

The virtual element method for the 3D resistive magnetohydrodynamic model

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Abstract

We present a four-field virtual element discretization for the time-dependent resistive Magnetohydrodynamics equations in three space dimensions, focusing on the semi-discrete formulation. The proposed method employs general polyhedral meshes and guarantees velocity and magnetic fields that are divergence free up to machine precision. We provide a full convergence analysis under suitable regularity assumptions, which is validated by some numerical tests.

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1 Introduction

Magnetohydrodynamics (MHD) is the physical-mathematical framework that describes the dynamics of magnetic fields in electrically conducting fluids, e.g., plasmas and ionized gases [17, 18]. Indeed, the electrically charged particles moving in a plasma generate an electromagnetic field that self-consistently interacts with the fluid motion through the Lorentz force acting on such particles. The theoretical framework of MHD has widely been used to develop predictive mathematical models, for example, in astrophysics (solar wind and space weather [41, Chapter 1], galactic jets [56]), geophysics (dynamo theory [57]), and nuclear fusion (design of fusion reactors, ignition in *inertial confinement fusion* [33, 43, 55]). Roughly speaking, MHD resorts on a strongly nonlinear coupling of a fluid flow submodel and an electromagnetic one. In a general setting, as the one considered in this work, the fluid flow submodel is based on the incompressible Navier-Stokes equations and the electromagnetic submodel on the Maxwell equations. A thorough description of the MHD models, their variants, derivation, and physical and mathematical properties can be found in many textbooks and review papers, e.g., [17, 18, 36, 65] just to mention a few.

Despite its ubiquity in scientific applications, which is reflected by the number of papers and books published in the last few decades, the computer resolution of the system of the MHD equations is still a formidable and challenging task and may pose pitfalls to the numerical scientists. Here, we shortly review some of the more critical points, with particular attention on the usefulness of polyhedral meshes and related schemes, always referring all interested readers to the rich technical literature for specific and detailed expositions.

First, vortical and/or shear flows may lead to large domain deformations, i.e., severe mesh deformations, also with the occurrence of non-planar mesh faces, in the Lagrangian and Arbitrary Lagrangian-Eulerian (ALE) numerical frameworks [49, 69]. Such mesh deformations are treated by remeshing the computational domain; then, by remapping the unknowns to the new mesh [68]. An optimal remap algorithm should be accurate, cheap, and feature-preserving, i.e., it should preserve the divergence-free constraints, the solution positivity, and conserve mass, momentum, and energy, cf. [22]. However, current remap algorithms can be too expensive, prone to significant accuracy losses, and not feature-preserving. Therefore, we infer that a good approximation method should

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be accurate and stable even in presence of severe mesh deformations in order to reduce the number of remeshing steps in a numerical simulation.

Next, low-order accurate algorithms can be affected by an excessive numerical diffusion and dispersion. For example, numerical diffusion can smear physical vorticity and shocks at an unacceptable level and even force the magnetic reconnection in ideal MHD models, which is clearly an artificial and physically meaningless effect. Numerical diffusion and dispersion can be reduced by increasing the order of the approximation and by using carefully selected polygonal or polyhedral meshes. In [38, 54], the finite difference time domain (FDTD) method [76] is applied to a grid of hexagonal prisms and yields much less numerical dispersion and anisotropy than on regular hexahedral grids where such method is normally considered.

The final major point that we want to consider is related to the divergence-free nature of both the fluid velocity and the magnetic flux fields. If such constraints are not accurately satisfied, possibly at the machine precision level, unreliable or even unphysical solutions may result from a numerical simulation. The consequence of the violation of the divergence-free constraint for the magnetic flux field has widely been investigated in the literature. It was seen that the numerical simulations are prone to significant errors, see, e.g., [18, 19, 34, 73], as fictitious forces and an unphysical behavior may appear, cf. [34]. For this reason, in the last two decades, a great effort has been devoted to the development of divergence-free numerical approximations. Just to mention a few possible approaches, the divergence-free constraint of the magnetic flux field can explicitly be enforced by introducing a Lagrange multiplier in the set of the unknowns, cf. [37]; by using a special flux limiter in the formulation of the numerical scheme, cf. [58]; by minimizing a special energy functional in a least squares finite element formulation, cf. [53]. However, such fixing strategies can result in costly inefficient schemes, they can be in conflict with each other or not suitable to polyhedral meshes. A major breakthrough was provided by realizing that “classical” numerical discretizations fail to reproduce the divergence-free nature of these fields if the discrete version of the divergence applied to the discrete version of the rotational operator does not annihilate at the zero machine precision. In fact, a small but not zero remainder can significantly accumulate over the many cycles of a long-time calculation, thus breaking the constraint. This understanding has led to the design of numerical approximation techniques in the framework of compatible/mimetic methods, as for example in [60, 68, 69]. A major drawback of compatible/mimetic discretizations is that they are only low-order accurate and can be too dispersive and diffusive. The discontinuous Galerkin (DG) method offers high-order accurate discretizations suitable to general polyhedral meshes that can reduce the overall numerical diffusion and dispersion. Nonetheless, DG methods are not compatible and may require a costly divergence-cleaning procedure based on solving an additional global equation, thus resulting in very expensive calculations. Finally, we note that the FEM literature for MHD models on tetrahedral and hexahedral meshes is very broad. Here, we limit ourselves to recalling the divergence-free schemes in [45, 48, 50], see the related work [70] as well, and the partial convergence analysis results from [46, 66]

In this work, we start exploring and presenting the design of numerical approximations of the MHD equations based on the virtual element method (VEM). Our goal is to address at least part of the issues discussed so far, but in a framework that makes it possible to address all of them in the future developments of this research project. The VEM, initially proposed in [4] for the numerical approximation of elliptic problems, is a Galerkin projection method such as the finite element method (FEM). The major difference between the VEM and the FEM is that in the formulation and practical implementation of the VEM we do not need an explicit knowledge of the basis functions that generate the finite element approximation spaces. In fact, all the bilinear forms and linear functionals of the discrete variational formulation are built using suitable polynomial projections that are always computable from a careful choice of the degrees of freedom. For these reasons, such approximation spaces and the method itself are dubbed as “virtual”. Local polynomial consistency and an additional stability term provide the well-posedness of the final discretization. Since the formulation of the method does not need a closed form of the basis functions, the resulting computational framework is extremely powerful and offers important advantages with respect to the FEM. We can build approximation spaces that are in principle of any order of accuracy and global regularity, suitable to general polyhedral meshes, even including nonconvex or nonconforming elements in 2D and 3D, and, importantly, satisfying other additional properties such as being part of a discrete de Rham chain. This last property has the major

consequence that both the numerical approximation of the fluid velocity and the magnetic flux field are intrinsically divergence free, a fact that we can verify numerically up to the machine precision.

The discrete divergence-free property in the VEM has been firstly analyzed in [6,7] (see also [3]) in the framework of the Stokes and Navier-Stokes problems. Later, it has been extended to other fluid flow models, a short list of representatives being [10, 15, 23–26, 31, 35, 44, 52, 61, 62, 74, 75]. On the other hand, De-Rahm complexes in the virtual element setting have been introduced in [13] and then improved (and applied to the analysis of magnetostatic problems) in [5, 11, 12]. The Maxwell’s equations were the target of [14], see also [28, 29], and their analysis hinges upon technical tools from [8]. Other interesting results along these directions are contained in [30] that paves the way for a VEM discretization of the Bianchi-Einstein equations.

In the present contribution we present a low-order four-field VEM formulation of the time-dependent MHD equations in three dimensions, in the spirit of [50]. Compared to [50], we further prove convergence of the semi-discrete scheme and employ a convenient divergence-free discretization of the velocity field (whereas the Taylor-Hood element is used in [50]). In order to discretize the unknown fields (pressure, and velocity, magnetic, and electric fields), we exploit the compatible VEM discrete spaces outlined above. Since the novelty of the method is in the space discretization approach, we focus on the semi-discrete version of the scheme, and leave the development of different time advancing techniques, together with the associated linear and nonlinear solvers, to future publications. The ensuing VEM scheme employs general polyhedral meshes and, differently from the majority of the FEM schemes available in the literature, guarantees that both the velocity and magnetic fields are divergence free (up to machine precision). An important achievement of the present contribution is that, under suitable regularity conditions on the exact solution, we are able to prove the linear convergence of the scheme in the “natural” norms of the problem. To the best of our knowledge, there are few convergence analyses of FEM schemes for the time-dependent MHD model in the literature, and none for the 4-field formulation here presented. For instance, in [51], a lowest order FEM (with discrete velocity that is not divergence free) has been designed and analyzed for the *stationary* MHD system. Thus, a major contribution of this work is that we also provide the details for the convergence analysis for the fully nonlinear model; such an analysis can be simplified to the FE setting, thus leading to possibly useful results also for FEM. Clearly, our analysis has the drawback of requiring a sufficiently regular exact solution; thus, our theoretical results should be intended as a guarantee that, at least in good conditions, the method delivers an accurate solution. In the final part of the article, we present some numerical tests that show the good performance of the proposed method.

The paper is organized as follows. After discussing some basic notation and definitions in Section 2, we review the continuous problem in Section 3. In Section 4, we introduce the velocity/pressure discrete fields and some theoretical results essential to the approximation of the convection term. In Section 5, after presenting the discrete spaces adopted for the electromagnetic fields and some related property, we state the discrete method. In Section 6, we show and prove the theoretical convergence results, which we validate in Section 7 with some numerical tests. Finally, in Section 8, we draw some conclusions and outline future developments.

bigskip

2 Notation

2.1 Mesh notation and functional spaces

Let $\mathcal{T} = \{\Omega_h\}_h$ be a family of mesh decompositions Ω_h of the computational domain Ω uniquely identified by the value of the mesh size parameter h . Every mesh Ω_h is a finite collection of polytopal elements K forming a finite covering of Ω , i.e., $\bar{\Omega} = \bigcup_{K \in \Omega_h} K$, planar faces \mathcal{F}_h , straight edges \mathcal{E}_h , and vertices \mathcal{V}_h . The mesh elements are nonoverlapping in the sense that the intersection in \mathbb{R}^3 of any pair of them, e.g., K and K' , has Lebesgue measure (volume) equal to zero, i.e., $|K \cap K'| = 0$ if $K \neq K'$. Accordingly, the intersection of the elemental boundaries of K and K' is either the empty set, or a set of common vertices, edges, or faces.

For every element K , we denote its volume, center, and diameter by $|K|$, $\mathbf{b}_K = (x_K, y_K, z_K)^T$,

and $h_K = \max_{\mathbf{x}, \mathbf{y} \in K} |\mathbf{x} - \mathbf{y}|$, and its set of faces, edges, and vertices by \mathcal{F}_h^K , \mathcal{E}_h^K , and \mathcal{V}_h^K , respectively. As usual, the maximum of the diameters h_K for $K \in \Omega_h$ is the mesh size, i.e., $h = \max_{K \in \Omega_h} h_K$. The boundary of every element K is denoted by ∂K and formed by a finite set of nonintersecting planar faces $F \in \mathcal{F}_h^K$ so that $\partial K = \bigcup_{F \in \mathcal{F}_h^K} F$. For every face F , we denote its area, center, and diameter by $|F|$, $\mathbf{b}_F = (x_F, y_F, z_F)^T$, and $h_F = \max_{\mathbf{x}, \mathbf{y} \in F} |\mathbf{x} - \mathbf{y}|$, and its set of edges and vertices by \mathcal{E}_h^F and \mathcal{V}_h^K , respectively. The boundary of every face F is the planar polygon ∂F , which is formed by a finite number of nonintersecting straight segments $e \in \mathcal{E}_h^F$ so that $\partial F = \bigcup_{e \in \mathcal{E}_h^F} e$. We denote the length of edge e by h_e and its center, which we shall also refer to as the edge mid-point, by $\mathbf{b}_e = (x_e, y_e, z_e)$.

We assume that all the meshes Ω_h of a given sequence $\{\Omega_h\}_h$ satisfy these conditions for $h \rightarrow 0$: there exists a real constant factor $\gamma \in (0, 1)$ that is independent of Ω_h and $h_K \in \Omega_h$ such that

(M₁) shape-regularity: all the elements $K \in \Omega_h$ and faces $F \in \mathcal{F}_h$ are “ γ -shape” regular [32];

(M₂) uniform scaling: $\gamma h_K \leq h_F$ for every face $F \in \mathcal{F}_h^K$ of every element $K \in \Omega_h$, and, analogously, $\gamma h_F \leq h_e$ for every edge $e \in \mathcal{E}_h^F$ of every $F \in \mathcal{F}_h^K$.

Throughout the paper, we shall refer to **(M₁)**-**(M₂)** as the *mesh regularity assumptions*. Such assumptions could be weakened [16, 20, 21, 27]; however, for the sake of presentation, we stick to a simpler setting.

On every face $F \in \mathcal{F}_h$, we define a local coordinate system (ξ_1, ξ_2) and denote the differentiation along ξ_1 and ξ_2 as ∂_{ξ_1} and ∂_{ξ_2} , respectively. The corresponding second order derivatives are denoted by $\partial_{\xi_1 \xi_1}^2$ and $\partial_{\xi_2 \xi_2}^2$. Then, we consider the two-dimensional vector-valued field $\mathbf{v} = (v_1, v_2) : F \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and the scalar field $v : F \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$, and let the face-based differential operators div_F , rot_F , curl_F and Δ_F applied to $\mathbf{v}(\xi_1, \xi_2)$ and $v(\xi_1, \xi_2)$ be defined as

$$\text{div}_F \mathbf{v} := \partial_{\xi_1} v_1 + \partial_{\xi_2} v_2, \quad \text{rot}_F \mathbf{v} := \partial_{\xi_2} v_1 - \partial_{\xi_1} v_2, \quad (1a)$$

$$\text{curl}_F v := (\partial_{\xi_2} v, -\partial_{\xi_1} v)^T, \quad \Delta_F v := \partial_{\xi_1 \xi_1}^2 v + \partial_{\xi_2 \xi_2}^2 v. \quad (1b)$$

We denote the partial derivatives along the directions x , y , and z by ∂_x , ∂_y , and ∂_z , respectively, and the corresponding second derivatives by ∂_{xx}^2 , ∂_{yy}^2 , and ∂_{zz}^2 . We define the Laplace, divergence, and curl operators of the three-dimensional field $\mathbf{v} = (v_1, v_2, v_3)^T$ as follows:

$$\begin{aligned} \Delta \mathbf{v} &:= \partial_{xx}^2 v_1 + \partial_{yy}^2 v_2 + \partial_{zz}^2 v_3, & \text{div } \mathbf{v} &:= \partial_x v_1 + \partial_y v_2 + \partial_z v_3, \\ \text{curl } \mathbf{v} &:= (\partial_y v_3 - \partial_z v_2, \partial_z v_1 - \partial_x v_3, \partial_x v_2 - \partial_y v_1)^T. \end{aligned}$$

2.2 Functional spaces

Consider the polygonal/polyhedral domain $D \subset \mathbb{R}^d$, $d = 2, 3$, with (Lipschitz) boundary ∂D . Throughout the paper, D can be either a mesh face F , a mesh element K , or the whole domain Ω .

Sobolev spaces According to [1], $L^2(D)$ denotes the Lebesgue space of real-valued square integrable functions defined on D ; $L_0^2(D)$ is the subspace of the functions in $L^2(D)$ having zero average on D ; for any integer number $s > 0$, $H^s(D)$ is the Sobolev space s of the real-valued functions in $L^2(D)$ with all weak partial derivatives of order up to s in $L^2(D)$; $[L^2(D)]^d$, $[L_0^2(D)]^d$, and $[H^s(D)]^d$ are the vector version of these spaces. The Sobolev spaces of noninteger orders are constructed by using the interpolation theory and the Sobolev spaces of negative orders by duality [1].

We denote the Sobolev space on ∂D by $H^s(\partial D)$ and recall that the functions in $H^s(D)$ admit a trace on ∂D when $s > 1/2$. Since D can be either a polygonal or a polyhedral domain, the bound $s < 3/2$ must also be valid. We denote the inner product in $L^2(D)$ and $H^s(D)$ by $(v, w)_D$ and $(v, w)_{s,D}$, respectively; we also denote the corresponding induced norms by $\|v\|_D$ and $\|v\|_{s,D}$, and the seminorm in $H^s(D)$ by $|v|_{s,D}$. When D is the whole computational domain, we prefer to

omit the subindex Ω and rather use the notation (v, w) , $\|v\|$, $\|v\|_s$, etc. instead of $(v, w)_\Omega$, $\|v\|_\Omega$, $\|v\|_{s,\Omega}$, etc.

In the light of these definitions, for a given $s > 0$, we introduce the functional spaces

$$H^s(\operatorname{div}, D) := \left\{ \mathbf{v} \in [H^s(D)]^3 \mid \operatorname{div} \mathbf{v} \in H^s(D) \right\}, \quad (2)$$

$$H^s(\mathbf{curl}, D) := \left\{ \mathbf{v} \in [H^s(D)]^3 \mid \mathbf{curl} \mathbf{v} \in [H^s(D)]^3 \right\}. \quad (3)$$

If $s = 0$, we write $H(\operatorname{div}, D)$ and $H(\mathbf{curl}, D)$ instead of $H^0(\operatorname{div}, D)$ and $H^0(\mathbf{curl}, D)$. Let \mathbf{n}_D be the unit vector orthogonal to the boundary ∂D and pointing out of D . According, e.g., to [64, Section 3.5], for the spaces (2) and (3), we can define the trace operators

$$\operatorname{tr}_{\operatorname{div}} : H(\operatorname{div}, D) \rightarrow H^{-\frac{1}{2}}(\partial D), \quad \operatorname{tr}_{\mathbf{curl}} : H(\mathbf{curl}, D) \rightarrow [H^{-\frac{1}{2}}(\partial D)]^d, \quad d = 2, 3,$$

which are such that

$$\operatorname{tr}_{\operatorname{div}}(\mathbf{v}) := \mathbf{n}_D \cdot \mathbf{v}, \quad \operatorname{tr}_{\mathbf{curl}}(\mathbf{v}) := \mathbf{n}_D \times \mathbf{v},$$

for all sufficiently smooth vector-valued field \mathbf{v} . These trace operators allow us to define the subspaces of $H^s(\operatorname{div}, D)$ and $H^s(\mathbf{curl}, D)$

$$H_0^s(\operatorname{div}, D) := \left\{ \mathbf{v} \in H^s(\operatorname{div}, D) \mid \operatorname{tr}_{\operatorname{div}} \mathbf{v} = 0 \right\},$$

$$H_0^s(\mathbf{curl}, D) := \left\{ \mathbf{v} \in H^s(\mathbf{curl}, D) \mid \operatorname{tr}_{\mathbf{curl}} \mathbf{v} = \mathbf{0} \right\}.$$

These subspaces incorporate the homogeneous boundary conditions in their definition. Finally, we define the norms for div and curl spaces: for every sufficiently smooth field \mathbf{v} ,

$$\|\mathbf{v}\|_{\mathbf{curl}}^2 := \|\mathbf{v}\|_0^2 + \|\mathbf{curl} \mathbf{v}\|_0^2, \quad \|\mathbf{v}\|_{\operatorname{div}}^2 := \|\mathbf{v}\|_0^2 + \|\operatorname{div} \mathbf{v}\|_0^2.$$

Bochner spaces Let $T > 0$ be a real number and $(X, \|\cdot\|_X)$ a normed space, where X can be either $L^2(\Omega)$ or $H^s(\Omega)$, $s \geq 0$. According to [40], the Bochner space $L^p(0, T; X)$ is the space of functions v such that the sublinear functional

$$\|v\|_{L^p(0, T; X)} = \begin{cases} \left(\int_0^T \|v(t)\|_X^p dt \right)^{1/p} & 1 \leq p < \infty, \\ \operatorname{ess\,sup}_{t \in [0, T]} \|v(t)\|_X & p = \infty, \end{cases}$$

is a *finite* norm for almost every $t \in [0, T]$. According to this notation, $C(0, T; X)$ is the space of the continuous functions from $[0, T]$ to X .

Polynomial spaces We denote the space of polynomials of degree $\ell = 0, 1$ defined on the element K , the face F and the edge e by $\mathbb{P}_\ell(K)$, $\mathbb{P}_\ell(F)$, and $\mathbb{P}_\ell(e)$, respectively. We set $\mathbb{P}_{-1}(K) = \mathbb{P}_{-1}(F) = \mathbb{P}_{-1}(e) = \{0\}$. The space $\mathbb{P}_1(K)$ is the span of the *scaled monomials* defined as:

$$m_0(\mathbf{x}) = 1, \quad m_1(\mathbf{x}) = \frac{x - x_K}{h_K}, \quad m_2(\mathbf{x}) = \frac{y - y_K}{h_K}, \quad m_3(\mathbf{x}) = \frac{z - z_K}{h_K} \quad \forall \mathbf{x} = (x, y, z)^T \in K.$$

The bases of $\mathbb{P}_1(F)$ and $\mathbb{P}_1(e)$ are defined in a similar way. We let $\mathbb{P}_\ell(\Omega_h)$ denote the space of the piecewise discontinuous polynomials of degree $\ell = 0, 1$ that are globally defined on Ω and such that $q|_K \in \mathbb{P}_1(K)$ for all elements $K \in \Omega_h$.

Orthogonal projections onto polynomial spaces In the forthcoming discrete formulation, given any ω either in Ω_h , \mathcal{F}_h , or \mathcal{E}_h , we shall use the polynomial projectors $\Pi_\ell^{0, \omega} : L^2(\omega) \rightarrow \mathbb{P}_\ell(\omega)$, $\ell = 0, 1$, and $\Pi_1^{\nabla, \omega} : H^1(\omega) \rightarrow \mathbb{P}_1(\omega)$ defined on all the mesh objects ω . The operator $\Pi_\ell^{0, \omega}$ is the orthogonal projection onto constant ($\ell = 0$) and linear ($\ell = 1$) polynomials with respect to the inner product in $L^2(\omega)$. The operator $\Pi_1^{\nabla, \omega}$ is the orthogonal projection onto linear polynomials

with respect to the semi-inner product in $H^1(\omega)$; we call it the *elliptic projection*. The elliptic projection $\Pi_\ell^{\nabla,\omega} v$ of a function $v \in H^1(\omega)$ is the linear polynomial solving the variational problem

$$(\nabla(\Pi_1^{\nabla,\omega} v - v), \nabla q)_\omega = 0 \quad \forall q \in \mathbb{P}_1(\omega) \quad \text{and} \quad \int_{\partial\omega} (\Pi_1^{\nabla,\omega} v - v) = 0. \quad (4)$$

The second condition in (4) is considered to ensure the uniqueness of the projection $\Pi_1^{\nabla,K} v$.

