ALEA, Lat. Am. J. Probab. Math. Stat. **20**, 825–860 (2023)
DOI: 10.30757/ALEA.v20-30

Quantitative Multidimensional Central Limit Theorems for Means of the Dirichlet-Ferguson Measure

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Abstract. The Dirichlet-Ferguson measure is a cornerstone in nonparametric Bayesian statistics and the study of the distributional properties of expectations with respect to such measure is an important line of research initiated in Cifarelli and Regazzini (1979a,b) and still very active, see Letac and Piccioni (2018) and Lijoi and Prünster (2009). In this paper we provide explicit upper bounds for the d_3, the d_2 and the convex distances between random vectors whose components are means of the Dirichlet-Ferguson measure and a random vector distributed according to the multivariate Gaussian law. These results are applied to the Gaussian approximation of linear transformations of random vectors with the Dirichlet distribution, yielding presumably optimal rates on the d_3 and the d_2 distances and presumably suboptimal rates on the convex and the Kolmogorov distances.

1. Introduction

Let $\mathbb{X}$ be a Polish space equipped with the Borel sigma-field $\mathcal{B}(\mathbb{X})$ and let $U \equiv \{U_n\}_{n\geq 1}$ be a sequence of exchangeable random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and taking values on $\mathbb{X}$. Let $\mathcal{P}(\mathbb{X})$ be the set of probability measures on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ endowed with the smallest sigma-field making the mappings $\mathcal{P}(\mathbb{X}) \ni \mu \mapsto \mu(B)$, $B \in \mathcal{B}(\mathbb{X})$, measurable. According to de Finetti's Representation Theorem, there exists a probability measure $\pi \in \mathcal{P}(\mathbb{X})$ such that

$$\mathbb{P}(U_1 \in B_1, \dots, U_n \in B_n) = \int_{\mathcal{P}(\mathbb{X})} \prod_{i=1}^{n} p(B_i)\pi(\mathrm{d}p), \quad \forall\, B_1, \dots, B_n \in \mathcal{B}(\mathbb{X}),\, n \geq 1.$$

Therefore there exists a random probability measure $\Pi : \Omega \to \mathcal{P}(\mathbb{X})$ such that, given Π, the U_i's are independent and identically distributed with law Π. The distribution of Π coincides with π and it acts as a nonparametric *prior* in Bayesian inference. The Dirichlet-Ferguson measure is a cornerstone in Bayesian Nonparametrics since it is the most notable example of prior Π, see the seminal paper Ferguson (1973) and the review article Lijoi and Prünster (2009) (see also Lemma 2.1).

Received by the editors February 21st, 2023; accepted April 20th, 2023.

2010 *Mathematics Subject Classification.* 60F05;60G57;60H07.

Key words and phrases. Central Limit Theorem; Dirichlet-Ferguson Measure; Gaussian Approximation; Stein's Method.

Let σ be a finite and positive measure on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ with total mass

$$\beta := \sigma(\mathbb{X}) \in (0, \infty).$$

Throughout this work we denote by η the Dirichlet-Ferguson measure with parameter σ, see Section 2 for a formal definition. The measure σ completely determines the law of η and, in a Bayesian context, has to be interpreted as a mathematical representation of the *a priori* information of the observer. It is well-known that Dirichlet-Ferguson measures are conjugate, see Ferguson (1973) and James et al. (2006). Indeed, for any $n \geq 1$ and $u_1, \ldots, u_n \in \mathbb{X}$, under the conditional probability measure $\mathbb{P}(\cdot \,|\, U_1 = u_1, \ldots, U_n = u_n)$, the law of η is that of a Dirichlet-Ferguson measure $\eta_{\mathbf{u}_n}$ with parameter $\sigma_{\mathbf{u}_n} := \sigma + \sum_{k=1}^{n} \varepsilon_{u_k}$. Here ε_x denotes the Dirac measure at $x \in \mathbb{X}$ and the parameter $\sigma_{\mathbf{u}_n}$ describes the *a posteriori* information of the observer after n samples $U_1 = u_1, \ldots, U_n = u_n$.

