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The concept of relation in Bolzano's *Betrachtungen* (1804)

0. Introduction

The "Considerations on some Objects of Elementary Geometry" or *Betrachtungen über einige Gegenstände der Elementargeometrie*[1] (henceforth BG) was issued in 1804 as first publication of the logical and mathematical Bernard Bolzano (1781-1848). The work aims at demonstrating the Euclidean notion of angle as well as the theory of parallel lines, by relying on a certain "theory of similarity", in view to avoid any recourse to the notion of plane. Bolzano's demonstration is expressly driven by a justificatory purpose with reference to basic geometrical notions, with the aim to account for their correctness at both conceptual and epistemological level. In this perspective, the BG anticipates the presentation of a specific method of demonstration, subsequently developed in 1810, within Bolzano's *Beyträge zu einer begründeteren Darstellung der Mathematik*[2] (henceforth BD), in the frame of which mathematical concepts are described as truths to be justified in their ground-consequence relation. Among the rules included in this method, stands out the requirement not to account for a concept (e.g. the angle) basing on a more complex concept (e.g. the plane); and also the requirement not to include, within the definition or demonstration of a specific concept, other concepts pertaining to different fields, meaning that e.g. the concept of movement is not to be included in the definition of any geometrical concepts. In this frame, the objects of elementary geometry are exactly treated as spatial objects, since belonging to the autonomous theory of space and hence dealing only with the spatial "dimension".

Precisely, Bolzano identifies geometry with a branch of applied mathematics, exactly as algebra, arithmetic and analysis, whereby all of them belong to pure mathematics; he does so in the wake of a renewed perspective of his time, within which geometry starts losing its paradigmatic statute within mathematics[3], in order to become nothing but mathematics applied to space. Notably the demonstrations developed in BG don't represent any valuable geometrical results[4]. Also, Bolzano's arguments about the parallel theory, which is the basis of V Euclidean postulate, don't fit into any debate on non-Euclidean geometries[5]: the Euclidean perspective is not even challenged. Nonetheless, it can be worth a close inspection on Bolzano's new approach in view to justify basic Euclidean notions and theories (e.g. the parallel theory), as relying on logical demonstrations rather than on axioms based on empirical evidence or intuitions (cfr BG I § 27). Of course Bolzano's attempt to justify geometry as a theory of space and, especially, as a conceptual science rather than an empirical science has been recognized as extremely valuable, in so far as able to disclose the topological properties of space, if not even "invent" the unknown domain of topology[6], while by all means providing seminal contributions to the development of dimension theory and of topology itself as a science. Even more, the BG insights can be considered a valuable source of logical arguments, anticipating key concepts at the basis of Bolzano's main logical notions and theories. Among these key concepts, the most fundamental one is

[1] BOLZANO, *Betrachtungen,* in: RUSS S., *The Mathematical Works of Bernard Bolzano*, 2004.

[2] BOLZANO, *Beyträge zu einer begründeteren Darstellung,* in: RUSS S., *The Mathematical Works of Bernard Bolzano*, 2004.

[3] KLINE, *Mathematical thought*, 1990.

[4] Cfr JOHNSON, *Prelude to Dimension Theory*, 1977.

[5] Cfr RUSS. S., *Geometry and Foundations*, 2004.

[6] Cfr JOHNSON, *Prelude to Dimension Theory*, p. 291.

exactly that of relation. The concept of relation is precisely at the basis of the concept of determination, which is a central concept in the frame of Bolzano's justification of geometrical objects, as well as of the space itself.

1. *The concept of Relation*

Bolzano doesn't challenge the Euclidean space, nonetheless he doesn't preserve any of the traditional logical justification of the Euclidean space, primarily refusing the notable Kantian perspective of a priori (pure) intuition of spatial properties. Neither geometrical objects (BG I §24) nor the most elementary spatial properties (e.g. distance) are a priori given (BG I §19), that is to say: both the space and the geometric objects contained in it must be someway "constituted". In this perspective, one crucial concept underlying all BG arguments is that of "relation", mainly exploited in its qualitative meaning, namely used in view to account for relations of equality, inequality and opposition between geometrical and spatial objects (i.e. points), with no need to rely on a strictly quantitative-metric perspective. Precisely, the relation of comparison is employed as a central notion, whereby the involved qualitative perspective expresses a certain logical valence able to support a refined quantitative framework, as well.

As it will be detailed, in the first section of BG the concept of relation is mainly applied to geometrical objects, through the main concepts of determination and similarity. The crucial and pervasive notion of determination is basically referred to the very conditions of possibility of geometrical objects. On the other side, the notion of similarity, inherited by Leibinz's perspective, is based on the concept of determination itself and therefrom employed for heuristic purposes. Also, the notion of similarity is exploited in view to account for the equivalence of distinct geometrical objects (e.g. two parallel straight lines), with the aim to avoid any empirical condition imposed by the traditional notion of congruence.

Both arguments concerning determination and similarity are crucial in Bolzano's geometrical investigations and further refined in subsequent works, particularly in *Die drei Probleme der Rectification, der Complanation und der Cubirung*[7] of 1817 (henceforth DP). Differently from BG, in this later work both concepts are also treated in the frame of quantitative-metric relations[8], e.g. with reference to the notion of analytic (continuous) function, to account for a new concept of curve derivation. Nonetheless, as it will be cleared, the basic logical significance of the concept of determination, as included in the concept of similarity, doesn't change: both within a qualitative and a quantitative-metric perspective the reference is to the possibility that two different but similar geometrical objects (e.g. two straight lines, or two curve lines) can be constituted and quantitatively compared basing on the same "rule of formation", which precisely in the work of 1817 is a continuous function[9]. What is relevant to highlight in this regard is the efficacy of the BG qualitative perspective, in the frame of which the concept of determination, by relying on qualitative relations, is still able to pave the way for more complex metric notions.

