Research Article

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A modular variable Orlicz inequality for the local maximal operator

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Abstract: In this note we prove a modular variable Orlicz inequality for the local maximal operator. This result generalizes several Orlicz and variable exponent modular inequalities that have appeared previously in the literature.

Keywords: Musielak–Orlicz spaces, local maximal operator, variable exponents, variable Lebesgue spaces, modular inequality

MSC 2010: 42B25, 46E30

Dedicated to Professor V. Kokilashvili on the occasion of his 80th birthday

1 Introduction

Given an open set $\Omega \subset \mathbb{R}^n$, the boundedness on $L^p(\Omega)$, $1 < p \leq \infty$, of the classical Hardy–Littlewood maximal operator defined for $f \in L^1_{loc}(\mathbb{R}^n)$ by

$$Mf(x) = \sup_{B \subset \mathbb{R}^n} \frac{1}{|B|} \int_{B \cap \Omega} |f(y)| dy \cdot \chi_B(x),$$

can be expressed both as a norm inequality and, equivalently, as a modular inequality. However, when working in more general variable Lebesgue spaces (see, e.g., [6, 9, 19]), in the Musielak–Orlicz spaces (see, e.g., [20]), also referred to as variable Orlicz spaces, or, e.g., in more recent "grand variable exponent Lebesgue spaces" (see [18, 13]), we must distinguish between *norm inequalities* and *modular inequalities*.

Define the *local* maximal operator $\mathcal{M}$ by

$$\mathcal{M}f(x) = \sup_{B \subset \Omega} \fint_B |f(y)| dy \cdot \chi_B(x), \quad x \in \Omega.$$

In this definition the balls B used to compute the maximal operator are contained in Ω, the domain of f. The purpose of this note is to prove a modular inequality of the form

$$\int_\Omega \Phi\big((\mathcal{M}f)^{p(x)}\big)\, dx \leq \int_\Omega \Lambda\big(|f|^{p(x)}\big)\, dx + C(\Omega, \Phi, n, p(\cdot)), \tag{1.1}$$

where Φ and Λ are Young functions or, more generally, continuous, increasing functions. In particular, we do not need convexity but will need to control the decay of Φ near the origin. The appearance of the

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extra term $C(\Omega, \Phi, n, p(\cdot))$ on the right-hand side of (1.1) is essentially necessary for modular inequalities like (1.1) to be true except in the trivial case when $p(x) = p$ almost everywhere. We refer the interested reader to the discussion and references in [3].

There is an extensive literature on modular inequalities for the maximal operator. These works include the results for Orlicz spaces (see [17, 4] and the references therein), Zygmund spaces (see [5] and its references), and the variable Lebesgue spaces (see [6, 9, 3] and their references). Many of these results are for the global maximal operator; when specialized to the local maximal operator, they yield special cases of our main result.

The remainder of this note is organized as follows. In Section 2 we state and prove our main result, Theorem 2.1, and in Section 3 we give a few illustrative examples to show the kinds of results which can be obtained as special cases of Theorem 2.1.

Throughout this paper we will use the notation for the variable Lebesgue spaces as given in [6], and also the basic results for these function spaces given there. The letters C, c denote the constants which may possibly change their value at each appearance; at several points, in order to highlight their dependence on the dimension n or the exponent $p(\cdot)$, etc., we write $c(n)$, $c(p(\cdot))$, etc.

2 A modular variable Orlicz inequality for the local maximal operator

Before stating our main result we give a few basic definitions about variable exponents. For further information see [7, 8, 6]. By an exponent function we mean a measurable function $p(\cdot) : \Omega \to [1, \infty)$. We define

$$p_- = \operatorname*{ess\,inf}_{x\in\Omega} p(x), \quad p_+ = \operatorname*{ess\,sup}_{x\in\Omega} p(x),$$

and we assume $1 \le p_- \le p_+ < \infty$. (Thus, while $p_- = 1$ is allowed, we will always assume that $p(\cdot)$ is uniformly bounded.)

