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Electrostatic Imaging

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Abstract.

Integral equations describe a new approach to the problem of Electrostatic Imaging (EI). These equations are derived and discussed in the scope of Potential Theory from Maxwell's equations in quasi-static approximation. They establish an analytical link, in closed form, between the boundary-measured observables and the proper electrical quantities inside the given finite volume. This is the novelty of this work. Such integral equations act as a sound mathematical framework overcoming and unifying the many different approaches based on the numerical solution of the differential Dirichlet's problem and the related various approximations and improvements. Therefore, they allow a self-consistent analysis (i.e. an analysis without any *ad hoc* hypothesis) about various factors affecting any electrostatic imaging technique. For instance, they allow a direct discussion about the role of electrode shapes; they take into account (self-consistently) the effects of inclusions in the given volume; they permit a direct evaluation of the theoretical space-resolving power; they help to assess the reliability of any inversion method applied to recovered experimental data, etc. Up to now, present numerical methods, although widely used, usually prevent from such a conceptually simple and self-consistent approach, and are generally long far from real experimental systems. This report describes how the relevant integral equations are obtained, discusses their scope, outlines the topics which require further investigation for facing the main experimental difficulties in order to definitely assess the reliability of the method.

Keywords: Electrical Impedance Tomography, Electrostatic Tomography, Electrical Imaging, Electric Tomography, Capacitance Tomography, Electric Sensors, Electrical 3d-Visualization.

1. Introduction

The astounding amount of works available in literature [(1)] assesses the present interest about the so-called Electrostatic Imaging (EI). Just for giving an idea of the attention paid to this method, e. g. in biophysics, a query about *Electric Tomography* in PubMed [(2)] of NCBI gave 3718 results at the time when this note was written. This non-invasive imaging technique is very attractive not only in health diagnostics. Enjoying both an effective applicability to any electrically reactive system and a dramatic instrumental simplicity with respect to more conventional imaging apparatuses, it is potentially useful in many different fields: volume imaging, proximity sensors, process and flow controls, geophysics, superconductivity, bioelectricity, material analysis, etc.

The variety of names indicating what is substantially the *same* method is a proof about its wide circulation. Just for giving some examples, instead of EI we often find: ERT, Electrical Resistivity Tomography, or ET, Electrical Tomography, or EIT, Electrical Impedance Tomography, or EST, Electrostatic Tomography, or ECT, Electrical Capacitance Tomography, etc…

Data interpretation in EI is more difficult than in other non-electric tomographic diagnostics, since the 'response' of the system under observation is global. This means that the boundary-measured values of the observables describing the response of the system are related neither to any emission volume nor to any given propagation direction. Therefore, e.g., the precious tool of Radon's transform is useless.

The early theoretical efforts (in ERT, for geophysical research [(3)]) seem to be the works of R. E. Langer [1] and L.B. Slichter [2] about a layered medium. First basics for the relevant mathematical inverse problem are attributed to a work of A. P. Calderon "appeared only in the form of a crudely typed conference proceedings" in 1980 [3, 4]. Today, it is impossible to cite some works concerned with different numerical methods without badly disregarding the many other excellent papers proposed during the last years: the author apologizes for this. Nevertheless, a rather comprehensive effort appears in [5] and in the references therein.

Experimental papers show the same rich situation: from the early works about the so-called "antivision technology", through the realm of EFS, Electric Field Sensing, there is an impressive amount of contributions. Once more apologizing for the suggested arbitrary choice, notable works are in [6, 7]. Just because one is facing such an uncontrollable amount of papers, the author is fully aware that any further proposed remark about EI could result outdated or even definitely obsolete. In spite of this, in the author's opinion, most of the examined theoretical and experimental efforts performed up to now in EI do not rely on a physically reliable picture useful for a sound foundation of the proposed results. The problem refers to the present state-of-the-art which, roughly speaking, shows an unsatisfactory two-faced scenario.

[1] A non-advanced (i.e. non correctly assessed) web query could result in about 1 million answers.
[2] http://www.ncbi.nlm.nih.gov/entrez/query.fcgi?CMD=search&DB=pubmed
[3] http://en.wikipedia.org/wiki/Electrical_resistivity_tomography

Indeed, a first class of works attempts to solve the inverse problem by sophisticated numerical procedures where the values of the relevant material functions inside the given volume are iteratively adapted in order to reproduce the experimental data within the required accuracy. In other words, the *inversion* problem is solved by an iterative procedure that attempts to solve the *forward* one.

Such efforts usually disregard the possible existence of sharp conductive boundaries between the media inside the volume into consideration and are often based on an exceedingly ease use of Maxwell's equations and of constitutive relations. When the presence of sharp conductive boundaries is taken into account, the used techniques are generally not self-consistent [8, 9, and 10].

On the other side of the scenario, the "almost-purely experimental side", there are many efforts which try to deduce the space distribution of matter by means of various lumped-circuit models simulating capacitive couplings, induction or potential coefficients, more or less arbitrary current paths, etc. Also this procedure is not fully satisfactory and these experimental works are often engaged in a desperate challenge for explaining a relatively multifaceted reality by inadequate theoretical tools.

Other contributions [11, 12] do not substantially change the situation depicted above.

The aim of this note is not to claim a further solution of the inverse problem. The author has a different and more modest goal: he searches for a mathematical formulation of EI clearly assessing the link between the measured values and the relevant quantities inside the volume we are dealing with. This approach starts from Maxwell equations and, self-consistently, develops a mathematical picture in the usual language of integral equations. Although this language today is not so trendy in the realm of electrical imaging, the author believes that it is very suitable for discussing more meaningfully about the difficulties in real measurements.

For instance, the space resolving-power, the existence of macroscopic spurious capacitive couplings, the importance of the electrode shapes, etc., cannot be easily discussed in the scope of NtD or DtN maps.

On the contrary, integral equations describing the response of the system under investigation allow an easier evaluation of the overall feasibility of the method. Indeed, they can take into account and discuss directly and self-consistently the various figures of merit that characterize the reliability of the chosen experimental lay out (e.g. the space resolving power, the sensitivity, the presence of macroscopic competitive couplings, etc.).

This work arose as the first necessary step towards the full feasibility analysis about EI imaging in reacting systems of very different nature, where instrumental simplicity is mandatory (e.g. process tracking in reactors or the imaging of the combustion field in solid rocket propulsion, etc.). Therefore, the maximum of generality is required, when possible.
Therefore, in the following, we search for integral relations linking self-consistently the boundary-measured observables to the proper electrical quantities of the medium inside a given finite volume. To do so, we start from Maxwell's equations in the scope of potential theory, with the minimum of hypotheses.