1.1 The notion of Determination

As well as never specified as a proper notion, the concept of determination (*Bestimmung*) is introduced and pervasively exploited in BG, not only in its defining or descriptive meaning but also and especially in its constitutive meaning, namely in the meaning of "cause" of a certain geometrical or spatial property. Precisely, in BG the concept of determination usually refers to the qualitative derivation of an objectual property from other objects' properties. Importantly in this frame, the concept of determination is always exploited within a qualitative perspective.

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[7] BOLZANO, *Die drey Probleme,* in: RUSS S., *The Mathematical Works of Bernard Bolzano*, 2004.

[8] SEBESTIK, *Logique et Mathématique*, 1992*,* pp. 33-55.

[9] RUSS., *Early Analysis*, pp. 151-152.

As mentioned, Bolzano intends to demonstrate the Euclidean theory of parallels, by avoiding any recourse to the notion of plane and to traditional parallel lines properties: on this purpose, in the first section of BG he develops a theory of similarity of triangles. From the outset, Bolzano points out the qualitative perspective of his demonstration, by defining the notion of angle as a qualitative rather than quantitative notion (BG I §6, §9). Hence, an angle is not primarily a quantity, i.e. an arithmetical sum of radians, in need for a plane to exist and to be possibly demonstrated; rather "we can think of an angle as two lines with a common endpoint" (BG I §6), without having to think of a surface or of a line motion. Within this frame, the angle is defined as a property, precisely as the predicative property (*Prädicat*) of two straight lines having one of their extreme points in common (BG I §1); also, it's defined as the property (*Eigenschaft*) of two directions (sharing one point: BG I §2). The outlined perspective is crucial and extremely insightful: far from being immediately defined as an object, the angle is above all a property; precisely the angle is a predicable property in so far as being an objectual property. The argument is worth a separated discussion, to be elsewhere developed. What is relevant to highlight for the present purposes is that the property "angle" is defined as "determined" if its sides (arms) are determined (BG I §3), that is to say: the property angle is caused by specific constitutive conditions. Here the geometrical property is described as determined within a constitutive rather than descriptive perspective. Hence, it's assumed a constitutive meaning of the concept of determination, going far beyond the meaning of predicate assignment to an object and rather encompassing the broader meaning of "formation" of an object property.

Interestingly, Bolzano's concept of determination is, in all its meanings, symmetrically associated with the concept of property. As for the concept of determination, the concept of property is never introduced as a proper notion in BG; nonetheless it's extensively discussed within Bolzano's Theory of Science[10] (WL § 80) as a relational notion directly associated with the notion of determination (WL §80.2). Precisely, in WL any attribute or property is defined as a relation-based determination. The notion of attribute or property is also significantly broken down in two different sub-notions: the variable property of an object, arising from external relations with respect to other objects; and the invariable or inner property (*Beschaffenheit*) of an object, possibly arising from internal or constitutive relations. Both kinds of relations are interestingly intertwined in WL arguments, in a way for them to deserve a separate discussion; suffices here to consider that the reference to the concept of property in BG can be considered as mainly concerning the "invariable" characterizing quality of a geometrical object, possibly deriving from its proper constitutive conditions. In this frame we can consider that the concept of property is variously mentioned throughout both BG sections, mostly as *Eigenschaft* (BG I § 2, §6, §11, §24, § 33, § 54, §57; BG II §1), namely in the meaning of characteristic or distinctive property of an object; and also as *Beschaffenheit* (BG II §39), namely in its meaning of proper quality or nature of an object.

The relation-based aspect of the concept of determination is clearly illustrated in BG. Exactly in this regard it's worth noticing that the geometrical object is constituted as a "system" of other geometrical (or spatial) objects: the angle is e.g. determined as the property (*Eigenschaft*) of a system of two lines sharing a point (BG I §1, §2, §7, §12), or rather as "everything perceivable" in the system of two directions (BG I §22). Importantly, the recurrent concept of system in BG plausibly indicates specific relational conditions between two or more terms, hence making the case for the determination itself to configure as a relational condition. In this frame, the reference to the "determining conditions" of a geometrical object coincides with the reference to certain constitutive conditions in so far as relational conditions of a system: e.g. the angle and, secondary, the length of the relating two lines (sides) suffice to determine the triangle as a relational system: if these are included, everything in the system is determined (BG I § 12). Indeed, such constitutive relations involve components or "pieces" of the object to be determined: the data (*Angabe*) on which the constitutive relations of an object develop (BG I § 10) are the "determining pieces" of that object. More precisely, it's assumed that specific determining conditions of an object depend on the specific (quantitative) properties of certain components of that object: e.g. two data, i.e. an angle and one of its sides, can be determining pieces of a triangle if the ratio

[10] Cfr. Bolzano, *Wissenschaftslehre,* in: Rusnock P., George R., *Bernard Bolzano: Theory of Science*, vol 3, 2014 (henceforth WL).

between the given side and the other side of the angle is also given (BG I §21). Clearly, the perspective underlying the BG concept of system is that of reciprocally influencing relational terms, making the case for a part-whole relation perspective: the properties of certain determining pieces cause and orient necessary relations between themselves, also engendering necessary systemic relational conditions, to which missing pieces of the same kind are subject. The same perspective is accounted in BG paragraphs dedicated to the theory of similarity, whereby components, or pieces of geometrical objects are also definitely referred to as parts (BG I §1, §16).