We will also assume that $p(\cdot) \in LH(\Omega)$, that is,

$$|p(x) - p(y)| \le \frac{C}{-\log|x-y|}, \quad x, y \in \Omega, \ \ |x-y| < \frac{1}{2},$$
$$|p(x) - p(y)| \le \frac{C}{\log(e+|x|)}, \quad x, y \in \Omega, \quad |y| \ge |x|.$$

Note that this condition is sufficient for $\mathcal{M} : L^{p(\cdot)}(\Omega) \to L^{p(\cdot)}(\Omega)$ and is the optimal pointwise regularity condition on $p(\cdot)$ for $\mathcal{M}$ to be bounded.

Theorem 2.1. *Given an open set $\Omega \subset \mathbb{R}^n$, $n \ge 2$, let $p(\cdot)$ be an exponent function such that $1 \le p_- \le p_+ < \infty$ and $p(\cdot) \in LH(\Omega)$. Let $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ be absolutely continuous, increasing, and such that $\Phi(0) = 0$ and*

$$\int_0^1 \frac{\Phi(\sigma^{p_-})}{\sigma} \frac{d\sigma}{\sigma} < \infty. \tag{2.1}$$

Then for every $f \in L^{p(\cdot)}(\Omega)$ such that

$$|f|_{p(\cdot),\Omega} := \int_\Omega |f(y)|^{p(y)} dy \le 1, \tag{2.2}$$

$$\int_\Omega \Phi((\mathcal{M}f)^{p(x)}) dx \le C(n, p(\cdot)) \int_\Omega \Lambda(|f|^{p(x)}) dx + C(\Omega, \Phi, n, p(\cdot)), \tag{2.3}$$

where $\Lambda(s) = \Phi(s) + R_{p_-,\Phi}(s)$ *and*

$$R_{p_-,\Phi}(s) = s^{\frac{1}{p_-}} \int_0^{s^{\frac{1}{p_-}}} \frac{\Phi(\sigma^{p_-})}{\sigma} \frac{d\sigma}{\sigma}.$$

Proof. First, we may assume without loss of generality that f is non-negative, bounded and has compact support. To prove the general case, note that given arbitrary f, $f_k(x) = \min(|f(x)|, k)\chi_{B(0,k)}(x)$ is bounded and has compact support, and f_k increases pointwise to $|f|$, so $|f_k|_{p(\cdot),\Omega} \le |f|_{p(\cdot),\Omega} \le 1$. Moreover, we have that $\mathcal{M}f_k(x)$ increases to $\mathcal{M}f(x)$. (See [6, Lemma 3.30]. This proof is for the global maximal operator but the same argument holds for the local maximal operator.) The general result then follows by the monotone convergence theorem.

Since $p(\cdot) \in LH(\Omega)$ and (2.2), we have that for every $x \in \Omega$ and every ball $B \subset \Omega$ containing x,

$$\left(\frac{1}{|B|}\int_B |f(y)|dy\right)^{p(x)} \le c\left(\frac{1}{|B|}\int_B |f(y)|^{\frac{p(y)}{p_-}}\,dy\right)^{p_-} + c(e+|x|)^{-np_-}.$$

The constants depend only on n and $p(\cdot)$ and are independent of f. This inequality was essentially proved in [2, Theorem 4.1]; for this precise version, see [6, Theorem 3.32]. Let $S(x) = c(e+|x|)^{-np_-}$. As an immediate consequence, if we take the supremum over all balls containing x and contained in Ω, we get

$$\mathcal{M}f(x)^{p(x)} \le c\mathcal{M}(|f(\cdot)|^{\frac{p(\cdot)}{p_-}})(x)^{p_-} + S(x), \quad x \in \Omega. \tag{2.4}$$

We note in passing that if $p_- > 1$, then $S \in L^1(\Omega)$, and from this inequality we can immediately deduce that $\mathcal{M}$ is bounded on $L^{p(\cdot)}(\Omega)$: see [2] for details.

