Reducing Fuzzy Description Logics into Classical Description Logics

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Abstract

In this paper we consider Description Logics (DLs), well-known logics for managing structured knowledge, with its fuzzy extension to deal with vague information.

While for fuzzy DLs correct and complete ad-hoc reasoning procedures have been given, the topic of this paper is to present a reasoning preserving transformation of fuzzy DLs into classical DLs. This has the considerable practical consequence that reasoning in fuzzy DLs is feasible using already existing DL systems.

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1 INTRODUCTION

In the last decade a substantial amount of work has been carried out in the context of *Description Logics* (DLs) [1]. DLs are a logical reconstruction of the so-called frame-based knowledge representation languages, with the aim of providing a simple well-established Tarski-style declarative semantics to capture the meaning of the most popular features of structured representation of knowledge. Nowadays, a whole family of knowledge representation systems has been build using DLs, which differ with respect to their expressiveness and their complexity, and they have been used for building a variety of applications (see the DL community home page http://dl.kr.org/).

Despite their growing popularity, relative little work has been carried out [1] in extending them to the management of uncertain information. This is a well-known and

[1]Comparing with other formalisms -notably logic programming (see, e.g. [9, 11], for an overview).

important issue whenever the real world information to be represented is of imperfect nature. In DLs, the problem has attracted the attention of some researchers and some frameworks have been proposed, which differ in the underlying notion of uncertainty, e.g. probability theory [5, 6, 8, 10, 15], possibility theory [7], metric spaces [13] and fuzzy theory [4, 16, 18, 19].

In this paper we consider the fuzzy extension of DLs towards the management of vague knowledge [16]. The choice of fuzzy set theory as a way of endowing a DL with the capability to deal with imprecision is motivated as fuzzy logics capture the notion of imprecise concept, i.e. a concept for which a clear and precise definition is not possible. Therefore, fuzzy DLs allow to express that a sentence, like "it is Cold", is not just true or false like in classical DLs, but has a degree of truth, which is taken from the real unit interval $[0, 1]$. The truth degree dictates to which extent a sentence is true.

From a computational point of view, the reasoning procedures in [16] are based on an ad-hoc tableaux calculus, similar to the ones presented for almost all DLs. Unfortunately, a drawback of the tableaux calculus in [16] is that any system, which would like to implement this fuzzy logic, has to be worked out from scratch.

The contribution of this paper is as follows. Primarily, we present a reasoning preserving transformation of fuzzy DLs into classical DLs. This has the considerable practical consequence that reasoning in fuzzy DLs is feasible using already existing DL systems. Secondarily, we allow the representation of so-called general terminological axioms, while in [16], the axioms were very limited in the form. To best of our knowledge, no algorithm has yet been worked out for general axioms in fuzzy DLs. Overall, our approach may be extended to more expressive DLs than the one we present here as well.

We proceed as follows. In the next section, we recall some fundamental notions about DLs. In Section 3 we recall fuzzy DLs. Section 4 is the main part of this paper, where we present our reduction of fuzzy DLs into classical DLs. Finally, Section 5 concludes the paper.

2 A QUICK LOOK TO DLs

Instrumental to our purpose, the specific DL we extend with "fuzzy" capabilities is $\mathcal{ALC}$, a significant representative of DLs (see, e.g. [1, 14]. $\mathcal{ALC}$ is sufficiently expressive to illustrate the main concepts introduced in this paper. More expressive DLs will be the subject of an extended work. Note that [16] considered $\mathcal{ALC}$ as well.

Consider three alphabets of symbols, for *concepts names* (denoted A), for *roles names* (denoted R) and *individual names* (denoted a and b) [2]. A *concept* (denoted C or D) of the language $\mathcal{ALC}$ is built inductively from concept names A and role names R according to the following syntax rule:

[2]Metavariables may have a subscript or a superscript.

$$
\begin{array}{rrl}
C, D & \longrightarrow & \top | \quad \text{(top concept)} \\
& & \bot | \quad \text{(bottom concept)} \\
& & A | \quad \text{(concept name)} \\
& & C \sqcap D | \quad \text{(concept conjunction)} \\
& & C \sqcup D | \quad \text{(concept disjunction)} \\
& & \neg C | \quad \text{(concept negation)} \\
& & \forall R.C | \quad \text{(universal quantification)} \\
& & \exists R.C \quad \text{(existential quantification)}\ .
\end{array}
$$

A *terminology*, $\mathcal{T}$, is a finite set of concept inclusions or role inclusions, called *terminological axioms*, τ, where given two concepts C and D, and two role names R and R', a terminological axiom is an expression of the form $C \sqsubseteq D$ (D subsumes C) or of the form $R \sqsubseteq R'$ (R' subsumes R) .

An *assertion*, α, is an expression of the form $a{:}C$ ("a is an instance of C"), or an expression $(a,b){:}R$ ("(a,b) is an instance of R").

A *Knowledge Base* (KB), $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$, is such that $\mathcal{T}$ and $\mathcal{A}$ are finite sets of terminological axioms and assertions, respectively.

An *interpretation* $\mathcal{I}$ is a pair $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ consisting of a non empty set $\Delta^{\mathcal{I}}$ (called the *domain*) and of an *interpretation function* $\cdot^{\mathcal{I}}$ mapping individuals into elements of $\Delta^{\mathcal{I}}$ (note that usually the *unique name assumption* [3] is considered, but it does not matter us here), concepts names into subsets of $\Delta^{\mathcal{I}}$ and roles names into subsets of $\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$.

The interpretation of complex concepts is defined inductively as usual:

$$
\begin{array}{rcl}
\top^{\mathcal{I}} & = & \Delta^{\mathcal{I}} \\
\bot^{\mathcal{I}} & = & \emptyset \\
(C \sqcap D)^{\mathcal{I}} & = & C^{\mathcal{I}} \cap D^{\mathcal{I}} \\
(C \sqcup D)^{\mathcal{I}} & = & C^{\mathcal{I}} \cup D^{\mathcal{I}} \\
(\neg C)^{\mathcal{I}} & = & \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}} \\
(\forall R.C)^{\mathcal{I}} & = & \{d \in \Delta^{\mathcal{I}} \mid \forall d'.(d,d') \notin R^{\mathcal{I}} \text{ or } d' \in C^{\mathcal{I}}\} \\
(\exists R.C)^{\mathcal{I}} & = & \{d \in \Delta^{\mathcal{I}} \mid \exists d'.(d,d') \in R^{\mathcal{I}} \text{ and } d' \in C^{\mathcal{I}}\}\ .
\end{array}
$$

A concept C is *satisfiable* iff there is an interpretation $\mathcal{I}$ such that $C^{\mathcal{I}} \neq \emptyset$. Two concepts C and D are *equivalent* (denoted $C \equiv D$) iff $C^{\mathcal{I}} = D^{\mathcal{I}}$, for all interpretations $\mathcal{I}$.