Exactly in this perspective, we can also consider geometrical determination as a process within which more basic systems of relations are able to cause or engender more complex systems of relations: the assumption is supported by the BG account of geometrical objects as a progressive hierarchical constitution of systemic relations. Overall, as rightly observed[11], the concept of determination identifying the engendering of relational systems, is to be treated within the metamathematical perspective subsequently developed in the *Allgemaine Mathesis* of 1810[12] (henceforth AM). Indeed, such relational-constitutive aspects of the concept of determination anticipate several logical, more precisely ontological arguments pertaining to the very conditions of possibility of geometrical and other mathematical objects, exactly as of any other objects in so far as objects of thought. A pertaining discussion can be elsewhere developed.

Also, the constitutive significance of Bolzano's notion of determination is all the more worth to be investigated in the frame of Bolzano's epistemology. By assuming the possibility for an object to be constituted by other distinct objects and also, by considering that specific determining conditions have other conditions as their consequence (BG I § 64); then it's also possible to assume the possibility to trace back the determining pieces or rather constitutive components of any (geometrical) object, within a reverse order of determining relations. This assumed perspective exactly allows for Bolzano's purpose of "conceptual correctness"[13], namely for the purpose of a correct definition of concepts, precisely in view to produce correct (truthful) judgements (BD). In this perspective, the correct assignment of predicative properties to an object, which is to say the correct predication of an object property, depends on the possibility to keep track of the determining conditions and precisely of all "determining pieces" constituting that very object. A similar perspective underlies Bolzano's BG as much as his BD and precisely his notable theory of grounding[14].

It's worth considering that Bolzano's notion of ground (*Grund*) is without question defined as an epistemological notion (BD II §2, §11, §12) and this is also the way it's generally discussed; nonetheless, Bolzano himself refers to the theory of ground-consequence (*Grund und Folge*) as a legacy of the ontology domain (BD I §13), in the frame of which it's made reference to the possibility that objects are "produced through some ground" as well as according to "certain general conditions" (BD I §13). This perspective exactly suggests how Bolzano's notion of grounding, clearly connected to the concept of determination, is worth to be considered as more than an epistemological notion and therefore worth to be investigated as an ontological notion, on both the constitutive and the heuristic sides. Interestingly in this regard, Bolzano considers the theory of ground-consequence to be developed in the frame of Aetiology (BD I §13), namely in the frame of the science investigating the correct definition of similar objects in so far as determined by similar determining conditions: "similar grounds have similar consequences" (BD I §13). Exactly in this direction, the BG builds a theory of geometric similarity, basing on the notion of determination and by all evidence assuming that any heuristic perspective, as the possibility of any epistemological correctness, must first rely on the possibility for an objectual property to be constituted and, secondary, on the possibility for the pertaining constitutive (determining) conditions to be traced back.

Bolzano's theory of similarity, as exposed within the BG, doesn't properly express any aetiological purpose, rather it expresses a justificatory purpose concerning similar objects, in a way to exclude aliens or

[11] Sebestik, *Logique et Mathématique,* p. 47.
[12] Bolzano, *Allgemeine Mathesis*, in: Maigné C. et Sebestik J., *Bernard Bolzano: Premiers écrits*, 2010.
[13] Russ, *Geometry and Foundations*, p. 20.
[14] Sebestik, *Logique et Mathématique,* p. 48.

empirical demonstrative concepts of similarity (BG I § 24). According to this theory, which can be considered as an interesting source of logical and ontological arguments concerning Bolzano's perspective on objects constitution, distinct objects can be demonstrated as similar in as much as both determined by similar determining conditions.

1.2 The notion of Similarity

Analogously to Leibniz's and Wolff's perspective on similarity[15], as clearly recognized by Bolzano himself (DP §31); precisely, analogously to the arguments described by Wolff as a reader of Leibniz[16], Bolzano develops his theory of similarity. The BG notion of similarity is developed as an expedient to demonstrate the theory of parallels[17], notably concerning the condition that through a point external to a straight line, exactly one straight line can be drawn parallel to the given one. Plausibly, Bolzano's justificatory perspective in this regard refers to the possibility to demonstrate the equality of certain spatial properties belonging to two distinct geometrical objects: the same inclination of two straight lines with different length.

Since refusing to accept any kind of intuitive or empirical evidence of equality, therefore any empirical aspect of the concept of comparison, Bolzano challenges the geometrical notion of congruence as a possible demonstration of the equality of geometrical objects, by considering this an empirical and superfluous notion (BG I § 49). Precisely, congruence is empirical in so far as involving the concept of superposition of geometrical objects, in turn based on the concept of motion on a plane. On the other side, the notion is superfluous in so far as any exact coincidence of spatial properties makes the two objects undistinguishable, both expressing identical spatial properties and therefore an identity of the geometrical object itself: this would nullify the meaning of congruence as a way to demonstrating the equality of two different objects, whereby both objects must remain distinct as a condition for comparison (BG II § 1). In the wake of Wolff's argument (BG I §24) and by exploiting the constitutive potential of the concept of determination, Bolzano considers that there is no need to rely on any empirical comparison in view to demonstrate the equality (isometry) of two distinct objects: suffices it to consider that the determining conditions, precisely the determining pieces of both objects are equal one another. On this basis Bolzano replaces the concept of congruence with the concept of "equal determining conditions", more precisely with the equality of determining pieces, as a necessary condition in view to demonstrate the "property" of a geometrical object to be equal to another one, hence excluding from the comparison any recourse to empirical concepts[18].