The goal of this work is to derive integral equations governing electrostatic imaging in order to assess:

- 'how and which' observables *have* to be measured at the boundary;
- 'how' these observables are self-consistently linked to the relevant quantities of the electrically reactive medium inside the volume;

A further goal is to discuss the scope of the obtained equations and to sketch the future work.

In the following, section 2 is concerned with the integral formulation of the electrostatic problem for EI. The third section points out some remarks about the theoretical relations obtained, and discusses their usefulness in real situations. In section 4, some appliance considerations are discussed to some extent, while section 5 briefly summarizes the conclusions and sketches a few guidelines for future work.

2. Integral formulation

In order to deal with a rather general situation, let us consider in space a compact volume V_0 ideally enclosed by the surface S_0 ; into this volume there is an unknown medium, of course electrically reactive, the nature of which needs not to be specified. This means that the medium inside V_0 can be "composed" by solid and/or liquid and/or gaseous phases, either neutral or ionized, as well as dielectric or conductive. Of course, the possible presence of metallic bodies and of sharp boundaries is not disregarded. Our goal, roughly speaking, is to get some information about what's happening

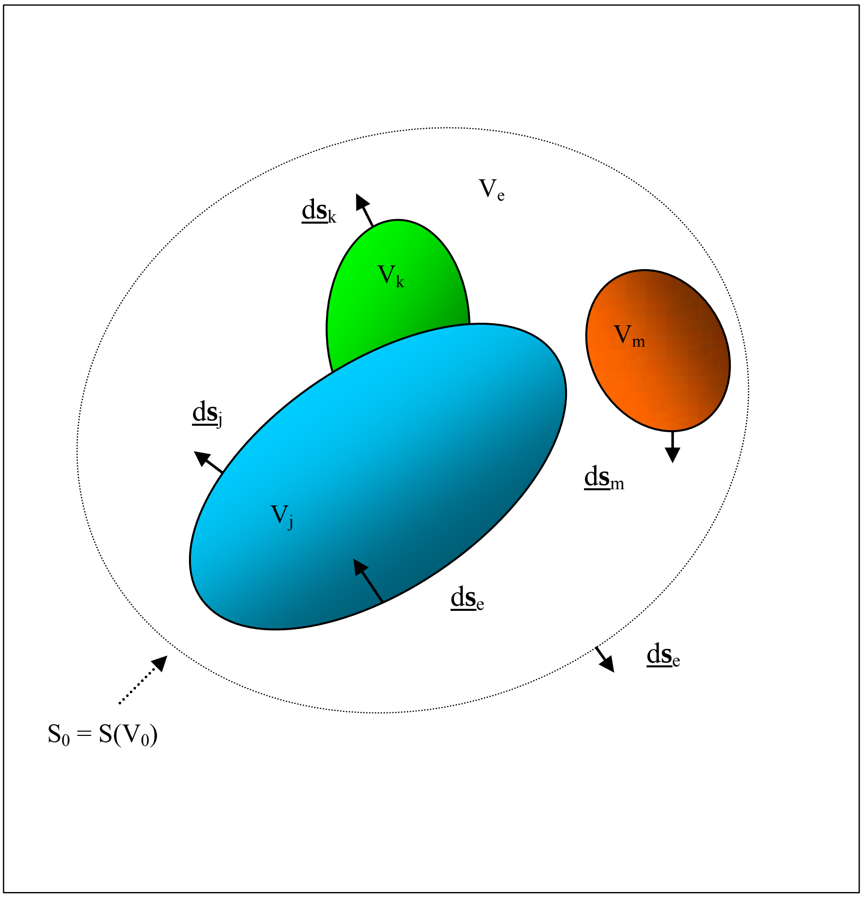

Fig. 1

inside V_0. For this aim, let us now firstly consider the simply connected piece of generic (but electrically reactive) material with volume V_j and surface $S_j = S(V_j)$ inside $S_0 = S(V_0)$, see Fig. 1.

This piece of material is assumed as the representative of any region, body or phase, etc., contained in the volume V_0. One possible starting point toward the integral formulation is the well-known Green's second identity, obtained by applying the divergence theorem to the auxiliary vector fields $\frac{\nabla'\varphi_j(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}$ and $\varphi_j(\mathbf{x}',t)\nabla'\frac{1}{|\mathbf{x}-\mathbf{x}'|}$, where the electric scalar potential φ_j is supposed continuous in V_j [4].

We obtain the relation:

$$G_j(\mathbf{x}) \equiv \int_{S_j} \frac{\mathbf{ds}_j' \cdot \nabla'\varphi_j'}{|\mathbf{x}'-\mathbf{x}|} + \int_{S_j} \varphi'_j \, \mathbf{ds}_j' \cdot \frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|^3} - \int_{V_j} \frac{\nabla'^2 \varphi_j'}{|\mathbf{x}'-\mathbf{x}|} \mathbf{dx}' = \begin{cases} 4\pi\varphi_j(\mathbf{x}) \ \text{if} \ \mathbf{x} \in V_j^0 \\ 2\pi\varphi_j(\mathbf{x}) \ \text{if} \ \mathbf{x} \in S_j \\ 0 \ \text{if} \ \mathbf{x} \notin \overline{V}_j \end{cases} \quad (1)$$

[4] For the sake of simplicity, from now on time dependence will be understood.

where V_j^0 is the internal part (5) (the 'interior') of V_j.

In (1) **dx'** is the volume element, $\mathbf{ds}'_j = \hat{\mathbf{n}}_j(\mathbf{x}')\,da(\mathbf{x}')$ is the vector surface-element. The local unit vector $\hat{\mathbf{n}}_j$, which is perpendicular to the closed surface $S_j = S(V_j)$, is directed outward with respect to V_j^0. Moreover, $\nabla\varphi_j \cdot \hat{\mathbf{n}}_j \equiv \left.\frac{\partial\varphi_j}{\partial\hat{\mathbf{n}}_j}\right|_0$ is supposed to be continuous on S_j, and the subscript "0" remind us that its value is calculated on the *inner* side of S_j.