Within this framework, natural questions to address concern the distributional properties of means with respect to η, i.e., the distributional properties of random variables of the form

$$I(\eta, h) := \int_{\mathbb{X}} h(x)\eta(\mathrm{d}x), \tag{1.1}$$

for some measurable function (or kernel) $h : \mathbb{X} \to \mathbb{R}$. This important line of research was initiated in Cifarelli and Regazzini (1979a,b). An appealing aspect of this topic is that distributional results concerning $I(\eta, h)$ are also of interest in research areas not related to Bayesian nonparametric inference, such as, for example, the growth of Young diagrams or the exponential representations of functions of negative imaginary parts. This fact was emphasized in Diaconis and Kemperman (1996) and further discussed in Kerov (1998). We remark that, thanks to the conjugacy property of the Dirichlet-Ferguson measures, the distributional properties of the *a priori* random mean $I(\eta, h)$ may be transferred to the *a posteriori* random mean

$$I(\eta_{\mathbf{u}_n}, h) \stackrel{\mathcal{L}}{=} I(\eta, h) \,|\, \{U_1 = u_1, \ldots, U_n = u_n\},$$

where the symbol $\stackrel{\mathcal{L}}{=}$ denotes equality in law.

In Cifarelli and Regazzini (1979b) the authors introduced a series of tools and techniques that, later in Cifarelli and Regazzini (1994), turned out to be fundamental to determine the probability distribution of $I(\eta, h)$. Some conditions formerly required in Cifarelli and Regazzini (1994) were successively relaxed in the papers Lijoi and Regazzini (2004) and Regazzini et al. (2002). Various moments formulas for $I(\eta, h)$ were obtained in Cifarelli and Melilli (2000), Hjort and Ongaro (2005) and Yamato (1980, 1984), and formulas for the covariance of functionals of η, even more general than $I(\eta, h)$, were proved in Flint and Torrisi (2023) and Peccati (2008). Let $t > 0$, $\widetilde{\sigma}$ a probability measure on $\mathbb{X} := \mathbb{R}^d$, η_t the Dirichlet-Ferguson measure with parameter $t\widetilde{\sigma}$ and μ_t the law of the random mean $I(\eta_t, h)$. In Letac and Piccioni (2018) the authors determine sufficient conditions on $t\widetilde{\sigma}$ and the kernel h which guarantee that the mapping $t \mapsto \mu_t$ is decreasing for the Strassen convex order on $(0, \infty)$.

All these achievements are exact results about functionals of the Dirichlet-Ferguson measure. It has to be noticed that the probability density (with respect to the Lebesgue measure) and the cumulative distribution function of $I(\eta, h)$, computed in Regazzini et al. (2002), are quite complicated to handle for practical purposes. Hence, it is of great interest to provide approximations of the law of $I(\eta, h)$. There have been various proposals in the literature. In Feigin and Tweedie (1989) the authors constructed a Harris ergodic Markov chain whose unique invariant probability measure is the law of $I(\eta, h)$. Later, it was proved in Guglielmi and Tweedie (2001) that such Markov chain, say $\{Y_n\}_{n\geq 1}$, is geometrically ergodic if $\int_{\mathbb{X}} |h(x)|\sigma(\mathrm{d}x) < \infty$ and uniform ergodic if the support of $\sigma \circ h^{-1}$ is bounded. Interestingly, the authors are also able to provide upper bounds on the total variation distance between the law of $I(\eta, h)$ and the law of Y_n. An algorithm for exact sampling from $I(\eta, h)$ has been formulated in Guglielmi et al. (2002). Another method to approximate the law of $I(\eta, h)$ has been proposed in Muliere and Tardella (1998). The idea relies on truncating the series representation of η at some random point in such a way that the (random) Prokhorov distance between η and its truncated version, say $\eta^{(\varepsilon)}$, is less than $\varepsilon > 0$ almost surely. In some cases it is possible to show that the closeness, with respect to the Prokhorov distance, between η and $\eta^{(\varepsilon)}$ induces closeness between the laws of $\int_{\mathbb{X}} h(x)\eta^{(\varepsilon)}(\mathrm{d}x)$ and $I(\eta, h)$. Numerical methods for approximating the distribution of $I(\eta, h)$ can be found in Regazzini et al. (2002) and Tamura (1988). The method provided in Tamura (1988) is based on the numerical inversion of the Laplace functional of the Gamma process which generates η (see Ferguson (1973) for details on this construction of η). Another numerical method is described in Regazzini et al. (2002). It consists in approximating, on a sufficiently large interval of the real line and with respect to the uniform metric, the cumulative distribution function of $I(\eta, h)$ when $\sigma \circ h^{-1}$ has a finite support.