We can now estimate as follows: for all $t > 0$,

$$\begin{aligned}\left|\left\{x \in \Omega : \mathcal{M}f(x)^{p(x)} > t\right\}\right| &\le \left|\left\{x \in \Omega : c\mathcal{M}(|f(\cdot)|^{\frac{p(\cdot)}{p_-}})(x)^{p_-} + S(x) > t\right\}\right| \\ &\le \left|\left\{x \in \Omega : c\mathcal{M}(|f(\cdot)|^{\frac{p(\cdot)}{p_-}})(x)^{p_-} > t/2\right\}\right| + \left|\left\{x \in \Omega : S(x) > t/2\right\}\right|.\end{aligned}$$

Since Φ is absolutely continuous, Φ' exists almost everywhere and we have that

$$\begin{aligned}\int_\Omega \Phi\big((\mathcal{M}f)^{p(x)}\big)dx &= \int_0^\infty \Phi'(t)\left|\left\{x \in \Omega : (\mathcal{M}f)^{p(x)} > t\right\}\right|dt \\ &\le \int_0^\infty \Phi'(t)\left|\left\{x \in \Omega : c(\mathcal{M}|f(\cdot)|^{\frac{p(\cdot)}{p_-}})(x)^{p_-} > t/2\right\}\right|dt + \int_0^\infty \Phi'(t)\left|\left\{x \in \Omega : S(x) > t/2\right\}\right|dt \\ &= A + B.\end{aligned}$$

We first prove that $B \le C(\Omega, \Phi, n, p(\cdot))$. By the definition of S,

$$\int_0^\infty \Phi'(t)\left|\left\{x \in \Omega : S(x) > t/2\right\}\right|dt = \int_\Omega \Phi(2S(x))\,dx \le C(\Omega, \Phi, n, p(\cdot)) + \int_{\{|x|>e\}} \Phi(c|x|^{-np_-})\,dx.$$

To estimate the second integral we switch to polar coordinates and use (2.1). By the change of variables $\sigma^{p_-} = cr^{-np_-}$ we get

$$\begin{aligned}\int_{\{x\in\Omega:|x|>e\}} \Phi(c|x|^{-np_-})\,dx &= \int_{S^{n-1}}\int_e^\infty \Phi(Cr^{-np_-})r^{n-1}\,dr\,d\theta = C(n)\int_e^\infty \Phi(Cr^{-np_-})r^n\,\frac{dr}{r} \\ &\le C(n,p(\cdot))\int_0^1 \frac{\Phi(\sigma^{p_-})}{\sigma}\frac{d\sigma}{\sigma} = C(n, \Phi, p(\cdot)).\end{aligned}$$

This completes the estimate for B.

To estimate A, define $\Phi_{p_-}(s) := \Phi(s^{p_-})$, so that $\Phi'_{p_-}(s) = \Phi'(s^{p_-})p_- s^{p_- -1}$. Then, if we make the change of variables $s := t^{\frac{1}{p_-}}$ in the definition of A, we get

$$A = p_- \int_0^\infty s^{p_- -1}\Phi'(s^{p_-})\big|\big\{x \in \Omega : (\mathcal{M}|f(\cdot)|^{\frac{p(\cdot)}{p_-}})(x) > s/c(p_-)\big\}\big|ds$$
$$= \int_0^\infty \Phi'_{p_-}(s)\big|\big\{x \in \Omega : (\mathcal{M}|f(\cdot)|^{\frac{p(\cdot)}{p_-}})(x) > s/c(p_-)\big\}\big|ds.$$