In this frame Bolzano develops his theories of equality and similarity: on the one side, objects whose determining pieces are equal are themselves equal (BG I § 18); on the other side, objects whose determining pieces are similar are themselves similar (BG I § 17). Importantly, Bolzano refines his arguments about the distinction between the concepts of similarity and equality, as respectively concerning the equality of objects in their qualitative and quantitative properties or, conversely, the inequality of objects, namely their difference, *salva qualitate* and *salva quantitate*. Overall, it's assumed that two distinct objects, while differing in some respects, still can feature equal qualitative or quantitative properties, allowing for both objects to be defined as similar or equal to one another. Importantly, in this frame, Bolzano considers that objects, e.g. triangles being both similar and equal are to be defined simply "equal" (BG II §11), since they are equal both in their qualitative and quantitative properties. The same perspective is confirmed by Bolzano's later arguments about similarity (DP §22), whereby similar and equal objects are to be better defined as "geometrical equal", namely as objects whose comparison results in an equality of "all their characteristics" (i.e. properties), ultimately traceable to an equality of their (intrinsic and extrinsic) distance properties. Exactly in this perspective, already in 1804 Bolzano considers that, within a geometrical investigation, while identical objects are as such in so far as compared with themselves, any two different objects are to be defined as being either equal or unequal (BG

[15] Cfr COSTANTINI F., Definitions by abstraction, 2024.
[16] RUSS, *Geometry and Foundations*, p. 17.
[17] SEBESTIK, *Logique et Mathématique,* pp. 46-47.
[18] RUSS, *Geometry and Foundations*, pp. 17-19.

II § 1) and nothing else. Significantly, according to this argument, unequal geometrical objects must also include similar objects, whose quantitative comparison, i.e. the comparison between their extrinsic quantitative properties, seems to be regarded as the only objective condition for their descriptive "determination".

On the background of this argument, Bolzano nonetheless defines his theories of equality and similarity as two different theories, respectively relying on the equality of components and on the equality of the intrinsic "characteristics" of different geometrical objects. As mentioned, the constitutive aspect of the concept of determination is crucial in this regard, being precisely involved in the specification of comparison terms.

In this frame, the equality of two geometrical objects is defined as the equality of their respective determining pieces (BG I §14), since the (geometrical) equality of reference data (*Angabe*) cannot but engender related equal components, as well as equal, intrinsic systemic relations; hence, for two triangles to be equal it's sufficient that two of their respective angles and one of their respective (relating) sides are equal to one another (BG I § 10). On the other side, two geometrical objects are defined as similar, "if all the characteristics which arise from the comparison of the parts of each one *among themselves*, are equal in both" (*wenn alle Merkmale, die aus der Vergleichung der Theile eines jeden unter sich hervorgehen, in beyden gleich sind*) (BG I § 16). As elsewhere confirmed (BG I §22), Bolzano's description refers to an equality of objects in regard to the "relational configuration" of their respective components, also expressly defined as parts; precisely, two similar triangles with similar angles α and a are required to be reciprocally proportionate in all their sides, "otherwise the comparison of the parts of angle α among themselves would not give rise to exactly the same idea as that of the parts of the angle a" (BG I §22). Hence, what is required for two triangles to be similar is that the proportion (ratio) between the sides of the first triangle equals the proportion (ratio) between the sides of the second triangle; namely, their similarity coincides with an equality of the way their respective parts intrinsically relate, which is to say they are required to be equal in the ratios governing their respective internal parts. Similar geometrical objects are therefore required to be equal, not in in their parts, rather in the "characteristics" or properties deriving from their intrinsic ratios. With reference to geometrical objects such a ratio exactly configures as an intrinsic proportion between distances, allowing for the geometrical equality of angles. Hence, Bolzano's statement that "*similar* angles are *equal*" (BG I §22) acquires significance, exactly as the statement that "in *similar* triangles, the angles opposite proportional sides are *equal*" (BG I §23).

Overall, as subsequently confirmed (DP §22), similarity is maintained as the equality between the intrinsic characteristics of geometrical objects, namely as the equality of that kind of characteristics, which don't need any extrinsic comparison in order to exist: differently from quantitative properties (e.g. distance), the way the parts of an object reciprocally relate is responsible for immediately recognizable characteristics or properties pertaining to that very object, independently from any extrinsic comparison. Hence similarity assumes an extrinsic comparison of the intrinsic characteristics or properties of two objects. Such characteristics or properties are expressed by the word *Merkmale* (BG I §16), also elsewhere used (BG I §16, §17, §22 – BG II §40) and always referring to some sorts of functional traits deriving from the relation between the parts of an object; nonetheless, they are also referred to as *Eigenschaften*: "if in two objects all properties (*Eigenschaften*), which can be observed from the comparison of the parts of each one among themselves are equal, then it only follows that the two things are similar" (BG I §24). Overall, either if *Merkmale* or *Eigenschaften,* Bolzano's references by all evidence concern specific properties "arising" from the relation between the parts of an object between themselves. The argument assumes that such characteristics, or properties, exactly emerge in so far as belonging to a whole, constituted by a specific relation of comparison between parts. In this regard, it's significant the reference to the notion of comparison, as also subsequently treated in the AM.