From the mathematical point of view, the singular integrals appearing in (1) cause no problems if properly treated. The single-layer term is continuous everywhere. The volume-integral converges even if its polarity is not weak, and it is continuous everywhere. The second term in the l.h.s. of (1), a 'double layer' term, converges if the surface fulfils Liapounov's conditions, but it is mandatory to remind us its behaviour when $\mathbf{x} \to \tilde{\mathbf{x}} \in S_j$ along the normal to the surface:

$$(\lim \mathbf{x} \to \tilde{\mathbf{x}}) \int_{S_j} \varphi'_j\, \mathbf{ds}'_j \cdot \frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|^3} = \int_{S_j} \varphi'_j\, \mathbf{ds}'_j \cdot \frac{\mathbf{x}'-\tilde{\mathbf{x}}}{|\mathbf{x}'-\tilde{\mathbf{x}}|^3} \pm 2\pi\, \varphi_j(\tilde{\mathbf{x}})$$

From the inner side (+ sign) or from the outer side (- sign), respectively.

So, the double layer term is continuous in V_j^0 and on S_j; it is also continuous outside V_j^0 and on S_j; but it suffers a discontinuity of $4\pi\varphi(\mathbf{x})$ when crossing S_j at the point **x**, from the inner side to the outer side of the surface [13]. In conclusion, (1) is meaningful also for $\mathbf{x} \in S_j$ and we can attribute $G_j(\mathbf{x}) = 2\pi\varphi_j(\mathbf{x})$ if $\mathbf{x} \in S_j$.

Relation (1) is all what is necessary to discuss mathematically about electrostatic imaging; from now on, physics will give full details in order to assess definitely the real problem.

Let us consider the 'quasi-static approximation' in Maxwell's equations for the fields. Just for avoiding misunderstandings, the term used in electrodynamics is 'longitudinal approximation', where reference is made to the fact that the macroscopic electric field **E** and the macroscopic magnetic induction **B** have a negligible 'transverse' (i. e. radiative) component. (Roughly speaking, this means that, although $\partial/\partial t \neq 0$, the minimum wavelength λ considered, e. g. that of the scalar potential, is much greater than the maximum characteristic length a of the system under consideration: (a/λ) << 1).

In this approximation, the equations governing the vector and the scalar potentials are amenable to those governing the static situation: indeed, if $\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \approx \mathbf{0}$ then $\mathbf{E} \cong -\nabla\varphi$ in the regions considered.

By taking also into account that $\nabla^2\varphi = -4\pi\rho$, where ρ is the volume density of any kind of charge (i.e. bound or conduction charge), we substitute for **E** and ρ in equation (1) and differentiate term by term with respect to time.

Equation (1) becomes:

$$\frac{\partial G_j(\mathbf{x})}{\partial t} \equiv -\int_{S_j} \frac{\mathbf{ds}'_j \cdot \frac{\partial \mathbf{E}'_j}{\partial t}}{|\mathbf{x}'-\mathbf{x}|} + \int_{S_j} \frac{\partial \varphi'_j}{\partial t}\, \mathbf{ds}'_j \cdot \frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|^3} + 4\pi \int_{V_j} \frac{\frac{\partial \rho'_j}{\partial t}}{|\mathbf{x}'-\mathbf{x}|}\, \mathbf{dx}' = \begin{cases} 4\pi \frac{\partial \varphi'_j}{\partial t} & \text{if } \mathbf{x} \in V_j^0 \\ 2\pi \frac{\partial \varphi'_j}{\partial t} & \text{if } \mathbf{x} \in S_j \\ 0 & \text{if } \mathbf{x} \notin \overline{V}_j \end{cases} \qquad (2)$$

Now, by considering the equation for the macroscopic magnetic induction **B**:

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}$$

5 $V_j \equiv \overline{V}_j = V_j^0 \cup S_j$ must be interpreted as the closure of V_j^0.

Where $\mathbf{J}(\mathbf{x},t)$ is the surface current-density of any kind (i. e. $\mathbf{J}(\mathbf{x},t)=\mathbf{J}_c(\mathbf{x},t)$ conduction current for conductive regions, and $\mathbf{J}(\mathbf{x},t)=\mathbf{J}_p(\mathbf{x},t)$ polarisation current for dielectric regions), we take advantage by assuming the suitable definition for the displacement vector **D**:

$$\frac{\partial \mathbf{D}}{\partial t}=4\pi\mathbf{J}+\frac{\partial \mathbf{E}}{\partial t} \tag{3}$$

Which enjoys the property:

$$\nabla\cdot\frac{\partial \mathbf{D}}{\partial t}=0 \tag{3a}$$

Indeed, from the continuity equation for the current, we can substitute for $-\nabla'\cdot\mathbf{J}'_j=\frac{\partial \rho'_j}{\partial t}$ in the volume integral of (2) so that this equation turns out to be:

$$\frac{\partial G_j(\mathbf{x})}{\partial t}\equiv-\int_{S_j}\frac{\mathbf{ds}'_j\cdot\frac{\partial \mathbf{D}'_j}{\partial t}}{|\mathbf{x}'-\mathbf{x}|}+\int_{S_j}\frac{\partial \varphi'_j}{\partial t}\mathbf{ds}'_j\cdot\frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|^3}-4\pi\int_{V_j}\mathbf{J}'_j\cdot\frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|^3}\mathbf{dx}'=\begin{cases}4\pi\frac{\partial \varphi'_j}{\partial t} & \text{if } \mathbf{x}\in V_j^0\\ 2\pi\frac{\partial \varphi'_j}{\partial t} & \text{if } \mathbf{x}\in S_j\\ 0 & \text{if } \mathbf{x}\notin \overline{V}_j\end{cases} \tag{4}$$

By means of relation (4), properly written by each region shown in Fig.1, we obtain the integral relation linking self-consistently the boundary-measured observables on $S_0\equiv S(V_0)$ to the proper electrical quantities of the medium inside the *whole* finite volume $\mathbf{V}_0$.

To this aim, it is enough that we repeat the whole reasoning above for an equation similar to (4) describing the potential φ_k in the region V_k also shown in fig. 1.