With an abuse of notation, we extend these definitions in a component-wise manner to the multidimensional projection operators $\Pi_\ell^{0,\omega} : [L^2(\omega)]^d \rightarrow [\mathbb{P}_\ell(\omega)]^d$ and $\Pi_1^{\nabla,\omega} : [H^1(\omega)]^d \rightarrow [\mathbb{P}_1(\omega)]^d$, $d = 2, 3$. In particular, we shall use the orthogonal projection $\Pi_0^{0,K} \nabla v$ of the gradient of a function $v \in H^1(K)$, which is the constant vector polynomial solving the variational problem:

$$(\Pi_0^{0,K} \nabla v - \nabla v, \mathbf{q})_K = 0 \quad \forall \mathbf{q} \in [\mathbb{P}_0(K)]^3.$$

For $\ell = 0, 1$, we also define the global projection operators $\Pi_\ell^0 : L^2(\Omega) \rightarrow \mathbb{P}_\ell(\Omega_h)$ and $\Pi_\ell^0 : [L^2(\Omega)]^d \rightarrow [\mathbb{P}_\ell(\Omega_h)]^d$ as the operators respectively satisfying $(\Pi_\ell^0 v)_{|K} = \Pi_\ell^{0,K}(v|_K)$ and $(\Pi_\ell^0 \mathbf{v})_{|K} = \Pi_\ell^{0,K}(\mathbf{v}|_K)$ for all mesh elements $K \in \Omega_h$.

Similarly, $\Pi_1^\nabla : H^1(\Omega) \rightarrow \mathbb{P}_1(\Omega_h)$ and $\Pi_1^\nabla : [H^1(\Omega)]^d \rightarrow [\mathbb{P}_1(\Omega_h)]^d$ are the global projection operators respectively satisfying $(\Pi_1^\nabla v)_{|K} = \Pi_1^{\nabla,K}(v|_K)$ and $(\Pi_1^\nabla \mathbf{v})_{|K} = \Pi_1^{\nabla,K}(\mathbf{v}|_K)$ for all mesh elements $K \in \Omega_h$.

Some names for constants In some of the forthcoming estimates, we shall occasionally write explicit constants and denote them with different symbols, depending on their meaning. Notably, we shall use the following notation:

- C_S denotes a constant depending on a Sobolev embedding;
- C_D denotes a constant depending on the shape of a domain;
- C_P denotes a constant appearing in a Poincaré inequality;
- C_I denotes a constant depending on an interpolation estimate with respect to functions in virtual element spaces;
- C_{inv} denotes a constant depending on an inverse estimate;
- C_{appr} denotes a constant depending on a polynomial approximation estimate.

We deem that this notation might help the reader to better follow some steps in the forthcoming proofs.

Henceforth, we use the letter “ C ” to denote a strictly positive constant that can take a different value at any occurrence. The constant C is independent of the mesh size parameter h but may depend on the other parameters of the differential problem and virtual element discretization such as the domain shape, the mesh regularity constant γ , and the coercivity and continuity constants of the bilinear forms used in the variational formulations that will be introduced in the next sections.

3 The continuous problem

Let $\Omega \subset \mathbb{R}^3$ be a polyhedral domain with Lipschitz continuous boundary $\partial\Omega$. We model the interaction of an electrically charged, incompressible fluid having velocity \mathbf{u} and pressure p with the self-consistently generated electric field \mathbf{E} and magnetic flux field \mathbf{B} . We denote the viscous Reynolds number by Re , the magnetic Reynolds number by Re_m , and the Hartman number by s . Furthermore, we define the electric current density

$$\mathbf{j} := \mathbf{E} + \mathbf{u} \times \mathbf{B}. \quad (5)$$

Henceforth, the subscript t as in \mathbf{u}_t and \mathbf{B}_t denotes the time derivative and $[0, T]$ is the time integration interval for a given final time $T > 0$.

The MHD problem reads as follows: *For all times $t \in (0, T]$, find \mathbf{u} , p , \mathbf{E} , and \mathbf{B} , such that*

$$\mathbf{u}_t + (\nabla \mathbf{u})\mathbf{u} - Re^{-1}\Delta \mathbf{u} - s\mathbf{j} \times \mathbf{B} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \quad (6a)$$

$$\mathbf{j} - Re_m^{-1} \mathbf{curl} \mathbf{B} = \mathbf{0} \quad \text{in } \Omega, \quad (6b)$$

$$\mathbf{B}_t + \mathbf{curl} \mathbf{E} = \mathbf{0} \quad \text{in } \Omega, \quad (6c)$$

$$\text{div} \mathbf{B} = 0 \quad \text{in } \Omega, \quad (6d)$$

$$\text{div} \mathbf{u} = 0 \quad \text{in } \Omega, \quad (6e)$$

where $(\nabla \mathbf{u})\mathbf{u} = \sum_{i,j} (\partial_j u_i) u_j$. The MHD equations (6) are completed by the set of initial conditions for the velocity and the magnetic flux fields

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \quad \mathbf{B}(\mathbf{x}, 0) = \mathbf{B}_0(\mathbf{x}), \quad \forall \mathbf{x} \in \Omega, \quad (7)$$

and the homogeneous boundary conditions

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{0}, \quad \mathbf{B}(\mathbf{x}, t) \cdot \mathbf{n}_\Omega = 0, \quad \mathbf{E}(\mathbf{x}, t) \times \mathbf{n}_\Omega = \mathbf{0} \quad \forall \mathbf{x} \in \partial\Omega, \quad \forall t \in [0, T]. \quad (8)$$

Equations (6a) and (6e) describe the hydrodynamic behavior of an electrically charged, incompressible fluid under the action of an external force \mathbf{f} and the electromagnetic force $\mathbf{j} \times \mathbf{B}$ multiplied by the Hartmann number, which acts as a coupling coefficient. The electromagnetic force is self-consistently generated by the electromagnetic fields \mathbf{E} and \mathbf{B} satisfying equations (6b), (6c), and (6d). These last three equations describe the electromagnetic submodel in the magneto-hydrodynamics approximation [17]. The incompressibility of the fluid velocity and the solenoidal nature of the magnetic field require that both \mathbf{u}_0 and \mathbf{B}_0 in (7) are divergence free.

The weak formulation of problem (6)-(8) reads as follows: *For almost every $t \in [0, T]$, find $(\mathbf{u}(t), p(t), \mathbf{E}(t), \mathbf{B}(t)) \in [H_0^1(\Omega)]^3 \times L_0^2(\Omega) \times H_0(\mathbf{curl}, \Omega) \times H_0(\text{div}, \Omega)$ such that*

$$(\mathbf{u}_t, \mathbf{v}) + Re^{-1}a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) + c(\mathbf{u}; \mathbf{u}, \mathbf{v}) - s(\mathbf{j} \times \mathbf{B}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in [H_0^1(\Omega)]^3, \quad (9a)$$

$$(\mathbf{j}, \mathbf{F}) - Re_m^{-1}(\mathbf{B}, \mathbf{curl} \mathbf{F}) = 0 \quad \forall \mathbf{F} \in H_0(\mathbf{curl}, \Omega), \quad (9b)$$

$$(\mathbf{B}_t, \mathbf{C}) + (\mathbf{curl} \mathbf{E}, \mathbf{C}) = 0 \quad \forall \mathbf{C} \in H_0(\text{div}, \Omega), \quad (9c)$$

$$b(\mathbf{u}, q) = 0 \quad \forall q \in L_0(\Omega), \quad (9d)$$

where \mathbf{j} is defined in (5). The bilinear forms $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$, and the trilinear form $c(\cdot; \cdot, \cdot)$ are defined as

$$a : [H^1(\Omega)]^3 \times [H^1(\Omega)]^3 \rightarrow \mathbb{R} : \quad a(\mathbf{u}, \mathbf{v}) := (\nabla \mathbf{u}, \nabla \mathbf{v}) = \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v},$$

$$b : [H^1(\Omega)]^3 \times L_0^2(\Omega) \rightarrow \mathbb{R} : \quad b(\mathbf{v}, q) := -(\text{div} \mathbf{v}, q) = - \int_{\Omega} q \text{div} \mathbf{v},$$

$$c : [H^1(\Omega)]^3 \times [H^1(\Omega)]^3 \times [H^1(\Omega)]^3 \rightarrow \mathbb{R} : \quad c(\mathbf{w}; \mathbf{u}, \mathbf{v}) := ((\nabla \mathbf{u})\mathbf{w}, \mathbf{v}) = \int_{\Omega} (\nabla \mathbf{u})\mathbf{w} \cdot \mathbf{v},$$

where $\mathbf{A} : \mathbf{B} = \sum_{ij} A_{ij} B_{ij}$ and $(\mathbf{A})\mathbf{b} \cdot \mathbf{c} = \sum_{ij} A_{ij} b_j c_i$ for the matrices $\mathbf{A} = (A_{ij})$ and $\mathbf{B} = (B_{ij})$, and the vectors $\mathbf{b} = (b_i)$ and $\mathbf{c} = (c_i)$. We also consider the local forms that are defined by splitting the above forms on the mesh elements $K \in \Omega_h$:

$$a(\mathbf{u}, \mathbf{v}) = \sum_{K \in \Omega_h} a^K(\mathbf{u}, \mathbf{v}) \quad \text{and} \quad a^K(\mathbf{u}, \mathbf{v}) = (\nabla \mathbf{u}, \nabla \mathbf{v})_K,$$

$$b(\mathbf{v}, q) = \sum_{K \in \Omega_h} b^K(\mathbf{v}, q) \quad \text{and} \quad b^K(\mathbf{v}, q) = -(\text{div} \mathbf{v}, q)_K,$$

$$c(\mathbf{v}; \mathbf{u}, \mathbf{w}) = \sum_{K \in \Omega_h} c^K(\mathbf{v}; \mathbf{u}, \mathbf{w}) \quad \text{and} \quad c^K(\mathbf{v}; \mathbf{u}, \mathbf{w}) = ((\nabla \mathbf{u})\mathbf{v}, \mathbf{w})_K.$$

Several (partial) results are available from the technical literature about the well-posedness of the MHD model in the strong and weak formulation, e.g., problems (6) and (9) and their variants, cf. [39, 42, 47, 63, 67, 72] and the citations therein. Some of these results concerning existence, uniqueness and stability of the solution fields $(\mathbf{u}, p, \mathbf{E}, \mathbf{B})$ have been derived under specific assumptions that may reduce the generality of the model. To the best of our knowledge, the full mathematical understanding of the MHD model is still an open issue and an active research area. This topic is beyond the scope of our work; thus, we shall simply assume that the MHD model is well-posed, at least in the setting that we are using in this paper.

4 Virtual element spaces for the incompressible flow equations

4.1 The discrete pressure space

We approximate the pressure unknown in the space of piecewise discontinuous, constant functions with zero average on Ω :

$$Q_h := \left\{ q_h \in L_0^2(\Omega) \mid q_h|_K \in \mathbb{P}_0(K) \ \forall K \in \Omega_h \right\} = \mathbb{P}_0(\Omega_h) \cap L_0^2(\Omega).$$

Every $q_h \in Q_h$ is uniquely determined by the set of constant values $(q_h|_K)$. Accordingly, we can approximate any scalar function $q \in L_0^2(\Omega)$ by its averages over the mesh elements.

4.2 The discrete velocity space

Here, we consider the “lowest order” version of the spaces introduced in [7, 15]. This includes an “enhancement” procedure that allows for the computation of an L^2 linear polynomial projector. We refer to such papers for a better understanding of the motivations behind this construction.

We define the nodal velocity space on a face $F \in \mathcal{F}_h$ as

$$\begin{aligned} \mathbf{W}_h(F) := \left\{ \mathbf{w}_h \in [H^1(F)]^3 \mid \mathbf{w}_h|_{\partial F} \in [C^0(\partial F)]^3, \mathbf{w}_h|_e \in [\mathbb{P}_1(e)]^3 \ \forall e \in \mathcal{E}_h^F, \right. \\ \Delta_F \mathbf{w}_h \in [\mathbb{P}_2(F)]^3, \\ (\mathbf{w}_{h,\tau} - \Pi_1^{\nabla, F} \mathbf{w}_{h,\tau}, \mathbf{q})_{0,F} = 0 \quad \forall \mathbf{q} \in [\mathbb{P}_2(F)]^2, \\ \left. (w_{h,\mathbf{n}} - \Pi_1^{\nabla, F} w_{h,\mathbf{n}}, q)_{0,F} = 0 \quad \forall q \in \mathbb{P}_2(F) \setminus \mathbb{R} \right\}. \end{aligned}$$

Above, we denoted the tangential and normal components of a given field \mathbf{w} by $\mathbf{w}_{h,\tau} = (\mathbf{n}_F \times \mathbf{w}) \times \mathbf{n}_F$ and $w_{h,\mathbf{n}} = \mathbf{n}_F \cdot \mathbf{w}$, respectively. We define the nodal velocity space on the whole boundary ∂K of a mesh element K as

$$\mathbf{W}_h(\partial K) := \left\{ \mathbf{w}_h \in [C^0(\partial K)]^3 \mid \mathbf{w}_h|_F \in \mathbf{W}_h(F) \quad \forall F \in \mathcal{F}_h^K \right\}.$$

Finally, we introduce the nodal velocity space on the element $K \in \Omega_h$ as follows:

$$\begin{aligned} \mathbf{W}_h(K) := \left\{ \mathbf{w}_h \in [H^1(K)]^3 \mid \mathbf{w}_h|_{\partial K} \in \mathbf{W}_h(\partial K), \right. \\ \left. \begin{cases} \Delta \mathbf{w}_h + \nabla s \in \mathbf{x} \times [\mathbb{P}_0(K)]^3 \text{ for some } s \in L_0^2(K), \\ \operatorname{div} \mathbf{w}_h \in \mathbb{P}_0(K), \end{cases} \right. \\ \left. (\mathbf{w}_h - \Pi_1^{\nabla, K} \mathbf{w}_h, \mathbf{x} \times \mathbf{q})_{0,K} = 0 \quad \forall \mathbf{q} \in [\mathbb{P}_0(K)]^3 \right\}. \end{aligned}$$

Every virtual element vector field $\mathbf{w}_h \in \mathbf{W}_h(K)$ is uniquely determined by the set of values $((\mathbf{w}_v)_{v \in \mathcal{V}_h^K}, (w_F)_{F \in \mathcal{F}_h^K})$, where

- $\mathbf{w}_v = \mathbf{w}_h(\mathbf{x}_v)$ is the value of \mathbf{w}_h at the vertex $v \in \mathcal{V}_h^K$ (with position vector \mathbf{x}_v);

- w_F is the scaled zero-th order moment of the normal component of \mathbf{w}_h on the face $F \in \mathcal{F}_h^K$, given by

$$w_F = \frac{1}{|F|} \int_F w_{h,\mathbf{n}}.$$

The unsolvence of these degrees of freedom can be proved as in [3, 15]. For any $\mathbf{w}_h \in \mathbf{W}_h(K)$, the elliptic projection $\Pi_1^{\nabla,K} \mathbf{w}_h \in [\mathbb{P}_1(K)]^3$, the L^2 -orthogonal projection $\Pi_1^{0,K} \mathbf{w}_h \in [\mathbb{P}_1(K)]^3$, and the divergence of \mathbf{w}_h are directly computable from such degrees of freedom. Finally, we define the global nodal velocity space by an H^1 -conforming coupling of the local degrees of freedom:

$$\mathbf{W}_h := \left\{ \mathbf{w}_h \in [H_0^1(\Omega)]^3 \mid \mathbf{w}_h|_K \in \mathbf{W}_h(K) \quad \forall K \in \Omega_h \right\}.$$

Remark 4.1. As in [7, 15], the couple of spaces $\mathbf{W}_h \times Q_h$ satisfies a discrete inf-sup condition, i.e., there exists $\beta > 0$ independent of h such that

$$\beta \leq \inf_{q_h \in Q_h} \sup_{\mathbf{v}_h \in \mathbf{W}_h} \frac{b(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_1 \|q_h\|}.$$

Furthermore, we also have the important property

$$\operatorname{div} \mathbf{W}_h \subseteq Q_h.$$

4.3 The virtual element bilinear forms $a_h(\cdot, \cdot)$ and $m_h(\cdot, \cdot)$

We use the elliptic projection operator $\Pi_1^{\nabla,K}$ to define the elemental bilinear form $a_h^K : \mathbf{W}_h(K) \times \mathbf{W}_h(K) \rightarrow \mathbb{R}$, which mimics the H^1 -inner product on the element K :

$$a_h^K(\mathbf{u}_h, \mathbf{v}_h) := (\nabla \Pi_1^{\nabla,K} \mathbf{u}_h, \nabla \Pi_1^{\nabla,K} \mathbf{v}_h)_K + S^K((I - \Pi_1^{\nabla,K}) \mathbf{u}_h, (I - \Pi_1^{\nabla,K}) \mathbf{v}_h) \quad \forall \mathbf{u}_h, \mathbf{v}_h \in \mathbf{W}_h(K). \quad (10)$$

Here, $S^K : \mathbf{W}_h(K) \times \mathbf{W}_h(K) \rightarrow \mathbb{R}$ can be any computable symmetric bilinear form such that there exist two positive constants σ_* and σ^* independent of h_K satisfying

$$\sigma_* |\mathbf{v}_h|_{1,K}^2 \leq S^K(\mathbf{v}_h, \mathbf{v}_h) \leq \sigma^* |\mathbf{v}_h|_{1,K}^2 \quad \forall \mathbf{v}_h \in \ker(\Pi_1^{\nabla,K}) \cap \mathbf{W}_h(K).$$

A “standard” choice for the stabilization is given by [15]

$$S^K(\mathbf{u}_h, \mathbf{v}_h) := h_K \sum_i \operatorname{dof}_i^K(\mathbf{u}_h) \operatorname{dof}_i^K(\mathbf{v}_h), \quad \forall \mathbf{u}_h, \mathbf{v}_h \in \mathbf{W}_h(K),$$

where the summation on the index i is carried over all the elemental degrees of freedom and $\operatorname{dof}_i^K(\mathbf{w}_h)$ is the bounded, linear functional defined on $\mathbf{W}_h(K)$ providing the i -th degree of freedom of $\mathbf{w}_h \in \mathbf{W}_h(K)$. The summation term in $S^K(\cdot, \cdot)$ is multiplied by h_K to have a consistent scaling for both terms of $a_h^K(\cdot, \cdot)$ on the right-hand side of (10) with respect to the element size.

The local bilinear form $a_h^K(\cdot, \cdot)$ satisfies the two fundamental properties of consistency and stability:

- **consistency:** for all $\mathbf{q} \in [\mathbb{P}_1(K)]^3$ and $\mathbf{w}_h \in \mathbf{W}_h(K)$,

$$a_h^K(\mathbf{q}, \mathbf{w}_h) = a^K(\mathbf{q}, \mathbf{w}_h); \quad (11)$$

- **stability:** for all $\mathbf{w}_h \in \mathbf{W}_h(K)$,

$$\alpha_* \|\mathbf{w}_h\|_{1,K}^2 \leq a_h^K(\mathbf{w}_h, \mathbf{w}_h) \leq \alpha^* \|\mathbf{w}_h\|_{1,K}^2, \quad (12)$$

with $\alpha_* = \min(1, \sigma_*)$ and $\alpha^* = \max(1, \sigma^*)$.

Finally, we define the global bilinear form $a_h : \mathbf{W}_h \times \mathbf{W}_h \rightarrow \mathbb{R}$ that we use in the virtual element formulation of the MHD model:

$$a_h(\mathbf{u}_h, \mathbf{v}_h) := \sum_{K \in \Omega_h} a_h^K(\mathbf{u}_h, \mathbf{v}_h) \quad \forall \mathbf{u}_h, \mathbf{v}_h \in \mathbf{W}_h.$$

In what follows, we use the discrete norm

$$\|\mathbf{v}_h\|_{\mathbf{W}_h}^2 := a_h(\mathbf{v}_h, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{W}_h.$$

We use the L^2 -projection operator $\Pi_1^{0,K}$ to define the bilinear form $m_h^K : \mathbf{W}_h(K) \times \mathbf{W}_h(K) \rightarrow \mathbb{R}$, which mimics the local L^2 -inner product on the element K :

$$m_h^K(\mathbf{u}_h, \mathbf{v}_h) := (\Pi_1^{0,K} \mathbf{u}_h, \Pi_1^{0,K} \mathbf{v}_h)_K + \tilde{S}^K \left((I - \Pi_1^{0,K}) \mathbf{u}_h, (I - \Pi_1^{0,K}) \mathbf{v}_h \right) \quad \forall \mathbf{u}_h, \mathbf{v}_h \in \mathbf{W}_h(K). \quad (13)$$

Here, $\tilde{S}^K : \mathbf{W}_h(K) \times \mathbf{W}_h(K) \rightarrow \mathbb{R}$ can be any computable bilinear form such that there exist two positive constants $\tilde{\sigma}_*$ and $\tilde{\sigma}^*$ independent of h_K satisfying

$$\tilde{\sigma}_* \|\mathbf{v}_h\|_{0,K}^2 \leq \tilde{S}^K(\mathbf{v}_h, \mathbf{v}_h) \leq \tilde{\sigma}^* \|\mathbf{v}_h\|_{0,K}^2 \quad \forall \mathbf{v}_h \in \ker(\Pi_1^{0,K}) \cap \mathbf{W}_h(K).$$

A “standard” choice for the stabilization term $\tilde{S}^K(\cdot, \cdot)$ is given by:

$$\tilde{S}^K(\mathbf{u}_h, \mathbf{v}_h) := h_K^3 \sum_i \text{dof}_i^K(\mathbf{u}_h) \text{dof}_i^K(\mathbf{v}_h), \quad \forall \mathbf{u}_h, \mathbf{v}_h \in \mathbf{W}_h(K),$$

where the summation on the index i is again on all the elemental degrees of freedom and $\text{dof}_i^K(\cdot)$ is the same functional used in the definition of the stabilization term $S^K(\cdot, \cdot)$. The summation term in $\tilde{S}^K(\cdot, \cdot)$ is multiplied by h_K^3 to have a consistent scaling for both terms of $m_h^K(\cdot, \cdot)$ on the right-hand side of (13) with respect to the element size. The stability bounds above are shown in [9].