To the best of our knowledge, the first Quantitative One-dimensional Central Limit Theorem for means with respect to the Dirichlet-Ferguson measure has been proved in Flint and Torrisi (2023). Since the purpose of this paper is to extend such result to d-dimensional, $d \in \mathbb{N} := \{1, 2, \ldots\}$, random vectors, whose components are expectations with respect to η, it is worthwhile to state the Gaussian approximation of $I(\eta, h)$ established in Flint and Torrisi (2023). Hereon, we denote by d_W the Wasserstein distance, by Lip(1) the set of Lipschitz functions $g : \mathbb{R} \to \mathbb{R}$ with Lipschitz constant at most 1 and by $N(0, \nu^2)$ a random variable with Gaussian law with mean 0 and variance ν^2.

Theorem 1.1 (Flint and Torrisi (2023))**.** *Let $h : \mathbb{X} \to \mathbb{R}$ be a measurable function such that*

$$\int_{\mathbf{X}} h(x)\,\sigma(\mathrm{d}x) = 0. \tag{1.2}$$

Then:
(i)

$$\begin{aligned} d_W(I(\eta, h), N(0,1)) &:= \sup_{g \in \mathrm{Lip}(1)} |\mathbb{E}[g(I(\eta, h))] - \mathbb{E}[g(N)]| \\ &\leq \sqrt{2/\pi}\Big|1 - \frac{1}{\beta(\beta+1)}\int_{\mathbb{X}} h(x)^2\sigma(\mathrm{d}x)\Big| + \frac{2}{\beta(\beta+1)(\beta+2)}\int_{\mathbb{X}} |h(x)|^3\,\sigma(\mathrm{d}x) \\ &\quad + \frac{2}{\beta^2(\beta+1)^2(\beta+2)}\int_{\mathbb{X}} |h(x)|\,\sigma(\mathrm{d}x)\int_{\mathbb{X}} h(x)^2\,\sigma(\mathrm{d}x). \end{aligned} \tag{1.3}$$

(ii) Let $\{\eta_M\}_{M\in\mathbb{N}}$ be a sequence of Dirichlet-Ferguson measures such that, for any $M \in \mathbb{N}$, η_M has parameter σ_M and $\beta_M := \sigma_M(\mathbb{X}) \to +\infty$, as $M \to +\infty$. For each $M \in \mathbb{N}$, let h_M be a function in $L^3(\mathbb{X}, \sigma_M)$, which satisfies (1.2) with h_M in place of h and σ_M in place of σ, and suppose that

$$\frac{1}{\beta_M^2}\int_{\mathbb{X}} h_M(x)^2\sigma_M(\mathrm{d}x) \to 1 \quad \text{and} \quad \frac{1}{\beta_M^3}\int_{\mathbb{X}} |h_M(x)|^3\sigma_M(\mathrm{d}x) \to 0, \quad \text{as } M \to +\infty.$$

Then the bound (1.3) holds, with η_M in place of η, h_M in place of h, σ_M in place of σ and β_M in place of β, and the upper bound goes to zero as $M \to +\infty$. Therefore

$$d_W(I(\eta_M, h_M), N(0,1)) \to 0, \quad \text{as } M \to +\infty$$

and

$$\int_{\mathbb{X}} h_M(x)\,\eta_M(\mathrm{d}x) \to N(0,1) \quad \text{in law, as } M \to +\infty.$$