We get:

$$\frac{\partial G_k(\mathbf{x})}{\partial t}\equiv-\int_{S_k}\frac{\mathbf{ds}'_k\cdot\frac{\partial \mathbf{D}'_k}{\partial t}}{|\mathbf{x}'-\mathbf{x}|}+\int_{S_k}\frac{\partial \varphi'_k}{\partial t}\mathbf{ds}'_k\cdot\frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|^3}-4\pi\int_{V_k}\mathbf{J}'_k\cdot\frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|^3}\mathbf{dx}'=\begin{cases}4\pi\frac{\partial \varphi'_k}{\partial t} & \text{if } \mathbf{x}\in V_k^0\\ 2\pi\frac{\partial \varphi'_k}{\partial t} & \text{if } \mathbf{x}\in S_k\\ 0 & \text{if } \mathbf{x}\notin \overline{V}_k\end{cases} \tag{4a}$$

Let us now define the function $\Phi_{jk}(\mathbf{x})$ obtained by summing up (4) and (4a):

$$4\pi\Phi_{jk}(\mathbf{x})\equiv G_j+G_k$$

We obtain:

$$4\pi\frac{\partial \Phi_{jk}}{\partial t}\equiv-\sum_{r=j,k}\int_{S_r/s_{jk}}\frac{\mathbf{ds}'_r\cdot\frac{\partial \mathbf{D}'_r}{\partial t}}{|\mathbf{x}'-\mathbf{x}|}+\sum_{r=j,k}\int_{S_r/s_{jk}}\frac{\partial \varphi'_r}{\partial t}\mathbf{ds}'_r\cdot\frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|^3}-4\pi\sum_{r=j,k}\int_{V_r}\mathbf{J}'_r\cdot\frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|^3}\mathbf{dx}' \tag{5}$$

In (5) s_{kj} (small and italic) is the surface portion shared by V_k and V_j ; S_k/s_{kj} indicates the surface $S(V_k)$ except this shared surface portion. The same convention is used for S_j/s_{kj}. Property (3a) has been used to eliminate the electric displacement contributions relevant to the opposite sides of s_{kj}, while continuity of the potential $\varphi_j(\mathbf{x})=\varphi_k(\mathbf{x})$ for $\mathbf{x}\in s_{kj}$ allows to eliminate the double-layer integrals. Moreover, function $\Phi_{jk}(\mathbf{x})$ is $\varphi_j(\mathbf{x})$ when $\mathbf{x}\in V_j^0$; it is $\varphi_k(\mathbf{x})$ when $\mathbf{x}\in V_k^0$, *and* it is continuous across s_{kj}.

In other words, (5) is valid for *any* $\mathbf{x} \in V^0_{kj}$, where $V^0_{kj} = V^0_k \cup V^0_j$ and for *any* **x** on s_{kj}, so we can use $\Phi_{jk}(\mathbf{x}) \equiv \Phi(\mathbf{x})$ instead of the various $\varphi(\mathbf{x})$, and **D** and **J** without any index.

When $\mathbf{x} \in S_{kj}$, where $S_{kj} = S(V_{kj})$, (5) becomes:

$$2\pi \frac{\partial \Phi}{\partial t} \equiv -\int_{S_{kj}} \frac{\mathbf{ds}' \cdot \frac{\partial \mathbf{D}'}{\partial t}}{|\mathbf{x}'-\mathbf{x}|} + \int_{S_{kj}} \frac{\partial \Phi'}{\partial t} \mathbf{ds}' \cdot \frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|^3} - 4\pi \int_{V_{kj}} \mathbf{J}' \cdot \frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|^3} \mathbf{dx}' \qquad (5a)$$

Where the vector area element **ds'** points outward with respect to V^0_{kj}.

Equation (5a) can easily be extended to the general case depicted in Fig. 1. It is enough to observe that S_0, the surface enclosing the four volumes V_j, V_k, V_m and V_e plays the role of S_{kj}, when the proper cancelling of the double-layer and of the electric displacement contributions on the opposite sides of all the surfaces encircling each volume has been taken into account. Therefore, we obtain the analogous of (5a), where we have rearranged terms for the sake of clarity:

$$-4\pi \int_{V_0} \mathbf{J}' \cdot \frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|^3} \mathbf{dx}' = 2\pi \frac{\partial \Phi(\mathbf{x})}{\partial t} - \int_{S_0} \frac{\partial \Phi'}{\partial t} \mathbf{ds}' \cdot \frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|^3} + \int_{S_0} \frac{\mathbf{ds}' \cdot \frac{\partial \mathbf{D}'}{\partial t}}{|\mathbf{x}'-\mathbf{x}|} \qquad (6)$$

For $\mathbf{x} \in S_0$, where $S_0 = S(V_0)$ and $V_0 = V_j \cup V_k \cup V_m \cup V_e$.

Equation (6) is the integral relation we had to search for.

It is a Fredholm's integral equation of the first kind, with antisymmetric kernel, for the unknown function **J**.

The r. h. s., in principle, is a known function: we can calculate it if we know *both* the surface distribution for the potential Φ on S_0 *and* the surface distribution of the vector $\frac{\partial \mathbf{D}}{\partial t}$ through S_0.

Therefore, for any $\mathbf{x} \in S_0$, we have *self-consistently* linked the results of *boundary* measurements to the integral of the space distribution of $\mathbf{J}' \cdot \frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|^3}$ *inside* the given volume V_0.

The boundary measurements must be performed by a discrete set of electrodes placed on the surface S_0. (In principle, an electrode is a perfect conductor sharing points with $\overline{V_0}$). Then, on the outer side of the electrode surface, $\frac{\partial \mathbf{D}}{\partial t} = 4\pi\, \mathbf{J}_c$, where $\mathbf{J}_c$ is the conduction current through the electrode. This is a measurable quantity.

In practice, as we shall see later, experimental efforts based on equation (6) will suffer several approximations and errors. For instance, approximations will arise when V_0 will be partitioned in order to solve equation (6) in the form of a linear system. Again, the finite dimensions of the electrodes will introduce further approximations. Moreover, competitive macroscopic capacitive couplings (see paragraph 4.4) could mask the measurements, etc.

In spite of this, equation (6) is self-consistent, i.e. without ad hoc hypotheses, and we can directly estimate the influence of any approximation assumed in order to perform the measure.

3. Remarks

3.1. Scope

It is not quite useless to emphasize the condition required for obtaining (6). Just because the problem is time dependent, the definition of the displacement vector **D** shows its suitability by the basic property $\nabla\cdot\frac{\partial \mathbf{D}}{\partial t}=0$. This not only made conveniently handy the required mathematical reasoning, but also teaches us that in static regime, $\partial/\partial t = 0$, we can obtain an integral relation of the form (8) *only if* we know a priori that $\nabla\cdot\mathbf{D}=0$ *everywhere* in V_0^0 when substituting the electric field.