If we adapt the standard proof of the weak $(1,1)$ inequality for the maximal operator (using Vitali's covering lemma: see [22, p. 7]), we have that for all $s > 0$,

$$\big|\big\{x \in \Omega : (\mathcal{M}|f(\cdot)|^{\frac{p(\cdot)}{p_-}})(x) > s/c(p_-)\big\}\big| \le c(p_-, n)\frac{1}{s} \int_{\{|f(\cdot)|^{\frac{p(\cdot)}{p_-}} > s\}} |f(x)|^{\frac{p(x)}{p_-}}\,dx.$$

Hence,

$$A \le c(p_-, n) \int_0^\infty \frac{\Phi'_{p_-}(s)}{s}\Bigg(\int_{\{|f(\cdot)|^{\frac{p(\cdot)}{p_-}} > s\}} |f(x)|^{\frac{p(x)}{p_-}}\,dx\Bigg)ds. \tag{2.5}$$

Define

$$\Psi_{p_-}(s) = \Phi_{p_-}(s) + s\int_0^s \frac{\Phi_{p_-}(\sigma)}{\sigma^2}d\sigma = \Phi(s^{p_-}) + s\int_0^s \frac{\Phi(\sigma^{p_-})}{\sigma^2}d\sigma;$$

then we have that

$$\left(\frac{\Psi_{p_-}(s)}{s}\right)' = \frac{\Phi'_{p_-}(s)s - \Phi_{p_-}(s)}{s^2} + \frac{\Phi_{p_-}(s)}{s^2} = \frac{\Phi'_{p_-}(s)}{s}.$$

Therefore, since Φ is increasing, Φ_{p_-} and $\Psi_{p_-}(s)/s$ are increasing as well.

Now define the function

$$F(s) = \int_{\{|f(\cdot)|^{\frac{p(\cdot)}{p_-}} > s\}} |f(x)|^{\frac{p(x)}{p_-}}\,dx;$$

since we have assumed that f is bounded and has compact support, F is well-defined and $F(s) = 0$ for all $s > 0$ sufficiently large. Thus, if we define

$$G(s) = \frac{\Psi_{p_-}(s)}{s} \int_{\{|f(\cdot)|^{\frac{p(\cdot)}{p_-}} > s\}} |f(x)|^{\frac{p(x)}{p_-}}\,dx = \frac{\Psi_{p_-}(s)}{s}F(s),$$

then $G(0) = 0$ and $G(s) \to 0$ as $s \to \infty$. (Indeed, $G(s) = 0$ for s large.)

Therefore, if we combine this with estimate (2.5) for A above and integrate by parts we get

$$A \le c(p_-, n)\int_0^\infty \left(\frac{\Psi_{p_-}(s)}{s}\right)' F(s)\,ds = cG(s)\Big|_{s=0}^{\infty} - c\int_0^\infty \frac{\Psi_{p_-}(s)}{s}dF(s) = -c\int_0^\infty \frac{\Psi_{p_-}(s)}{s}\,dF(s),$$

and by [11, Lemma 4.3] (see also the final remark in [12]),

$$= \int_{\{x\in\Omega:|f(x)|^{\frac{p(x)}{p_-}} > 0\}} \Psi_{p_-}\big(|f(x)|^{\frac{p(x)}{p_-}}\big)\,dx = \int_\Omega \Psi_{p_-}(|f(x)|^{\frac{p(x)}{p_-}})\,dx$$
$$= \int_\Omega \Phi\big(|f(x)|^{p(x)}\big)\,dx + \int_\Omega R_{p_-,\Phi}\big(|f(x)|^{p(x)}\big)\,dx = \int_\Omega \Lambda\big(|f(x)|^{p(x)}\big)\,dx.$$

If we combine the estimates for A and B we get (2.3). This completes the proof. □

3 Some illustrative examples

In this section we give some specific examples of Φ and Λ in Theorem 2.1. First, if we fix $p(\cdot)$, $p_- \geq 1$, and let $\Phi(s) = s^a$, then (2.1) holds if $ap_- > 1$. In this case it is easy to see that $R_{p_-,\Phi}(s) \approx s^a$, so we can take $\Lambda(s) = \Phi(s)$ (at the expense of a slightly larger constant). In particular, if $p_- > 1$, then we can take $\Phi(s) = s$. This gives us the modular inequality

$$\int_\Omega \mathcal{M}f(x)^{p(x)}\,dx \leq C \int_\Omega |f(x)|^{p(x)}\,dx + C;$$

for this result, see [6, Theorem 3.33].