An interpretation $\mathcal{I}$ *satisfies* an assertion $a{:}C$ (resp. $(a,b){:}R$) iff $a^{\mathcal{I}} \in C^{\mathcal{I}}$ (resp. $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in R^{\mathcal{I}}$), while $\mathcal{I}$ *satisfies* a terminological axiom $C \sqsubseteq D$ iff $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$. The satisfiability of role inclusions $R \sqsubseteq R'$ is similar.

Furthermore, an interpretation $\mathcal{I}$ *satisfies* (is a *model* of) a terminology $\mathcal{T}$ (resp. a set of assertions $\mathcal{A}$) iff $\mathcal{I}$ satisfies each element in $\mathcal{T}$ (resp. $\mathcal{A}$), while $\mathcal{I}$ *satisfies* (is a *model* of) a KB $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$ iff $\mathcal{I}$ satisfies both $\mathcal{T}$ and $\mathcal{A}$. Finally, given a KB $\mathcal{K}$ and an assertion α we say that $\mathcal{K}$ *entails* α, denoted $\mathcal{K} \models \alpha$, iff each model of $\mathcal{K}$ satisfies α.

Example 1 *Consider the following KB $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$, where*

$$
\begin{array}{rcl}
\mathcal{T} & = & \{A{:}= \forall R.\neg B\} \\
\mathcal{A} & = & \{a{:}\forall R.C\}\ .
\end{array}
$$

[3] $a^{\mathcal{I}} \neq b^{\mathcal{I}}$, if $a \neq b$.

Consider the assertion

$$\alpha = a{:}A \sqcup \exists R.(B \sqcap C) \ .$$

It can be shown that $\mathcal{K} \models \alpha$ holds. In fact, consider a model $\mathcal{I}$ of $\mathcal{K}$. Then either $a^{\mathcal{I}} \in A^{\mathcal{I}}$ or $a^{\mathcal{I}} \notin A^{\mathcal{I}}$. In the former case, $\mathcal{I}$ satisfies α. In the latter case, as $\mathcal{I}$ satisfies $\mathcal{T}$, $a^{\mathcal{I}} \notin (\forall R.\neg B)^{\mathcal{I}}$, i.e. $a^{\mathcal{I}} \in (\exists R.B)^{\mathcal{I}}$ holds. But, $\mathcal{I}$ satisfies $\mathcal{A}$ as well, i.e. $a^{\mathcal{I}} \in (\forall R.C)^{\mathcal{I}}$ and, thus, $a^{\mathcal{I}} \in (\exists R.(B \sqcap C))^{\mathcal{I}}$. Therefore, $\mathcal{I}$ satisfies α, which concludes.

Finally, note that there exists decision procedures for the satisfiability and the entailment problems in $\mathcal{ALC}$ (see, e.g. [1]) and there are implemented reasoners like, for instance, RACER [4] or FACT [5], which allow to reason in quit more expressive DLs as $\mathcal{ALC}$. This concludes this part.

3 A QUICK LOOK TO FUZZY DLs

We recall here the main notions related to fuzzy DLs, taken from [16]. Worth noting is that we deal with general terminological axioms of the form $C \sqsubseteq D$, while in [16] the terminological component is restricted in the form, i.e. in [16] a terminology, $\mathcal{T}$, is a finite set of concept definitions and concept inclusions, where (*i*) for a concept name A and a concept C, a concept definition is an expression of the form $A{:}= C$, while a concept inclusion is an expression of the form $A \sqsubseteq C$; and (*ii*) $\mathcal{T}$ is such that no concept name A appears more than once on the left hand side of a terminological axiom $\tau \in \mathcal{T}$ and that no cyclic definitions are present in $\mathcal{T}$ [6]. In this work, we do not impose these restrictions on the terminological component.

For convenience, we call the fuzzy extension of $\mathcal{ALC}$, $\mu\mathcal{ALC}$. The main idea underlying $\mu\mathcal{ALC}$ is that an assertion $a{:}C$, rather being interpreted as either true or false, will be mapped into a truth value $c \in [0,1]$. The intended meaning is that c indicates to which extend (how certain it is that) 'a is a C'. Similarly for role names.

Formally, a *μinterpretation* is a pair $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, where $\Delta^{\mathcal{I}}$ is the *domain* and $\cdot^{\mathcal{I}}$ is an *interpretation function* mapping

- individuals as for the classical case;
- a concept C into a function $C^{\mathcal{I}}\colon \Delta^{\mathcal{I}} \to [0,1]$; and
- a role R into a function $R^{\mathcal{I}}\colon \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to [0,1]$.

If C is a concept then $C^{\mathcal{I}}$ will naturally be interpreted as the *membership degree function* (μ_C in 'fuzzy notation') of the fuzzy concept (set) C w.r.t. $\mathcal{I}$, i.e. if $d \in \Delta^{\mathcal{I}}$ is an object of the domain $\Delta^{\mathcal{I}}$ then $C^{\mathcal{I}}(d)$ gives us the degree of being the object d an element of the fuzzy concept C under the μinterpretation $\mathcal{I}$. Similarly for roles.

The definition of concept *equivalence* is like for $\mathcal{ALC}$. Two concepts C and D are equivalent iff $C^{\mathcal{I}} = D^{\mathcal{I}}$, for all μinterpretations $\mathcal{I}$.

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[4] http://www.cs.concordia.ca/∼haarslev/racer/

[5] http://www.cs.man.ac.uk/∼horrocks/FaCT/

[6] We say that A *directly uses* primitive concept B in $\mathcal{T}$, if there is $\tau \in \mathcal{T}$ such that A is on the left hand side of τ and B occurs in the right hand side of τ. Let *uses* be the transitive closure of the relation directly uses in $\mathcal{T}$. $\mathcal{T}$ is *cyclic* iff there is A such that A uses A in $\mathcal{T}$.