For instance, if we know a priori that inside V_0 there may be only dielectric materials, then $\mathbf{J}(\mathbf{x}, t) = \mathbf{J}_p(\mathbf{x}, t) = \frac{\partial \mathbf{P}}{\partial t}$, where **P** is the polarization vector. Then for the displacement **D** in V_0^0 we set $\mathbf{D} = \mathbf{E}+4\pi\mathbf{P}$, which is the usual 'static' definition for the displacement vector which enjoys the property $\nabla\cdot\mathbf{D}=0$ when "free", or "conduction" charges are absent.

In this case, equation (6) becomes:

$$-4\pi\int_{V_0}\mathbf{P}(\mathbf{x}')\cdot\frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|^3}\mathbf{dx}' = 2\pi\Phi(\mathbf{x}) - \int_{S_0}\Phi'\,\mathbf{ds}'\cdot\frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|^3} + \int_{S_0}\frac{\mathbf{ds}'\cdot\mathbf{D}'}{|\mathbf{x}'-\mathbf{x}|} \quad \text{for } \mathbf{x}\in S_0\,; \tag{7}$$

Here, the quantities to be measured on S_0, i.e. at the electrodes, are the potential distribution Φ and the surface distribution of free-charge (i.e. conduction charge) density.

On the contrary, in the static situation when $\partial/\partial t = 0$, suppose that V_m be a conductor and V_e a dielectric. Then $\nabla\cdot\mathbf{D}\neq 0$ in V_0^0, due to the presence of conduction charge unbalance on the surface of the conductor. Therefore, in static conditions, a single-layer term appears in the r.h.s. of (7):

$$-\int_{S_e}\frac{\mathbf{D}_e'\cdot\mathbf{ds}_e' + \mathbf{D}_m'\cdot\mathbf{ds}_m'}{|\mathbf{x}-\mathbf{x}'|} = -4\pi\int_{S_m}\frac{\sigma_{cm}'\,da'}{|\mathbf{x}-\mathbf{x}'|}$$

Where σ_{cm} is the free-charge surface-density on the conductor. This single layer term depends from both the detailed form of the m-surface and the surface-charge density distribution on it. The net result is that with this extra term equation (7) becomes useless. Therefore, a trivial but important conclusion is that if a region V_0^0 can contain macroscopic conductive bodies (or conductive clusters, or conductive regions with sharp boundaries, etc.) equation (7) cannot be obtained in *static* conditions.

It also trivially clear that in this situation, where the shape and the position of the metal fragment (e.g. our m-body of Fig. 1) is unknown, the solving of the forward problem is an almost desperate business.

3.2. Pyroelectricity, f.e.m. sources, scrape-off region

An important point about (6) requires a definite assessment and it is concerned with the possible presence of f.e.m. sources inside the volume under consideration.

The crucial question is the following: can equation (6) describe this occurrence in an exhaustive and adequate manner?

Note that this question is particularly important in many reactive systems: for instance in biophysics, or in systems where the so-called pyroelectric phenomena are possible. From a general point of view, also the problem of a proper description for phenomena occurring in any scrape-off region at the inner boundary of V_0 is of crucial importance.

All these points require further and careful investigations.

3.3. Space limitations
Referring to the term on the l.h.s. of equation (6) or (7), it is worth observing that, for any $\mathbf{x} \in S_0$, our "probing arm" has the form $\frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}-\mathbf{x}'|^3}$. Therefore, the probing of large volumes is difficult.

3.4. Another integral relation
In quasi-static approximation, we can ask what kind of information we get by measuring only the surface distribution of the vector $\frac{\partial \mathbf{D}}{\partial t}$ on S_0. The answer comes from $\nabla \cdot \frac{\partial \mathbf{D}}{\partial t} = 0$. In fact, by the divergence theorem applied to the vector $\frac{\frac{\partial \mathbf{D}'}{\partial t}}{|\mathbf{x}'-\mathbf{x}|}$ we obtain:

$$\int_{V_0} \frac{\partial \mathbf{D}'}{\partial t} \cdot \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} \mathbf{dx}' = \int_{S_0} \frac{\mathbf{ds}' \cdot \frac{\partial \mathbf{D}'}{\partial t}}{|\mathbf{x}'-\mathbf{x}|} \tag{8}$$

To make an example, from (8) it is clear that measuring the surface distribution of the currents collected by several electrodes on S_0 offers some (approximated) information about $\partial \mathbf{D}/\partial t$ inside V_0.
Of course, in static condition the same equation holds only if $\nabla \cdot \mathbf{D} = 0$ in V_0, see point 3.1.
In passing, it is worth observing that when we have to take into account only conductive materials, the situation is just the opposite with respect to the general case. In fact, an equation similar to (8) arises in steady-current regimes when the media *completely* filling V_0 are conductors, even if they have different conductivities. In this case, the sole current is conductive $\mathbf{J} = \mathbf{J}_c$ and $\nabla \cdot \mathbf{J}_c = 0$ everywhere, since $\partial/\partial t = 0$. By applying the divergence theorem to the vector $\frac{\mathbf{J}_c'}{|\mathbf{x}'-\mathbf{x}|}$ we again obtain an equation of the form (8) with $\mathbf{J}_c$ in the place of $\partial \mathbf{D}/\partial t$.

3.5. Further remarks
Without boring the reader, many other similar considerations are possible according to the foreseen particular situation. But it also clear that only quasi-static approximation enjoys the maximum of generality in facing unpredictable events such as failures, breakings, cracks, inclusions, etc.
As a final and trivial remark, relation (4) can be obtained also by applying the divergence theorem to the sole auxiliary vector field $\varphi_j(\mathbf{x}',t)\nabla'\frac{1}{|\mathbf{x}-\mathbf{x}'|}$, provided that $-\frac{\partial}{\partial t}\nabla'\varphi_j' \equiv \frac{\partial \mathbf{E}_j'}{\partial t} = \frac{\partial \mathbf{D}_j'}{\partial t} - 4\pi \mathbf{J}_j'$ is substituted and rearranged in the resulting relation. Otherwise, the choice of Green's second identity could be more familiar, being the usual way to demonstrate the solution of Dirichlet's inner problem by means of the Green's function.