As mentioned, the relation-based constitutivity of the determination assumes a particular significance within Bolzano's notion of similarity. As the constitutive potential of systemic relations allows for the possibility to engender similar objects with equal properties, by applying the same kind of ratio between their internal parts; then we can affirm that Bolzano's notion of similarity assumes the possibility for different but similar objects to be determined by identical formal conditions. In fact, by outlining a perspective within which two different (unequal) objects are equal as for the qualitative properties stemming from the comparison between their own parts, namely as for the distinctive properties characterizing their respective internal relations, Bolzano clearly assumes an identity of the formal aspects characterizing such internal relations, in a way that “everything which can be observed in the one system by the comparison of its parts is also equal in the other” (BG I §24). It’s assumed that in two similar objects, while their respective determining pieces or parts can change e.g. in size, nonetheless the *kind* of ratio between such parts must always be equal to itself, i.e. it must be identical in both objects. Hence, all similar objects are equal in their characteristic properties as properties arising from identical formal aspects of their internal relations; still, they can be unequal objects. Exactly in this perspective, Bolzano plausibly understands similar objects as “variations” of one and the same “formal object”. Interestingly, all the above-mentioned arguments make the case for a formal ontology within Bolzano’s theory, to be investigated also with respect to the mature arguments developed in his WL. Regardless any possible doubts about the originality of Bolzano’s notion of similarity[19], it’s worth to highlight how the BG recognizes the importance of such formal aspects concerning the relation-based constitutivity of geometrical objects (as possibly any kind of relation-based constitutivity), by referring to the notion of “genus” as to the generality of certain kinds of determining conditions, while on the other side referring to the concrete variations of such formal constitutivity in terms of “species”. In this frame, differently from usual proofs of similarity within Eucliedean geometry, mainly relying on the inductive comparison between individual objects, Bolzano focuses on the role of a genus (abstract or formal object), able to vary in different species (concrete objects) and therefore able to “demonstrate what is true of the genus without using an induction from the individual species” (BG II §24), since “volumes of similar solids vary as the cubes of similar sides”, exactly as “surfaces vary as surfaces; lines (curved) vary as lines” (BG II §24). Interestingly, Bolzano’s arguments appear as a prelude to the differential geometry perspective.

Nonetheless, Bolzano’s notion of similarity, at least in BG doesn’t exceed the purpose to conceptually justify the Euclidean elementary geometry, being therefore devoted to the demonstration of similarity between two straight lines (BG I §30), as well as between two parallel lines (BG I §40), or rather to the demonstration of the property of two lines to be parallels (BG I §54). Moreover, in BG Bolzano significantly resorts to his theory of similarity for heuristic purposes, namely in view to account for the determination of “possible objects” (BG I §24). Even if there’s no a-priori knowledge of any determinate spatial objects, nonetheless it’s possible to assume a-priori necessary (formal) properties of objects, by relying on similarity relations, which end up being translated into a quantitative comparison (equality or inequality) between ratios. In this frame, Bolzano’s arguments suggest that specific relational conditions can account as much for the constitution of (possible) geometrical objects, as for the constitution of space itself.

2. *The space as relation*

In the second section of BG Bolzano’s both concepts of determination and similarity are applied in view to justify the geometrical object “straight line”, by means of more basic geometrical or rather “spatial” objects, i.e. that of distance and direction. The same arguments are exploited in view to provide a simpler demonstration of the object “triangle”. Also in the second BG section, Bolzano’s justificatory purpose (BG II §4) is able to disclose innovative notions, particularly valuable under a logical point of view.

As mentioned, Bolzano doesn’t refuse the notion of Euclidean space, nonetheless he refuses the notion of an a-priori space (BG I §19), hence refusing the Kantian logical perspective and precisely Kant’s notion of (pure a-priori) intuition as the more fitting justificatory perspective in this regard. Overall, Bolzano’s rejects any possibility of a-priori given determinations of space; rather, possibly influenced by the Leibnizian perspective, he considers the science of space as a pure conceptual science[20]. Far from being empirical objects, lines and surfaces are “objects of thought” (BG II §6), i.e. they are conceptual determinations, according to Bolzano’s arguments. Geometrical objects are in fact assumed as relation-based determinations, exactly in so far as conceptual determinations. Precisely, Bolzano’s arguments refer to the notion of space itself as

[19] RUSS, *Geometry and Foundations*, p. 16-17.
[20] Cfr. JOHNSON, *Prelude to Dimension Theory*, 1977; RUSS, *Geometry and Foundations*, p. 15.

consituted by relations between primitive elements[21], in a way which would make particularly interesting to investigate a possible parallelism between Bolzano's perspective and Leibniz's *Analysis Situs*.

By relying on the notion of point, as primitive notion of elementary geometry, in place of the notion of line, the BG provides an important theoretical innovation within Euclidean geometry. The point is exactly defined as a primitive spatial mark (*Merkmals eines raumes - semeion*), which is itself not part of the space (BG II §5). In this perspective, the point is intended as a local sign[22] within Euclidean tridimensionality and, far from being any proper spatial object, is in fact to be intended as a sort of imagery object (BG II §5), functioning as a mere relational term. Hence, the point isn't any elementary portion of spatiality; rather it is a conceptual expedient, able to support the constitution of any possible spatial determination in as much as relation-based determination. Points are assumed as equal between themselves, since the equality relation is assumed as condition for any spatial determination (BG II §3), as for the determination of a point itself (BG II §6). Nonetheless, as mentioned, relations between proper spatial properties can generally vary as equality and inequality relations (BG II §1).