The interpretation function $\cdot^{\mathcal{I}}$ has also to satisfy the following equations: for all $d \in \Delta^{\mathcal{I}}$,

$$\begin{array}{rcl}
\top^{\mathcal{I}}(d) & = & 1 \\
\bot^{\mathcal{I}}(d) & = & 0 \\
(C \sqcap D)^{\mathcal{I}}(d) & = & \min(C^{\mathcal{I}}(d), D^{\mathcal{I}}(d)) \\
(C \sqcup D)^{\mathcal{I}}(d) & = & \max(C^{\mathcal{I}}(d), D^{\mathcal{I}}(d)) \\
(\neg C)^{\mathcal{I}}(d) & = & 1 - C^{\mathcal{I}}(d) \\
(\forall R.C)^{\mathcal{I}}(d) & = & \inf_{d' \in \Delta^{\mathcal{I}}}\{\max(1 - R^{\mathcal{I}}(d, d'), C^{\mathcal{I}}(d'))\} \\
(\exists R.C)^{\mathcal{I}}(d) & = & \sup_{d' \in \Delta^{\mathcal{I}}}\{\min(R^{\mathcal{I}}(d, d'), C^{\mathcal{I}}(d'))\} \ .
\end{array}$$

These equations are the standard interpretation of conjunction, disjunction, negation and quantification, respectively for fuzzy sets [20] (see also [12, 18]). Nonetheless, some conditions deserve an explanation.

- The semantics of $\exists R.C$ is the result of viewing $\exists R.C$ as the open first order formula $\exists y.R(x, y) \wedge \bar{C}(y)$ (where $\bar{C}$ is the translation of C into first-order logic) and $\exists$ is viewed as a disjunction over the elements of the domain;
- Similarly, the semantics of $\forall R.C$ is related to $\forall y.\neg R(x, y) \vee \bar{C}(y)$, where $\forall$ is viewed as a conjunction over the elements of the domain.

As for the classical DLs, dual relationships between concepts hold: e.g. $(C \sqcap D) \equiv \neg(\neg C \sqcup \neg D)$ and $(\forall R.C) \equiv \neg(\exists R.\neg C)$, but $C \sqcap (\neg C \sqcup D) \not\equiv D$.

A *μassertion* (denoted $\mu\alpha$) is an expression of the form $\langle \alpha \geq c_1 \rangle$, $\langle \alpha > c_2 \rangle$, $\langle \alpha' \leq c_2 \rangle$ or $\langle \alpha' < c_1 \rangle$, where α is an $\mathcal{ALC}$ assertion, $c_1 \in (0, 1]$ and $c_2 \in [0, 1)$, but α' is an $\mathcal{ALC}$ assertion of the form $a{:}C$ only. For coherence, we do not allow μassertions of the form $\langle (a, b){:}R \leq c \rangle$ or $\langle (a, b){:}R < c \rangle$ as they relate to 'negated roles', which is not part of classical $\mathcal{ALC}$.

From a semantics point of view, a μassertion $\langle \alpha \leq c \rangle$ constrains the truth value of α to be less or equal to c (similarly for $\geq, >$ and $<$). So, a μinterpretation $\mathcal{I}$ *satisfies* $\langle a{:}C \geq c \rangle$ (resp. $\langle (a, b){:}R \geq c \rangle$) iff $C^{\mathcal{I}}(a^{\mathcal{I}}) \geq c$ (resp. $R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \geq c$). Similarly for $>, \leq$ and $<$. Note that, e.g. $\langle a{:}\neg C \geq c \rangle$ and $\langle a{:}C \leq 1 - c \rangle$ are satisfied by the same set of μinterpretations, i.e.

$$\mathcal{I} \text{ satisfies } \langle a{:}\neg C \geq c \rangle \text{ iff } \mathcal{I} \text{ satisfies } \langle a{:}C \leq 1 - c \rangle \ . \quad (1)$$

Concerning terminological axioms, a $\mu\mathcal{ALC}$ terminological axiom is, as for the classical DL $\mathcal{ALC}$, of the form $C \sqsubseteq D$, where C and D are $\mathcal{ALC}$ concepts, or of the form $R \sqsubseteq R'$, where R and R' are role names. From a semantics point of view, a μinterpretation $\mathcal{I}$ *satisfies* $C \sqsubseteq D$ iff for all $d \in \Delta^{\mathcal{I}}, C^{\mathcal{I}}(d) \leq D^{\mathcal{I}}(d)$. Similarly, μinterpretation $\mathcal{I}$ *satisfies* $R \sqsubseteq R'$ iff for all $\{d, d'\} \subseteq \Delta^{\mathcal{I}}, R^{\mathcal{I}}(d, d') \leq R'^{\mathcal{I}}(d, d')$.

A *μKnowledge Base* (μKB) is pair $\mu\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$, where $\mathcal{T}$ and $\mathcal{A}$ are finite sets of terminological axioms and μassertions, respectively. A μinterpretation $\mathcal{I}$ *satisfies* (is a *model* of) a terminology $\mathcal{T}$ (resp. a set of μassertions $\mathcal{A}$) iff $\mathcal{I}$ satisfies each element in $\mathcal{T}$ (resp. $\mathcal{A}$), while $\mathcal{I}$ *satisfies* (is a *model* of) a KB $\mu\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$ iff $\mathcal{I}$ satisfies both $\mathcal{T}$ and $\mathcal{A}$.

Given a μKB $\mu\mathcal{K}$, and a μassertion $\mu\alpha$, we say that $\mu\mathcal{K}$ *entails* $\mu\alpha$, denoted $\mu\mathcal{K} \models \mu\alpha$, iff each model of $\mu\mathcal{K}$ satisfies $\mu\alpha$. For instance, if $c' > 1 - c$ then

$$\{\langle (a, b){:}R \geq c' \rangle, \langle a{:}\forall R.C \geq c \rangle\} \models \langle a{:}C \geq c \rangle \ . \quad (2)$$

Finally, given $\mu\mathcal{K}$ and an $\mathcal{ALC}$ assertion α, it is of interest to compute α's best lower and upper truth value bounds. The *greatest lower bound* of α w.r.t. $\mu\mathcal{K}$ (denoted $glb(\mu\mathcal{K}, \alpha)$) is

$$glb(\mu\mathcal{K}, \alpha) = \sup\{c : \mu\mathcal{K} \models \langle \alpha \geq c \rangle\} \ ,$$

while the *least upper bound* of α with respect to $\mu\mathcal{K}$ (denoted $lub(\mu\mathcal{K}, \alpha)$) is

$$lub(\mu\mathcal{K}, \alpha) = \inf\{c : \mu\mathcal{K} \models \langle \alpha \leq c \rangle\}$$

where $\sup \emptyset = 0$ and $\inf \emptyset = 1$. Determining the *lub* and the *glb* is called the *Best Truth Value Bound* (BTVB) problem. Note that

$$lub(\Sigma, a{:}C) = 1 - glb(\Sigma, a{:}\neg C) \ , \tag{3}$$

i.e. the *lub* can be determined through the *glb* (and vice-versa). The same reduction to *glb* does not hold for $lub(\Sigma, (a, b){:}R)$ as $(a, b){:}\neg R$ is not an expression of our language.[7]