4. Appliance considerations

4.1. Basic relations
We refer to equations (6) and (8) as to the basic relations for electrostatic imaging.
For commodity, we rewrite them here, by using their Fourier-transformed versions (note that Fourier-transformed quantities are in italic):

$$\frac{4\pi}{i\omega}\int_{V_0} \boldsymbol{J}' \cdot \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} \mathbf{dx}' = 2\pi \Phi(\mathbf{x}) - \int_{S_0} \Phi' \, \mathbf{ds}' \cdot \frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|^3} + \int_{S_0} \frac{\mathbf{ds}' \cdot \boldsymbol{D}'}{|\mathbf{x}'-\mathbf{x}|} \tag{9a}$$

$$\int_{V_0} \boldsymbol{D}' \cdot \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} \mathbf{dx}' = \int_{S_0} \frac{\mathbf{ds}' \cdot \boldsymbol{D}'}{|\mathbf{x}'-\mathbf{x}|} \tag{9b}$$

Where $\mathbf{x} \in S_0$, and the r.h.s. contain measurable quantities.

An obvious remark: we need two equations because the space distribution of $\boldsymbol{J}$ alone is not enough for obtaining information about the space distribution of matter inside V_0. In fact, $\boldsymbol{J} = \boldsymbol{J}_c + \boldsymbol{J}_p = (\sigma + i\omega\chi)\boldsymbol{E}$ [6], where $\sigma(\mathbf{x})$ and $\chi(\mathbf{x})$ are the electric conductivity and the electric susceptibility functions, respectively. So we need also $\boldsymbol{D} = \frac{4\pi}{i\omega}\boldsymbol{J} + \boldsymbol{E}$ to say something about the space distribution of σ and/or χ. The same reasoning applies to the static case, if relations of the form (9a) and (9b) are allowed (see point 3.1).

4.2. Discretization

As said before for equation (6), relations (9a) and (9b), are Fredholm's integral equations of the first kind with antisymmetric kernel in a 3-dimensional domain. I disregard here the formidable task related to the problem of the existence, uniqueness and stability of the solution, which, to the author's knowledge, is again waiting for a definite assessment, by mathematicians [14].

In spite of this, it is spontaneous to ask ourselves if it is possible to extract some information about what is happening inside $S_0 = S(V_0)$ by partitioning the given volume V_0. Of course, at this point, any careful attention must be paid to the questions related to Hadamard's criteria about ill-posed problems, to the use of ill-conditioned matrices, to the (otherwise unavoidable) discontinuous sampling of boundary data, etc.

The l.h.s. in equation (9a), or in equation (9b), is of the form: $\int_V \boldsymbol{\eta} \cdot \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} \mathbf{dx}'$, where $\boldsymbol{\eta}$ indicates either $\boldsymbol{J}$ or $\boldsymbol{D}$ and V is the relevant volume. Let us make a partition of the volume V_0 in M connected volumes V_m. For any V_m, we set $\boldsymbol{\eta} = \overline{\boldsymbol{\eta}}_m + \delta\boldsymbol{\eta}_m$, where the volume average is $\overline{\boldsymbol{\eta}}_m$. By disregarding the fluctuation, the integral above can be approximated by:

$$\int_V \boldsymbol{\eta}' \cdot \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} \mathbf{dx}' \approx \sum_{m=1}^{M} \int_{V_m} \boldsymbol{\eta}' \cdot \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} \mathbf{dx}' \approx \sum_{m=1}^{M} \overline{\boldsymbol{\eta}}_m \cdot \int_{V_m} \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} \mathbf{dx}' = \sum_{m=1}^{M} \overline{\boldsymbol{\eta}}_m \cdot \int_{S_m} \frac{\mathbf{ds}_m'}{|\mathbf{x}-\mathbf{x}'|}$$

Now, suppose that there are n electrodes *on* S_0 (by definition, an electrode is a *perfect* conductor sharing points with $\overline{V}_0$) so that $S_0 = S_R \bigcup_{k=1}^{n} S_k$, where S_R is the "remaining" part of S_0, (i.e. the portion of S_0 free from electrodes), and S_k is the surface of the k-th electrode facing V_0^0. Let us consider any point $\mathbf{x} \in S_j$: since any electrode is equipotential, the double layer on the r.h.s. of (9a) can be approximated by:

$$-\int_{S_0} \Phi' \, \mathbf{ds}' \cdot \frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|^3} = -\sum_{k=1}^{n} \Phi_k \, \Omega_k(\mathbf{x}) - \int_{S_R} \Phi' \, \mathbf{ds}' \cdot \frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|^3} \approx -\sum_{k=1}^{n} \Phi_k \, \Omega_k(\mathbf{x})$$

Where $\Omega_k(\mathbf{x})$ is the solid angle subtended by the k^{th}-electrode at the point $\mathbf{x} \in S_j$. Note that the whole contribution of S_R is *brutally* disregarded. Moreover, it is worth observing that for flat electrodes, the electrode k = j contributes nothing. With the same mind, we can approximate the single layer in the r.h.s. of (9a) as:

$$\int_{S_0} \frac{\mathbf{ds}' \cdot \boldsymbol{D}'}{|\mathbf{x}'-\mathbf{x}|} \approx -\frac{4\pi}{i\omega} \sum_{k=1}^{n} \int_{S_k} \frac{j'_{\perp k} \, da'}{|\mathbf{x}'-\mathbf{x}|} + \int_{S_R} \frac{\mathbf{ds}' \cdot \boldsymbol{D}'}{|\mathbf{x}'-\mathbf{x}|} \approx -\frac{4\pi}{i\omega} \sum_{k=1}^{n} \int_{S_k} \frac{j'_{\perp k} \, da'}{|\mathbf{x}'-\mathbf{x}|}$$

Where $j_{\perp k}(\mathbf{x})$ is the (Fourier-transformed) current density on S_k *entering* V_0 *from* the k^{th} electrode. Since experimentally we deal with currents, for $j_{\perp k}(\mathbf{x})$ we are forced to write: $j_{\perp k}(\mathbf{x}) = \frac{\iota_k}{S_k} + \delta j_{\perp k}(\mathbf{x})$, where ι_k is the

[6] In the scope of Linear Response Theory: i.e. for linear media.

current leaving the k^th^ electrode *towards* V_0, (i.e. $\frac{\iota_k}{S_k}$ is the surface-average of the current density) and $\delta j_{\perp k}$ is the fluctuation. Once more, we brutally disregard this fluctuation, as well as the whole contribution of S_R.
In conclusion, a *rough* discretized version of (9a) is:

$$\sum_{m=1}^{M} \bar{\boldsymbol{J}}_m \cdot \int_{S_m} \frac{\mathbf{ds}'_m}{|\mathbf{x}-\mathbf{x}'|} \approx \frac{i\omega}{2}\Phi(\mathbf{x}) - \frac{i\omega}{4\pi}\sum_{k=1}^{n}\Phi_k\,\Omega_k(\mathbf{x}) - \sum_{k=1}^{n}\int_{S_k}\frac{j'_{\perp k}\,da'}{|\mathbf{x}'-\mathbf{x}|} \qquad (10)$$

where $\mathbf{x} \in S_j$, and j: 1→ n.