Overall, Bolzano's relation-based approach to the notion of space brings about important theoretical consequences, both within and outside the scope of Bolzano's theoretical system. It's particularly worth considering how the BG introduces key concepts lying at the basis of Bolzano's subsequent notion of collection (*Inbegriff*), allowing for an interpretation of it as a fundamentally relation-based notion. Exactly in this frame, already in DP, the line and all geometrical objects in general are identified with (infinite) collections of points (DP § 11-note)[23]. On the other side, the BG relation-based arguments have been recognized as paving the way for important theoretical paths outside Bolzano's thought and precisely concerning the topological properties of space[24], whereby the geometric continuum is developed in the wake of the arithmetical continuum (DP).
In this frame, Bolzano's investigations on the relational aspects of space primarily rely on the notion of system of points as the most elementary (simplest) geometrical object (BG II §1), on the basis of which all possible geometrical objects can be progressively determined.

2.1 The first geometrical object

While the point is defined as the simplest (non-determined, hence non-provable) spatial object (BD II §5); the system of two points is instead defined as the simplest namely most elementary geometrical object, in so far as expressing a determination, i.e. "something distinguishable" (BG II §6) with respect to some other objects. Significantly in this regard, the differential relation between two equal points, namely the relation between two different (unequal) locations or positions, is able to express the first (invariable) space property, able to "ground" the constitutive order of any other geometrical determinations, e.g. the angle, the triangle and the straight line.

The description of the system of two points (BG II §6, §7) conveys insightful and anticipatory arguments concerning Bolzano's mature notions of proposition in itself and especially of idea in itself. The "conception" of a relation between two points *a, b* can determine predicates, which couldn't arise by conceiving each single point independently (BG II §6). As mentioned, such predicates sure enough refer to (objective) predicable properties, in as much as identifying certain objectual properties (they can be considered predicable properties exactly in so far as being objectual properties). Precisely, the above-mentioned predicates arise as "something which can be *perceived* in the relation of these points to one another" and specifically from the relation of point *b* to point *a* (BG II §6). More precisely, what is perceived is the "whole concept" of the relation of *b* to *a* (BG II §7) containing two specific spatial predicates or properties: distance and direction.

[21] Sebestik, *Logique et Mathématique,* pp. 33-36.
[22] Sebestik, *Logique et Mathématique,* p. 40.
[23] Sebestik, *Logique et Mathématique*, p. 48.
[24] Cfr. Johnson, *Prelude to Dimension Theory*, 1977.

The property of "distance of point *b* from *a"* (BG II §6) is generally defined as that belonging to *b* in relation to *a,* in such a way that it's independent of the specific point *a* and therefore can "equally" belong to *b* in relation to another point *α*. On the other side, the property of "direction of *b* lying from *a*" (BG II §6) is generally defined as that belonging to *b* in its exclusive relation to *a*, "after having separated from the concept of this relation what is already in the concept of the distance *b* to *a*", namely after considering the possibility for the very point *a* to be also at a different distance from *b* (e.g. even farer), which is to say after considering the possibility for *a* to be a different point.

The whole concept of a system of points *a, b* is the concept of a definite distance with a specific direction (BG II §7), able to determine the spatial location of point *b.* In this perspective, by expressing the property of an oriented segment, this concept has been recognized as expressing the property of the modern concept of vector[25]. Interestingly, the system *a, b* is a "whole" concept, composed by the concepts of distance and direction. Both of them are of course not a priori given in the system *a, b*, still they have a "real" i.e. objective content, also belonging to other systems of points, different from *a, b* (BG II §7). Interestingly, the possibility for both distance and direction of system *a, b* to be objective properties again depends on the possibility of their "variation"; namely on the possibility for their intrinsic properties to hold also for other systems, different from *a, b* (BG II § 11) but similar with respect to them. By relying on the equality between the intrinsic properties of all distances and directions, the notion of similarity is assumed in view to demonstrate the (objective) invariability of distance and directions as objectual properties, whereby all distances and directions are "produced" by similar, i.e. formally identical determining conditions, respectively relying on bilateral and unilateral relations between certain points. Exactly in this perspective, same distances can be observed at different directions, while same directions can be observed at different distances (BG II § 8).

The BG definition of distance and direction as objective geometrical properties appears as an important and insightful argument, far from obscure, notwithstanding the qualitative perspective of the arguments. Differently for the metric definition of the same notions within DP[26], the BG arguments rely on qualitative relations of identity, equality, inequality and opposition between points; nonetheless, exactly this qualitative perspective, opening to possible investigations also in the frame of Leibniz's analysis situs, is able to express the significant logical value of elementary spatial relations.

Importantly, Bolzano exactly exploits these qualitative relations between points, in view to demonstrate the determination of angles: an angle is defined as the property of a "system" of two directions (BG I §1; BG I § 22), namely as the product of the relation between two different (unequal) directions. Precisely, the angle is defined as a system of two non-opposite directions sharing one point (BG II §12, §13, §14). Also, the triangle is defined as a system of three points determined by specific distances and directions (BG II §17, §18). Finally, and most significantly, the straight line is demonstrated as determined qualitative relations between points: by basing on the definition of "situated point" between two opposite directions (BG II §10, §24, §25), the straight line is defined as an object "containing all points" between two opposite directions (BG II § 26, 30); hence, far from containing only a system of points, the line is specifically able to contain "an infinite number" of points (BG II § 27). In this frame, the straight line is justified independently from the notions of plane or space extension, whereas the plane is itself assumed as constituted by relations between systems of points (BG II § 43), in a way to outline the theoretical premises for the modern general topology[27]. It's also worth considering how the perspective underlying the definition of straight line (an object containing a situated point between opposite directions, as well as an infinite number of other points) shares the same logical assumption underlying Bolzano's arithmetical definition of the intermediate value theory, as a prelude to the development of the arithmetical continuum, as much as of the geometric continuum. Interestingly, both these notions are based on the fundamental concept of collection of points.