Finally, note that, $\Sigma \models_{\mathcal{L}} \langle \alpha \geq n \rangle$ iff $glb(\Sigma, \alpha) \geq n$, and similarly $\Sigma \models_{\mathcal{L}} \langle \alpha \leq n \rangle$ iff $lub(\Sigma, \alpha) \leq n$ hold. Concerning roles, note that $\Sigma \models_{\mathcal{L}} \langle (a, b){:}R \geq n \rangle$ iff $\langle (a, b){:}R \geq m \rangle \in \Sigma$ with $m \geq n$. Therefore,

$$glb(\Sigma, R(a, b)) \quad = \quad \max\{n : \langle R(a, b) \geq n \rangle \in \Sigma\} \ . \tag{4}$$

Concerning the entailment problem, it is quite easily verified that the entailment problem can be reduced to the unsatisfiability problem:

$$\langle \mathcal{T}, \mathcal{A} \rangle \models \langle \alpha \geq n \rangle \text{ iff } \langle \mathcal{T}, \mathcal{A} \cup \{\langle \alpha < n \rangle\} \rangle \text{ is not satisfiable} \ , \tag{5}$$

$$\langle \mathcal{T}, \mathcal{A} \rangle \models \langle \alpha \leq n \rangle \text{ iff } \langle \mathcal{T}, \mathcal{A} \cup \{\langle \alpha > n \rangle\} \rangle \text{ is not satisfiable} \ . \tag{6}$$

In [16] decision procedures for the satisfiability, the entailment and the BTVB problem are given for $\mu\mathcal{ALC}$, but with the already discussed restrictions on the form of terminological axioms and terminologies.

Example 2 *Similarly to Example 1, consider* $\mu\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$, *where*

$$\begin{array}{rcl} \mathcal{T} & = & \{A{:}= \forall R.\neg B\} \\ \mathcal{A} & = & \{\langle a{:}\forall R.C \geq 0.7 \rangle\} \ . \end{array}$$

Consider the assertion

$$\alpha = a{:}A \sqcup \exists R.(B \sqcap C) \ .$$

It can be shown that

$$\begin{array}{rcl} glb(\mu\mathcal{K}, \alpha) & = & 0.5 \\ lub(\mu\mathcal{K}, \alpha) & = & 1 \end{array}$$

hold. In fact, for any model $\mathcal{I}$ *of* $\mu\mathcal{K}$, *we have that*

$$(A \sqcup \exists R.(B \sqcap C))^{\mathcal{I}}(a^{\mathcal{I}}) \geq \max(c, \min(0.7, 1 - c)) \ , \tag{7}$$

for any $c \in [0, 1]$. *Indeed, let* $\mathcal{I}$ *be a model of* $\mu\mathcal{K}$. *Assume that* $(A \sqcup \exists R.(B \sqcap C))^{\mathcal{I}}(a^{\mathcal{I}}) = w$. *Consider* $c \in [0, 1]$. *Then either* $A^{\mathcal{I}}(a^{\mathcal{I}}) \geq c$ *or* $A^{\mathcal{I}}(a^{\mathcal{I}}) < c$. *In the former case,*

[7] Of course, $lub(\Sigma, (a, b){:}R) = 1 - glb(\Sigma, (a, b){:}\neg R)$ holds, where $(\neg R)^{\mathcal{I}}(d, d') = 1 - R^{\mathcal{I}}(d, d')$.

it follows that $w \geq c$. In the latter case, as $\mathcal{I}$ satisfies $\mathcal{T}$, from $A^{\mathcal{I}}(a^{\mathcal{I}}) < c$ it follows that $(\forall R.\neg B)^{\mathcal{I}}(a^{\mathcal{I}}) < c$. But, $\forall R.\neg B \equiv \neg\exists R.B$ and, thus, $(\exists R.B)^{\mathcal{I}}(a^{\mathcal{I}}) > 1-c$. Therefore, there is $d \in \Delta^{\mathcal{I}}$ such that $R^{\mathcal{I}}(a^{\mathcal{I}}, d) > 1-c$ and $B^{\mathcal{I}}(d) > 1-c$. But, $\mathcal{I}$ satisfies $\mu\mathcal{A}$, i.e. $(\forall R.C)^{\mathcal{I}}(a^{\mathcal{I}}) \geq 0.7$. By definition, this means that $\inf_{d' \in \Delta^{\mathcal{I}}}\{\max(1 - R^{\mathcal{I}}(a^{\mathcal{I}}, d'), C^{\mathcal{I}}(d'))\} \geq 0.7$ and, in particular, for $d' = d$, $\max(1 - R^{\mathcal{I}}(a^{\mathcal{I}}, d), C^{\mathcal{I}}(d)) \geq 0.7$ holds. Therefore, $1 - R^{\mathcal{I}}(a^{\mathcal{I}}, d) < 0.7$ (i.e., $R^{\mathcal{I}}(a^{\mathcal{I}}, d) > 0.3$) implies $C^{\mathcal{I}}(d)) \geq 0.7$. As a consequence, from $R^{\mathcal{I}}(a^{\mathcal{I}}, d) > 1-c$, for $c \leq 0.7$ it follows that $C^{\mathcal{I}}(d) \geq 0.7$ (see also Equation 2). Therefore, $(\exists R.(B \sqcap C))^{\mathcal{I}}(a^{\mathcal{I}}) \geq \min(0.7, 1-c)$ and, thus, $w \geq \max(c, \min(0.7, 1-c))$, which proofs (7).

Finally, as for any $c \in [0,1]$, $\max(c, \min(0.7, 1-c)) \geq 0.5$ and there is no $c' > 0.5$ such that for all $c \in [0,1]$, $\max(c, \min(0.7, 1-c)) \geq c'$, by (7), $glb(\mu\mathcal{K}, \alpha) = 0.5$ follows.

The proof of $lub(\mu\mathcal{K}, \alpha) = 1$ is easy.

4 MAPPING $\mu\mathcal{ALC}$ INTO $\mathcal{ALC}$

Our aim is to map $\mu\mathcal{ALC}$ knowledge bases into satisfiability and entailment preserving classical $\mathcal{ALC}$ knowledge bases. An immediate consequence is then that (*i*) we have reasoning procedures for $\mu\mathcal{ALC}$ with general terminological axioms, which are still unknown; and (*ii*) we can rely on already implemented reasoners to reason in $\mu\mathcal{ALC}$.