The local bilinear form $m_h^K(\cdot, \cdot)$ satisfies the two fundamental properties of consistency and stability:

- **consistency:** for all $\mathbf{q} \in [\mathbb{P}_1(K)]^3$ and $\mathbf{w}_h \in \mathbf{W}_h(K)$,

$$m_h^K(\mathbf{q}, \mathbf{w}_h) = (\mathbf{q}, \mathbf{w}_h)_{0,K}; \quad (14)$$

- **stability:** for all $\mathbf{w}_h \in \mathbf{W}_h(K)$,

$$\mu_* \|\mathbf{w}_h\|_{0,K}^2 \leq m_h^K(\mathbf{w}_h, \mathbf{w}_h) \leq \mu^* \|\mathbf{w}_h\|_{0,K}^2, \quad (15)$$

with $\mu_* = \min(1, \tilde{\sigma}_*)$ and $\mu^* = \max(1, \tilde{\sigma}^*)$.

Finally, we define the global inner product $m_h : \mathbf{W}_h \times \mathbf{W}_h \rightarrow \mathbb{R}$ that we use in the virtual element formulation of the MHD model:

$$m_h(\mathbf{u}_h, \mathbf{v}_h) := \sum_{K \in \Omega_h} m_h^K(\mathbf{u}_h, \mathbf{v}_h).$$

Associated with $m_h(\cdot, \cdot)$, we define the corresponding discrete norm

$$\|\mathbf{v}_h\|_{m_h}^2 := m_h(\mathbf{v}_h, \mathbf{v}_h).$$

4.3.1 Interpolation in \mathbf{W}_h and a stability result

We define the (energy) interpolant $\mathbf{u}_I \in \mathbf{W}_h$ of a sufficiently smooth vector-valued field \mathbf{u} as the unique function in \mathbf{W}_h sharing the same degrees of freedom of \mathbf{u} . We have the following local and global interpolation property; see [9].

Lemma 4.2. *Let $\mathbf{u} \in [H^2(\Omega)]^3$ and $\mathbf{u}_I \in \mathbf{W}_h$ be its degrees of freedom interpolant. Then, a real, positive constant C_I independent of h exists such that*

$$\|\mathbf{u} - \mathbf{u}_I\|_{0,K} + h_K \|\mathbf{u} - \mathbf{u}_I\|_{1,K} \leq C_I h_K^2 |\mathbf{u}|_{2,K} \quad \forall K \in \Omega_h; \quad \|\mathbf{u} - \mathbf{u}_I\| + h \|\mathbf{u} - \mathbf{u}_I\|_1 \leq C_I h^2 |\mathbf{u}|_2.$$

The constant C_I depends on the mesh parameter γ .

An important consequence of the definition of the interpolant \mathbf{u}_I is that, if \mathbf{u} is divergence free, then (see also Remark 4.1)

$$b(\mathbf{u}_I, q_h) = b(\mathbf{u}, q_h) = 0 \quad \forall q_h \in Q_h \implies \operatorname{div} \mathbf{u}_I = 0. \quad (16)$$

We conclude this section with a technical lemma that will be useful in Section 6.

Lemma 4.3. *We have a stability bound for the L^∞ -norm of the elemental L^2 polynomial projection of \mathbf{u}_I : for all $\ell \in \mathbb{N}$,*

$$\|\mathbf{\Pi}_\ell^0 \mathbf{u}_I\|_{L^\infty} \leq C_{\text{inv}} C_D (C_I |\mathbf{u}|_{W^{1,3}} + \|\mathbf{u}\|_{L^\infty}).$$

Proof. On each element $K \in \Omega_h$, standard manipulations imply

$$\begin{aligned} \|\mathbf{\Pi}_\ell^0 \mathbf{u}_I\|_{L^\infty(K)} &\leq \|\mathbf{\Pi}_\ell^0 (\mathbf{u} - \mathbf{u}_I)\|_{L^\infty(K)} + \|\mathbf{\Pi}_\ell^0 \mathbf{u}\|_{L^\infty(K)} \leq C_{\text{inv}} h_K^{-\frac{3}{2}} (\|\mathbf{u} - \mathbf{u}_I\|_{0,K} + \|\mathbf{u}\|_{0,K}) \\ &\leq C_{\text{inv}} (C_I h_K^{-\frac{1}{2}} |\mathbf{u}|_{1,K} + h_K^{-\frac{3}{2}} \|\mathbf{u}\|_{0,K}) \leq C_{\text{inv}} (C_I C_D |\mathbf{u}|_{W^{1,3}(K)} + C_D \|\mathbf{u}\|_{L^\infty(K)}). \end{aligned}$$

Taking the maximum over all elements gives the assertion. \square

4.4 The discrete trilinear forms

We introduce the local continuous and discrete trilinear forms on the element K :

$$\begin{aligned} c^K(\mathbf{v}; \mathbf{u}, \mathbf{w}) &:= \int_K (\nabla \mathbf{u}) \mathbf{v} \cdot \mathbf{w} \quad \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in [H^1(K)]^3, \\ c_h^K(\mathbf{v}_h; \mathbf{u}_h, \mathbf{w}_h) &:= \int_K (\mathbf{\Pi}_0^{0,K} \nabla \mathbf{u}_h) \mathbf{\Pi}_1^{0,K} \mathbf{v}_h \cdot \mathbf{\Pi}_1^{0,K} \mathbf{w}_h \quad \forall \mathbf{u}_h, \mathbf{v}_h, \mathbf{w}_h \in \mathbf{W}_h(K). \end{aligned}$$

We also define their skew-symmetric counterparts

$$\begin{aligned} \tilde{c}^K(\mathbf{v}; \mathbf{u}, \mathbf{w}) &:= \frac{1}{2} (c^K(\mathbf{v}; \mathbf{u}, \mathbf{w}) - c^K(\mathbf{v}; \mathbf{w}, \mathbf{u})) \quad \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in [H^1(K)]^3, \quad (17) \\ \tilde{c}_h^K(\mathbf{v}_h; \mathbf{u}_h, \mathbf{w}_h) &:= \frac{1}{2} (c_h^K(\mathbf{v}_h; \mathbf{u}_h, \mathbf{w}_h) - c_h^K(\mathbf{v}_h; \mathbf{w}_h, \mathbf{u}_h)) \quad \forall \mathbf{u}_h, \mathbf{v}_h, \mathbf{w}_h \in \mathbf{W}_h(K), \quad (18) \end{aligned}$$

and the associated global skew-symmetric trilinear forms

$$\begin{aligned} \tilde{c}(\mathbf{v}; \mathbf{u}, \mathbf{w}) &= \sum_{K \in \Omega_h} \tilde{c}^K(\mathbf{v}; \mathbf{u}, \mathbf{w}) \quad \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in [H^1(\Omega)]^3, \\ \tilde{c}_h(\mathbf{v}_h; \mathbf{u}_h, \mathbf{w}_h) &= \sum_{K \in \Omega_h} \tilde{c}_h^K(\mathbf{v}_h; \mathbf{u}_h, \mathbf{w}_h) \quad \forall \mathbf{u}_h, \mathbf{v}_h, \mathbf{w}_h \in \mathbf{W}_h. \end{aligned}$$

If $\mathbf{v} \in [H_0^1(\Omega)]^3$ is divergence free, then

$$\tilde{c}(\mathbf{v}; \mathbf{u}, \mathbf{w}) = c(\mathbf{v}; \mathbf{u}, \mathbf{w}).$$

In the remainder of the section, we discuss three properties of these trilinear forms. For the sake of exposition, we consider the extension of the virtual element bilinear and trilinear forms to $[H^1(K)]^3$ and $[H^1(\Omega)]^3$.

The first property is the continuity of the trilinear form, which can be proved as in [7, Proposition 3.3].

Lemma 4.4. *The following continuity bound is valid:*

$$|\tilde{c}_h(\mathbf{v}_h; \mathbf{u}_h, \mathbf{w}_h)| \lesssim |\mathbf{u}_h|_1 |\mathbf{v}_h|_1 |\mathbf{w}_h|_1,$$

where the hidden constant is independent of h .

The second property is the measure of the variational crime perpetrated in the discretization of the trilinear form. The proof is a modification of the proof of [7, Lemma 4.3].

Lemma 4.5. *Given $\mathbf{w} \in [H_0^1(\Omega)]^3$ and $\mathbf{v} \in [H_0^1(\Omega)]^3 \cap [H^2(\Omega)]^3$, we have the following bound*

$$|\tilde{c}(\mathbf{v}; \mathbf{v}, \mathbf{w}) - \tilde{c}_h(\mathbf{v}; \mathbf{v}, \mathbf{w})| \leq C_C h \|\mathbf{v}\|_2^2 |\mathbf{w}|_1,$$

where the constant C_C depends only on the constant of inverse estimates, approximation bounds, and Sobolev embedding inequalities, but is independent of h_K .

Proof. A straightforward manipulation of relations (17)-(18) defining the skew-symmetric trilinear forms yields

$$\begin{aligned} |\tilde{c}(\mathbf{v}; \mathbf{v}, \mathbf{w}) - \tilde{c}_h(\mathbf{v}; \mathbf{v}, \mathbf{w})| &\leq \frac{1}{2} (|c(\mathbf{v}; \mathbf{v}, \mathbf{w}) - c_h(\mathbf{v}; \mathbf{v}, \mathbf{w})| + |c(\mathbf{v}; \mathbf{w}, \mathbf{v}) - c_h(\mathbf{v}; \mathbf{w}, \mathbf{v})|) \\ &= \frac{1}{2} (\mathsf{T}_1 + \mathsf{T}_2). \end{aligned} \quad (19)$$

We estimate the two terms on the right-hand side separately. We begin with the first one. Using the definition of the orthogonal projection $\mathbf{\Pi}_1^0 \mathbf{v}$, and adding and subtracting $(\nabla \mathbf{v}) \mathbf{v} \cdot \mathbf{\Pi}_1^0 \mathbf{w}$ to the integral argument, we write

$$\begin{aligned} \mathsf{T}_1 &= \left| \int_{\Omega} (\nabla \mathbf{v}) \mathbf{v} \cdot \mathbf{w} - \int_{\Omega} (\mathbf{\Pi}_0^0 \nabla \mathbf{v}) \mathbf{\Pi}_1^0 \mathbf{v} \cdot \mathbf{\Pi}_1^0 \mathbf{w} \right| \\ &= \left| \int_{\Omega} (\nabla \mathbf{v}) \mathbf{v} \cdot \mathbf{w} - \int_{\Omega} (\mathbf{\Pi}_0^0 \nabla \mathbf{v}) \mathbf{v} \cdot \mathbf{\Pi}_1^0 \mathbf{w} \right| \\ &\leq \left| \int_{\Omega} (\nabla \mathbf{v}) \mathbf{v} \cdot (\mathbf{w} - \mathbf{\Pi}_1^0 \mathbf{w}) \right| + \left| \int_{\Omega} (\nabla \mathbf{v} - \mathbf{\Pi}_0^0 \nabla \mathbf{v}) \mathbf{v} \cdot \mathbf{\Pi}_1^0 \mathbf{w} \right| =: \mathsf{T}_{1,1} + \mathsf{T}_{1,2}. \end{aligned}$$

Again, we focus on the two terms on the right-hand side separately. Using the Hölder inequality, polynomial approximation properties, and the Sobolev embedding theorem gives

$$\mathsf{T}_{1,1} \leq \|\nabla \mathbf{v}\|_{L^4} \|\mathbf{v}\|_{L^4} \|\mathbf{w} - \mathbf{\Pi}_1^0 \mathbf{w}\| \leq C_{\text{appr}} h \|\nabla \mathbf{v}\|_{L^4} \|\mathbf{v}\|_{L^4} \|\mathbf{w}\|_1 \leq C_S^2 C_{\text{appr}} h \|\mathbf{v}\|_1 \|\mathbf{v}\|_2 \|\mathbf{w}\|_1.$$

Next, for each element $K \in \Omega_h$, we apply a polynomial inverse inequality, the Hölder inequality, and standard manipulations:

$$\|\mathbf{\Pi}_1^0 \mathbf{w}\|_{L^4(K)} \leq C_{\text{inv}} h_K^{-\frac{3}{4}} \|\mathbf{w}\|_{0,K} \leq C_{\text{inv}} h_K^{-\frac{3}{4}} \|1\|_{L^4(K)} \|\mathbf{w}\|_{L^4(K)} \leq C_{\text{inv}} C_D \|\mathbf{w}\|_{L^4(K)}.$$

This entails

$$\begin{aligned} \mathsf{T}_{1,2} &\leq \sum_{K \in \Omega_h} \|\nabla \mathbf{v} - \mathbf{\Pi}_0^0 \nabla \mathbf{v}\|_{0,K} \|\mathbf{v}\|_{L^4(K)} \|\mathbf{\Pi}_1^0 \mathbf{w}\|_{L^4(K)} \\ &\leq C_{\text{inv}} C_D C_{\text{appr}} h \sum_{K \in \Omega_h} |\mathbf{v}|_{2,K} \|\mathbf{v}\|_{L^4(K)} \|\mathbf{w}\|_{L^4(K)}. \end{aligned}$$

We apply a sequential $\ell^2 - \ell^4 - \ell^4$ Hölder inequality and the Sobolev embedding theorem twice, and deduce

$$\mathsf{T}_{1,2} \leq C_{\text{inv}} C_D C_{\text{appr}} h |\mathbf{v}|_2 \|\mathbf{v}\|_{L^4} \|\mathbf{w}\|_{L^4} \leq C_{\text{appr}} C_{\text{inv}} C_D C_S^2 h \|\mathbf{v}\|_2 \|\mathbf{v}\|_1 \|\mathbf{w}\|_1.$$

This step concludes the derivation of a bound on the term T_1 in (19).

Next, we focus on the term T_2 . Adding and subtracting $(\mathbf{\Pi}_0^0 \nabla \mathbf{w}) \mathbf{v} \cdot \mathbf{v}$ yields

$$\begin{aligned} T_2 &= \left| \int_{\Omega} (\nabla \mathbf{w}) \mathbf{v} \cdot \mathbf{v} - \int_{\Omega} (\mathbf{\Pi}_0^0 \nabla \mathbf{w}) \mathbf{v} \cdot \mathbf{\Pi}_1^0 \mathbf{v} \right| \\ &\leq \left| \int_{\Omega} (\nabla \mathbf{w} - \mathbf{\Pi}_0^0 \nabla \mathbf{w}) \mathbf{v} \cdot \mathbf{v} \right| + \left| \int_{\Omega} (\mathbf{\Pi}_0^0 \nabla \mathbf{w}) \mathbf{v} \cdot (\mathbf{v} - \mathbf{\Pi}_1^0 \mathbf{v}) \right| =: T_{2,1} + T_{2,2}. \end{aligned}$$

We derive the bounds on the terms $T_{2,1}$ and $T_{2,2}$ by using the approximation properties of the orthogonal projections, and theoretical tools similar to those used for the bound on T_1 :

$$\begin{aligned} T_{2,1} &= \left| \int_{\Omega} (\nabla \mathbf{w}) (\mathbf{v} \cdot \mathbf{v} - \mathbf{\Pi}_0^0 \mathbf{v} \cdot \mathbf{v}) \right| \leq |\mathbf{w}|_1 \|\mathbf{v} \cdot \mathbf{v} - \mathbf{\Pi}_0^0 (\mathbf{v} \cdot \mathbf{v})\| \leq C_{\text{appr}} h |\mathbf{w}|_1 |\mathbf{v} \cdot \mathbf{v}|_1 \\ &\leq C_{\text{appr}} h |\mathbf{w}|_1 \|\mathbf{v}\|_{W^{1,4}}^2 \leq C_{\text{appr}} C_S^2 h \|\mathbf{v}\|_2^2 |\mathbf{w}|_1. \end{aligned}$$

On the other hand, given \mathbf{v}_π the best piecewise linear approximant in L^4 of \mathbf{v} over Ω_h , we also have

$$\begin{aligned} T_{2,2} &\leq \sum_{K \in \Omega_h} |\mathbf{w}|_{1,K} \|\mathbf{v}\|_{L^4(K)} \|\mathbf{v} - \mathbf{\Pi}_1^0 \mathbf{v}\|_{L^4(K)} \\ &\leq \sum_{K \in \Omega_h} |\mathbf{w}|_{1,K} \|\mathbf{v}\|_{L^4(K)} (\|\mathbf{v} - \mathbf{v}_\pi\|_{L^4(K)} + \|\mathbf{v}_\pi - \mathbf{\Pi}_1^0 \mathbf{v}\|_{L^4(K)}). \end{aligned}$$

Observe that

$$\begin{aligned} \|\mathbf{v}_\pi - \mathbf{\Pi}_1^{0,K} \mathbf{v}\|_{L^4(K)} &\leq C_{\text{inv}} h_K^{-\frac{3}{4}} (\|\mathbf{v} - \mathbf{v}_\pi\|_{0,K} + \|\mathbf{v} - \mathbf{\Pi}_1^{0,K} \mathbf{v}\|_{0,K}) \\ &\leq 2C_{\text{inv}} h_K^{-\frac{3}{4}} \|\mathbf{v} - \mathbf{v}_\pi\|_{0,K} \leq 2C_{\text{inv}} C_D \|\mathbf{v} - \mathbf{v}_\pi\|_{L^4(K)} \\ &\leq 2C_{\text{inv}} C_D C_{\text{appr}} h_K |\mathbf{v}|_{W^{1,4}(K)}. \end{aligned}$$

Inserting this bound above and using polynomial approximation estimates yield

$$T_{2,2} \leq (1 + 2C_{\text{inv}} C_D) C_{\text{appr}} h \sum_{K \in \Omega_h} |\mathbf{w}|_{1,K} \|\mathbf{v}\|_{L^4(K)} \|\mathbf{v}\|_{W^{1,4}(K)}.$$

We apply a sequential $\ell^2 - \ell^4 - \ell^4$ Hölder inequality and the Sobolev embedding theorem, and deduce

$$T_{2,2} \leq (1 + 2C_{\text{inv}} C_D) C_{\text{appr}} h |\mathbf{w}|_1 \|\mathbf{v}\|_{L^4} \|\mathbf{v}\|_{W^{1,4}} \leq (1 + 2C_{\text{inv}} C_D) C_{\text{appr}} C_S^2 h \|\mathbf{v}\|_1 \|\mathbf{v}\|_2 \|\mathbf{w}\|_1.$$

Collecting all the bounds together yields the assertion. \square

The last property measures the distance between the continuous and the discrete solutions through the discrete trilinear form. The proof differs from that of [7, Lemma 4.4], since here we are interested in the analysis of the time-dependent case. For this reason, we also use some techniques from [71, Lemma 4.1]. Further, we require some extra regularity on the exact velocity solution \mathbf{u} to (6).