The aim of the paper is to measure how much the law of a d-dimensional random vector of the form

$$\mathbf{I}(\eta, \mathbf{h}) := (I(\eta, h_1), \ldots, I(\eta, h_d)), \quad \mathbf{h} := (h_1, \ldots, h_d)$$

is far from a centered d-dimensional Gaussian random vector with covariance matrix Σ, denoted by

$$\mathbf{N} := (N_1, \ldots, N_d).$$

Throughout this paper Σ can be positive semi-definite or positive definite. We quantify the distance between $\mathbf{I}(\eta, \mathbf{h})$ and $\mathbf{N}$ providing explicit upper bounds on $\mathfrak{d}(\mathbf{I}(\eta, \mathbf{h}), \mathbf{N})$ for three different metrics $\mathfrak{d}$: the d_3, the d_2 and the convex distance d_c. Such metrics differ among each other for the degree of regularity required on the test functions involved in their definitions. Specifically, both the d_3 and the d_2 distances are defined in terms of smooth test functions g, but the definition of d_3 requires $g \in C^3(\mathbb{R}^d)$ while the definition of d_2 requires $g \in C^2(\mathbb{R}^d)$. Here $C^k(\mathbb{R}^d)$, $k \in \mathbb{N}$, denotes the set of functions $g : \mathbb{R}^d \to \mathbb{R}$ which admit continuous partial derivatives up to the order k. The convex distance d_c is instead based on test functions which are indicators of measurable convex sets, and it has to be considered as a standard multivariate counterpart of the Kolmogorov distance d_K. In fact, the convex distance is often preferable to the Kolmogorov distance since d_c enjoys a number of invariance properties not satisfied by d_K. For instance, $d_c(L\mathbf{X}, L\mathbf{Y}) = d_c(\mathbf{X}, \mathbf{Y})$, for any invertible affine operator $L : \mathbb{R}^d \to \mathbb{R}^d$ and d-dimensional random vectors $\mathbf{X}$, $\mathbf{Y}$, see Bentkus (2003), Nourdin et al. (2022) and Schulte and Yukich (2019).

We emphasize that, although the d_2 distance requires less regularity than the d_3 distance, the Quantitative Multidimensional Central Limit Theorem in the d_2 distance (see Theorem 4.2) holds under the more restrictive assumption that the covariance matrix Σ is positive definite, while the Quantitative Multidimensional Central Limit Theorem in the d_3 distance (see Theorem 4.1) holds, more generally, for a positive semi-definite covariance matrix Σ. Applying Theorems 4.1 and Theorem 4.2 to linear transformations of random vectors distributed according to the Dirichlet law, we obtain presumably optimal rates which, roughly speaking, are of order $O(C_M^{-1})$, as $M \to +\infty$, where C_M is a normalizing sequence (see Theorem 5.3 and the Examples 1 and 2). The convex distance d_c is arguably more interesting than the d_3 and the d_2 distances since, as already mentioned, it has very nice properties. However, it is harder to deal with d_c because the underlying test functions are discontinuous. Consequently, the application of the Quantitative Multidimensional Central Limit Theorem in the d_c distance (see Theorem 4.4) to linear transformations of random vectors distributed according to the Dirichlet law yields a presumably suboptimal rate which, roughly speaking, is of order $O(C_M^{-1/2})$, as $M \to +\infty$, where C_M is a normalizing sequence (see again Theorem 5.3 and the Examples 1 and 2).