Before we are going to formally present the mapping, we first illustrate the basic idea we rely on. Our mapping relies on ideas presented in [2, 3].

Assume we have a μKB, $\mu\mathcal{K} = \langle \emptyset, \mathcal{A} \rangle$, where $\mathcal{A} = \{\mu\alpha_1, \mu\alpha_2, \mu\alpha_3, \mu\alpha_4\}$ and

$$\begin{array}{rcl} \mu\alpha_1 & = & \langle a{:}A \geq 0.4 \rangle \\ \mu\alpha_2 & = & \langle a{:}A \leq 0.7 \rangle \\ \mu\alpha_3 & = & \langle a{:}B \leq 0.2 \rangle \\ \mu\alpha_4 & = & \langle b{:}B \leq 0.1 \rangle \ . \end{array}$$

Let us introduce some new concepts, namely $A_{\geq 0.4}$, $A_{\leq 0.7}, B_{\leq 0.2}$ and $B_{\leq 0.1}$. Informally, the concept $A_{\geq 0.4}$ represents the set of individuals, which are instance of A with degree $c \geq 0.4$, while $A_{\leq 0.7}$ represents the set of individuals, which are instance of A with degree $c \leq 0.7$. Similarly, for the other concepts. Of course, we have to consider also the relationships among the introduced concepts. For instance, we need the terminological axiom

$$B_{\leq 0.1} \quad \sqsubseteq \quad B_{\leq 0.2} \ .$$

This axiom dictates that if a truth value is ≤ 0.1 then it is also ≤ 0.2. We may represent, thus, the μassertion $\mu\alpha_1$ with the $\mathcal{ALC}$ assertion $a{:}A_{\geq 0.4}$, indicating that a is an instance of A with a degree ≥ 0.4. Similarly, $\mu\alpha_2$ may be mapped into $a{:}A_{\geq 0.7}$, $\mu\alpha_3$ may be mapped into $a{:}B_{\geq 0.2}$, while $\mu\alpha_4$ may be mapped into $b{:}B_{\geq 0.1}$. From a semantics point of view, let us consider the so-called *canonical model* [1] $\mathcal{I}$ of the resulting classical $\mathcal{ALC}$ KB, i.e.

$$\mathcal{I} = \{A_{\geq 0.4}(a), A_{\leq 0.7}(a), B_{\leq 0.2}(a), B_{\leq 0.1}(b), B_{\leq 0.2}(b)\} \ .$$

It is then easily verified that, from $\mathcal{I}$ a model $\mathcal{I}'$ of $\mu\mathcal{K}$ can easily be built and, vice-versa, if $\mathcal{I}'$ is a model of $\mu\mathcal{K}$, then a model like $\mathcal{I}$ above can be obtained as well. Therefore, our transformation of $\mu\mathcal{K}$ into an $\mathcal{ALC}$ KB, at least for the above case, is satisfiability preserving. This illustrates our basic idea.

Let us now proceed formally. Consider a μKB $\mu\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$. Let $\mathcal{A}^{\mu\mathcal{K}}$ and $\mathcal{R}^{\mu\mathcal{K}}$ be the set of concept names and concept roles occurring in $\mu\mathcal{K}$. Of course, both $|\mathcal{A}^{\mu\mathcal{K}}|$ and $|\mathcal{R}^{\mu\mathcal{K}}|$ are linearly bounded by $|\mu\mathcal{K}|$. Consider

$$\begin{aligned} X^{\mu\mathcal{K}} &= \{0, 0.5, 1\} \cup \{c : \langle \alpha \geq c \rangle \in \mathcal{A}\} \\ &\quad \cup \{1 - c : \langle \alpha \leq c \rangle \in \mathcal{A}\} \end{aligned}$$

from which we define

$$N^{\mu\mathcal{K}} = X^{\mu\mathcal{K}} \cup \{1 - c : c \in X^{\mu\mathcal{K}}\} \ . \tag{8}$$

Note that $|N^{\mu\mathcal{K}}|$ is linearly bounded by $|\mathcal{A}|$. Essentially, with $N^{\mu\mathcal{K}}$ we collect from $\mu\mathcal{K}$ all the relevant numbers we require for the transformation. Without loss of generality, we may assume that $N^{\mu\mathcal{K}} = \{c_1, \ldots, c_{|N^{\mu\mathcal{K}}|}\}$ and $c_i < c_{i+1}$, for $1 \leq i \leq |N^{\mu\mathcal{K}}| - 1$. Note that $c_1 = 0$ and $c_{|N^{\mu\mathcal{K}}|} = 1$.

For each $c \in N^{\mu\mathcal{K}}$, for each relation $\bowtie \in \{\geq, >, \leq, <\}$, for each $A \in \mathcal{A}^{\mu\mathcal{K}}$ and for each $R \in \mathcal{R}^{\mu\mathcal{K}}$, consider a new concept name $A_{\bowtie c}$ and new role names $R_{\geq c}$ and $R_{>c}$, but we do not consider $A_{<0}, A_{>1}$ and $R_{>1}$ (which are not needed). There are as many as $(4|N^{\mu\mathcal{K}}| - 2)|\mathcal{A}^{\mu\mathcal{K}}|$ new concept names and $(2|N^{\mu\mathcal{K}}| - 1)|\mathcal{R}^{\mu\mathcal{K}}|$ new role names. Note that we do not require new role names $R_{\leq c}$ and $R_{<c}$, as e.g. expressions of the form $\langle (a, b){:}R \leq c \rangle$ are not part of our language.