Lemma 4.6. *Let $\mathbf{u} \in W^{2,3}(\Omega)$ and $\mathbf{e}_h^{\mathbf{u}} := \mathbf{u}_h - \mathbf{u}_I$, where \mathbf{u}_I is the degrees of freedom interpolant of \mathbf{u} ; see Lemma 4.2. Then, for every positive ε , there exist two positive constants C_1 and C_2 independent of h such that*

$$|\tilde{c}_h(\mathbf{u}; \mathbf{u}, \mathbf{e}_h^{\mathbf{u}}) - \tilde{c}_h(\mathbf{u}_h; \mathbf{u}_h, \mathbf{e}_h^{\mathbf{u}})| \leq \varepsilon |\mathbf{e}_h^{\mathbf{u}}|_1^2 + \left(1 + \frac{1}{\varepsilon}\right) C_1 R_1(\mathbf{u}) \|\mathbf{e}_h^{\mathbf{u}}\|^2 + \left(1 + \frac{1}{\varepsilon}\right) C_2 R_2(\mathbf{u}) h^2,$$

where

$$R_1(\mathbf{u}) := \|\mathbf{u}\|_{W^{2,3}}^2 + \|\mathbf{u}\|_{W^{2,3}} + \|\mathbf{u}\|_{W^{1,\infty}} + 1, \quad R_2(\mathbf{u}) := \|\mathbf{u}\|_{W^{2,3}}^4 + \|\mathbf{u}\|_{W^{2,3}}^2 \|\mathbf{u}\|_{W^{1,\infty}}^2.$$

Proof. The triangle inequality implies

$$\begin{aligned} & |\tilde{c}_h(\mathbf{u}; \mathbf{u}, \mathbf{e}_h^{\mathbf{u}}) - \tilde{c}_h(\mathbf{u}_h; \mathbf{u}_h, \mathbf{e}_h^{\mathbf{u}})| \\ & \leq |\tilde{c}_h(\mathbf{u}; \mathbf{u}, \mathbf{e}_h^{\mathbf{u}}) - \tilde{c}_h(\mathbf{u}_h; \mathbf{u}, \mathbf{e}_h^{\mathbf{u}})| + |\tilde{c}_h(\mathbf{u}_h; \mathbf{u}, \mathbf{e}_h^{\mathbf{u}}) - \tilde{c}_h(\mathbf{u}_h; \mathbf{u}_h, \mathbf{e}_h^{\mathbf{u}})| =: \mathsf{T}_1 + \mathsf{T}_2. \end{aligned}$$

We estimate the two terms on the right-hand side separately and begin by splitting T_1 into the sum of two further contributions, due to the definition of the skew symmetric form $\tilde{c}_h(\cdot; \cdot, \cdot)$:

$$\mathsf{T}_1 \leq \frac{1}{2} (|c_h(\mathbf{u}; \mathbf{u}, \mathbf{e}_h^{\mathbf{u}}) - c_h(\mathbf{u}_h; \mathbf{u}, \mathbf{e}_h^{\mathbf{u}})| + |c_h(\mathbf{u}; \mathbf{e}_h^{\mathbf{u}}, \mathbf{u}) - c_h(\mathbf{u}_h; \mathbf{e}_h^{\mathbf{u}}, \mathbf{u})|) =: \frac{1}{2}(\mathsf{T}_{1,1} + \mathsf{T}_{1,2}).$$

Using standard manipulations, we get

$$\begin{aligned} \mathsf{T}_{1,1} &= \left| \int_{\Omega} (\mathbf{\Pi}_0^0 \nabla \mathbf{u})(\mathbf{u} - \mathbf{u}_h) \cdot \mathbf{\Pi}_1^0 \mathbf{e}_h^{\mathbf{u}} \right| \leq \|\mathbf{\Pi}_0^0 \nabla \mathbf{u}\|_{L^\infty} \|\mathbf{u} - \mathbf{u}_h\| \|\mathbf{\Pi}_1^0 \mathbf{e}_h^{\mathbf{u}}\| \\ &\leq |\mathbf{u}|_{W^{1,\infty}} (\|\mathbf{u} - \mathbf{u}_I\| + \|\mathbf{e}_h^{\mathbf{u}}\|) \|\mathbf{e}_h^{\mathbf{u}}\| \leq C_I h |\mathbf{u}|_{W^{1,\infty}} |\mathbf{u}|_1 \|\mathbf{e}_h^{\mathbf{u}}\| + |\mathbf{u}|_{W^{1,\infty}} \|\mathbf{e}_h^{\mathbf{u}}\|^2 \\ &\leq C_I h^2 |\mathbf{u}|_{W^{1,\infty}}^2 \|\mathbf{u}\|_1^2 + (|\mathbf{u}|_{W^{1,\infty}} + C_I) \|\mathbf{e}_h^{\mathbf{u}}\|^2 \\ &\leq C_I C_S^2 h^2 |\mathbf{u}|_{W^{1,\infty}}^2 \|\mathbf{u}\|_{W^{2,3}}^2 + (|\mathbf{u}|_{W^{1,\infty}} + C_I) \|\mathbf{e}_h^{\mathbf{u}}\|^2. \end{aligned}$$

As for the term $\mathsf{T}_{1,2}$, we write

$$\begin{aligned} \mathsf{T}_{1,2} &= \left| \int_{\Omega} (\mathbf{\Pi}_0^0 \nabla \mathbf{e}_h^{\mathbf{u}})(\mathbf{\Pi}_1^0(\mathbf{u} - \mathbf{u}_h)) \cdot \mathbf{u} \right| \leq |\mathbf{e}_h^{\mathbf{u}}|_1 \|\mathbf{u} - \mathbf{u}_h\| \|\mathbf{u}\|_{L^\infty} \\ &\leq |\mathbf{e}_h^{\mathbf{u}}|_1 (\|\mathbf{u} - \mathbf{u}_I\| + \|\mathbf{e}_h^{\mathbf{u}}\|) \|\mathbf{u}\|_{L^\infty} \leq C_I h |\mathbf{e}_h^{\mathbf{u}}|_1 |\mathbf{u}|_1 \|\mathbf{u}\|_{L^\infty} + |\mathbf{e}_h^{\mathbf{u}}|_1 \|\mathbf{e}_h^{\mathbf{u}}\| \|\mathbf{u}\|_{L^\infty} \\ &\leq \frac{\varepsilon}{2} |\mathbf{e}_h^{\mathbf{u}}|_1^2 + \frac{1}{\varepsilon} C_I^2 h^2 |\mathbf{u}|_1^2 \|\mathbf{u}\|_{L^\infty}^2 + \frac{1}{\varepsilon} \|\mathbf{u}\|_{L^\infty}^2 \|\mathbf{e}_h^{\mathbf{u}}\|^2 \\ &\leq \frac{\varepsilon}{2} |\mathbf{e}_h^{\mathbf{u}}|_1^2 + \frac{1}{\varepsilon} C_I^2 C_S^2 h^2 \|\mathbf{u}\|_{W^{2,3}}^4 + \frac{1}{\varepsilon} \|\mathbf{u}\|_{L^\infty}^2 \|\mathbf{e}_h^{\mathbf{u}}\|^2. \end{aligned}$$

Next, we focus on the term T_2 . Note that

$$\tilde{c}_h^K(\mathbf{u}_h; \mathbf{u}_h, \mathbf{e}_h^{\mathbf{u}}) = \tilde{c}_h^K(\mathbf{u}_h; \mathbf{u}_I, \mathbf{e}_h^{\mathbf{u}}), \quad (20)$$

which is a consequence of the skew symmetry of $\tilde{c}_h^K(\cdot; \cdot, \cdot)$ and (16).

We split the term T_2 into two contributions, due to the definition of the skew symmetric form $\tilde{c}_h(\cdot; \cdot, \cdot)$. Recalling (20), we write

$$\mathsf{T}_2 \leq \frac{1}{2} (|c_h(\mathbf{u}_h; \mathbf{u}, \mathbf{e}_h^{\mathbf{u}}) - c_h(\mathbf{u}_h; \mathbf{u}_I, \mathbf{e}_h^{\mathbf{u}})| + |c_h(\mathbf{u}_h; \mathbf{e}_h^{\mathbf{u}}, \mathbf{u}) - c_h(\mathbf{u}_h; \mathbf{e}_h^{\mathbf{u}}, \mathbf{u}_I)|) =: \frac{1}{2}(\mathsf{T}_{2,1} + \mathsf{T}_{2,2}). \quad (21)$$

Using standard manipulations as above, we get

$$\mathsf{T}_{2,1} \leq |c_h(\mathbf{e}_h^{\mathbf{u}}; \mathbf{u} - \mathbf{u}_I, \mathbf{e}_h^{\mathbf{u}})| + |c_h(\mathbf{u}_I; \mathbf{u} - \mathbf{u}_I, \mathbf{e}_h^{\mathbf{u}})| =: \mathsf{T}_{2,1,1} + \mathsf{T}_{2,1,2}.$$

Preliminary, for all $K \in \Omega_h$, we observe that

$$\|\mathbf{\Pi}_0^0 \nabla(\mathbf{u} - \mathbf{u}_I)\|_{L^\infty(K)} \leq C_D h_K^{-\frac{3}{2}} |\mathbf{u} - \mathbf{u}_I|_{1,K} \leq C_D C_I h_K^{-\frac{1}{2}} |\mathbf{u}|_{2,K} \leq C_D^2 C_I |\mathbf{u}|_{W^{2,3}(K)}.$$

By taking the maximum over all the elements on both sides entails

$$\|\mathbf{\Pi}_0^0 \nabla(\mathbf{u} - \mathbf{u}_I)\|_{L^\infty} \leq C_D^2 C_I |\mathbf{u}|_{W^{2,3}}.$$

Then, we have

$$\begin{aligned} \mathsf{T}_{2,1,1} &\leq \int_{\Omega} |\mathbf{\Pi}_0^0 \nabla(\mathbf{u} - \mathbf{u}_I)| |\mathbf{\Pi}_1^0 \mathbf{e}_h^{\mathbf{u}}| |\mathbf{\Pi}_1^0 \mathbf{e}_h^{\mathbf{u}}| \leq \|\mathbf{\Pi}_0^0 \nabla(\mathbf{u} - \mathbf{u}_I)\|_{L^\infty} \|\mathbf{e}_h^{\mathbf{u}}\|^2 \\ &\leq C_D^2 C_I |\mathbf{u}|_{W^{2,3}} \|\mathbf{e}_h^{\mathbf{u}}\|^2. \end{aligned}$$

Next, we apply Lemma 4.3 and write

$$\|\Pi_1^{0,K} \mathbf{u}_I\|_{L^\infty(K)} \leq C_{\text{inv}} C_D (C_I |\mathbf{u}|_{W^{1,3}} + \|\mathbf{u}\|_{L^\infty}).$$

By means of the above inequality, we deduce the bound

$$\begin{aligned} \mathbb{T}_{2,1,2} &\leq \int_{\Omega} |\Pi_0^0 \nabla(\mathbf{u} - \mathbf{u}_I)| |\Pi_1^0 \mathbf{u}_I| |\mathbf{e}_h^{\mathbf{u}}| \leq \|\Pi_0^0 \nabla(\mathbf{u} - \mathbf{u}_I)\| \|\Pi_1^0 \mathbf{u}_I\|_{L^\infty} \|\mathbf{e}_h^{\mathbf{u}}\| \\ &\leq C_I h |\mathbf{u}|_2 C_{\text{inv}} C_D (C_I + 1) (\|\mathbf{u}\|_{W^{1,3}} + \|\mathbf{u}\|_{L^\infty}) \|\mathbf{e}_h^{\mathbf{u}}\| \\ &\leq C_I^2 C_{\text{inv}}^2 C_D^2 (C_I + 1)^2 h^2 |\mathbf{u}|_2^2 (\|\mathbf{u}\|_{W^{1,3}}^2 + \|\mathbf{u}\|_{L^\infty}^2) + \frac{1}{4} \|\mathbf{e}_h^{\mathbf{u}}\|^2 \\ &\leq C_I^2 C_{\text{inv}}^2 C_D^2 (C_I + 1)^2 (C_S^2 + 1) h^2 \|\mathbf{u}\|_{W^{2,3}}^4 + \frac{1}{4} \|\mathbf{e}_h^{\mathbf{u}}\|^2. \end{aligned}$$

Next, we focus on the term $\mathbb{T}_{2,2}$ appearing on the right-hand side of (21):

$$\begin{aligned} \mathbb{T}_{2,2} &\leq \left| \int_{\Omega} (\Pi_0^0 \nabla \mathbf{e}_h^{\mathbf{u}}) \mathbf{u}_h \cdot \Pi_1^0(\mathbf{u} - \mathbf{u}_I) \right| \\ &\leq \left| \int_{\Omega} (\Pi_0^0 \nabla \mathbf{e}_h^{\mathbf{u}}) \mathbf{e}_h^{\mathbf{u}} \cdot \Pi_1^0(\mathbf{u} - \mathbf{u}_I) \right| + \left| \int_{\Omega} (\Pi_0^0 \nabla \mathbf{e}_h^{\mathbf{u}}) \mathbf{u}_I \cdot \Pi_1^0(\mathbf{u} - \mathbf{u}_I) \right| =: \mathbb{T}_{2,2,1} + \mathbb{T}_{2,2,2}. \end{aligned}$$

We have the bounds

$$\begin{aligned} \mathbb{T}_{2,2,1} &\leq \sum_{K \in \Omega_h} \|\Pi_0^{0,K} \nabla \mathbf{e}_h^{\mathbf{u}}\|_{0,K} \|\mathbf{e}_h^{\mathbf{u}}\|_{0,K} \|\Pi_1^{0,K}(\mathbf{u} - \mathbf{u}_I)\|_{L^\infty(K)} \\ &\leq \sum_{K \in \Omega_h} |\mathbf{e}_h^{\mathbf{u}}|_{1,K} \|\mathbf{e}_h^{\mathbf{u}}\|_{0,K} C_{\text{inv}} h_K^{-\frac{3}{2}} \|\mathbf{u} - \mathbf{u}_I\|_{0,K} \\ &\leq \sum_{K \in \Omega_h} |\mathbf{e}_h^{\mathbf{u}}|_{1,K} \|\mathbf{e}_h^{\mathbf{u}}\|_{0,K} C_{\text{inv}} C_I h_K^{-\frac{1}{2}} |\mathbf{u}|_{1,K} \leq C_{\text{inv}} C_I C_D |\mathbf{u}|_{W^{1,3}} \sum_{K \in \Omega_h} |\mathbf{e}_h^{\mathbf{u}}|_{1,K} \|\mathbf{e}_h^{\mathbf{u}}\|_{0,K} \\ &\leq C_{\text{inv}} C_I C_D |\mathbf{u}|_{W^{1,3}} |\mathbf{e}_h^{\mathbf{u}}|_1 \|\mathbf{e}_h^{\mathbf{u}}\| \leq \frac{\varepsilon}{4} |\mathbf{e}_h^{\mathbf{u}}|_1^2 + \frac{1}{\varepsilon} C_{\text{inv}}^2 C_I^2 C_D^2 |\mathbf{u}|_{W^{2,3}}^2 \|\mathbf{e}_h^{\mathbf{u}}\|^2 \end{aligned}$$

and, with similar computations,

$$\begin{aligned} \mathbb{T}_{2,2,2} &\leq \sum_{K \in \Omega_h} |\mathbf{e}_h^{\mathbf{u}}|_{1,K} \|\mathbf{u}_I\|_{0,K} \|\Pi_1^{0,K}(\mathbf{u} - \mathbf{u}_I)\|_{L^\infty(K)} \\ &\leq \sum_{K \in \Omega_h} |\mathbf{e}_h^{\mathbf{u}}|_{1,K} (1 + C_I h_K) \|\mathbf{u}\|_{1,K} C_{\text{inv}} C_I h_K^{\frac{1}{2}} |\mathbf{u}|_{2,K} \\ &\leq \sum_{K \in \Omega_h} |\mathbf{e}_h^{\mathbf{u}}|_{1,K} (1 + C_I h_K) \|\mathbf{u}\|_{1,K} C_{\text{inv}} C_I C_D h_K |\mathbf{u}|_{W^{2,3}(K)} \\ &\leq \frac{\varepsilon}{4} |\mathbf{e}_h^{\mathbf{u}}|_1^2 + \frac{1}{\varepsilon} (1 + C_I h_K)^2 C_{\text{inv}}^2 C_I^2 C_D^2 h^2 \|\mathbf{u}\|_1^2 |\mathbf{u}|_{W^{2,3}}^2 \\ &\leq \frac{\varepsilon}{4} |\mathbf{e}_h^{\mathbf{u}}|_1^2 + \frac{1}{\varepsilon} (1 + C_I h_K)^2 C_{\text{inv}}^2 C_I^2 C_D^2 h^2 \|\mathbf{u}\|_{W^{2,3}}^4. \end{aligned}$$

We collect the bounds on all the terms above and the assertion follows. \square

5 Virtual element spaces for the electromagnetic equations

5.1 A pivot nodal space

Here, we introduce a nodal virtual element space that will be instrumental in defining polynomial projectors for edge spaces. Consider the mesh face $F \in \mathcal{F}_h$ and the translated position vector $\mathbf{x}_F = \mathbf{x} - \mathbf{b}_F$ for all $\mathbf{x} \in F$ (we recall that \mathbf{b}_F is the center of F). We define the nodal virtual element space on F as

$$V_h^{\text{node}}(F) := \left\{ v_h \in C^0(\bar{F}) \mid \Delta v_h \in \mathbb{P}_0(F), v_h|_e \in \mathbb{P}_1(e) \forall e \in \mathcal{E}_h^F, \int_F \nabla v_h \cdot \mathbf{x}_F = 0 \right\}.$$

Then, for all elements $K \in \Omega_h$, we define the local nodal space as:

$$V_h^{\text{node}}(K) := \left\{ v_h \in C^0(\overline{K}) \mid \Delta v_h = 0, v_h|_F \in V_h^{\text{node}}(F) \forall F \in \mathcal{F}_h^K \right\}.$$

The virtual element functions in the local space $V_h^{\text{node}}(K)$ for all $K \in \Omega_h$ are uniquely determined by the set of their vertex values over ∂K ; see, e.g., [12] for a proof of their unisolvence. We introduce the global nodal space

$$V_h^{\text{node}} := \{v_h \in C^0(\overline{\Omega}) \mid v_h|_K \in V_h^{\text{node}}(K) \forall K \in \Omega_h\}, \quad (22)$$

which will be used in the definitions of the edge interpolant \mathbf{E}_H in the forthcoming Section 5.4.

5.2 The edge space

We define the virtual element edge space on the mesh face $F \in \mathcal{F}_h$ as

$$\mathbf{V}_h^{\text{edge}}(F) := \left\{ \mathbf{E}_h \in [L^2(F)]^2 \mid \operatorname{div}_F \mathbf{E}_h \in \mathbb{P}_0(F), \operatorname{rot}_F \mathbf{E}_h \in \mathbb{P}_0(F), \right. \\ \left. \mathbf{E}_h \cdot \mathbf{t}_e \in \mathbb{P}_0(e) \forall e \in \mathcal{E}_h^F, \int_F \mathbf{E}_h \cdot \mathbf{x}_F = 0 \right\},$$

where div_F and rot_F are defined in (1), and again $\mathbf{x}_F = \mathbf{x} - \mathbf{b}_F$.

The local edge space on an element $K \in \Omega_h$ is defined as

$$\mathbf{V}_h^{\text{edge}}(K) := \left\{ \mathbf{E}_h \in [L^2(K)]^3 \mid \mathbf{E}_{h,\tau_F} \in \mathbf{V}_h^{\text{edge}}(F) \forall F \in \mathcal{F}_h^K, \mathbf{E}_h \cdot \mathbf{t}_e \text{ continuous at each edge } e, \right. \\ \left. \begin{cases} \operatorname{curl} \operatorname{curl} \mathbf{E}_h \in [\mathbb{P}_0(K)]^3 \\ \operatorname{div} \mathbf{E}_h = 0 \end{cases}, \right. \\ \left. \int_K \operatorname{curl} \mathbf{E}_h \cdot (\mathbf{x}_K \times \mathbf{q}) = 0 \forall \mathbf{q} \in [\mathbb{P}_0(K)]^3 \right\},$$

where $\mathbf{x}_K = \mathbf{x} - \mathbf{b}_K$ for all $\mathbf{x} \in K$, \mathbf{b}_K is the center of K , and $\mathbf{E}_{h,\tau_F} = (\mathbf{n}_F \times \mathbf{v}_h) \times \mathbf{n}_F$.

Each element $\mathbf{E}_h \in \mathbf{V}_h^{\text{edge}}(K)$ is uniquely determined by the set of constant values $(E_e)_{e \in \mathcal{E}_h^K}$, where $E_e = \mathbf{E}_h \cdot \mathbf{t}_e$ is the tangential component of \mathbf{E}_h along the elemental edge e . We take these values as the degrees of freedom of \mathbf{E}_h and refer the reader to [12] for the proof of their unisolvence in $\mathbf{V}_h^{\text{edge}}(K)$. A major consequence of such a space definition is that $\mathbf{\Pi}_0^{0,K} \mathbf{E}_h$ is computable from the degrees of freedom of \mathbf{E}_h for all $\mathbf{E}_h \in \mathbf{V}_h^{\text{edge}}(K)$.

We define the global edge space $\mathbf{V}_h^{\text{edge}}$ by an $H(\operatorname{curl}, \Omega)$ -conforming coupling of the degrees of freedom, so that

$$\mathbf{V}_h^{\text{edge}} := \left\{ \mathbf{E}_h \in H_0(\operatorname{curl}, \Omega) \mid \mathbf{E}_h|_K \in \mathbf{V}_h^{\text{edge}}(K) \forall K \in \Omega_h \right\}. \quad (23)$$

This definition includes the homogeneous tangential boundary conditions on $\partial\Omega$.

5.2.1 Virtual element inner product

According to [12, 14], we equip the virtual element edge space $\mathbf{V}_h^{\text{edge}}$ with an inner product mimicking the L^2 inner product. Notably, we first introduce the local bilinear form

$$[\mathbf{E}_h, \mathbf{F}_h]_{\text{edge},K} := \left(\mathbf{\Pi}_0^{0,K} \mathbf{E}_h, \mathbf{\Pi}_0^{0,K} \mathbf{F}_h \right)_K + S_{\text{edge}}^K \left((I - \mathbf{\Pi}_0^{0,K}) \mathbf{E}_h, (I - \mathbf{\Pi}_0^{0,K}) \mathbf{F}_h \right) \\ \forall \mathbf{E}_h, \mathbf{F}_h \in \mathbf{V}_h^{\text{edge}}(K), \quad (24)$$

where $S_{\text{edge}}^K(\cdot, \cdot)$ can be any computable, symmetric bilinear form such that there exist two positive constants γ_* and γ^* independent of h_K satisfying

$$\gamma_* |\mathbf{E}_h|_{0,K}^2 \leq S_{\text{edge}}^K(\mathbf{E}_h, \mathbf{E}_h) \leq \gamma^* |\mathbf{E}_h|_{0,K}^2 \quad \forall \mathbf{E}_h \in \ker(\mathbf{\Pi}_0^{0,K}) \cap \mathbf{V}_h^{\text{edge}}(K). \quad (25)$$

An explicit stabilization satisfying (25) was introduced in [12, formula (4.8)] and analyzed in [8, Proposition 5.5], and reads

$$S_{\text{edge}}^K(\mathbf{E}_h, \mathbf{F}_h) := h_K^2 \sum_{F \in \mathcal{F}_h^K} \sum_{e \in \mathcal{E}_h^F} (\mathbf{E}_h \cdot \mathbf{t}_e, \mathbf{F}_h \cdot \mathbf{t}_e)_e \quad \forall \mathbf{E}_h, \mathbf{F}_h \in \mathbf{V}_h^{\text{edge}}(K).$$

The summation term in $S_{\text{edge}}^K(\cdot, \cdot)$ is multiplied by h_K^2 to have a consistent scaling for both terms of the edge inner product (24) with respect to the element size. The bilinear form (24) is computable on all elements $K \in \Omega_h$ using the degrees of freedom of $\mathbf{E}_h, \mathbf{F}_h \in \mathbf{V}_h^{\text{edge}}(K)$, and has the two crucial properties of consistency and stability:

- **consistency:** for all $\mathbf{q} \in [\mathbb{P}_0(K)]^3$ and all $\mathbf{E}_h \in \mathbf{V}_h^{\text{edge}}(K)$,

$$[\mathbf{q}, \mathbf{E}_h]_{\text{edge}, K} = (\mathbf{q}, \mathbf{E}_h)_K; \quad (26)$$

- **stability:** for all $\mathbf{E}_h \in \mathbf{V}_h^{\text{edge}}(K)$,

$$\eta_* \|\mathbf{E}_h\|_{0, K}^2 \leq [\mathbf{E}_h, \mathbf{E}_h]_{\text{edge}, K} \leq \eta^* \|\mathbf{E}_h\|_{0, K}^2, \quad (27)$$

with $\eta_* = \min(1, \gamma_*)$ and $\eta^* = \max(1, \gamma^*)$.