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[25] SEBESTIK, *Logique et Mathématique*, pp. 40 - 42.
[26] SEBESTIK, *Logique et Mathématique*, p. 41.
[27] Cfr. JOHNSON, *Prelude to Dimension Theory,* 1977; SEBESTIK, *Logique et Mathématique*, p. 58.

The BG of course doesn't contain any reference to Bolzano's articulated notion of collection (*Inbegriff*), except for some mentions of plurality (*Vielheit*) with respect to the concept of numerical relation (BG I §6, §24). Nonetheless, the concept of system identifies specific arguments pertaining to the concept of collection as a whole, which are by all evidence anticipatory of the notion of whole as it's treated within the AM. Exactly in this perspective, Bolzano makes the case for a further demonstration of the straight line (beyond the concepts of distance and direction), by relying on the argument of the whole as determined by several equal parts, namely by basing on the argument of the constitutive determination as a measure relation (BG II §31, §33). This argument is worth to be considered in detail.

2.2 The measure relation

By relying on the concept of situated point (BG II § 30), Bolzano attains the definition of parts composing the straight line within the given interval between points *a* and *b*, in a way to define such a straight line as a whole made of parts (BG II § 31). Such parts are identified with smaller straight lines featuring equal distance and opposite directions, namely they are defined as equal parts (BG II §33, §34), while each of them is also defined as a "unit" (BG II § 33), i.e. a specific object implying the relation with one or more equal objects of the same kinds. Bolzano considers all such equal parts as determining pieces of the straight line, assuming an equal ratio and also an addition relation between them. Precisely, the "whole" straight line is determined by a sum between one of its parts, to be considered as a unit*,* and the number *n* of all remaining parts. In this frame, the straight line is defined as a *Grösse*, namely as a magnitude or quantity (BG II §33, §35) expressed by the sum between a unit and a number of parts equal to the unit or measure (BG I §6; BG II § 35). Elsewhere in the BG the *Grösse* is also defined as a number or a "plurality" of equal parts (BG I §6). It's worth to notice how a certain constitutive aspect of Bolzano's early notion of *Grösse* is assumed in BG, within the same perspective subsequently outlined in the AM, where this notion is defined as a whole determined by a measure unit and a number (or more numbers) reciprocally related by a rule, e.g the addition rule (AM III §7, AM III §20). In this frame the discrete *Grösse* is precisely described as an aggregate constituted by equal parts (AM III §15). Overall, it's especially worth to highlight how such a constitutive aspect of the early notion of *Grösse* relies on a measure relation, which is defined within a qualitative perspective, i.e. by simply assuming the possibility of a "measure of equality".

Interestingly, the equality relation, clearly assumed as reference criterion for any measure relation, underlies most of BG arguments on the background of both theories of equality and similarity. In both cases, as well as qualitatively defined, the equality criterion is applied to define a quantitative comparison between spatial and geometrical objects. As confirmed in later arguments (DP § 21), the equality-based measure is exploited to define geometrical equality, i.e. the equality relation between two geometrical objects, whereby either each determining part of the one object must be compared with a determining part of the other, or both must be compared with a (equal) common unit of measurement (BG I §24).

Also, the equality-based measure is condition for the "determination" of any primitive spatial property, in as much as quantitative properties. Bolzano remarks that there is no a-priori idea of any spatial or geometrical property: no ideas of any distance or length as of any "determinate kind" of separation between two points (BG I §19; BG I §24). Nonetheless, it's possible to have an idea of the (equal, therefore also unequal) relations between different systems of points, considering that every separation between two points is similar (BG I § 20). On this basis, since we have no a priori idea "of any determinate separation of two points" (distance), "there is nothing we can do but to note the ratios of different distances to one another" (BG I §24). More specifically, for any a priori knowledge of spatial objects to be possible, "this must be valid for every unit of measurement adopted" (BG I §24). According to this argument, within a science of space, primitive space properties as distances, are to be conceived as quantitative properties; nonetheless, we cannot asses any a-priori quantitative property concerning the separation of certain two points, but only the pertaining qualitative property; namely, we can grasp the property of distance as a bilateral relation between two different space locations (a system of two points), but we cannot objectively define “what kind” of distance it is, unless we measure it with another similar (equal or unequal) distance.

It’s also interesting to consider how the measure relation is involved in the determination of the property of similarity, assuming the concept of similarity as a quantitative inequality *salva qualitate*. In BG similarity is, in fact, accounted as the equal quality of two ratios, both concerning unequal objects, whereby the ratio itself can be assumed as nothing but a measure relation. In this frame, similarity seems to emerge as an identity of “the way” two terms of two different ratios equal parts of the other respective two terms of the ratios. Exactly such a concept of measure is by Bolzano exploited for heuristic purposes. In fact, as for an unknown distance (BG II § 39), the unknown length of a straight line (BG II § 41) is determined as the property (*Bechaffenheit*) of an object fitting a certain ratio in as much as this ratio is term of a certain equation; precisely it can be determined as that object able to quantitatively relate to another object, “in the same way” two objects of the same kind reciprocally relate, which is to say that it’s determined on condition of a specific, given “kind” of measure.