A similar relation can be written for equation (9b).

Relation (10) is a linear system for the 3M unknowns $\bar{\boldsymbol{J}}_{m,x}, \bar{\boldsymbol{J}}_{m,y}, \bar{\boldsymbol{J}}_{m,z}$, m: 1→M, which needs 3M 'observation points' $\mathbf{x} \in S_0$ with known potential and current density values. Of course, for a given partition of V_0, the observation points must be carefully chosen in order that the determinant of the characteristic 3M×3M matrix is different from zero.

Usually, the 'jth-observation point' is on the jth-electrode, so that $\Phi(\mathbf{x}_j) = \Phi_j$, and j: 1→3M = n. In spite of this, it is worth noting that the required number of observation points, strictly speaking, is *not* related to the number of electrodes facing the partitioned V_0. This originates from the fact that different observation points can be set on the same electrode, if it is large enough, so that potential and current values remain the same, but geometric quantities can vary significantly. Of course, it is also trivially clear that observation points ill located are unsuitable. For instance, observation points located according to a non-uniform surface distribution, or located on a jth electrode far from either a given V_m or a given kth electrode (k ≠ j) can cause serious errors. (See remark 3.2).

At the end of this paragraph, it is important to remark that relation (10) is only a coarse-grained approximation of the integral equation (9a).

Indeed, the whole contribution of S_R, when dealing with both potential and surface current density values, is disregarded.

Let's now consider the approximation about the term containing the potential value.

If the vessel containing V_0 is a conductor, its potential can be assumed as the reference value, and no approximation arises from disregarding the S_R contribution. Nevertheless, in this case, a serious problem arises from the strong capacity coupling of any electrode with the vessel itself, see section 4.4.

On the contrary, if the outer vessel is a dielectric, it cannot be assumed equipotential, and disregarding the contribution of S_R is completely arbitrary. The distribution of the potential on the surface S_R must be evaluated, e.g. by some interpolation. Moreover, in this case, there is a strong capacity coupling between the electrodes: this competitive effect must be taken into account.

Even less justifiable is the disregarding of the surface current density distribution on S_R: also, in this case a suitable and careful interpolation is required in order to avoid that the *"current tapping"* procedure is meaningless.

4.3. Lumped parameters

Sometimes, instead of (10), and similar relations mentioned above, reference is made to different operative schemes based on macroscopic parameters involved by a more or less well-defined "capacitance" or "capacitive-coupling" coefficients. Anyway, disregarding here a proper and rigorous treatment, one could note that relation (10) can be written in a form amenable to "lumped" quantities by surface averaging any term over the surface of the given j-electrode. Indeed, by taking into account that any electrode is equipotential, we obtain:

$$\sum_{m=1}^{M} \bar{\boldsymbol{J}}_m \cdot \mathbf{a}_{m,j} = \frac{i\omega}{2}\Phi_j - \frac{i\omega}{4\pi}\sum_{k=1}^{n}\Phi_k\,\Omega_{k,j} - \sum_{k=1}^{n}\iota_k\,b_{k,j} \qquad (11)$$

Where:

$$\mathbf{a}_{m,j} = \frac{1}{S_j}\int_{S_j} da \int_{S_m} \frac{\mathbf{ds}'_m}{|\mathbf{x}-\mathbf{x}'|}\ ; \qquad \Omega_{k,j} = \frac{1}{S_j}\int_{S_j} \Omega_{\boldsymbol{k}}(\mathbf{x})\, da\ ; \qquad b_{k,j} = \frac{1}{S_j S_k}\int_{S_j} da \int_{S_k} \frac{da'}{|\mathbf{x}'-\mathbf{x}|}\ ;$$

Relation (11) shows no advantages with respect to (10); on the contrary, it generates a real disadvantage by linking the number of volumes partitioning V_0 and the number of electrodes facing V_0. In fact, (11) requires 3M electrodes for writing down a linear system equivalent to that arising from (10).

4.4. Competitive capacitive couplings

A trivial but very important note: generally speaking, in the case of a conductive vessel surrounding V_0, the various ι_k cannot be measured directly, since there is at least an unavoidable spurious capacitive coupling C_k which originates from the pass-through from the electrode facing V_0 to the outside region beyond the vessel thickness. Note that when the electrodes are embedded in the vessel the competitive coupling has its maximum.

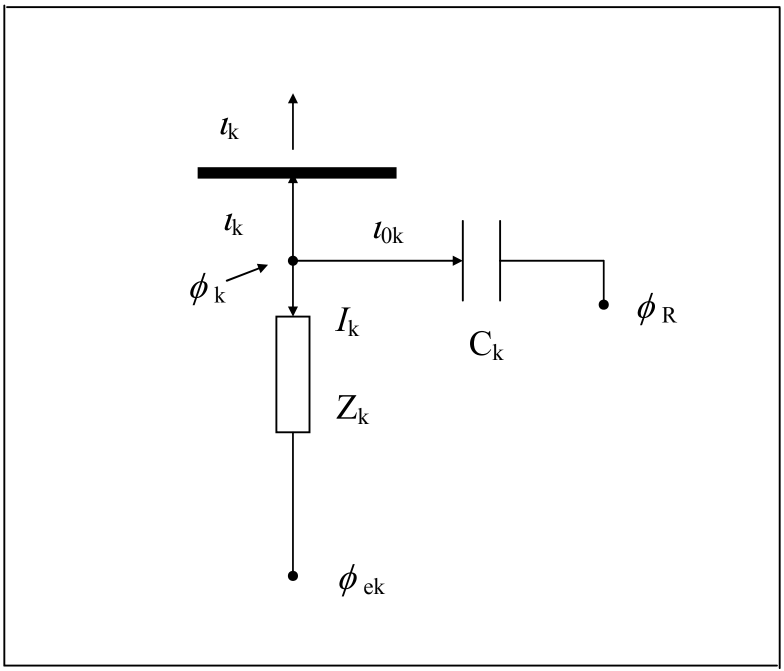

Fig. 2

For instance, an equivalent circuit for a given k-electrode is sketched in fig. 2. Z_k indicates the generic impedance loading the k^{th} electrode, Φ_{ek} is a known potential value and Φ_R is the (reference) potential of the vessel. Similar reasoning, with proper differences taken into account, is applicable to the case of dielectric vessels, where the main macroscopic spurious coupling is the intra-electrode one.