Finally, the global inner product over the virtual element edge space $\mathbf{V}_h^{\text{edge}}(K)$ is given by adding all the elemental contributions:

$$[\mathbf{E}_h, \mathbf{F}_h]_{\text{edge}} := \sum_{K \in \Omega_h} [\mathbf{E}_h, \mathbf{F}_h]_{\text{edge}, K}.$$

In what follows, we shall use the following discrete norm:

$$\|\cdot\|_{\mathbf{V}_h^{\text{edge}}}^2 := [\cdot, \cdot]_{\text{edge}}.$$

5.3 The face space

We define the virtual element face space on an element $K \in \Omega_h$ as

$$\mathbf{V}_h^{\text{face}}(K) := \left\{ \mathbf{B}_h \in [L^2(K)]^3 \mid \mathbf{B}_h \cdot \mathbf{n}_F \in \mathbb{P}_0(F) \forall F \in \mathcal{F}_h^K, \right. \\ \left. \begin{cases} \text{div } \mathbf{B}_h \in \mathbb{P}_0(K), \\ \text{curl } \mathbf{B}_h \in [\mathbb{P}_0(K)]^3, \end{cases} \int_K \mathbf{B}_h \cdot (\mathbf{x}_K \times \mathbf{q}) = 0 \forall \mathbf{q} \in [\mathbb{P}_0(K)]^3 \right\},$$

where $\mathbf{x}_K = \mathbf{x} - \mathbf{b}_K$.

Each element $\mathbf{B}_h \in \mathbf{V}_h^{\text{face}}(K)$ is uniquely determined by its normal components on the elemental faces, i.e., the set of constant values of $B_F = (\mathbf{B}_h \cdot \mathbf{n}_F)_{F \in \mathcal{F}_h^K}$. We take these values as the degrees of freedom of \mathbf{B}_h and refer the reader to [12] for the proof of their unisolvence. A major consequence of such a space definition is that $\mathbf{\Pi}_0^{0, K} \mathbf{B}_h$ and the divergence of \mathbf{B}_h are computable from the degrees of freedom of \mathbf{B}_h for all $\mathbf{B}_h \in \mathbf{V}_h^{\text{face}}(K)$.

We define the global face space $\mathbf{V}_h^{\text{face}}$ by an $H(\text{div})$ -coupling of the degrees of freedom:

$$\mathbf{V}_h^{\text{face}} := \left\{ \mathbf{B}_h \in H_0(\text{div}, \Omega) \mid \mathbf{B}_h|_K \in \mathbf{V}_h^{\text{face}}(K) \forall K \in \Omega_h \right\}.$$

This definition includes the homogeneous normal boundary conditions on $\partial\Omega$.

5.3.1 Virtual element inner product

From [13,14], we recall the discretization of the L^2 -inner product for the local face virtual element space, which reads as

$$[\mathbf{B}_h, \mathbf{C}_h]_{\text{face},K} := \left(\Pi_0^{0,K} \mathbf{B}_h, \Pi_0^{0,K} \mathbf{C}_h \right)_{0,K} + S_{\text{face}}^K \left((I - \Pi_0^{0,K}) \mathbf{B}_h, (I - \Pi_0^{0,K}) \mathbf{C}_h \right) \quad \forall \mathbf{B}_h, \mathbf{C}_h \in \mathbf{V}_h^{\text{face}}(K), \quad (28)$$

where $S_{\text{face}}^K(\cdot, \cdot)$ can be any computable, symmetric bilinear form such that there exist two positive constants $\tilde{\gamma}_*$ and $\tilde{\gamma}^*$ independent of h_K satisfying

$$\tilde{\gamma}_* \|\mathbf{B}_h\|_{0,K}^2 \leq S_{\text{face}}^K(\mathbf{B}_h, \mathbf{B}_h) \leq \tilde{\gamma}^* \|\mathbf{B}_h\|_{0,K}^2 \quad \forall \mathbf{B}_h \in \ker(\Pi_0^{0,K}) \cap \mathbf{V}_h^{\text{face}}(K). \quad (29)$$

An explicit stabilization satisfying (29) was introduced in [12, formula (4.8)] analyzed in [8, Proposition 5.2], and reads

$$S_{\text{face}}^K(\mathbf{B}_h, \mathbf{C}_h) := h_K \sum_{F \in \mathcal{F}_h^K} (\mathbf{n}_F \cdot \mathbf{B}_h, \mathbf{n}_F \cdot \mathbf{C}_h)_{0,F} \quad \forall \mathbf{B}_h, \mathbf{C}_h \in \mathbf{V}_h^{\text{face}}(K).$$

The bilinear form (28) is computable on all elements $K \in \Omega_h$ using the degrees of freedom of $\mathbf{B}_h, \mathbf{C}_h \in \mathbf{V}_h^{\text{face}}(K)$, and has the two crucial properties of consistency and stability:

- **consistency:** for all $\mathbf{q} \in [\mathbb{P}_0(K)]^3$ and all $\mathbf{B}_h \in \mathbf{V}_h^{\text{face}}(K)$,

$$[\mathbf{q}, \mathbf{B}_h]_{\text{face},K} = (\mathbf{q}, \mathbf{B}_h)_K;$$

- **stability:** for all $\mathbf{B}_h \in \mathbf{V}_h^{\text{face}}(K)$:

$$\chi_* \|\mathbf{B}_h\|_{0,K}^2 \leq [\mathbf{B}_h, \mathbf{B}_h]_{\text{face},K} \leq \chi^* \|\mathbf{B}_h\|_{0,K}^2, \quad (30)$$

with $\chi_* = \min(1, \tilde{\gamma}_*)$ and $\chi^* = \max(1, \tilde{\gamma}^*)$.

Finally, the inner product over the global virtual element space $\mathbf{V}_h^{\text{face}}$ is given by summing all the elemental contributions:

$$[\mathbf{B}_h, \mathbf{C}_h]_{\text{face}} := \sum_{K \in \Omega_h} [\mathbf{B}_h, \mathbf{C}_h]_{\text{face},K}.$$

In what follows, we shall use the following discrete norm:

$$\|\cdot\|_{\mathbf{V}_h^{\text{face}}}^2 := [\cdot, \cdot]_{\text{face}}.$$

5.4 Interpolations and exact sequence properties

As thoroughly discussed, e.g., in [12], the above spaces satisfy fundamental exact sequence properties. Introduce an edge approximation operator $\mathcal{I}^{\mathcal{E}h} : H(\mathbf{curl}, \Omega) \rightarrow \mathbf{V}_h^{\text{edge}}$ as follows. The edge

$$\begin{array}{ccc} H(\mathbf{curl}, \Omega) & \xrightarrow{\mathbf{curl}} & H(\text{div}, \Omega) \\ \downarrow \mathcal{I}^{\mathcal{E}h} & & \downarrow \mathcal{I}^{\mathcal{F}h} \\ \mathbf{V}_h^{\text{edge}} & \xrightarrow{\mathbf{curl}} & \tilde{\mathbf{V}}_h^{\text{face}} \end{array}$$

Figure 1: Commuting diagram for the edge and face virtual element functions.

approximant $\mathcal{I}^{\mathcal{E}^h}(\mathbf{E}) := \mathbf{E}_{II} \in \mathbf{V}_h^{\text{edge}}$ of a given vector field $\mathbf{E} \in H(\mathbf{curl}, \Omega)$ is defined as the solution to the variational problem

$$[\mathbf{curl} \mathbf{E}_{II}, \mathbf{curl} \mathbf{F}_h]_{\text{face}} = (\mathbf{curl} \mathbf{E}, \mathbf{curl} \mathbf{F}_h) \quad \forall \mathbf{F}_h \in \mathbf{V}_h^{\text{edge}}, \quad (31a)$$

$$[\mathbf{E}_{II}, \nabla r_h]_{\text{edge}} = (\mathbf{E}, \nabla r_h) \quad \forall r_h \in V_h^{\text{node}}, \quad (31b)$$

where the spaces $\mathbf{V}_h^{\text{edge}}$ and V_h^{node} are defined in (23) and (22), respectively. The following approximation bound is valid [14, Proposition 5].

Lemma 5.1. *Let $\mathbf{E} \in H(\mathbf{curl}, \Omega)$ and consider the approximant $\mathbf{E}_{II} \in \mathbf{V}_h^{\text{edge}}$ defined as in (31). Then, there exists a real, positive constant C_I independent of h such that*

$$\|\mathbf{E} - \mathbf{E}_{II}\|_{\mathbf{curl}} \leq C_I h (\|\mathbf{E}\|_{1,\Omega} + \|\mathbf{curl} \mathbf{E}\|_{0,\Omega} + h \|\mathbf{curl} \mathbf{E}\|_{1,\Omega}).$$

Introduce a face approximation operator $\mathcal{I}^{\mathcal{F}^h} : H(\text{div}, \Omega) \rightarrow \mathbf{V}_h^{\text{face}}$ as follows. The face approximant

$$\mathcal{I}^{\mathcal{F}^h}(\mathbf{B}) := \mathbf{B}_{II} \in \tilde{\mathbf{V}}_h^{\text{face}} := \{\mathbf{C}_h \in \mathbf{V}_h^{\text{face}} \mid \text{div} \mathbf{C}_h = 0\}$$

of a given vector field $\mathbf{B} \in H(\text{div}, \Omega)$ is defined as the solution to the variational problem

$$[\mathbf{B}_{II}, \mathbf{C}_h]_{\text{face}} = (\mathbf{B}, \mathbf{C}_h) \quad \forall \mathbf{C}_h \in \tilde{\mathbf{V}}_h^{\text{face}}. \quad (32)$$

We have that $\tilde{\mathbf{V}}_h^{\text{face}} = \mathbf{curl}(\mathbf{V}_h^{\text{edge}})$; see [12, equation (4.35)].

We recall the following approximation result [14, Lemma 7].

Lemma 5.2. *Let $\mathbf{B} \in [H^1(\Omega)]^3$ be a divergence-free magnetic field and $\mathbf{B}_{II} \in \mathbf{V}_h^{\text{face}}$ be its approximant defined as in (32). Then, there exists a real, positive constant C_I independent of h such that*

$$\|\mathbf{B} - \mathbf{B}_{II}\| \leq C_I h \|\mathbf{B}\|_1.$$

Additionally, we have a commuting property involving the operators defined in (31) and (32), and the curl operator, which we report in the next lemma, whose proof can be found in [14, Proposition 6]. The interpolation diagram is shown in Figure 1.

Lemma 5.3. *Let $\mathbf{E}_{II} \in \mathbf{V}_h^{\text{edge}}$ and $(\mathbf{curl} \mathbf{E})_{II} \in \mathbf{V}_h^{\text{face}}$ be the approximant of \mathbf{E} and $\mathbf{curl} \mathbf{E} \in H(\text{div}, \Omega)$ defined in (31) and (32), respectively. Then, the following commuting property is valid:*

$$\mathbf{curl}(\mathbf{E}_{II}) = (\mathbf{curl} \mathbf{E})_{II} \quad \forall \mathbf{E} \in H(\mathbf{curl}, \Omega).$$

5.5 The semi-discrete MHD model

First, note that $-(\mathbf{j} \times \mathbf{B}, \mathbf{v}) = (\mathbf{j}, \mathbf{v} \times \mathbf{B})$ in equation (9a). To discretize this term, we consider the elemental bilinear operator $\chi_{h,K} : \mathbf{W}_h(K) \times \mathbf{V}_h^{\text{face}}(K) \rightarrow \mathbf{V}_h^{\text{edge}}(K)$. This operator denotes the discretization of the cross product “ $\mathbf{v}_h \times \mathbf{B}_h$ ” for $\mathbf{v}_h \in \mathbf{W}_h(K)$ and $\mathbf{B}_h \in \mathbf{V}_h^{\text{face}}(K)$ through the elemental edge function $\chi_{h,K}(\mathbf{v}_h, \mathbf{B}_h) \in \mathbf{V}_h^{\text{edge}}(K)$. For the sake of notation, we also introduce the global operator

$$\chi_h : \mathbf{W}_h \times \mathbf{V}_h^{\text{face}} \rightarrow \Pi_{K \in \Omega_h} \mathbf{V}_h^{\text{edge}}(K) \quad \text{such that} \quad \chi_h(\mathbf{v}_h, \mathbf{B}_h)|_K = \chi_{h,K}(\mathbf{v}_h|_K, \mathbf{B}_h|_K)$$

for the global fields $\mathbf{v}_h \in \mathbf{W}_h$ and $\mathbf{B}_h \in \mathbf{V}_h^{\text{face}}$. We emphasize that $\chi_h(\mathbf{v}_h, \mathbf{B}_h)$ does not belong to $\mathbf{V}_h^{\text{edge}}$ since it is not $H(\mathbf{curl}, \Omega)$ -conforming. However, to simplify several equations below, with an abuse of notation, we let

$$[\mathbf{F}_h, \chi_h(\mathbf{v}_h, \mathbf{B}_h)]_{\text{edge}} := \sum_{K \in \Omega_h} [\mathbf{F}_h, \chi_{h,K}(\mathbf{v}_h, \mathbf{B}_h)]_{\text{edge},K}.$$

Using this bilinear operator, we define the “broken” virtual element edge function

$$\mathbf{j}_h = \mathbf{E}_h + \chi_h(\mathbf{u}_h, \mathbf{B}_h),$$

which can be considered as the electric current density of the discrete MHD model. For the sake of simplicity, we take $s = 1$ in (6a) and (9a).

Several choices for χ_h are possible. Among them, we pick

$$\chi_h(\mathbf{v}_h, \mathbf{B}_h)|_K := \Pi_0^{0,K} \mathbf{v}_h \times \Pi_0^{0,K} \mathbf{B}_h. \quad (33)$$

The semi-discrete virtual element method for the MHD equations reads as: *For every $t \in (0, T)$, find $(\mathbf{u}_h, p_h, \mathbf{E}_h, \mathbf{B}_h) \in \mathbf{W}_h \times Q_h \times \mathbf{V}_h^{\text{edge}} \times \mathbf{V}_h^{\text{face}}$ such that*

$$\begin{aligned} m_h(\mathbf{u}_{h,t}, \mathbf{v}_h) + Re^{-1} a_h(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) + \tilde{c}_h(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h) + [\mathbf{j}_h, \chi_h(\mathbf{v}_h, \mathbf{B}_h)]_{\text{edge}} \\ = (\mathbf{f}, \Pi_1^0 \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{W}_h \subset [H_0^1(\Omega)]^3, \end{aligned} \quad (34a)$$

$$[\mathbf{j}_h, \mathbf{F}_h]_{\text{edge}} - Re_m^{-1} [\mathbf{B}_h, \mathbf{curl} \mathbf{F}_h]_{\text{face}} = 0 \quad \forall \mathbf{F}_h \in \mathbf{V}_h^{\text{edge}} \subset H_0(\mathbf{curl}, \Omega), \quad (34b)$$

$$[\mathbf{B}_{h,t}, \mathbf{C}_h]_{\text{face}} + [\mathbf{curl} \mathbf{E}_h, \mathbf{C}_h]_{\text{face}} = 0 \quad \forall \mathbf{C}_h \in \mathbf{V}_h^{\text{face}} \subset H_0(\text{div}, \Omega), \quad (34c)$$

$$b(\mathbf{u}_h, q_h) = 0 \quad \forall q_h \in Q_h \subset L_0(\Omega). \quad (34d)$$

We provide the semi-discrete scheme (34) with the discrete initial conditions $\mathbf{u}_h(0) = \mathbf{u}_{h,0}$ and $\mathbf{B}_h(0) = \mathbf{B}_{h,0}$, where $\mathbf{u}_{h,0}$ and $\mathbf{B}_{h,0}$ are the degrees of freedom interpolants in \mathbf{W}_h and $\mathbf{V}_h^{\text{face}}$ of the continuous initial conditions \mathbf{u}_0 and \mathbf{B}_0 in (7). Provided that \mathbf{B} is sufficiently regular, from [8, Proposition 4.5], the degrees of freedom interpolant of \mathbf{B} in the face space $\mathbf{V}_h^{\text{face}}(K)$ satisfies

$$\|\mathbf{B}(0) - \mathbf{B}_{h,0}\|_{0,K} \lesssim h_K. \quad (35)$$

Remark 5.4. *Recalling that $\text{div} \mathbf{u}_h \in Q_h$ and $\mathbf{curl} \mathbf{E}_h \in \mathbf{V}_h^{\text{face}}$, the third and fourth equations provide*

$$\mathbf{B}_{h,t} + \mathbf{curl} \mathbf{E}_h = \mathbf{0}, \quad \text{div} \mathbf{u}_h = 0 \quad \text{in } \Omega. \quad (36)$$

As a consequence, by setting the initial data such that $\text{div} \mathbf{B}_{h,0} = 0$, the first equation in (36) implies $\text{div} \mathbf{B}_h = 0$ at all times. Thus, the proposed scheme satisfies both solenoidal constraints exactly.

We conclude this section by presenting the following stability result on the solution to the semi-discrete scheme (34) and briefly commenting on the existence of a (unique) solution to (34).

Proposition 5.5. *Let \mathbf{u}_h , \mathbf{E}_h , and \mathbf{B}_h be solutions to (34). Then, the following stability estimates are valid:*

$$\frac{1}{2} \frac{d}{dt} m_h(\mathbf{u}_h, \mathbf{u}_h) + \frac{1}{2} Re_m^{-1} \frac{d}{dt} [\mathbf{B}_h, \mathbf{B}_h]_{\text{face}} + Re^{-1} a_h(\mathbf{u}_h, \mathbf{u}_h) + [\mathbf{j}_h, \mathbf{j}_h]_{\text{edge}} = (\mathbf{f}, \Pi_1^0 \mathbf{u}_h) \quad \forall t \in (0, T)$$

and

$$\begin{aligned} \max_{0 \leq t \leq T} (m_h(\mathbf{u}_h, \mathbf{u}_h) + Re_m^{-1} [\mathbf{B}_h, \mathbf{B}_h]_{\text{face}}) + Re^{-1} \int_0^T a_h(\mathbf{u}_h, \mathbf{u}_h) + 2 \int_0^T [\mathbf{j}_h, \mathbf{j}_h]_{\text{edge}} \\ \leq m_h(\mathbf{u}_{h,0}, \mathbf{u}_{h,0}) + Re_m^{-1} [\mathbf{B}_{h,0}, \mathbf{B}_{h,0}]_{\text{face}} + \alpha_*^{-1} C_P Re \int_0^T \|\mathbf{f}\|_0. \end{aligned}$$

Proof. The first estimate follows picking $\mathbf{v}_h = \mathbf{u}_h$ in (34a), $\mathbf{F}_h = \mathbf{E}_h$ in (34b), and $\mathbf{C}_h = \mathbf{B}_h$ in (34c), multiplying (34b) by Re_m , and summing up the three resulting equations.

As for the second one, we observe that

$$\begin{aligned} (\mathbf{f}, \Pi_1^0 \mathbf{u}_h) &\leq \frac{1}{2} \alpha_*^{-1} C_P Re \|\mathbf{f}\|_0^2 + \frac{1}{2} \alpha_* C_P^{-1} Re^{-1} \|\Pi_1^0 \mathbf{u}_h\|^2 \leq \frac{1}{2} \alpha_*^{-1} C_P Re \|\mathbf{f}\|_0^2 + \frac{1}{2} \alpha_* Re^{-1} |\mathbf{u}_h|_1^2 \\ &\leq \frac{1}{2} \alpha_*^{-1} C_P Re \|\mathbf{f}\|_0^2 + \frac{1}{2} Re^{-1} a_h(\mathbf{u}_h, \mathbf{u}_h). \end{aligned}$$

Next, we absorb the second term on the right-hand side in the left-hand side of the first bound. The assertion follows by integrating in time the first bound between 0 and any time $0 < t \leq T$, multiplying by two both sides, and taking the maximum over the time t . \square

Existence and uniqueness of a solution to the semi-discrete problem (34) follows from the standard ODE theory in finite dimensions. Here, we only sketch the proof. First, we eliminate the pressure variable and equation (34d) by restricting the (test and trial) velocity space to divergence-free functions. Next, (34b) at the initial time $t = 0$ together with the definition of \mathbf{j}_h allows us to define an initial condition $\mathbf{E}_{h,0}$ for the electric field, depending on the initial conditions $\mathbf{u}_{h,0}$ and $\mathbf{B}_{h,0}$ of the velocity and magnetic fields. Then, we can substitute (34b) with the same equation derived in time:

$$[\mathbf{j}_{h,t}, \mathbf{F}_h]_{\text{edge}} - Re_m^{-1} [\mathbf{B}_{h,t}, \mathbf{curl} \mathbf{F}_h]_{\text{face}} = 0 \quad \forall \mathbf{F}_h \in \mathbf{V}_h^{\text{edge}}.$$

Recalling the definition of \mathbf{j}_h , deriving in time once, and applying (34c), we obtain

$$[\mathbf{E}_{h,t}, \mathbf{F}_h]_{\text{edge}} + Re_m^{-1} [\mathbf{curl} \mathbf{E}_h, \mathbf{curl} \mathbf{F}_h]_{\text{face}} + [\partial_t \chi_h(\mathbf{v}_h, \mathbf{B}_h), \mathbf{F}_h]_{\text{edge}} = 0 \quad \forall \mathbf{F}_h \in \mathbf{V}_h^{\text{edge}}. \quad (37)$$

Substituting (34b) with (37) and recalling that $\mathbf{E}_{h,0}$ is known yield an equivalent system, which is a first order Cauchy problem on the finite-dimensional variable $(\mathbf{u}_h, \mathbf{B}_h, \mathbf{E}_h)$: for all \mathbf{v}_h in \mathbf{W}_h , \mathbf{F}_h in $\mathbf{V}_h^{\text{edge}}$, and \mathbf{C}_h in $\mathbf{V}_h^{\text{face}}$,

$$\begin{cases} m_h(\mathbf{u}_{h,t}, \mathbf{v}_h) + Re^{-1} a_h(\mathbf{u}_h, \mathbf{v}_h) + \tilde{c}_h(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h) + [\mathbf{j}_h, \chi_h(\mathbf{v}_h, \mathbf{B}_h)]_{\text{edge}} &= (\mathbf{f}, \Pi_1^0 \mathbf{v}_h), \\ [\mathbf{E}_{h,t}, \mathbf{F}_h]_{\text{edge}} + Re_m^{-1} [\mathbf{curl} \mathbf{E}_h, \mathbf{curl} \mathbf{F}_h]_{\text{face}} + [\partial_t \chi_h(\mathbf{v}_h, \mathbf{B}_h), \mathbf{F}_h]_{\text{edge}} &= 0, \\ [\mathbf{B}_{h,t}, \mathbf{C}_h]_{\text{face}} + [\mathbf{curl} \mathbf{E}_h, \mathbf{C}_h]_{\text{face}} &= 0. \end{cases} \quad (38)$$

The term

$$[\partial_t \chi_h(\mathbf{v}_h, \mathbf{B}_h), \mathbf{F}_h]_{\text{edge}} = [\Pi_0^{0,K} \mathbf{u}_{h,t} \times \Pi_0^{0,K} \mathbf{B}_h, \mathbf{F}_h]_{\text{edge}} + [\Pi_0^{0,K} \mathbf{u}_h \times \Pi_0^{0,K} \mathbf{B}_{h,t}, \mathbf{F}_h]_{\text{edge}}$$

can be interpreted as a quartic function of $(\mathbf{u}_h, \mathbf{B}_h, \mathbf{E}_h)$. In fact, using the algebraic form of the first and third equation in (38), and inverting the corresponding ‘‘mass’’ matrices, give that $\mathbf{u}_{h,t}$ can be interpreted as a cubic function of $(\mathbf{u}_h, \mathbf{B}_h, \mathbf{E}_h)$ and $\mathbf{B}_{h,t}$ as a linear function of \mathbf{E}_h . This implies that $\Pi_0^{0,K} \mathbf{u}_{h,t} \times \Pi_0^{0,K} \mathbf{B}_h$ and $\Pi_0^{0,K} \mathbf{u}_h \times \Pi_0^{0,K} \mathbf{B}_{h,t}$ can be interpreted as a quartic and quadratic functions of $(\mathbf{u}_h, \mathbf{B}_h, \mathbf{E}_h)$.

