar{x}|}$. Let $\tilde{\eta} : [0, +\infty) \to [0, +\infty)$

be the function defined by $\tilde{\eta}(t) := \eta^*(tv)$ where v is a (arbitrarily chosen) unitary vector in $\mathbb{R}^d$. For every $\varepsilon > 0$ let $\tilde{f}_\varepsilon : [0, +\infty) \to [0, +\infty)$ with $\tilde{f}_\varepsilon(0) = 0$ and derivative given by

$$\tilde{f}'_\varepsilon(t) := \begin{cases} 1 & \text{if } 0 \le t \le |\bar{x}| \\ 1 - \varepsilon & \text{if } |\bar{x}| \le t \le \hat{t} \\ 1 & \text{elsewhere.} \end{cases}$$

We set $f_\varepsilon(x) := \tilde{f}_\varepsilon(|x|)$ and let $\mu_\varepsilon := f_\varepsilon^\sharp(\eta^* \, \mathrm{d}x)$ be the push-forward of the measure $\eta^* \, \mathrm{d}x$ through f_ε. Trivially, $\mu_\varepsilon(\mathbb{R}^d) = \|\eta\|_{L^1}$; moreover, notice that μ_ε is absolutely continuous with respect to the Lebesgue maesure, and, for ε sufficiently small, its density takes values in $[0, 1]$. By assumption there exists $r^* > 0$ such that k is strictly decreasing in $(0, r^*)$; for every $x \in \mathbb{R}^d$ we set

$$E_\varepsilon(x) := \{y \in \mathbb{R}^d \, : \, |f_\varepsilon(x) - f_\varepsilon(y)| < |x - y| < r^*\}. \tag{A.5}$$

By the very definition of push-forward, for ε small enough we have

$$\begin{aligned} I(\eta^*, \eta^*) &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} k(|f_\varepsilon^{-1}(x) - f_\varepsilon^{-1}(y)|) \, \mathrm{d}\mu_\varepsilon(y) \, \mathrm{d}\mu_\varepsilon(x) \\ &< \int_{\mathbb{R}^d} \int_{f_\varepsilon(E_\varepsilon(x))} k(|x - y|) \, \mathrm{d}\mu_\varepsilon(y) \, \mathrm{d}\mu_\varepsilon(x) \\ &\quad + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d \setminus f_\varepsilon(E_\varepsilon(x))} k(|x - y|) \, \mathrm{d}\mu_\varepsilon(y) \, \mathrm{d}\mu_\varepsilon(x) = I(\mu_\varepsilon, \mu_\varepsilon) \, , \end{aligned} \tag{A.6}$$

where the inequality is strict by the very definition of $E_\varepsilon(x)$ in (A.5), using also that

$$\int_{\mathbb{R}^d} \int_{f_\varepsilon(E_\varepsilon(x))} \mathrm{d}\mu_\varepsilon(y) \, \mathrm{d}\mu_\varepsilon(x) \ge \int_{f_\varepsilon(B_{\hat{t}} \setminus B_{|\bar{x}|})} \int_{f_\varepsilon(E_\varepsilon(x))} \mathrm{d}\mu_\varepsilon(y) \, \mathrm{d}\mu_\varepsilon(x) > 0.$$

By (A.6) and (A.2) we get the desired contradiction. □

Now we provide a stability result for the Riesz inequality.

Theorem A.4 (A stability case). *Assume that k is strictly decreasing in a neighborhood of the origin. Let $m > 0$ and let $\{E_n\}_{n \in \mathbb{N}} \subset \mathcal{M}_{\mathrm{f}}(\mathbb{R}^d)$ with $|E_n| \equiv m$ be such that $\mathcal{K}(E_n) \to \mathcal{K}(B^m)$. Then, there exists a sequence $\{\tau_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^d$ such that $\chi_{E_n}(\cdot - \tau_n) \to \chi_{B^m}$ strongly in $L^1(\mathbb{R}^d)$.*