The presence of such dangerous capacitive couplings, to the author's knowledge, is usually disregarded in both numerical works and experimental layouts, in spite of the fact that they represent a serious limitation to the applicability of EI, especially in the case of conductive vessels, unless a proper experimental lay out is planned.

At this point, it's also worth observing that these parasitic couplings are relevant to the macroscopic electric circuit, and have nothing to do with the naïf modelling of "contact interactions" at the electrode surfaces, e.g. see paragraph 2.2 in [15] ("Modelling the electrodes").

Moreover, just referring to [15], if we had to consider the general case of a reacting media, the problem of the scrape-off region is really difficult, and its modelling is a very hard business, well beyond the scope of any simple "contact impedance" approximation.

5. Conclusions and future work

Integral equations (6) and (8) are basic relations for electric imaging.

They act in the scope of quasi-static approximation and connect rigorously the results of proper boundary-measurements to the space distributions of the total current density **J** (or of the displacement vector **D**) inside the given volume V_0. Disregarding here the important remark already mentioned in *3.2.*, they are self-consistent and rather general; for instance, they require no ad-hoc hypotheses, approximations, or iterative procedures to simulate the effect of inclusions in the given volume.

In spite of these advantages, it is advisable to point here some topics that require more investigation and work.

For the sake of brevity, from now on, let us refer to equation (6) (and/or to its Fourier-transform (9a)) only, since (6) and (8) are of the same type.

5.1. Existence and uniqueness of solution

First of all, there is the question about existence and uniqueness of solution for equation (6). To the author's knowledge, this question remains open. It is important to note that here reference is made to the general 3-D situation, disregarding any simplifying assumption, e.g. the assumption about the convexity of the related domain, which some thirty years ago allowed the fruitful discussion e.g. described by Smirnov [16]. It is worth observing that, in principle, the topic of existence and of uniqueness of solution for the inversion problem is twofold: from one hand it is mandatory to assess the existence and the uniqueness of the solution for equation (6) (or for equation (9a), which is the Fourier-transform of (6)); from the other hand, what is the link of this solution with the 'solution' offered by the system (10) resulting from discretization?

Referring to this last aspect, and completely disregarding the brutal approximations assumed, we could answer this question by saying that 'any good partition, which is arbitrary, shows a different appearance of the solution for (6), *provided* that this solution exists and is unique.'

Another point: the obvious suggestion is that, for a given shape of V_0, several partitions should be taken into consideration for improving the usefulness of relation (10).

As a last remark, it is useful to remind that, according to the system one is dealing with, the class of potentially interesting outcomes is usually forecast by defining the deviations from an assumed 'standard' or 'regular' situation. This helps to suggest the best choices amongst the many potentially useful partitions and electrode layouts.

5.2. Validity of equations and approximations

A further short discussion is relevant to the difference between integral equation (6) and its discretized and approximate form (10).

Indeed, relation (10) is nothing but a very 'natural' (i.e. 'spontaneous', "direct", ecc.) approximation of relation (9a) which, disregarding the weak requirements for Fourier transform, stems on fundamental equation (6). This equation (6) is the equation that "must hold".

Of course, I do not believe that (6) can be demonstrated to be wrong. In fact, (6) results from Green's identities (i.e., in essence, from one of the two mathematical proposition[7] which, together, fully describe the whole electrostatics: $\nabla^2 \frac{1}{|\mathbf{x}-\mathbf{x}'|} = -4\pi\delta(\mathbf{x}-\mathbf{x}')$). To this proposition we must add the continuity properties for vector fields and currents stated by Maxwell's equations in quasi-static approximation. In passing, let us note that (9b) is coming from the divergence theorem coupled with the same above cited continuities.

Therefore, whatever method is used to determine the space distribution of matter inside the volume of interest, the resulting situation, in principle, must *necessarily* be compatible with relation (6).

Then the basic question arises: if the picture resulting from experimental data and the inversion method is unavoidably affected by errors and approximations, how can we test this picture by means of the supposed exact, but purely theoretical, relation (6) ?

[7] The other proposition is the validity of superposition for electrostatic forces.

Indeed, let us assume that (9a) is exact, while the collected data are managed by equation (10) and, obviously, suffer experimental approximations.

Roughly speaking, these approximations are of three kinds: the first type comprises microscopic effects which are difficult to forecast and to describe, see paragraph 3.2. The second kind groups the "familiar" sources of experimental error which can be accounted for by the usual procedures, see e.g. the textbook: Philip R. Bevington 1969, Data Reduction and Error Analysis for the Physical Sciences, McGraw-Hill Book Company, New York.

Eventually, the third type of errors arise from the macroscopic approximations connected with the denumerable set of electrodes, the finite size of their surfaces, the more or less fine meshing of the volume of interest, the spurious capacity couplings masking the experimental effects, and from the *brutal* cuts used to obtain (10). All these *macroscopic* approximations can be evaluated by term by term comparison with the exact (9a), starting from a given, i.e.known, situation.

This means that, when we use (10), we can improve the experimental layout in order to approach a meaningful accuracy.

So, we get the point: relation (10) can act as control formula in the sense discussed above: no matter what method is used to determine the space distribution of the "content" inside the volume of interest, the resulting situation, in principle, must necessarily fulfil, within the total experimental error, the approximated quasi-static relation (10).

A trivial observation: the computation of geometric quantities in (10), although not particularly arduous, can be rather lengthy especially for fine-grained partitions, since some care must be used in order to avoid the apparent singularities. Otherwise, for a given shape of V_0 , i.e. for a definite vessel, and for given shapes and positions of the electrodes and/or of the observation points, the computation of double layers, single layers and solid angles is made once for each interesting partition, and need not to be repeated.

Finally, just for the sake of completeness, it is important to remind that, for a given partition of V_0 , the positions of the observation points must result in $\mathrm{Det} \neq 0$ for the characteristic $3M \times 3M$ matrix of the discrete system. The required recipe is simple: 'for a given partition, *any* observation point must 'see' *any* elemental volume V_m in a manner different with respect to the manner of *any* other observation point.'

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