Consequently, (38) is a first order Cauchy problem in $(\mathbf{u}_h, \mathbf{B}_h, \mathbf{E}_h)$ with quartic right-hand side. Proposition 5.5 asserts that the solutions to the system (if they exist) cannot blow up in finite time. Thus, the nonlinearity on the right-hand side is uniformly Lipschitz. For finite times, existence and uniqueness follow from the standard ODE theory in finite dimension; see, e.g., [59, Section 9.2].

6 Convergence analysis of the virtual element approximation

Introduce the ‘‘discrete errors’’

$$\mathbf{e}_h^{\mathbf{u}} := \mathbf{u}_h - \mathbf{u}_I, \quad \mathbf{e}_h^{\mathbf{E}} := \mathbf{E}_h - \mathbf{E}_{II}, \quad \mathbf{e}_h^{\mathbf{B}} := \mathbf{B}_h - \mathbf{B}_{II}, \quad (39)$$

and the virtual element edge fields

$$\mathbf{j}_I = \mathbf{E}_{II} + \chi_h(\mathbf{u}_I, \mathbf{B}_h) \quad (40)$$

and

$$\mathbf{e}_h^{\mathbf{j}} = \mathbf{j}_h - \mathbf{j}_I = (\mathbf{E}_h + \chi_h(\mathbf{u}_h, \mathbf{B}_h)) - (\mathbf{E}_{II} + \chi_h(\mathbf{u}_I, \mathbf{B}_h)) = \mathbf{e}_h^{\mathbf{E}} + \chi_h(\mathbf{e}_h^{\mathbf{u}}, \mathbf{B}_h). \quad (41)$$

These quantities are bounded as in the following theorem, whose proof can be found in Section 6.1.

Theorem 6.1. *Let $(\mathbf{u}(t), p(t), \mathbf{E}(t), \mathbf{B}(t))$ in $[H_0^1(\Omega)]^3 \times L_0^2(\Omega) \times H_0(\mathbf{curl}, \Omega) \times H_0(\text{div}, \Omega)$ for almost every $t \in (0, T)$ be the solution to the variational formulation (9) under the assumptions in Section 3. Furthermore, assume that the following regularity properties*

$$\mathbf{u} \in [W^{2,\infty}(\Omega)]^3, \quad \mathbf{u}_t \in [H^1(\Omega)]^3, \quad \mathbf{B} \in [H^1(\Omega) \cap L^\infty(\Omega)]^3 \quad \text{and} \quad \mathbf{j} \in [L^\infty(\Omega)]^3 \quad (42)$$

hold uniformly in time.

Let $(\mathbf{u}_h(t), p_h(t), \mathbf{E}_h(t), \mathbf{B}_h(t))$ in $\mathbf{W}_h \times Q_h \times \mathbf{V}_h^{\text{edge}} \times \mathbf{V}_h^{\text{face}}$ for every $t \in (0, T)$ be the solution to the virtual element method (34). Then, a positive constant C exists that is independent of h such that

$$\|\mathbf{e}_h^{\mathbf{u}}(t)\| + \|\mathbf{e}_h^{\mathbf{B}}(t)\| + \left(\int_0^t \|\mathbf{e}_h^{\mathbf{j}}(s)\|^2 \right)^{\frac{1}{2}} \leq C(\|\mathbf{e}_h^{\mathbf{u}}(0)\| + \|\mathbf{e}_h^{\mathbf{B}}(0)\| + h) \quad \text{for every } t \in (0, T],$$

where $\mathbf{e}_h^{\mathbf{u}}$, $\mathbf{e}_h^{\mathbf{B}}$, and $\mathbf{e}_h^{\mathbf{j}}$ are defined in (39)-(41). The constant C depends on the parameters of the discretization, the final time T , and the regularity of the MHD solution fields $\mathbf{u}(t)$, $\mathbf{E}(t)$, $\mathbf{B}(t)$, and $\mathbf{B}_h(t)$.

In view of Theorem 6.1, we have the following convergence result.

Theorem 6.2. Let $(\mathbf{u}(t), p(t), \mathbf{E}(t), \mathbf{B}(t))$ in $[H_0^1(\Omega)]^3 \times L_0^2(\Omega) \times H_0(\mathbf{curl}, \Omega) \times H_0(\mathbf{div}, \Omega)$ for almost every $t \in (0, T)$ be the solution to the variational formulation (9) under the assumptions in Section 3. Let the regularity assumptions of Theorem 6.1 be valid and assume that the initial conditions \mathbf{u}_0 , \mathbf{B}_0 , and \mathbf{E}_0 of the fields \mathbf{u} , \mathbf{B} , and \mathbf{E} belong to $[H^1(\Omega)]^3$. Let $(\mathbf{u}_h(t), p_h(t), \mathbf{E}_h(t), \mathbf{B}_h(t))$ in $\mathbf{W}_h \times Q_h \times \mathbf{V}_h^{\text{edge}} \times \mathbf{V}_h^{\text{face}}$ for every $t \in (0, T)$ be the solution to the virtual element method (34). Then, a positive constant C exists, independent of h , such that

$$\|\mathbf{u}(t) - \mathbf{u}_h(t)\| + \|\mathbf{B}(t) - \mathbf{B}_h(t)\| + \left(\int_0^t \|\mathbf{E}(s) - \mathbf{E}_h(s)\|^2 \right)^{\frac{1}{2}} \leq Ch \quad (43)$$

for almost every $t \in (0, T]$. The constant C depends on the parameters of the discretization, the final time T , and the regularity of the MHD solution fields $\mathbf{u}(t)$, $\mathbf{E}(t)$, $\mathbf{B}(t)$, and $\mathbf{B}_h(t)$.

Proof. We add and subtract \mathbf{u}_I (see Lemma 4.2), \mathbf{E}_{II} (see (31)), and \mathbf{B}_{II} (see (32)) to the left-hand side of (43) and get

$$\begin{aligned} & \|\mathbf{u}(t) - \mathbf{u}_h(t)\| + \|\mathbf{B}(t) - \mathbf{B}_h(t)\| + \left(\int_0^t \|\mathbf{E}(s) - \mathbf{E}_h(s)\|^2 \right)^{\frac{1}{2}} \\ & \leq \left[\|\mathbf{u}(t) - \mathbf{u}_I(t)\| + \|\mathbf{B}(t) - \mathbf{B}_{II}(t)\| + \left(\int_0^t \|\mathbf{E}(s) - \mathbf{E}_{II}(s)\|^2 \right)^{\frac{1}{2}} \right] \\ & \quad + \left[\|\mathbf{e}_h^{\mathbf{u}}(t)\| + \|\mathbf{e}_h^{\mathbf{B}}(t)\| + \left(\int_0^t \|\mathbf{e}_h^{\mathbf{E}}(s)\|^2 \right)^{\frac{1}{2}} \right] := S_1 + S_2. \end{aligned}$$

An upper bound on the term S_1 can be shown using the interpolation properties of Lemmas 4.2, 5.2, and 5.1.

Observe that $\mathbf{e}_h^{\mathbf{u}}(0) = 0$. Furthermore, the triangle inequality, (35), and Lemma 5.2 give

$$\|\mathbf{e}_h^{\mathbf{B}}(0)\| \leq \|\mathbf{B}(0) - \mathbf{B}_{h,0}\| + \|\mathbf{B}(0) - \mathbf{B}_{II}(0)\| \lesssim h.$$

With this at hand, Theorem 6.1 allows us to show a direct bound on the terms involving the velocity and magnetic fields appearing in S_2 . For the term involving the electric field we proceed as follows: using the triangle inequality, the definition of χ_h in (33), the bound on $\|\mathbf{e}_h^{\mathbf{u}}\|$ from (42), the stability estimates on the discrete face element bilinear forms (30), and the bound on $[\mathbf{B}_h, \mathbf{B}_h]_{\text{face}}$ in Proposition 5.5, we arrive at

$$\begin{aligned} & \left(\int_0^t \|\mathbf{e}_h^{\mathbf{E}}(s)\|^2 \right)^{\frac{1}{2}} \lesssim \left(\int_0^t \left(\|\mathbf{e}_h^{\mathbf{j}}(s)\|^2 + \|\chi_h(\mathbf{e}_h^{\mathbf{u}}(s), \mathbf{B}_h(s))\|^2 \right) \right)^{\frac{1}{2}} \\ & \leq h + \left(\int_0^t \|\mathbf{e}_h^{\mathbf{u}}(s)\|^2 \|\mathbf{B}_h(s)\|^2 \right)^{\frac{1}{2}} \lesssim h + \|\mathbf{e}_h^{\mathbf{u}}\|_{L^\infty(0,t;L^2(\Omega))} \|\mathbf{B}_h\|_{L^\infty(0,t;\mathbf{V}_h^{\text{face}})} \lesssim h. \end{aligned}$$

The final hidden constant depends on the square root of the instant time t . \square

6.1 Proof of Theorem 6.1

Proof. Throughout, for presentation's sake, we assume that $Re_m = 1$ and $Re = 1$. The general assertion can be proved by means of minor modifications but note that the final constant in Theorem 6.1 will depend on Re_m and Re ; treating convection-dominated cases is beyond the scope of this work.

First, we add and subtract \mathbf{E}_{II} and $\chi_h(\mathbf{u}_I, \mathbf{B}_h)$ to $\mathbf{j}_h = \mathbf{E}_h + \chi_h(\mathbf{u}_h, \mathbf{B}_h)$ so that

$$\mathbf{j}_h = \mathbf{E}_h + \chi_h(\mathbf{u}_h, \mathbf{B}_h) = (\mathbf{E}_h - \mathbf{E}_{II}) + \chi_h(\mathbf{u}_h - \mathbf{u}_I, \mathbf{B}_h) + \mathbf{E}_{II} + \chi_h(\mathbf{u}_I, \mathbf{B}_h), \quad (44)$$

where \mathbf{E}_{II} and \mathbf{u}_I are defined in (31) and Section 4.3.1 (see also Lemma 4.2), respectively. We substitute (44) in (34a), add and subtract \mathbf{u}_I , and find that

$$\begin{aligned} & m_h((\mathbf{u}_h - \mathbf{u}_I)_t, \mathbf{v}_h) + a_h(\mathbf{u}_h - \mathbf{u}_I, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) \\ & \quad + [\mathbf{E}_h - \mathbf{E}_{II} + \chi_h(\mathbf{u}_h - \mathbf{u}_I), \chi_h(\mathbf{v}_h, \mathbf{B}_h)]_{\text{edge}} \\ & = (\mathbf{f}, \Pi_1^0 \mathbf{v}_h) - m_h(\mathbf{u}_{I,t}, \mathbf{v}_h) - a_h(\mathbf{u}_I, \mathbf{v}_h) - \tilde{c}_h(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h) \\ & \quad - [\mathbf{E}_{II} + \chi_h(\mathbf{u}_I, \mathbf{B}_h), \chi_h(\mathbf{v}_h, \mathbf{B}_h)]_{\text{edge}}. \end{aligned} \quad (45)$$

Similarly, we use (44) in (34b), add and subtract \mathbf{B}_{II} defined in (32), and find that

$$\begin{aligned} & [\mathbf{E}_h - \mathbf{E}_{II} + \chi_h(\mathbf{u}_h - \mathbf{u}_I, \mathbf{B}_h), \mathbf{F}_h]_{\text{edge}} - [\mathbf{B}_h - \mathbf{B}_{II}, \mathbf{curl} \mathbf{F}_h]_{\text{face}} \\ & \quad = -[\mathbf{E}_{II} + \chi_h(\mathbf{u}_I, \mathbf{B}_h), \mathbf{F}_h]_{\text{edge}} + [\mathbf{B}_{II}, \mathbf{curl} \mathbf{F}_h]_{\text{face}}. \end{aligned} \quad (46)$$

Then, we use Lemma 5.3, (32) twice and (9c), and find that, for all $\mathbf{C}_h \in \tilde{\mathbf{V}}_h^{\text{face}}$,

$$[\mathbf{curl} \mathbf{E}_{II}, \mathbf{C}_h]_{\text{face}} = [(\mathbf{curl} \mathbf{E})_{II}, \mathbf{C}_h]_{\text{face}} = (\mathbf{curl} \mathbf{E}, \mathbf{C}_h) = -(\mathbf{B}_t, \mathbf{C}_h) = -[(\mathbf{B}_t)_{II}, \mathbf{C}_h]_{\text{face}}.$$

Since the derivation in time and the interpolation commute, e.g., $(\mathbf{B}_t)_{II} = \mathbf{B}_{II,t}$, we find

$$[\mathbf{B}_{II,t}, \mathbf{C}_h]_{\text{face}} + [\mathbf{curl} \mathbf{E}_{II}, \mathbf{C}_h]_{\text{face}} = 0.$$

Finally, we subtract this equation to (34c) and obtain, for all $\mathbf{C}_h \in \tilde{\mathbf{V}}_h^{\text{face}}$,

$$[(\mathbf{B}_h - \mathbf{B}_{II})_t, \mathbf{C}_h]_{\text{face}} + [\mathbf{curl}(\mathbf{E}_h - \mathbf{E}_{II}), \mathbf{C}_h]_{\text{face}} = 0. \quad (47)$$

Next, we take $\mathbf{v}_h = \mathbf{e}_h^{\mathbf{u}}$ in (45), $\mathbf{F}_h = \mathbf{e}_h^{\mathbf{E}}$ in (46), and $\mathbf{C}_h = \mathbf{e}_h^{\mathbf{B}}$ in (47). This choice of \mathbf{C}_h is admissible since $\text{div} \mathbf{e}_h^{\mathbf{B}} = 0$ for every $t \geq 0$ thanks to Remark 5.4 and the fact that \mathbf{B}_{II} has zero divergence by definition. We further observe $b(\mathbf{e}_h^{\mathbf{u}}, p_h) = 0$ since

$$\mathbf{e}_h^{\mathbf{u}} \in \mathbf{Z}_h := \{\mathbf{v}_h \in \mathbf{W}_h \mid (\text{div} \mathbf{v}_h, q_h) = 0 \quad \forall q_h \in Q_h\};$$

see also (16). Equations (45), (46), and (47) thus become

$$\begin{aligned} & m_h(\mathbf{e}_{h,t}^{\mathbf{u}}, \mathbf{e}_h^{\mathbf{u}}) + a_h(\mathbf{e}_h^{\mathbf{u}}, \mathbf{e}_h^{\mathbf{u}}) + [\mathbf{e}_h^{\mathbf{E}} + \chi_h(\mathbf{e}_h^{\mathbf{u}}, \mathbf{B}_h), \chi_h(\mathbf{e}_h^{\mathbf{u}}, \mathbf{B}_h)]_{\text{edge}} \\ & \quad = (\mathbf{f}, \Pi_1^0 \mathbf{e}_h^{\mathbf{u}}) - m_h(\mathbf{u}_{I,t}, \mathbf{e}_h^{\mathbf{u}}) - a_h(\mathbf{u}_I, \mathbf{e}_h^{\mathbf{u}}) - \tilde{c}_h(\mathbf{u}_h; \mathbf{u}_h, \mathbf{e}_h^{\mathbf{u}}) \\ & \quad \quad - [\mathbf{E}_{II} + \chi_h(\mathbf{u}_I, \mathbf{B}_h), \chi_h(\mathbf{e}_h^{\mathbf{u}}, \mathbf{B}_h)]_{\text{edge}}; \end{aligned} \quad (48)$$

$$\begin{aligned} & [\mathbf{e}_h^{\mathbf{E}} + \chi_h(\mathbf{e}_h^{\mathbf{u}}, \mathbf{B}_h), \mathbf{e}_h^{\mathbf{E}}]_{\text{edge}} - [\mathbf{e}_h^{\mathbf{B}}, \mathbf{curl} \mathbf{e}_h^{\mathbf{E}}]_{\text{face}} = -[(\mathbf{E}_{II} + \chi_h(\mathbf{u}_I, \mathbf{B}_h)), \mathbf{e}_h^{\mathbf{E}}]_{\text{edge}} \\ & \quad \quad + [\mathbf{B}_{II}, \mathbf{curl} \mathbf{e}_h^{\mathbf{E}}]_{\text{face}}; \end{aligned} \quad (49)$$

$$[\mathbf{e}_{h,t}^{\mathbf{B}}, \mathbf{e}_h^{\mathbf{B}}]_{\text{face}} + [\mathbf{curl} \mathbf{e}_h^{\mathbf{E}}, \mathbf{e}_h^{\mathbf{B}}]_{\text{face}} = 0. \quad (50)$$

We add (50) to (49), so that

$$\begin{aligned} [\mathbf{e}_h^{\mathbf{E}} + \chi_h(\mathbf{e}_h^{\mathbf{u}}, \mathbf{B}_h), \mathbf{e}_h^{\mathbf{E}}]_{\text{edge}} + [\mathbf{e}_{h,t}^{\mathbf{B}}, \mathbf{e}_h^{\mathbf{B}}]_{\text{face}} &= -[(\mathbf{E}_{II} + \chi_h(\mathbf{u}_I, \mathbf{B}_h)), \mathbf{e}_h^{\mathbf{E}}]_{\text{edge}} \\ &\quad + [\mathbf{B}_{II}, \mathbf{curl} \mathbf{e}_h^{\mathbf{E}}]_{\text{face}}. \end{aligned}$$

Then, we add this equation to (48) and write the resulting equation as

$$\begin{aligned} \text{LHS} &:= m_h(\mathbf{e}_{h,t}^{\mathbf{u}}, \mathbf{e}_h^{\mathbf{u}}) + a_h(\mathbf{e}_h^{\mathbf{u}}, \mathbf{e}_h^{\mathbf{u}}) + [\mathbf{e}_{h,t}^{\mathbf{B}}, \mathbf{e}_h^{\mathbf{B}}]_{\text{face}} + [\mathbf{e}_h^{\mathbf{E}} + \chi_h(\mathbf{e}_h^{\mathbf{u}}, \mathbf{B}_h), \mathbf{e}_h^{\mathbf{E}} + \chi_h(\mathbf{e}_h^{\mathbf{u}}, \mathbf{B}_h)]_{\text{edge}} \\ &= (\mathbf{f}, \Pi_1^0 \mathbf{e}_h^{\mathbf{u}}) - m_h(\mathbf{u}_{I,t}, \mathbf{e}_h^{\mathbf{u}}) - a_h(\mathbf{u}_I, \mathbf{e}_h^{\mathbf{u}}) - \tilde{c}_h(\mathbf{u}_h; \mathbf{u}_h, \mathbf{e}_h^{\mathbf{u}}) \\ &\quad - [\mathbf{E}_{II} + \chi_h(\mathbf{u}_I, \mathbf{B}_h), \chi_h(\mathbf{e}_h^{\mathbf{u}}, \mathbf{B}_h)]_{\text{edge}} \\ &\quad - [(\mathbf{E}_{II} + \chi_h(\mathbf{u}_I, \mathbf{B}_h), \mathbf{e}_h^{\mathbf{E}})]_{\text{edge}} + [\mathbf{B}_{II}, \mathbf{curl} \mathbf{e}_h^{\mathbf{E}}]_{\text{face}} =: \text{RHS}. \end{aligned} \quad (51)$$