The generalization of Theorem 1.1 to the multidimensional setting is far from being trivial. To enlight this fact, we mention that the proof of Theorem 1.1 is based on the Malliavin-Stein method (we refer the reader to the book Nourdin and Peccati (2012) for an introduction on this method and to the seminal papers Nourdin and Peccati (2009) and Peccati et al. (2010) for the Gaussian approximation of functionals of the Wiener and the Poisson measures by this method). Indeed, the proof of Theorem 1.1 combines a Mecke-type formula for η (proved in Ferguson (1973) and later extended in Last (2020)) with the celebrated Stein equation (see Stein (1972)), according to which, given a Lipschitz function g with Lipschitz constant at most 1, there exists a twice differentiable function f such that

$$g(x) - \mathbb{E}[g(N)] = xf(x) - f'(x), \quad x \in \mathbb{R}$$

and

$$\|f\|_\infty := \sup_{x\in\mathbb{R}} |f(x)| \leq 2\|g'\|_\infty, \quad \|f'\|_\infty \leq \sqrt{2/\pi}\|g'\|_\infty, \quad \|f''\|_\infty \leq 2\|g'\|_\infty.$$

In the multivariate setting, the (partial) differential equation involved in the Stein equation is of second order (see formula (3.3)), increasing the needed degree of smoothness by one, over what is required in the one-dimensional case. For this reason, when combining the multivariate Stein equation with a suitable "corrected" integration by parts formula for functionals of $\mathbf{I}(\eta, \mathbf{h})$ (see Lemma 3.8) we need to replace d_W with the smoother d_2 distance. The proof of the Quantitative Multidimensional Central Limit Theorem in the d_3 distance is based on the so-called "smart path" method and again the afore-mentioned "corrected" integration by parts formula for functionals of $\mathbf{I}(\eta, \mathbf{h})$. The proof of the Quantitative Multidimensional Central Limit Theorem in the d_c distance essentially combines a version of Stein's method for the multidimensional Gaussian approximation (which is different from that one considered in the proof of Theorem 4.2, see Lemma 3.6) with an integration by parts formula for functionals of η (see Lemma 3.1) and a remarkable smoothing lemma for the convex distance (see Lemma 3.9).

The article is organized as follows. In Section 2 we define the Dirichlet-Ferguson measure and recall some of its properties. In Section 3 we give some preliminaries, e.g., we state an integration by parts formula related to a suitable gradient operator, which plays an important role in the proofs of our main results, we define the d_3, the d_2 and the convex distances, we describe the multidimensional Stein method and we state a crucial "corrected" integration by parts formula for functionals of $\mathbf{I}(\eta, \mathbf{h})$. The Quantitative Multidimensional Central Limit Theorems are stated and proved in Section 4. Applications of these theorems to affine trasformations of random vectors with the Dirichlet law are provided in Section 5. We include an Appendix where we report a number of technical proofs.

2. The Dirichlet-Ferguson measure

A random probability measure is a measurable mapping from an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to $\mathcal{P}(\mathbb{X})$. Let $k \geq 2$ be a positive integer and

$$\Delta_{k-1} := \{(x_1, \dots, x_k) : \ x_1, \dots, x_k \geq 0, \ x_1 + \dots + x_k = 1\}$$

the $(k-1)$-dimensional simplex in $\mathbb{R}^k$. For $\alpha = (\alpha_1, \dots, \alpha_k) \in (0, \infty)^k$, the Dirichlet distribution with parameter α is the probability measure on Δ_{k-1} defined by

$$\mathrm{Dir}[\alpha](A) := \frac{\Gamma(\sum_{i=1}^k \alpha_i)}{\prod_{i=1}^k \Gamma(\alpha_i)} \int_{\mathbb{R}^{k-1}} \mathbf{1}_A(x_1, \dots, x_k) \left(\prod_{i=1}^k x_i^{\alpha_i - 1} \right) \mathrm{d}x_1 \cdots \mathrm{d}x_{k-1},$$

for each Borel set $A \in \mathcal{B}(\Delta_{k-1})$. Here $\Gamma(\cdot)$ is the Euler gamma function.