Proof. The proof is based on a concentration compactness argument *à la* Lions [27]. Roughly speaking, such a method consists in showing that it is impossible to split E_n into two sets (with measure bounded away from zero) whose mutual distance diverges. We can assume without loss of generality that $m = 1$. Let $\{A_n^1\}_{n \in \mathbb{N}}$, $\{A_n^2\}_{n \in \mathbb{N}}$ be two sequences of open sets and let $\lambda \in [\frac{1}{2}, 1]$ be such that, up to a (not relabelled) subsequence

$$\begin{aligned} &|E_n \cap A_n^1| \to \lambda, \quad |E_n \cap A_n^2| \to 1 - \lambda, \\ &\mathrm{dist}(A_n^1, A_n^2) \to +\infty \qquad \text{as } n \to +\infty. \end{aligned}$$

Let $\tau \in \mathbb{R}^d$ be such that $B^{1-\lambda}(\tau) \cap B^\lambda = \emptyset$, set $\hat{E} := B^{1-\lambda}(\tau) \cup B^\lambda$ and notice that $|\hat{E}| = 1$. By Riesz inequality we have

$$\mathcal{K}(B^1) = \limsup_{n \to +\infty} \mathcal{K}(E_n) \le \mathcal{K}(B^\lambda) + \mathcal{K}(B^{1-\lambda}) \le \mathcal{K}(\hat{E}) \le \mathcal{K}(B^1),$$

which, in view of Proposition A.3 implies that, up to a translation, $\hat{E} = B^1$, i.e., $\lambda = 1$. Once proven that it is impossible to split (any subsequence of) E_n into two sets whose mutual distance diverges, arguing as in the proof of [27, Lemma I.1], the tight convergence of χ_{E_n}, up to subsequences and to translations, follows. More precisely, there exists a sequence of translations $\{\tau_n\}_{n \in \mathbb{N}}$ and a probability measure with density ρ such that, up to a subsequence, $\chi_{E_n}(\cdot - \tau_n) \overset{*}{\rightharpoonup} \rho$ tightly. Since $\mathcal{K}$ is invariant by translations and continuous with respect to the tight convergence of characteristic functions, we deduce that $I(\rho, \rho) = \mathcal{K}(B^1)$, which together with Proposition A.3, yields $\rho = \chi_E$ for some $E \in \mathcal{M}_{\mathrm{f}}(\mathbb{R}^d)$. By Proposition A.2 we get that E is a ball. Since the limit is uniquely determined, we conclude that the whole sequence $\{\chi_{E_n - \tau_n}\}$ tightly converges to the characteristic function of a ball.

□

We conclude with two lemmas that have been used in this paper. In these results we replace the assumption $K \in L^1_{\mathrm{loc}}(\mathbb{R}^d; [0, +\infty))$ by the weaker assumption $K \in L^1_{\mathrm{loc}}(\mathbb{R}^d \setminus \{0\}; [0, +\infty))$.

Lemma A.5. *Let $R > 0$ and let $F \in \mathcal{M}_{\mathrm{f}}(\mathbb{R}^d)$ with $F \subset B_R$. Then*

$$\int_{B_R \setminus F} K(y)\, \mathrm{d}y \geq \int_{B_R \setminus B^{|F|}} K(y)\, \mathrm{d}y \,.$$

Lemma A.6. *Let $s, m > 0$. Then, for all $\rho \in L^1(\mathbb{R}^d; [0, 1])$ with $\|\rho\|_{L^1} \leq m$ and with $supp(\rho) \subseteq \mathbb{R}^d \setminus B_s$, we have*

$$\int_{\mathbb{R}^d} \rho(y) K(y)\, \mathrm{d}y \leq \int_{A_{s,R(m,s)}} K(y)\, \mathrm{d}y,$$

where A_{s_1,s_2} denotes the annulus $B_{s_2} \setminus B_{s_1}$ for all $0 < s_1 < s_2$, and $R(m, s) = (\frac{m}{\omega_d} + s^d)^{\frac{1}{d}}$ (so that $|A_{s,R(m,s)}| = m$).

The proofs of Lemmas A.5 and A.6 are easy consequences of standard rearrangement techniques and are left to the reader.

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