More generally, if $p_- = 1$ and we assume that Φ satisfies the growth condition

$$\int_0^s \frac{\Phi(\sigma)}{\sigma}\,\frac{d\sigma}{\sigma} \leq \frac{c\Phi(cs)}{s}, \tag{3.1}$$

then $R_{p_-,\Phi}(s) \leq c\Phi(cs)$, and we get the modular inequality

$$\int_\Omega \Phi\big(\mathcal{M}f(x)^{p(x)}\big)\,dx \leq C \int_\Omega \Phi(c|f(x)|^{p(x)})\,dx + C. \tag{3.2}$$

This in turn implies that we get the Orlicz space inequality

$$\|\mathcal{M}f(\cdot)^{p(\cdot)}\|_{L^\Phi(\Omega)} \leq C\||f(x)|^{p(x)}\|_{L^\Phi(\Omega)}.$$

Condition (3.1) is a natural one to assume since if Φ holds, then we have the modular inequality

$$\int_\Omega \Phi\big(\mathcal{M}f(x)\big)\,dx \leq C \int_\Omega \Phi(c|f(x)|)\,dx; \tag{3.3}$$

see [17, Theorem 1.2.1]. Inequality (3.2) is a generalization of this result. In particular, notice that if $p(\cdot) = 1$, then (by simply applying Hölder's inequality) in our proof we can take $S(x) = 0$ in (2.4), and so the constant term in (2.3) becomes zero. Thus (3.2) becomes (3.3).

For an arbitrary choice of Φ in Theorem 2.1, if $p(\cdot) \equiv 1$, inequality (2.3) reduces to a result in [14, Proposition 3.1].

If $p_- = 1$ and $\Phi(s) = s\chi_{(1,\infty)}(s)$, then $R_{p_-,\Phi}(s) = \max\big(0, \log(s)\big)$. Hence, (2.3) becomes

$$\int_{\{\mathcal{M}f \geq 1\}} \mathcal{M}f(x)^{p(x)}\,dx \leq C \int_{\{|f| \geq 1\}} |f(x)|^{p(x)}\,dx + C \int_\Omega |f(x)|^{p(x)} \log^+ |f(x)|^{p(x)} dx + C. \tag{3.4}$$

This is a generalization of Stein's $L \log L$ estimate for the maximal operator [21]. In fact, if $p(\cdot) = 1$ and $|\Omega| < \infty$, then we can recover this result from (3.4).

If we take $p_- = 1$, then for any $\epsilon > 0$, the function

$$\Phi(s) = \frac{s}{\log(e + s^{-1})^{1+\epsilon}}$$

satisfies (2.1), and we have that

$$R_{p_-,\Phi}(s) \approx \frac{s}{\log(e + s^{-1})^{\epsilon}}.$$

If we take $\Lambda(s)$ equal to the same function, then we get

$$\int_\Omega \frac{\mathcal{M}f(x)^{p(x)}}{\log(e + \mathcal{M}f(x)^{-p(x)})^{1+\epsilon}}\,dx \leq C \int_\Omega \frac{|f(x)|^{p(x)}}{\log(e + |f(x)|^{-p(x)})^{\epsilon}}\,dx + C.$$

This yields a generalization of some of the results in [11].

Finally, we remark that the results in [15, 16] (see also the references therein) assume the convexity of Φ, so our results cannot be recovered from theirs since we only assume Φ is continuous and increasing.

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