For an integer $k \geq 2$, we denote by $\mathfrak{P}_k(\mathbb{X})$ the set of partitions $\{X_1, \dots, X_k\}$ of $\mathbb{X}$ with $X_i \in \mathcal{B}(\mathbb{X})$ and $\sigma(X_i) > 0$, for any $i \in [k]$, where, for ease of notation, for $n \in \mathbb{N}$, we set $[n] := \{1, \dots, n\}$. We say that a random probability measure $\eta : \Omega \to \mathcal{P}(\mathbb{X})$ is a Dirichlet-Ferguson measure with parameter σ (see Feng (2010) and Ferguson (1973)) if, for each integer $k \geq 2$ and $\{X_1, \dots, X_k\} \in \mathfrak{P}_k(\mathbb{X})$,

$$(\eta(X_1), \dots, \eta(X_k)) \stackrel{\mathcal{L}}{=} \mathrm{Dir}[(\sigma(X_1), \dots, \sigma(X_k))].$$

2.1. *The Dirichlet-Ferguson measure as de Finetti's measure of a sequence of exchangeable random variables.* Let $U = \{U_i\}_{i \geq 1}$ be a sequence of $\mathbb{X}$-valued random variables, defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that

$$U_1 \quad \text{has law} \quad \sigma/\beta, \quad U_\ell \mid \{U_1 = u_1, \dots, U_{\ell-1} = u_{\ell-1}\} \quad \text{has law} \quad \frac{\sigma + \sum_{s=1}^{\ell-1} \varepsilon_{u_s}}{\beta + \ell - 1}, \quad \ell \geq 2. \tag{2.1}$$

In particular, for every $n \in \mathbb{N}$ and every $1 \leq j_1 < \dots < j_n < +\infty$,

$$\mathbb{P}(U_{j_1} \in \mathrm{d}x_1, \dots, U_{j_n} \in \mathrm{d}x_n) = \prod_{\ell=1}^n \frac{\sigma(\mathrm{d}x_\ell) + \sum_{s=1}^{\ell-1} \varepsilon_{x_s}(\mathrm{d}x_\ell)}{\beta + \ell - 1}, \tag{2.2}$$

where we set $\sum_{s=1}^0 := 0$. Therefore the sequence U is exchangeable.

The next lemma, which is proved in Blackwell and MacQueen (1973), characterizes the Dirichlet-Ferguson measure η as the de Finetti measure of U.

Lemma 2.1. *Let*

$$\mathbb{P}_n(B, \omega) := \frac{1}{n} \sum_{i=1}^n \mathbf{1}_B(U_i(\omega)), \quad n \in \mathbb{N}, \, B \in \mathcal{B}(\mathbb{X})$$

be the empirical measure of the sequence U previously defined. Then:

(i) As $n \to +\infty$, $\mathbb{P}_n$ converges $\mathbb{P}$-a.s. to the Dirichlet-Ferguson measure η with parameter σ.

(ii) Given η, the random variables $U_1, U_2, \ldots$ are independent and identically distributed with law η, i.e., the law of η is the de Finetti measure of U.

Throughout this paper we shall consider means of the Dirichlet-Ferguson measure η of the form (1.1) with h satisfying (1.2). By Lemma 2.1 one immediately has that if $h \in L^1(\mathbb{X}, \sigma)$ then $\mathbb{E}[h(U_1) \,|\, \eta] = I(\eta, h)$, and therefore

$$\mathbb{E}[I(\eta, h)] = \frac{1}{\beta} \int_{\mathbb{X}} h(x)\sigma(\mathrm{d}x) = 0. \tag{2.3}$$

For later purposes, we also remark that by the isometry formula (8) in Peccati (2008) one has

$$\mathbb{E}[I(\eta, h)^2] = \frac{1}{\beta+1}\mathbb{E}[h(U_1)^2] = \frac{1}{\beta(\beta+1)} \int_{\mathbb{X}} h(x)^2\sigma(\mathrm{d}x), \quad \forall\, h \in L^2(\mathbb{X}, \sigma) \text{ satisfying (1.2)}. \tag{2.4}$$