We reformulate LHS and RHS in (51) as

$$\begin{aligned} \text{LHS} &= \frac{1}{2} \frac{d}{dt} m_h(\mathbf{e}_h^{\mathbf{u}}, \mathbf{e}_h^{\mathbf{u}}) + \frac{1}{2} \frac{d}{dt} [\mathbf{e}_{h,t}^{\mathbf{B}}, \mathbf{e}_h^{\mathbf{B}}]_{\text{face}} + a_h(\mathbf{e}_h^{\mathbf{u}}, \mathbf{e}_h^{\mathbf{u}}) + [\mathbf{e}_h^{\mathbf{j}}, \mathbf{e}_h^{\mathbf{j}}]_{\text{edge}}, \\ \text{RHS} &= (\mathbf{f}, \Pi_1^0 \mathbf{e}_h^{\mathbf{u}}) - m_h(\mathbf{u}_{I,t}, \mathbf{e}_h^{\mathbf{u}}) - a_h(\mathbf{u}_I, \mathbf{e}_h^{\mathbf{u}}) - \tilde{c}_h(\mathbf{u}_h; \mathbf{u}_h, \mathbf{e}_h^{\mathbf{u}}) - [\mathbf{j}_I, \mathbf{e}_h^{\mathbf{j}}]_{\text{edge}} \\ &\quad + [\mathbf{B}_{II}, \mathbf{curl} \mathbf{e}_h^{\mathbf{E}}]_{\text{face}}. \end{aligned} \quad (52)$$

Next, we observe that

- equation (9a) with $\mathbf{v} = \mathbf{e}_h^{\mathbf{u}} \in [H_0^1(\Omega)]^3$ can be rewritten as

$$(\mathbf{f}, \mathbf{e}_h^{\mathbf{u}}) = (\mathbf{u}_t, \mathbf{e}_h^{\mathbf{u}}) + a(\mathbf{u}, \mathbf{e}_h^{\mathbf{u}}) + \tilde{c}(\mathbf{u}; \mathbf{u}, \mathbf{e}_h^{\mathbf{u}}) + (\mathbf{j}, \mathbf{e}_h^{\mathbf{u}} \times \mathbf{B}), \quad (53)$$

since $b(\mathbf{e}_h^{\mathbf{u}}, p) = 0$ as $\text{div} \mathbf{e}_h^{\mathbf{u}} = 0$ and $\tilde{c}(\mathbf{u}; \mathbf{u}, \mathbf{e}_h^{\mathbf{u}}) = c(\mathbf{u}; \mathbf{u}, \mathbf{e}_h^{\mathbf{u}})$ since $\text{div} \mathbf{u} = 0$ and $\mathbf{u} \in [H_0^1(\Omega)]$;

- using the definition of \mathbf{B}_{II} in (32), and (9b) with $\mathbf{F} = \mathbf{e}_h^{\mathbf{E}} \in H_0(\mathbf{curl}, \Omega)$ yields

$$[\mathbf{B}_{II}, \mathbf{curl} \mathbf{e}_h^{\mathbf{E}}]_{\text{face}} = (\mathbf{B}, \mathbf{curl} \mathbf{e}_h^{\mathbf{E}}) = (\mathbf{j}, \mathbf{e}_h^{\mathbf{E}}). \quad (54)$$

We substitute (53) and (54) in RHS defined in (52), add and subtract $(\mathbf{j}, \mathbf{e}_h^{\mathbf{u}} \times \mathbf{B}_h)$, use (39), and finally obtain

$$\begin{aligned} \text{RHS} &:= (\mathbf{f}, \Pi_1^0 \mathbf{e}_h^{\mathbf{u}}) - m_h(\mathbf{u}_{I,t}, \mathbf{e}_h^{\mathbf{u}}) - a_h(\mathbf{u}_I, \mathbf{e}_h^{\mathbf{u}}) - \tilde{c}_h(\mathbf{u}_h; \mathbf{u}_h, \mathbf{e}_h^{\mathbf{u}}) - [\mathbf{j}_I, \mathbf{e}_h^{\mathbf{j}}]_{\text{edge}} + (\mathbf{j}, \mathbf{e}_h^{\mathbf{E}}) \\ &= [(\mathbf{u}_t, \mathbf{e}_h^{\mathbf{u}}) - m_h(\mathbf{u}_{I,t}, \mathbf{e}_h^{\mathbf{u}})] + [a(\mathbf{u}, \mathbf{e}_h^{\mathbf{u}}) - a_h(\mathbf{u}_I, \mathbf{e}_h^{\mathbf{u}})] + [\tilde{c}(\mathbf{u}; \mathbf{u}, \mathbf{e}_h^{\mathbf{u}}) - \tilde{c}_h(\mathbf{u}_h; \mathbf{u}_h, \mathbf{e}_h^{\mathbf{u}})] \\ &\quad + (\mathbf{j}, \mathbf{e}_h^{\mathbf{u}} \times \mathbf{B}) - [\mathbf{j}_I, \mathbf{e}_h^{\mathbf{j}}]_{\text{edge}} + (\mathbf{j}, \mathbf{e}_h^{\mathbf{E}}) + (\mathbf{f}, \Pi_1^0 \mathbf{e}_h^{\mathbf{u}}) - (\mathbf{f}, \mathbf{e}_h^{\mathbf{u}}) \\ &= [(\mathbf{u}_t, \mathbf{e}_h^{\mathbf{u}}) - m_h(\mathbf{u}_{I,t}, \mathbf{e}_h^{\mathbf{u}})] + [\tilde{c}(\mathbf{u}; \mathbf{u}, \mathbf{e}_h^{\mathbf{u}}) - \tilde{c}_h(\mathbf{u}_h; \mathbf{u}_h, \mathbf{e}_h^{\mathbf{u}})] + (\mathbf{j}, \mathbf{e}_h^{\mathbf{E}} + \mathbf{e}_h^{\mathbf{u}} \times \mathbf{B}) \\ &\quad - [\mathbf{j}_I, \mathbf{e}_h^{\mathbf{j}}]_{\text{edge}} + [a(\mathbf{u}, \mathbf{e}_h^{\mathbf{u}}) - a_h(\mathbf{u}_I, \mathbf{e}_h^{\mathbf{u}})] + [(\mathbf{f}, \Pi_1^0 \mathbf{e}_h^{\mathbf{u}}) - (\mathbf{f}, \mathbf{e}_h^{\mathbf{u}})] \\ &= [(\mathbf{u}_t, \mathbf{e}_h^{\mathbf{u}}) - m_h(\mathbf{u}_{I,t}, \mathbf{e}_h^{\mathbf{u}})] + [\tilde{c}(\mathbf{u}; \mathbf{u}, \mathbf{e}_h^{\mathbf{u}}) - \tilde{c}_h(\mathbf{u}_h; \mathbf{u}_h, \mathbf{e}_h^{\mathbf{u}})] + [(\mathbf{j}, \mathbf{e}_h^{\mathbf{u}} \times (\mathbf{B} - \mathbf{B}_h))] \\ &\quad + [(\mathbf{j}, \mathbf{e}_h^{\mathbf{E}} + \mathbf{e}_h^{\mathbf{u}} \times \mathbf{B}_h) - [\mathbf{j}_I, \mathbf{e}_h^{\mathbf{j}}]_{\text{edge}}] + [a(\mathbf{u}, \mathbf{e}_h^{\mathbf{u}}) - a_h(\mathbf{u}_I, \mathbf{e}_h^{\mathbf{u}})] + [(\mathbf{f}, \Pi_1^0 \mathbf{e}_h^{\mathbf{u}}) - (\mathbf{f}, \mathbf{e}_h^{\mathbf{u}})] \\ &=: \text{T}_1 + \text{T}_2 + \text{T}_3 + \text{T}_4 + \text{T}_5 + \text{T}_6. \end{aligned} \quad (55)$$

We estimate the six terms T_j , $j = 1, \dots, 6$, separately, and we eventually apply Grönwall's inequality to derive the assertion of the theorem.

Estimate of \mathbf{T}_1 We split \mathbf{T}_1 into a summation on the elemental contributions; note that $\mathbf{u}_{I,t} = \mathbf{u}_{t,I}$; use the consistency property (14) with the piecewise linear discontinuous polynomial approximation $\mathbf{u}_{t,\pi}$ of \mathbf{u}_t ; use the stability property (15), the Cauchy-Schwarz inequality, and the Young inequality, and obtain:

$$\begin{aligned}
\mathbf{T}_1 &= \sum_{K \in \Omega_h} |(\mathbf{u}_t, \mathbf{e}_h^{\mathbf{u}})_K - m_h^K(\mathbf{u}_{I,t}, \mathbf{e}_h^{\mathbf{u}})| = \sum_{K \in \Omega_h} |(\mathbf{u}_t - \mathbf{u}_{t,\pi}, \mathbf{e}_h^{\mathbf{u}})_K - m_h^K(\mathbf{u}_{t,I} - \mathbf{u}_{t,\pi}, \mathbf{e}_h^{\mathbf{u}})| \\
&\leq \sum_{K \in \Omega_h} (\|\mathbf{u}_t - \mathbf{u}_{t,\pi}\|_K + \mu^* \|\mathbf{u}_{t,I} - \mathbf{u}_{t,\pi}\|_K) \|\mathbf{e}_h^{\mathbf{u}}\|_K \leq (\|\mathbf{u}_t - \mathbf{u}_{t,\pi}\| + \mu^* \|\mathbf{u}_{t,I} - \mathbf{u}_{t,\pi}\|) \|\mathbf{e}_h^{\mathbf{u}}\| \\
&\leq ((1 + \mu^*) \|\mathbf{u}_t - \mathbf{u}_{t,\pi}\| + \mu^* \|\mathbf{u}_t - \mathbf{u}_{t,I}\|) \|\mathbf{e}_h^{\mathbf{u}}\| \\
&\leq \frac{1}{2\varepsilon} \left((1 + \mu^*)^2 \|\mathbf{u}_t - \mathbf{u}_{t,\pi}\|^2 + (\mu^*)^2 \|\mathbf{u}_t - \mathbf{u}_{t,I}\|^2 \right) + \varepsilon \|\mathbf{e}_h^{\mathbf{u}}\|^2 \\
&\leq h^2 C_{\mathbf{T}_1}(\varepsilon) + \varepsilon \|\mathbf{e}_h^{\mathbf{u}}\|^2,
\end{aligned}$$

with $C_{\mathbf{T}_1}(\varepsilon) := (C_\pi^2(1 + \mu^*)^2 + C_I^2(\mu^*)^2) |\mathbf{u}_t|_1^2 / 2\varepsilon$.

Estimate of \mathbf{T}_2 We add and subtract $\tilde{c}_h(\mathbf{u}; \mathbf{u}, \mathbf{e}_h^{\mathbf{u}})$, use the triangle inequality, and write

$$\begin{aligned}
\mathbf{T}_2 &= |\tilde{c}(\mathbf{u}; \mathbf{u}, \mathbf{e}_h^{\mathbf{u}}) - \tilde{c}_h(\mathbf{u}_h; \mathbf{u}_h, \mathbf{e}_h^{\mathbf{u}})| \leq |\tilde{c}(\mathbf{u}; \mathbf{u}, \mathbf{e}_h^{\mathbf{u}}) - \tilde{c}_h(\mathbf{u}; \mathbf{u}, \mathbf{e}_h^{\mathbf{u}})| + |\tilde{c}_h(\mathbf{u}; \mathbf{u}, \mathbf{e}_h^{\mathbf{u}}) - \tilde{c}_h(\mathbf{u}_h; \mathbf{u}_h, \mathbf{e}_h^{\mathbf{u}})| \\
&=: \mathbf{T}_{2,1} + \mathbf{T}_{2,2}.
\end{aligned}$$

We control $\mathbf{T}_{2,1}$ by using Lemma 4.5 and the Young inequality:

$$\mathbf{T}_{2,1} \leq C_C h \|\mathbf{u}\|_2^2 |\mathbf{e}_h^{\mathbf{u}}|_1 \leq \frac{C_C^2}{4\varepsilon} h^2 \|\mathbf{u}\|_2^4 + \varepsilon |\mathbf{e}_h^{\mathbf{u}}|_1^2,$$

where the constant C_C is independent of h . Similarly, we control $\mathbf{T}_{2,2}$ using Lemma 4.6:

$$\mathbf{T}_{2,2} \leq \varepsilon |\mathbf{e}_h^{\mathbf{u}}|_1^2 + \left(1 + \frac{1}{\varepsilon}\right) C_1 R_1(\mathbf{u}) \|\mathbf{e}_h^{\mathbf{u}}\|^2 + \left(1 + \frac{1}{\varepsilon}\right) C_2 R_2(\mathbf{u}) h^2,$$

where the constants C_1 and C_2 are independent of h . We end up with

$$\mathbf{T}_2 \leq h^2 C_{\mathbf{T}_2}(\varepsilon) + \left(1 + \frac{1}{\varepsilon}\right) C_1 R_1(\mathbf{u}) \|\mathbf{e}_h^{\mathbf{u}}\|^2 + 2\varepsilon |\mathbf{e}_h^{\mathbf{u}}|_1^2,$$

where

$$C_{\mathbf{T}_2}(\varepsilon) := \frac{C_C^2}{4\varepsilon} \|\mathbf{u}\|_2^4 + \left(1 + \frac{1}{\varepsilon}\right) C_2 R_2(\mathbf{u}).$$

Estimate of \mathbf{T}_3 By adding and subtracting \mathbf{B}_H defined in (32), we obtain

$$\mathbf{T}_3 = (\mathbf{j}, \mathbf{e}_h^{\mathbf{u}} \times (\mathbf{B} - \mathbf{B}_h)) = (\mathbf{j}, \mathbf{e}_h^{\mathbf{u}} \times (\mathbf{B} - \mathbf{B}_H)) + (\mathbf{j}, \mathbf{e}_h^{\mathbf{u}} \times (\mathbf{B}_H - \mathbf{B}_h)) =: \mathbf{T}_{3,1} + \mathbf{T}_{3,2}.$$

We estimate the two terms $\mathbf{T}_{3,1}$ and $\mathbf{T}_{3,2}$ separately. Notably, we write

$$\begin{aligned}
|\mathbf{T}_{3,1}| &\leq \|\mathbf{j}\|_{L^\infty} \|\mathbf{e}_h^{\mathbf{u}}\| \|\mathbf{B} - \mathbf{B}_H\| \leq \frac{\|\mathbf{j}\|_{L^\infty}}{2} \left(\|\mathbf{e}_h^{\mathbf{u}}\|^2 + \|\mathbf{B} - \mathbf{B}_H\|^2 \right) \\
&\leq \frac{\|\mathbf{j}\|_{L^\infty}}{2} \left(\|\mathbf{e}_h^{\mathbf{u}}\|^2 + C_H^2 h^2 |\mathbf{B}|_1^2 \right)
\end{aligned}$$

and

$$|\mathbf{T}_{3,2}| \leq \|\mathbf{j}\|_{L^\infty} \|\mathbf{e}_h^{\mathbf{u}}\| \|\mathbf{B}_H - \mathbf{B}_h\| \leq \|\mathbf{j}\|_{L^\infty} \|\mathbf{e}_h^{\mathbf{u}}\| \|\mathbf{e}_h^{\mathbf{B}}\| \leq \frac{1}{2} \|\mathbf{j}\|_{L^\infty} (\|\mathbf{e}_h^{\mathbf{u}}\|^2 + \|\mathbf{e}_h^{\mathbf{B}}\|^2).$$

Collecting the two above estimates, we find that

$$\mathbf{T}_3 \leq h^2 C_{\mathbf{T}_3} + \|\mathbf{j}\|_{L^\infty} \|\mathbf{e}_h^{\mathbf{u}}\|^2 + \frac{1}{2} \|\mathbf{j}\|_{L^\infty} \|\mathbf{e}_h^{\mathbf{B}}\|^2,$$

where

$$C_{\mathbf{T}_3} := \frac{C_H^2}{2} \|\mathbf{j}\|_{L^\infty} |\mathbf{B}|_1^2.$$

Estimate of \mathbf{T}_4 Recalling that $\mathbf{e}_h^{\mathbf{j}} = \mathbf{e}_h^{\mathbf{E}} + \mathbf{e}_h^{\mathbf{u}} \times \mathbf{B}_h$, we use (by now) standard manipulations based on the consistency and stability properties (26) and (27), and get

$$\begin{aligned}
\mathbf{T}_4 &= (\mathbf{j}, \mathbf{e}_h^{\mathbf{j}}) - [\mathbf{j}_I, \mathbf{e}_h^{\mathbf{j}}]_{\text{edge}} = \sum_{K \in \Omega_h} \left((\mathbf{j}, \mathbf{e}_h^{\mathbf{j}})_K - [\mathbf{j}_I, \mathbf{e}_h^{\mathbf{j}}]_{\text{edge}, K} \right) \\
&= \sum_{K \in \Omega_h} \left((\mathbf{j} - \mathbf{j}_\pi, \mathbf{e}_h^{\mathbf{j}})_K - [\mathbf{j}_I - \mathbf{j}_\pi, \mathbf{e}_h^{\mathbf{j}}]_{\text{edge}, K} \right) \leq \sum_{K \in \Omega_h} \left(\|\mathbf{j} - \mathbf{j}_\pi\|_{0,K} + \eta^* \|\mathbf{j}_I - \mathbf{j}_\pi\|_{0,K} \right) \|\mathbf{e}_h^{\mathbf{j}}\|_{0,K} \\
&\leq \sum_{K \in \Omega_h} \left((1 + \eta^*) \|\mathbf{j} - \mathbf{j}_\pi\|_{0,K} + \eta^* \|\mathbf{j} - \mathbf{j}_I\|_{0,K} \right) \|\mathbf{e}_h^{\mathbf{j}}\|_{0,K} \leq \left((1 + \eta^*) \|\mathbf{j} - \mathbf{j}_\pi\| + \eta^* \|\mathbf{j} - \mathbf{j}_I\| \right) \|\mathbf{e}_h^{\mathbf{j}}\| \\
&\leq \frac{1}{2\varepsilon} \left((1 + \eta^*)^2 \|\mathbf{j} - \mathbf{j}_\pi\|^2 + (\eta^*)^2 \|\mathbf{j} - \mathbf{j}_I\|^2 \right) + \varepsilon \|\mathbf{e}_h^{\mathbf{j}}\|^2 \\
&\leq \frac{(1 + \eta^*)^2 C_{\text{appr}}^2}{2\varepsilon} h^2 \|\mathbf{j}\|_1^2 + \frac{(\eta^*)^2}{2\varepsilon} \|\mathbf{j} - \mathbf{j}_I\|^2 + \varepsilon \|\mathbf{e}_h^{\mathbf{j}}\|^2. \tag{56}
\end{aligned}$$

Above, \mathbf{j}_π stands for the piecewise constant best approximant of \mathbf{j} .

Then, we transform $\|\mathbf{j} - \mathbf{j}_I\|$ by using the triangle inequality and adding and subtracting $\mathbf{\Pi}_h^{0,K} \mathbf{u}_I$ and $\mathbf{\Pi}_h^{0,K} \mathbf{B}$:

$$\begin{aligned}
\|\mathbf{j} - \mathbf{j}_I\| &= \|\mathbf{E} + \mathbf{u} \times \mathbf{B} - (\mathbf{E}_I + \mathbf{\Pi}_h^{0,K} \mathbf{u}_I \times \mathbf{\Pi}_h^{0,K} \mathbf{B}_h)\| \\
&\leq \|\mathbf{E} - \mathbf{E}_I\| + \|\mathbf{u} \times \mathbf{B} - \mathbf{\Pi}_h^{0,K} \mathbf{u}_I \times \mathbf{\Pi}_h^{0,K} \mathbf{B}_h\| \\
&\leq C_I h |\mathbf{E}| + \|(\mathbf{u} - \mathbf{\Pi}_h^{0,K} \mathbf{u}_I) \times \mathbf{B}\| + \|\mathbf{\Pi}_h^{0,K} \mathbf{u}_I \times (\mathbf{B} - \mathbf{\Pi}_h^{0,K} \mathbf{B}_h)\| \\
&\quad + \|\mathbf{\Pi}_h^{0,K} \mathbf{u}_I \times (\mathbf{\Pi}_h^{0,K} \mathbf{B} - \mathbf{\Pi}_h^{0,K} \mathbf{B}_h)\| \\
&=: C_I h |\mathbf{E}| + \mathbf{T}_{4,1} + \mathbf{T}_{4,2} + \mathbf{T}_{4,3}. \tag{57}
\end{aligned}$$

Next, we estimate separately the three terms $\mathbf{T}_{4,1}$, $\mathbf{T}_{4,2}$, and $\mathbf{T}_{4,3}$. We control $\mathbf{T}_{4,1}$ by adding and subtracting $\mathbf{\Pi}_h^{0,K} \mathbf{u}$, and using the triangle inequality:

$$\begin{aligned}
\mathbf{T}_{4,1} &= \|(\mathbf{u} - \mathbf{\Pi}_h^{0,K} \mathbf{u}_I) \times \mathbf{B}\| \leq \|\mathbf{B}\|_{L^\infty} \|\mathbf{u} - \mathbf{\Pi}_h^{0,K} \mathbf{u}_I\| \\
&\leq \|\mathbf{B}\|_{L^\infty} \left(\|\mathbf{u} - \mathbf{\Pi}_h^{0,K} \mathbf{u}\| + \|\mathbf{\Pi}_h^{0,K} (\mathbf{u} - \mathbf{u}_I)\| \right) \leq \|\mathbf{B}\|_{L^\infty} h (C_{\text{appr}} + C_I) \|\mathbf{u}\|_1.
\end{aligned}$$

We control $\mathbf{T}_{4,2}$ by using Lemma 4.3:

$$\begin{aligned}
\mathbf{T}_{4,2} &= \|\mathbf{\Pi}_h^{0,K} \mathbf{u}_I \times (\mathbf{B} - \mathbf{\Pi}_h^{0,K} \mathbf{B}_h)\| \leq \|\mathbf{\Pi}_h^{0,K} \mathbf{u}_I\|_{L^\infty} \|\mathbf{B} - \mathbf{\Pi}_h^{0,K} \mathbf{B}_h\| \\
&\leq (C_{\text{inv}} C_I h_K^{\frac{1}{2}} + C_{\text{inv}} C_D C_S) \|\mathbf{u}\|_2 h C_{\text{appr}} |\mathbf{B}|_1.
\end{aligned}$$

We control $\mathbf{T}_{4,3}$ by adding and subtracting \mathbf{B}_I defined in (32) and using again Lemma 4.3:

$$\begin{aligned}
\mathbf{T}_{4,3} &= \|\mathbf{\Pi}_h^{0,K} \mathbf{u}_I \times (\mathbf{\Pi}_h^{0,K} (\mathbf{B} - \mathbf{B}_h))\| \leq \|\mathbf{\Pi}_h^{0,K} \mathbf{u}_I\|_{L^\infty} \|\mathbf{B} - \mathbf{B}_h\| \\
&\leq \|\mathbf{\Pi}_h^{0,K} \mathbf{u}_I\|_{L^\infty} \left(\|\mathbf{B} - \mathbf{B}_I\| + \|\mathbf{B}_I - \mathbf{B}_h\| \right) \\
&\leq (C_{\text{inv}} C_I h_K^{\frac{1}{2}} + C_{\text{inv}} C_D C_S) \|\mathbf{u}\|_2 (h C_I |\mathbf{B}|_1 + \|\mathbf{e}_h^{\mathbf{B}}\|).
\end{aligned}$$

We collect the three above estimates in (57):

$$\begin{aligned}
&\|\mathbf{j} - \mathbf{j}_I\| \\
&\leq h \left(C_I |\mathbf{E}|_1 + (C_{\text{appr}} + C_I) \|\mathbf{B}\|_{L^\infty} \|\mathbf{u}\|_1 + (C_{\text{inv}} C_I h_K^{\frac{1}{2}} + C_{\text{inv}} C_D C_S) (C_{\text{appr}} + C_I) \|\mathbf{u}\|_2 |\mathbf{B}|_1 \right) \\
&\quad + (C_{\text{inv}} C_I h_K^{\frac{1}{2}} + C_{\text{inv}} C_D C_S) \|\mathbf{u}\|_2 \|\mathbf{e}_h^{\mathbf{B}}\|,
\end{aligned}$$

and this inequality in (56), and obtain the following upper bound for T_4 :

$$\mathsf{T}_4 \leq h^2 C_{\mathsf{T}_4}(\varepsilon) + \frac{(\eta^*)^2}{\varepsilon} (C_{\text{inv}} C_I h_K^{\frac{1}{2}} + C_{\text{inv}} C_D C_S)^2 \|\mathbf{u}\|_2^2 \|\mathbf{e}_h^{\mathbf{B}}\|^2 + \varepsilon \|\mathbf{e}_h^{\mathbf{j}}\|^2,$$

where $C_{\mathsf{T}_4}(\varepsilon) \approx \varepsilon^{-1}$ collects all the constants above and depends also on $\|\mathbf{B}\|$, $\|\mathbf{j}\|_1$, $\|\mathbf{E}\|_1$, $\|\mathbf{B}\|_{L^\infty}$, and $\|\mathbf{u}\|_2$.