It’s worth considering how the same qualitative aspect of the concept of measure, as assumed in BG, is also expressed by Bolzano’s later notion of *Grösse,* within a logical rather than arithmetical perspective. According to Bolzano’s mature arguments, first provided in WL (§ 87), the *Grösse* is defined as a kind of object, for which only one of the following two possible relations holds: either a measure of equality or a measure of inequality with another object of the same kind, whereby in the latter case one of the objects is to be considered as a whole, containing a part equal to the other object. Interestingly, the equality relation is central in this argument, echoing the BG perspective, though in a different way with respect to the argument concerning the early notion of *Grösse*. In the light of arguments concerning the notion of property (WL §80) and differently from the early notion, the later notion of *Grösse* seems to be more an account of a variable, external relation rather than an invariable (internal), constitutive relation. Nonetheless, the fascinating assumption underlying this argument is exactly that there is no *Grösse* property in absence of a relation, precisely an equality relation.

Overall, in BG arguments it’s worth to highlight a certain logical, also ontological valence of the measure relation and, particularly, of the equality relation, in the frame of the determination of properties as a relation-based condition. It’s interesting how in BG the measure relation is condition for the determination of quantitative properties, within the spatial and geometrical domains, as both descriptive and objectual properties, respectively on account of intrinsic and extrinsic ratios. It’s even more interesting how specific BG arguments concerning the relation-based constitutivity of an objectual property as a whole, make the case for the measure criterion to be interpreted as a constitutive criterion itself, whereby any reference to the concept of whole, as well as any reference to equality relations between wholes, induce to exceed the quantitative perspective as such, in view to embrace a broad logical perspective. Exactly in this frame, as mentioned, the concept of determination outlined in BG is worth to be investigated in the light of Bolzano’s *Allgemeine Mathesis*[28]. Also, if we consider the assumption of “determination as measurability”, the BG arguments induce a reflection about Bolzano’s notion of *Grösse* within an ontological, possibly formal-ontological perspective, precisely relying on the equality criterion.

3. Conclusion

Driven by justificatory purposes, Bolzano’s first published geometrical investigations account for a logical rather than mathematical perspective on elementary geometry. The treatment of quantitative notions in the frame of qualitative relations (e.g. identity, equality, inequality, opposition) is interesting, not only for the disclosing of a certain “logic” underlying subsequent metric notions in geometry (DP), but also for the genetical understanding of basic geometrical and spatial notions, also able to justify a very modern perspective of space constitutivity. Exactly such genetical analysis of basic geometrical and spatial notions is able to introduce crucial concepts of Bolzano’s logic, most of them defined in later arguments. Particularly, the

[28] SEBESTIK, *Logique et Mathématique*, p. 47

concept of determination is central within the overall discussion, as a relation-based concept referring to the attribution as much as the constitution of objectual properties. In this frame, the outlined perspective configures as a fascinating intertwining of relation-based determinations, precisely of what Bolzano's later arguments define as internal and external attributes, or properties.

The constitutive aspect of the concept of determination is especially interesting, since adding a certain ontological significance to a descriptive significance. On this basis, this concept is all the more interesting in clearly conveying a heuristic valence, which makes the case for Bolzano's key notion of grounding to be investigated not only at epistemological but also at ontological level. It's exactly within this heuristic perspective that the constitutive meaning of determination is exploited in the frame of the theory of geometrical similarity, being able to account for the qualitative equality of objects in the face of their quantitative inequality. Here the relation-based condition of determination is assumed as a quantitative relation, i.e. a ratio; while, more interestingly, the notion of similarity is assumed as an identity of formal determining conditions, i.e. as an identity of the kind of constitutive relations, which determine different geometrical objects. Such assumption outlines a certain formal-ontological perspective, which is also worth to be investigated in the frame of Bolzano's logical notion of variation. Overall, it's relevant to consider that, in being accounted as a ratio, the relation-based condition of geometrical and spatial (quantitative) properties is indeed accounted as a measure relation, ultimately relying on an equality criterion. Then, in this frame, the concept of determination as measurability makes the case for a significant ontological aspect of Bolzano's central notion of *Grösse*, precisely opening to a possible understanding of this notion within a formal ontological perspective.

Although poorly explored, the *Betrachtungen* of 1804 is source of insightful arguments concerning Bolzano's logic, whose explanatory potential can be able to counterbalance any possible lack of scientific value of this work from a mathematical point of view.

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***Abstract* (English)**

Driven by justificatory purposes, Bolzano's first published geometrical investigations account for a logical rather than mathematical perspective on elementary geometry. Basic geometrical notions are treated within a qualitative perspective ultimately relying on the key concept of relation. Particularly, the central notion of determination is assumed as a relation-based notion, able to account for the constitution of geometrical objects as of space itself. Although poorly explored, the *Betrachtungen* of 1804 is a source of insightful arguments concerning Bolzano's logic, whose explanatory potential can be able to counterbalance any possible lack of scientific value of this work from a mathematical point of view.

Keywords: relation, geometry, space, ontology, Bolzano

***Abstract* (italiano)**

La prima pubblicazione di Bolzano presenta uno studio sulla geometria elementare in una prospettiva logica. Nello specifico, nozioni geometriche basilari vengono trattate in una prospettiva qualitativa che si fonda, in ultima analisi, sul concetto chiave di relazione. È dato particolare risalto all'aspetto relazionale della nozione di determinazione, centrale in tutta la discussione, nonché in grado di spiegare la costituzione degli oggetti geometrici, quanto dello spazio stesso. Sebbene poco studiata, l'analisi condotta nelle *Betrachtungen* del 1804 è fonte di argomenti altamente esplicativi riguardo alla logica di Bolzano, in grado di controbilanciare la possibile mancanza di valore scientifico di questo lavoro da un punto di vista matematico.

Parole chiave: relazione, geometria, spazio, ontologia, Bolzano

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