3. Preliminaries

In this section we give the following preliminaries. (i) We introduce a gradient operator and state a related integration by parts formula. (ii) We define the d_3, the d_2 and the d_c distances between probability laws on $\mathbb{R}^d$. As already mentioned in the introduction, such metrics will be exploited to measure how much the law of $\mathbf{I}(\eta, \mathbf{h})$ is far from the law of $\mathbf{N}$. (iii) We recall two slightly different solutions to the multivariate Stein equation, which will be exploited to prove the Quantitative Multidimensional Central Limit Theorems with respect to the d_2 and the d_c distances (see Theorems 4.2 and 4.4). (iv) We give three preliminary lemmas: the multivariate Gaussian integration by parts formula, a "corrected" integration by parts formula, which plays a key role in the proofs of Theorems 4.1 and 4.2, a remarkable smoothing formula for the convex distance d_c.

It has to be mentioned that in the specialized literature the symbols d_3 and d_2 usually denote smooth Wasserstein distances; these metrics are induced by test functions which are different from the ones involved in the Definitions 3.2 and 3.3. We refer the reader to Gaunt and Li (2023) for more insight into smooth Wasserstein distances.

3.1. *An integration by parts formula.* For a measurable mapping $\mathbf{G} : \mathcal{P}(\mathbb{X}) \to \mathbb{R}^d$, $d \in \mathbb{N}$, $\mathbf{G} = (G_1, \ldots, G_d)$, $(x, t) \in \mathbb{X} \times [0, 1]$ and $\mu \in \mathcal{P}(\mathbb{X})$, we define the gradient of $\mathbf{G}$ as

$$D_{(x,t)}\mathbf{G}(\mu) := (G_1\big((1-t)\mu + t\varepsilon_x\big) - G_1(\mu), \ldots, G_d\big((1-t)\mu + t\varepsilon_x\big) - G_d(\mu)). \tag{3.1}$$

The next lemma states that the operator $I(\eta, \cdot)$ is the adjoint of the gradient D.

Hereon, we denote by $\widehat{\sigma}$ the probability measure on $\mathbb{X} \times [0, 1]$ defined by

$$\widehat{\sigma}(\mathrm{d}x, \mathrm{d}t) := \sigma(\mathrm{d}x)(1-t)^{\beta-1}\mathrm{d}t$$

and we denote by $\langle \cdot, \cdot \rangle_{L^2(\mathbb{X}\times[0,1],\widehat{\sigma})}$ the inner product on $L^2(\mathbb{X} \times [0, 1], \widehat{\sigma})$.

Lemma 3.1. *Let $G : \mathcal{P}(\mathbb{X}) \to \mathbb{R}$ be a measurable function such that $G(\eta) \in L^2(\Omega, \mathbb{P})$ and assume that the kernel $h \in L^2(\mathbb{X}, \sigma)$ satisfies (1.2). Then*

$$\mathbb{E}[\langle h(\cdot), D.G(\eta)\rangle_{L^2(\mathbb{X}\times[0,1],\widehat{\sigma})}] = \mathbb{E}[G(\eta)I(\eta, h)]. \tag{3.2}$$

It is worthwhile to mention that formula (3.2) indeed holds under more general conditions on the kernel h and the functional G (see Ferguson (1973) and Last (2020)). However, the assumptions of Lemma 3.1 are the minimal ones for our purposes.

3.2. *Distances between probability measures on* $\mathbb{R}^d$. We start introducing some notation. For any $g \in C^1(\mathbb{R}^d)$, we set

$$\|g\|_{\mathrm{Lip}} := \sup_{x \in \mathbb{R}^d} \|\nabla g(x)\|,$$

where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^d$ and ∇ is the usual gradient of a smooth function; for any $g \in C^2(\mathbb{R}^d)$, we set

$$M_2(g) := \sup_{x \in \mathbb{R}^d} \|\mathrm{Hess}\, g(x)\|_{\mathrm{op}},$$

where $\mathrm{Hess}\, g(x)$ is the Hessian matrix of g at x and the operator norm of a $d \times d$ real matrix A is defined by $\|A\|_{\mathrm{op}} := \sup_{x:\, \|x\|=1} \|Ax\|$; for $k \in \mathbb{N}$ and $g \in C^k(\mathbb{R}^d)$, we set

$$\|g^{(k)}\|_\infty := \max_{1 \le i_1, \dots, i_k \le d} \sup_{x \in \mathbb{R}^d} \left| \frac{\partial^k g(x)}{\partial x_{i_1} \dots \partial x_{i_k}} \right|.$$