Let $\mathcal{T}(N^{\mu\mathcal{K}})$ be the following terminology relating the newly introduced concept names and role names: $\mathcal{T}(N^{\mu\mathcal{K}})$ is the smallest terminology such that for each $1 \leq i \leq |N^{\mu\mathcal{K}}| - 1$, for each $2 \leq j \leq |N^{\mu\mathcal{K}}|$, for each $A \in \mathcal{A}^{\mu\mathcal{K}}$ and for each $R \in \mathcal{R}^{\mu\mathcal{K}}$, $\mathcal{T}(N^{\mu\mathcal{K}})$ contains

$$\begin{aligned} A_{\geq c_{i+1}} &\sqsubseteq A_{>c_i} \\ A_{>c_i} &\sqsubseteq A_{\geq c_i} \\[1em] A_{<c_j} &\sqsubseteq A_{\leq c_j} \\ A_{\leq c_i} &\sqsubseteq A_{<c_{i+1}} \\[1em] A_{\geq c_j} \sqcap A_{<c_j} &\sqsubseteq \bot \\ A_{>c_i} \sqcap A_{\leq c_i} &\sqsubseteq \bot \\[1em] \top &\sqsubseteq A_{\geq c_j} \sqcup A_{<c_j} \\ \top &\sqsubseteq A_{>c_i} \sqcup A_{\leq c_i} \ . \end{aligned}$$

The first two groups reflect the $\geq, <, \leq, >$ ordering among the newly introduced concepts, while the third group identifies 'disjointness' conditions. For instance, among these terminological axioms we may have $A_{\geq 0.4} \sqcap A_{<0.4} \sqsubseteq \bot$ indicating that it cannot be that an individual a is an instance of the concept name A with degree ≥ 0.4 and degree < 0.4. The last group establishes the complimentarily relationships among the new concepts, e.g. $A_{\geq 0.4} \sqcup A_{<0.4} \equiv \top$. Note that $\mathcal{T}(N^{\mu\mathcal{K}})$ contains $8|\mathcal{A}^{\mu\mathcal{K}}|(|N^{\mu\mathcal{K}}| - 1)$ terminological axioms involving the newly introduced concepts names.

The terminological axioms in $\mathcal{T}(N^{\mu\mathcal{K}})$ relating the newly introduced role names are quite similar to the above axioms:

$$\begin{aligned} R_{\geq c_{i+1}} &\sqsubseteq R_{>c_i} \\ R_{>c_i} &\sqsubseteq R_{\geq c_i} \end{aligned}$$

Note that $\mathcal{T}(N^{\mu\mathcal{K}})$ contains $2|\mathcal{R}^{\mu\mathcal{K}}|(|N^{\mu\mathcal{K}}|-1)$ terminological axioms involving the newly introduced role names. Please note also that in case we would like to allow expressions of the form $\langle (a,b){:}R \leq c\rangle$ and $\langle (a,b){:}R < c\rangle$, then we need new role names $R_{\leq c}$ and $R_{<c}$ (excluding $R_{<0}$), and terminological axioms $R_{<c_j} \sqsubseteq R_{\leq c_j}$, $R_{\leq c_i} \sqsubseteq R_{<c_{i+1}}$, $R_{\geq c_j} \sqcap R_{<c_j} \sqsubseteq \bot^r$, $R_{>c_i} \sqcap R_{\leq c_i} \sqsubseteq \bot^r$, $\top^r \sqsubseteq R_{\geq c_j} \sqcup R_{<c_j}$ and $\top^r \sqsubseteq R_{>c_i} \sqcup R_{\leq c_i}$. In particular, note that 'role conjunction', 'role disjunction' and a 'bottom role' and a 'top role' are needed.

Example 3 *Consider Example 2. Then $N^{\mu\mathcal{K}}$ is*

$$N^{\mu\mathcal{K}} = \{0, 0.3, 0.5, 0.7, 1\} \ ,$$

while $\mathcal{A}^{\mu\mathcal{K}} = \{A, B, C\}$ and $\mathcal{R}^{\mu\mathcal{K}} = \{R\}$. Below, we provide an excerpt of the terminology $\mathcal{T}(N^{\mu\mathcal{K}})$:

$$\begin{array}{rcl}
\mathcal{T}(N^{\mu\mathcal{K}}) & = & \{A_{\geq 1} \sqsubseteq A_{>0.7}, A_{\geq 0.7} \sqsubseteq A_{>0.5}, \ldots\} \\
& \cup & \{A_{>0.7} \sqsubseteq A_{\geq 0.7}, A_{>0.5} \sqsubseteq A_{\geq 0.5}, \ldots\} \\
& \cup & \{A_{<0.3} \sqsubseteq A_{\leq 0.3}, A_{<0.5} \sqsubseteq A_{\leq 0.5}, \ldots\} \\
& \cup & \{A_{\leq 0} \sqsubseteq A_{<0.3}, A_{\leq 0.3} \sqsubseteq A_{<0.5}, \ldots\} \\
& \cup & \{A_{\geq 0.3} \sqcap A_{<0.3} \sqsubseteq \bot, \ldots\} \\
& \cup & \{A_{>0} \sqcap A_{\leq 0} \sqsubseteq \bot, \ldots\} \\
& \cup & \{\top \sqsubseteq A_{\geq 0.3} \sqcup A_{<0.3}, \ldots\} \\
& \cup & \{\top \sqsubseteq A_{>0} \sqcup A_{\leq 0}, \ldots\} \\
& \cup & \{B_{\geq 1} \sqsubseteq B_{>0.7}, \ldots\} \\
& \cup & \ldots \\
& \vdots & \\
& \cup & \{R_{\geq 1} \sqsubseteq R_{>0.7}, \ldots\} \\
& \cup & \{R_{>0.7} \sqsubseteq R_{\geq 0.7}, \ldots\} \ .
\end{array}$$

This concludes the management of the newly introduced concept names and role names.

We proceed now with the mapping of the μassertions in a μKB into $\mathcal{ALC}$ assertions. We define two mappings σ and ρ, defined as follows. Let $\mu\alpha$ be a μassertion. Then σ maps a μassertion into a classical $\mathcal{ALC}$ assertion, using ρ, as follows. In the following, we assume that $c \in [0,1]$ and $\bowtie \in \{\geq, >, \leq, <\}$.

$$\sigma(\mu\alpha) = \left\{ \begin{array}{lll} a{:}\rho(C, \bowtie c) & \text{if} & \mu\alpha = \langle a{:}C \bowtie c\rangle \\ (a,b){:}\rho(R, \bowtie c) & \text{if} & \mu\alpha = \langle (a,b){:}R \bowtie c\rangle \ . \end{array} \right.$$

We extend σ to a set of μassertions $\mathcal{A}$ point-wise, i.e. $\sigma(\mathcal{A}) = \{\sigma(\mu\alpha) | \mu\alpha \in \mathcal{A}\}$.