Estimate of T_5 We proceed with (by now) standard techniques based on the consistency and stability properties (11) and (12): given \mathbf{u}_π the piecewise linear best approximant of \mathbf{u} ,

$$\begin{aligned} \mathsf{T}_5 &= \sum_{K \in \Omega_h} [a^K(\mathbf{u}, \mathbf{e}_h^{\mathbf{u}}) - a_h^K(\mathbf{u}_I, \mathbf{e}_h^{\mathbf{u}})] = \sum_{K \in \Omega_h} [a^K(\mathbf{u} - \mathbf{u}_\pi, \mathbf{e}_h^{\mathbf{u}}) - a_h^K(\mathbf{u}_I - \mathbf{u}_\pi, \mathbf{e}_h^{\mathbf{u}})] \\ &\leq \sum_{K \in \Omega_h} [|\mathbf{u} - \mathbf{u}_\pi|_{1,K} + \sigma^* (|\mathbf{u} - \mathbf{u}_I|_{1,K} + |\mathbf{u} - \mathbf{u}_\pi|_{1,K})] |\mathbf{e}_h^{\mathbf{u}}|_{1,K} \\ &\leq h ((1 + \sigma^*) C_{\text{appr}} + \sigma^* C_I) \|\mathbf{u}\|_2 |\mathbf{e}_h^{\mathbf{u}}|_1 \leq h^2 C_{\mathsf{T}_5}(\varepsilon) + \varepsilon |\mathbf{e}_h^{\mathbf{u}}|_1^2, \end{aligned}$$

where

$$C_{\mathsf{T}_5}(\varepsilon) := \frac{1}{4\varepsilon} ((1 + \sigma^*) C_{\text{appr}} + \sigma^* C_I)^2 \|\mathbf{u}\|_1^2.$$

Estimate of T_6 Standard manipulations based on the Poincaré and the Young inequalities yield

$$\begin{aligned} \mathsf{T}_6 &= (\mathbf{f}, \Pi_1^0 \mathbf{e}_h^{\mathbf{u}}) - (\mathbf{f}, \mathbf{e}_h^{\mathbf{u}}) = (\mathbf{f} - \Pi_1^0 \mathbf{f}, \Pi_1^0 \mathbf{e}_h^{\mathbf{u}} - \mathbf{e}_h^{\mathbf{u}}) \\ &\leq \|\mathbf{f} - \Pi_1^0 \mathbf{f}\| \|\mathbf{e}_h^{\mathbf{u}} - \Pi_1^0 \mathbf{e}_h^{\mathbf{u}}\| \leq C_{\text{appr}}^2 h^2 \|\mathbf{f}\|_1 |\mathbf{e}_h^{\mathbf{u}}|_1 \leq h^2 C_{\mathsf{T}_6} + \varepsilon |\mathbf{e}_h^{\mathbf{u}}|_1^2, \end{aligned}$$

where

$$C_{\mathsf{T}_6} := \frac{1}{4} h^2 C_{\text{appr}}^2 \|\mathbf{f}\|_1^2.$$

Use of the Grönwall's inequality and final step Combining the above bounds on the term T_i , $i = 1, \dots, 6$, we get

$$\sum_{i=1}^6 \mathsf{T}_i \leq S_1(\varepsilon) h^2 + S_2(\varepsilon) \|\mathbf{e}_h^{\mathbf{u}}\|^2 + 5\varepsilon |\mathbf{e}_h^{\mathbf{u}}|_1^2 + S_3(\varepsilon) \|\mathbf{e}_h^{\mathbf{B}}\|^2 + \varepsilon \|\mathbf{e}_h^{\mathbf{j}}\|^2,$$

where

$$S_1(\varepsilon) = \sum_{j=1}^6 C_{\mathsf{T}_j}(\varepsilon), \quad S_3(\varepsilon) = \frac{1}{2} \|\mathbf{j}\|_{L^\infty} + \frac{(\eta^*)^2}{\varepsilon} (C_{\text{inv}} C_I h_K^{\frac{1}{2}} + C_{\text{inv}} C_D C_S)^2 \|\mathbf{u}\|_2^2,$$

$$S_2(\varepsilon) = \varepsilon + (1 + 1/\varepsilon) C_1 R_1(\mathbf{u}) + \|\mathbf{j}\|_{L^\infty}.$$

The terms $S_j(\varepsilon)$, $j = 1, 2, 3$, depend on the time instant t , since they depend on \mathbf{u} , \mathbf{B} , and \mathbf{j} . For the presentation's sake, we do not express this dependence explicitly in our notation.

We substitute the above bound in (55) and the resulting inequality in (51). Next, we pick

$$\varepsilon = \tilde{\varepsilon} = \min(\alpha_*^{-1}, \eta_*^{-1})/10$$

and, at every time instant $t \in (0, T]$, we obtain

$$\begin{aligned} \text{LHS} &\leq 2 [S_1(\tilde{\varepsilon}) h^2 + S_2(\tilde{\varepsilon}) \|\mathbf{e}_h^{\mathbf{u}}\|^2 + S_3(\tilde{\varepsilon}) \|\mathbf{e}_h^{\mathbf{B}}\|^2] \\ &\leq 2 [S_1(\tilde{\varepsilon}) h^2 + \mu_*^{-1} S_2(\tilde{\varepsilon}) m_h(\mathbf{e}_h^{\mathbf{u}}, \mathbf{e}_h^{\mathbf{u}}) + \chi_*^{-1} S_3(\tilde{\varepsilon}) [\mathbf{e}_h^{\mathbf{B}}, \mathbf{e}_h^{\mathbf{B}}]_{\text{face}}], \end{aligned}$$

where μ_* and χ_* are the stability constants defined in (15) and (30), respectively.

By integrating in time both sides between 0 and t , we find that

$$\begin{aligned} & \| \mathbf{e}_h^{\mathbf{u}}(t) \|_{m_h}^2 + \| \mathbf{e}_h^{\mathbf{B}}(t) \|_{\mathbf{V}_h^{\text{face}}}^2 + \int_0^t \| \mathbf{e}_h^{\mathbf{u}}(s) \|_{\mathbf{W}_h}^2 + \int_0^t \| \mathbf{e}_h^{\mathbf{j}}(s) \|_{\mathbf{V}_h^{\text{edge}}}^2 \\ & \leq \| \mathbf{e}_h^{\mathbf{u}}(0) \|_{m_h}^2 + \| \mathbf{e}_h^{\mathbf{B}}(0) \|_{\mathbf{V}_h^{\text{face}}}^2 + h^2 \int_0^t \mathcal{G}_1(s) + \int_0^t \mathcal{G}_2(s) \left(\| \mathbf{e}_h^{\mathbf{u}}(s) \|_{m_h}^2 + \| \mathbf{e}_h^{\mathbf{B}}(s) \|_{\mathbf{V}_h^{\text{face}}}^2 \right), \end{aligned}$$

where

$$\mathcal{G}_1(s) = 2S_1(\tilde{\varepsilon}), \quad \mathcal{G}_2(s) = 2\mu_*^{-1}S_2(\tilde{\varepsilon}) + 2\chi_*^{-1}S_3(\tilde{\varepsilon}).$$

The assertion of the theorem follows from an application of Grönwall's inequality, i.e.,

$$\begin{aligned} & \| \mathbf{e}_h^{\mathbf{u}}(t) \|_{m_h}^2 + \| \mathbf{e}_h^{\mathbf{B}}(t) \|_{\mathbf{V}_h^{\text{face}}}^2 + \int_0^t \| \mathbf{e}_h^{\mathbf{u}}(s) \|_{\mathbf{W}_h}^2 + \int_0^t \| \mathbf{e}_h^{\mathbf{j}}(s) \|_{\mathbf{V}_h^{\text{edge}}}^2 \\ & \leq C_G(t) \left(\| \mathbf{e}_h^{\mathbf{u}}(0) \|_{m_h}^2 + \| \mathbf{e}_h^{\mathbf{B}}(0) \|_{\mathbf{V}_h^{\text{face}}}^2 \right) \\ & \quad + h^2 C_G(t) \int_0^t \mathcal{G}_1(s) \quad \text{with} \quad C_G(t) = \exp \left(\int_0^t \mathcal{G}_2(s) \right), \end{aligned}$$

taking the square root on both sides of the above inequality, and using the stability properties (15), (27), and (30) on both sides. \square

Remark 6.3. *A linear error bound in the $L^2(0, T; L^2(\Omega))$ norm for the pressure variable could be derived using techniques based on the discrete inf-sup condition and a suitable error bound on the time derivatives of the velocity and magnetic field.*

7 Numerical experiments

In this section, we validate the theoretical results obtained in Theorem 6.2 and Remark 6.3 with numerical experiments on a manufactured solution problem. We also show that the discrete velocity and magnetic fields are divergence free up to the conditioning of the final system. To deal with time derivatives, we used an implicit Euler scheme. As for the treatment of the nonlinear terms, see (34a) and (34b), we employed a fixed-point strategy. In our experiments, only few fixed-point iterations were required for the convergence of the method at each time step (typically only 3 nonlinear iterations were sufficient). The resulting linear systems are always solved with a direct solver [2]. Moreover, to speed up the execution and reduce the computational time, the assembling and solving of the linear system at hand were run in parallel.

Given the computational domain $\Omega := [0, 1]^3$, $Re = Re_m = s = 1$, and the final time $T = 1$, we consider the following exact solution:

$$\begin{aligned} \mathbf{B} & := \begin{bmatrix} 4y^3 - 4z^3 - t(24y - 24z) \\ -3x^2 + 6t \\ -3y^2 + 6t \end{bmatrix}, \\ \mathbf{E} & := \begin{bmatrix} 6y \\ 12z^2 - 24t \\ 12y^2 - 24t + 6x \end{bmatrix}, \\ \mathbf{u} & := \begin{bmatrix} \sin(\pi x) \cos(\pi y) \cos(\pi z) \cos(t) \\ \cos(\pi x) \sin(\pi y) \cos(\pi z) \cos(t) \\ -2 \cos(\pi x) \cos(\pi y) \sin(\pi z) \cos(t) \end{bmatrix}, \\ p & := \left(x^2 + yz + z - \frac{13}{12} \right) \cos(t). \end{aligned}$$

We compute the right-hand side \mathbf{f} in (6a) accordingly. Observe that we are required to add a nonhomogeneous current density on the right-hand side of (6b).

Since the discrete fields \mathbf{B}_h , \mathbf{E}_h , and \mathbf{u}_h are not available in closed form, we measure approximate relative L^2 -errors involving suitable polynomial projections. On the other hand, the discrete pressure p_h is piecewise constant over Ω_h . Thence, we compute the following four error quantities:

$$\frac{\|\mathbf{u} - \Pi_1^0 \mathbf{u}_h\|}{\|\mathbf{u}\|}; \quad \frac{\|\mathbf{E} - \Pi_0^0 \mathbf{E}_h\|}{\|\mathbf{E}\|}; \quad \frac{\|\mathbf{B} - \Pi_0^0 \mathbf{B}_h\|}{\|\mathbf{B}\|}; \quad \frac{\|p - p_h\|}{\|p\|}. \quad (58)$$

For all quantities, the expected convergence rate is linear.

To verify that \mathbf{B}_h and \mathbf{u}_h are divergence free up to machine precision, we compute the L^2 norm of their divergence, i.e., $\text{div}(\mathbf{B}_h)$ and $\text{div}(\mathbf{u}_h)$, which are computable using only the degrees of freedom.

We consider three families of meshes on Ω :

- **tetra**: Delaunay tetrahedral meshes; see Figure 2(a);
- **cube**: structured meshes consisting of cubes; see Figure 2(b);
- **vor**: Voronoi tessellations optimized by the Lloyd algorithm; see Figure 2(c).

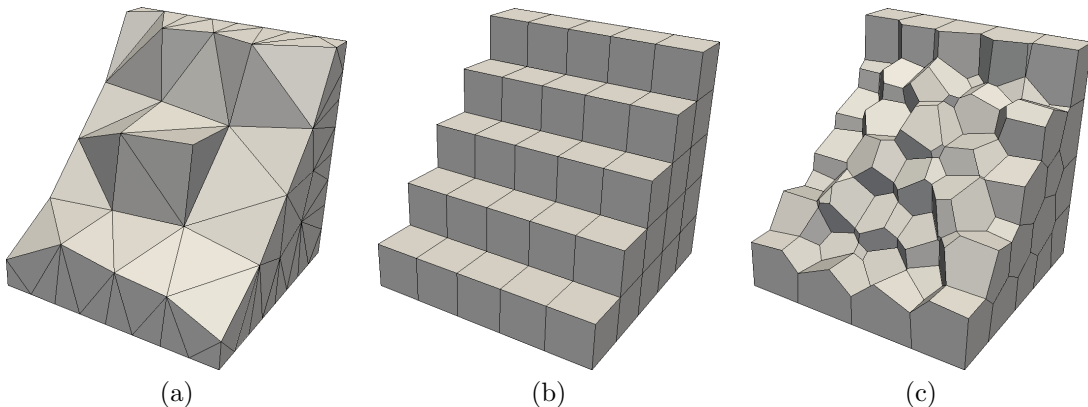


Figure 2: Three representatives of the three family of meshes decomposing the unit cube Ω . (a): Delaunay mesh; (b) cube mesh; (c) Voronoi-Lloyd mesh.

To better appreciate the linear trend of all field errors, we refine simultaneously in space and time: for each mesh family, we consider four meshes with decreasing mesh-size and use a uniform time discretization of steps $1/4$, $1/8$, $1/16$, and $1/32$.

In Figure 3, we display the errors in (58) for simultaneous space and time refinements. As expected, we observe linear convergence. Interestingly, we observe improved convergence by half an order for the velocity field when employing **cube** and **vor** meshes, and even a full order improved convergence when employing **tetra** meshes. This is in accordance with the fact that the velocity space contains \mathbb{P}_1 vector polynomials on each element and we are measuring the L^2 norm of the approximation errors. However that may be, it is an improvement over what is predicted in Theorem 6.2.

In Tables 1 and 2, we report the L^2 norm of the divergence of \mathbf{B}_h and \mathbf{u}_h , respectively, for each type of meshes and at each space/time refinement step. We refer to such refinement levels as **level 1**, **2**, **3**, and **4**, where **level 1** corresponds to the coarsest mesh and the largest time step. Throughout the refinement process, the value of the norm of the divergence of both fields remains below $1e-14$ and $1e-11$, respectively, for all choices of space and time refinement. The growth of such values corresponds to the deterioration of the condition number of the matrices appearing in the linear systems.

8 Conclusions

We constructed a virtual element method for the approximation of the solutions to the 3D time-dependent resistive magnetohydrodynamic model. The scheme guarantees “exactly” divergence-free velocity and magnetic fields and is suitable to general polyhedral meshes. The convergence

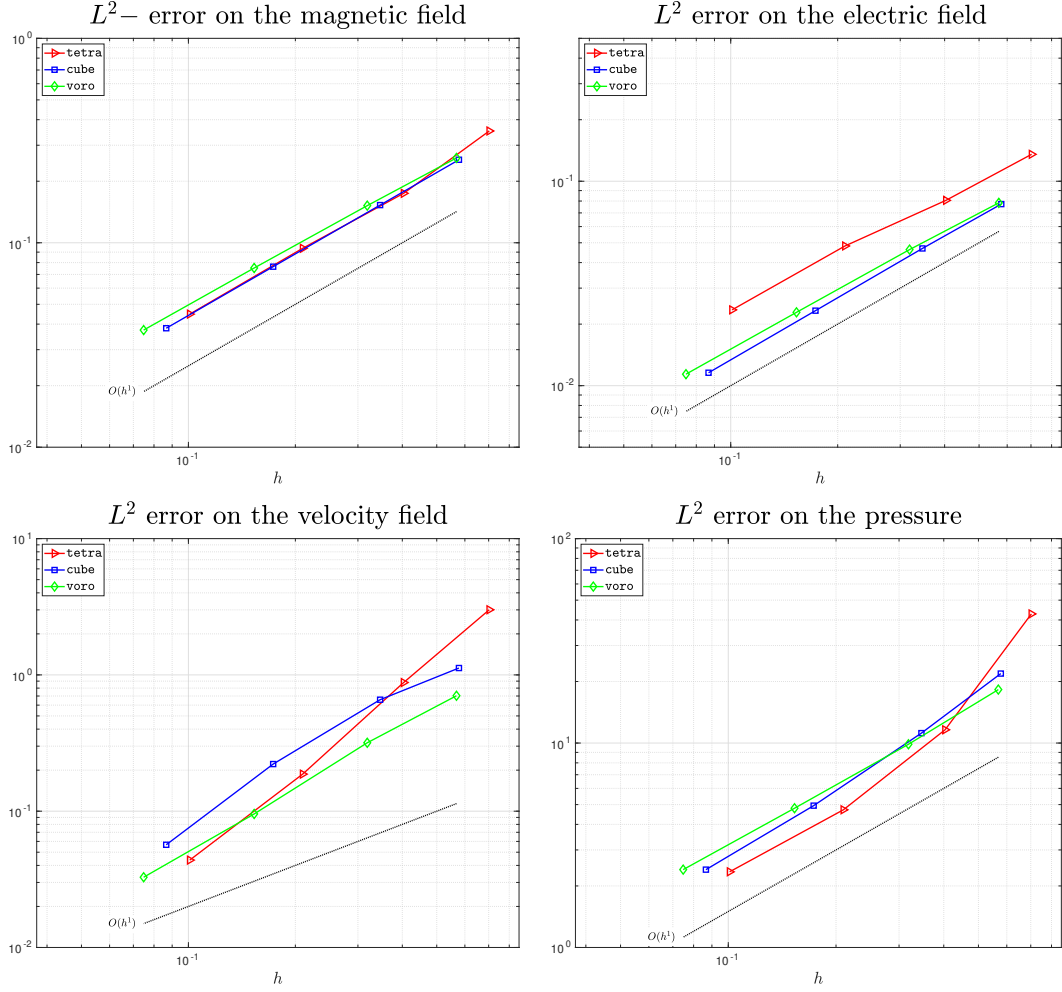


Figure 3: Errors (58) for simultaneous mesh refinements, using `tetra`, `cube`, and `voro` meshes.

analysis hinges upon the exact sequence structure of the employed virtual element spaces. We devoted particular attention to the treatment of the two nonlinear terms: one is the usual convective term appearing in the Navier-Stokes equations; the second couples the velocity, magnetic, and electric fields. We also validated the theoretical convergence results with a numerical experiment on a manufactured solution. Future works will thoroughly cope with the numerical performance of the method: we shall present more involved and realistic benchmarks and investigate different approaches to deal with the nonlinear terms. Further investigations will also cover the design of suitable (direct, iterative, and parallel) solvers to improve the computational efficiency in solving the final linear systems.

	$\ \operatorname{div} \mathbf{B}_h\ $			
	level 1	level 2	level 3	level 4
<code>tetra</code>	4.6977e-13	7.8962e-13	3.3856e-12	1.6680e-11
<code>cube</code>	1.1798e-13	2.2284e-13	6.6499e-13	2.2580e-12
<code>voro</code>	2.9645e-11	6.4690e-13	1.7873e-11	5.3376e-11

Table 1: L^2 norm of $\operatorname{div} \mathbf{B}_h$ for several space and time refinement steps.

	$\ \operatorname{div} \mathbf{u}_h\ $			
	level 1	level 2	level 3	level 4
tetra	7.6676e-16	2.0355e-15	1.0726e-14	6.1851e-14
cube	1.1714e-15	1.9226e-15	7.3605e-15	4.0470e-14
voro	2.7855e-16	3.2315e-15	1.6780e-14	8.8101e-14

Table 2: L^2 norm of $\operatorname{div} \mathbf{u}_h$ for several space and time refinement steps.

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