The mapping ρ encodes the idea we have previously presented in a simplified example and is inductively defined on the structure of concepts and roles. For roles, we have simply

$$\rho(R, \bowtie c) = R_{\bowtie c} \ .$$

So, for instance the μassertion $\langle (a,b){:}R \geq c\rangle$ is mapped into the $\mathcal{ALC}$ assertion $(a,b){:}R_{\geq c}$. Concerning concepts, we have the following inductive definitions: for $\top$

$$\rho(\top, \bowtie c) = \begin{cases} \top & \text{if} \quad \bowtie c = \geq c \\ \top & \text{if} \quad \bowtie c = > c, c < 1 \\ \bot & \text{if} \quad \bowtie c = > 1 \\ \top & \text{if} \quad \bowtie c = \leq 1 \\ \bot & \text{if} \quad \bowtie c = \leq c, c < 1 \\ \bot & \text{if} \quad \bowtie c = < c \ . \end{cases}$$

For $\bot$,

$$\rho(\bot, \bowtie c) = \begin{cases} \top & \text{if} \quad \bowtie c = \geq 0 \\ \bot & \text{if} \quad \bowtie c = \geq c, c > 0 \\ \bot & \text{if} \quad \bowtie c = > c \\ \top & \text{if} \quad \bowtie c = \leq c \\ \top & \text{if} \quad \bowtie c = < c, c > 0 \\ \bot & \text{if} \quad \bowtie c = < 0 \ . \end{cases}$$

For concept name A,

$$\rho(A, \bowtie c) = A_{\bowtie c} \ .$$

For concept conjunction $C \sqcap D$,

$$\rho(C \sqcap D, \bowtie c) = \begin{cases} \rho(C, \bowtie c) \sqcap \rho(D, \bowtie c) & \text{if} \quad \bowtie \in \{\geq, >\} \\ \rho(C, \bowtie c) \sqcup \rho(D, \bowtie c) & \text{if} \quad \bowtie \in \{\leq, <\} \ . \end{cases}$$

For concept disjunction $C \sqcup D$,

$$\rho(C \sqcup D, \bowtie c) = \begin{cases} \rho(C, \bowtie c) \sqcup \rho(D, \bowtie c) & \text{if} \quad \bowtie \in \{\geq, >\} \\ \rho(C, \bowtie c) \sqcap \rho(D, \bowtie c) & \text{if} \quad \bowtie \in \{\leq, <\} \ . \end{cases}$$

For concept negation $\neg C$,

$$\rho(\neg C, \bowtie c) = \rho(C, \neg \bowtie 1 - c) \ .$$

where $\neg \geq = \leq$, $\neg < = >$, $\neg \leq = \geq$ and $\neg < = >$. For instance, the μassertion $\langle a{:}\neg C \geq c \rangle$ is mapped into the $\mathcal{ALC}$ assertion $a{:}C_{\leq 1-c}$.

For existential quantification $\exists R.C$,

$$\rho(\exists R.C, \bowtie c) = \begin{cases} \exists \rho(R, \bowtie c).\rho(C, \bowtie c) & \text{if} \quad \bowtie \in \{\geq, >\} \\ \forall \rho(R, - \bowtie c).\rho(C, \bowtie c) & \text{if} \quad \bowtie \in \{\leq, <\} \ . \end{cases}$$

where $- \leq = >$ and $- < = \geq$. For instance, the μassertion $\langle a{:}\exists R.C \geq c \rangle$ is mapped into the $\mathcal{ALC}$ assertion $a{:}\exists R_{\geq c}.C_{\geq c}$, while $\langle a{:}\exists R.C \leq c \rangle$ is mapped into $a{:}\forall R_{>c}.C_{\leq c}$.

Finally, for universal quantification $\forall R.C$,

$$\rho(\forall R.C, \bowtie c) = \begin{cases} \forall \rho(R, + \bowtie 1 - c).\rho(C, \bowtie c) & \text{if} \quad \bowtie \in \{\geq, >\} \\ \exists \rho(R, \neg \bowtie 1 - c).\rho(C, \bowtie c) & \text{if} \quad \bowtie \in \{\leq, <\} \ . \end{cases}$$

where $+ \geq = >$ and $+ > = \geq$. For instance, the μassertion $\langle a{:}\forall R.C \geq 0.7 \rangle$ in Example 2 is mapped into the $\mathcal{ALC}$ assertion $a{:}\forall R_{>0.3}.C_{\geq 0.7}$, while $\langle a{:}\forall R.C \leq c \rangle$ is mapped into $a{:}\exists R_{\geq 1-c}.C_{\leq c}$.

It is easily verified that for a set of μassertions $\mathcal{A}$, $|\sigma(\mathcal{A})|$ is linearly bounded by $|\mathcal{A}|$.

We conclude with the reduction of a terminological axiom τ in a terminology $\mathcal{T}$ of a μKB $\mu\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$ into a $\mathcal{ALC}$ terminology, $\kappa(\mu\mathcal{K}, \tau)$. Note that a terminological axiom in $\mu\mathcal{ALC}$ is reduced into a *set* of $\mathcal{ALC}$ terminological axioms. As for σ, we extend κ to a terminology $\mathcal{T}$ point-wise, i.e. $\kappa(\mu\mathcal{K}, \mathcal{T}) = \cup_{\tau \in \mathcal{T}} \kappa(\mu\mathcal{K}, \tau)$. $\kappa(\mu\mathcal{K}, \tau)$ is defined as follows.

For a concept specialization $C \sqsubseteq D$,

$$\begin{array}{rcl} \kappa(C \sqsubseteq D) & = & \bigcup_{c \in N^{\mu\mathcal{K}}, \bowtie \in \{\geq, >\}} \{\rho(C, \bowtie c) \sqsubseteq \rho(D, \bowtie c)\} \\ & & \bigcup_{c \in N^{\mu\mathcal{K}}, \bowtie \in \{\leq, <\}} \{\rho(D, \bowtie c) \sqsubseteq \rho(C, \bowtie c)\} \ . \end{array}$$

For instance, by relying on the μKB $\mu\mathcal{K}$ in Example 2, it can be verified that $\kappa(\mu\mathcal{K}, \mathcal{T})$ contains the $\mathcal{ALC}$ terminological axioms (e.g. for $c = 0.3$) $A_{\geq 0.3} \sqsubseteq \forall R_{>0.7}.B_{\leq 0.7}$ and $\exists R_{\geq 0.7}.B_{\geq 0.7} \sqsubseteq A_{\leq 0.3}$.

For a role specialization $R \sqsubseteq R'$,

$$\kappa(R \sqsubseteq R') \quad = \quad \bigcup_{c \in N^{\mu\mathcal{K}}, \bowtie \in \{\geq, >\}} \{\rho(R, \bowtie c) \sqsubseteq \rho(R', \bowtie c)\} \ .$$

Note that $|\kappa(\mu\mathcal{K}, \mathcal{T})|$ contains at most $6|\mathcal{T}||N^{\mu\mathcal{K}}|$ terminological axioms.

We have now all the ingredients to complete the reduction of a μKB into an $\mathcal{ALC}$ KB. Let $\mu\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$ be μKB. The *reduction* of $\mu\mathcal{K}$ into an $\mathcal{ALC}$ KB, denoted $\mathcal{K}(\mu\mathcal{K})$, is defined as

$$\mathcal{K}(\mu\mathcal{K}) = \langle \mathcal{T}(N^{\mu\mathcal{K}}) \cup \kappa(\mu\mathcal{K}, \mathcal{T}), \sigma(\mathcal{A}) \rangle \ .$$

Note that $|\mathcal{K}(\mu\mathcal{K})|$ is $O(|\mu\mathcal{K}|^2)$.

Example 4 *Consider the μKB of Example 2. We have already shown an excerpt of its reduction into $\mathcal{ALC}$ during this section. Due to space limitations, the whole reduction of $\mu\mathcal{K}$ cannot be represented in this paper. However, we have seen that $\mu\mathcal{K} \models \langle \alpha \geq 0.5 \rangle$, which means that the μKB $\mu\mathcal{K}' = \langle \mathcal{T}, \mathcal{A} \cup \{\langle \alpha < 0.5 \rangle\} \rangle$ is not satisfiable. Let us verify that indeed our reduction is satisfiability preserving, by verifying that $\mathcal{K}(\mu\mathcal{K}')$ is not satisfiable as well. First, let us note that $\sigma(\langle \alpha < 0.5 \rangle)$ is the assertion*

$$\sigma(\langle \alpha < 0.5 \rangle) = a{:}A_{<0.5} \sqcap \forall R_{\geq 0.5}.(B_{<0.5} \sqcup C_{<0.5}) \ . \tag{9}$$

We proceed similarly as for Example 2. We show that any model $\mathcal{I}$ satisfying $\mathcal{K}(\mu\mathcal{K}')$, where (9) *has been removed, does not satisfy* (9)*. Therefore, there cannot be any model of $\mathcal{K}(\mu\mathcal{K}')$. Indeed, as $A_{\geq 0.5} \sqcap A_{<0.5} \sqsubseteq \perp$ and $\top \sqsubseteq A_{\geq 0.5} \sqcup A_{<0.5}$ occur in the terminology of $\mathcal{K}(\mu\mathcal{K}')$, we have that either $a^{\mathcal{I}}$ is an instance of $(A_{\geq 0.5})^{\mathcal{I}}$ or $a^{\mathcal{I}}$ is an instance of $(A_{<0.5})^{\mathcal{I}}$. In the former case, $\mathcal{I}$ does not satisfy* (9)*. In the latter case, we note that the terminological axiom $\forall R.\neg B \sqsubseteq A$ belongs to $\mathcal{T}$ and, thus, $\rho(A, < 0.5) \sqsubseteq \rho(\forall R.\neg B, < 0.5)$, i.e. $A_{<0.5} \sqsubseteq \exists R_{>0.5}.B_{>0.5}$), belongs to the terminology of $\mathcal{K}(\mu\mathcal{K}')$. Therefore, as $a^{\mathcal{I}}$ is an instance of $(A_{<0.5})^{\mathcal{I}}$, $a^{\mathcal{I}}$ has an $(R_{>0.5})^{\mathcal{I}}$ successor d which is an instance of $(B_{>0.5})^{\mathcal{I}}$. But then, as $\langle a{:}\forall R.C \geq 0.7 \rangle$ occurs in $\mu\mathcal{K}$ and, thus, $a{:}\forall R_{>0.3}.C_{\geq 0.7}$ occurs in $\mathcal{K}(\mu\mathcal{K}')$, and $R_{>0.5} \sqsubseteq R_{>0.3}$ is axiom of $\mathcal{K}(\mu\mathcal{K}')$, it follows that d is also an instance of $(C_{\geq 0.7})^{\mathcal{I}}$. Now, it can easily verified that $a^{\mathcal{I}}$ cannot be an instance of $(\forall R_{\geq 0.5}.(B_{<0.5} \sqcup C_{<0.5}))^{\mathcal{I}}$ as $a^{\mathcal{I}}$ has an $(R_{\geq 0.5})^{\mathcal{I}}$ successor d ($(R_{>0.5})^{\mathcal{I}} \subseteq (R_{\geq 0.5})^{\mathcal{I}}$), which is neither an instance of $(B_{<0.5})^{\mathcal{I}}$ ($(B_{<0.5})^{\mathcal{I}} \cap (B_{>0.5})^{\mathcal{I}} = \emptyset$) nor of $(C_{<0.5})^{\mathcal{I}}$ ($(C_{<0.5})^{\mathcal{I}} \cap (C_{\geq 0.7})^{\mathcal{I}} = \emptyset$). Therefore, $\mathcal{I}$ does not satisfy* (9).

The following theorem can be shown, which establishes that our reduction is satisfiability preserving.

Theorem 1 *Let $\mu\mathcal{K}$ be μKB. Then $\mu\mathcal{K}$ is satisfiable iff the $\mathcal{ALC}$ KB $\mathcal{K}(\mu\mathcal{K})$ is satisfiable.*

Theorem 1, together with Equations (5) and (6), gives us also the possibility to reduce the entailment problem in $\mu\mathcal{ALC}$, to an entailment problem in $\mathcal{ALC}$.

Finally, concerning the BTVB problem, Equation (4) solves straightforwardly the case for 'role assertions'. On the other hand, for assertions of the form $a{:}C$, we have to solve the case of the *glb* only, as from it the *lub* can derived (see Equation 3). In [16] it has been shown that $glb(\mu\mathcal{K}, a{:}C) \in N^{\mu\mathcal{K}}$. Therefore, by a binary search on $N^{\mu\mathcal{K}}$, the value of $glb(\mu\mathcal{K}, \alpha)$ can be determined in at most $\log |N^{\mu\mathcal{K}}|$ entailment tests in $\mu\mathcal{ALC}$ and, thus, entailment tests in $\mathcal{ALC}$. Therefore, the BTVB problem can be reduced to $\mathcal{ALC}$ as well.

5 CONCLUSION

We have presented a reasoning preserving transformation of $\mu\mathcal{ALC}$ into classical $\mathcal{ALC}$, where general terminological axioms are allowed. This gives us immediately a new method to reason in $\mu\mathcal{ALC}$ by means of already existing DL systems.

Our primary line of future work consists in extending $\mu\mathcal{ALC}$ to more expressive DLs. Another line consists in applying our method to a generalization of $\mu\mathcal{ALC}$ in a lattice-theoretic way, i.e. in place of $[0, 1]$ we allow the use of any arbitrary (complete) lattice as truth-value set, like in [17].

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