Definition 3.2. The distance d_3 between the laws of two $\mathbb{R}^d$-valued random vectors $\mathbf{X}$ and $\mathbf{Y}$ such that $\mathbb{E}[\|\mathbf{X}\|^2], \mathbb{E}[\|\mathbf{Y}\|^2] < \infty$, written $d_3(\mathbf{X}, \mathbf{Y})$, is given by

$$d_3(\mathbf{X}, \mathbf{Y}) := \sup_{g \in \mathcal{H}_3} |\mathbb{E}[g(\mathbf{X})] - \mathbb{E}[g(\mathbf{Y})]|,$$

where $\mathcal{H}_3$ indicates the collection of all functions $g \in C^3(\mathbb{R}^d)$ such that $\|g^{(2)}\|_\infty \le 1$ and $\|g^{(3)}\|_\infty \le 1$.

Definition 3.3. The distance d_2 between the laws of two $\mathbb{R}^d$-valued random vectors $\mathbf{X}$ and $\mathbf{Y}$ such that $\mathbb{E}[\|\mathbf{X}\|], \mathbb{E}[\|\mathbf{Y}\|] < \infty$, written $d_2(\mathbf{X}, \mathbf{Y})$, is given by

$$d_2(\mathbf{X}, \mathbf{Y}) := \sup_{g \in \mathcal{H}_2} |\mathbb{E}[g(\mathbf{X})] - \mathbb{E}[g(\mathbf{Y})]|,$$

where $\mathcal{H}_2$ indicates the collection of all functions $g \in C^2(\mathbb{R}^d)$ such that $\|g\|_{\mathrm{Lip}} \le 1$ and $M_2(g) \le 1$.

Note that both the d_3 and the d_2 distances are defined in terms of smooth test functions. In contrast, the convex distance d_c, that we are going to define soon, is based on test functions which are indicators of measurable convex sets.

Definition 3.4. The convex distance d_c between the laws of two $\mathbb{R}^d$-valued random vectors $\mathbf{X}$ and $\mathbf{Y}$, written $d_c(\mathbf{X}, \mathbf{Y})$, is given by

$$d_c(\mathbf{X}, \mathbf{Y}) := \sup_{g \in \mathcal{I}} |\mathbb{E}[g(\mathbf{X})] - \mathbb{E}[g(\mathbf{Y})]|,$$

where $\mathcal{I}$ denotes the collection of all indicator functions of measurable convex sets in $\mathbb{R}^d$.

We conclude this subsection emphasizing that, if $d_j(\mathbf{X}_n, \mathbf{X}) \to 0$, as $n \to +\infty$, where $j = 2, 3, c$ and $\mathbf{X}_n$, $\mathbf{X}$ are random vectors with values in $\mathbb{R}^d$, then $\mathbf{X}_n$ converges in law to $\mathbf{X}$, as $n \to +\infty$, see Peccati and Zheng (2010) and Schulte and Yukich (2019).

3.3. *The multidimensional Stein equation.* Hereon, we consider the Hilbert-Schmidt inner product and the Hilbert-Schmidt norm on the class of $d \times d$ real matrices, which are defined, respectively, by

$$\langle A, B \rangle_{H.S.} := Tr(AB^T) = \sum_{i,j=1}^d a_{ij} b_{ij} \quad \text{and} \quad \|A\|_{H.S.} = \sqrt{\langle A, A \rangle_{H.S.}}$$

for every pair of matrices $A = (a_{ij})_{1 \le i,j \le j}$ and $B = (b_{ij})_{1 \le i,j \le d}$, where the symbols $Tr(A)$ and A^T denote, respectively, the trace and the transpose of the matrix A. For any $g \in C^2(\mathbb{R}^d)$, we define the quantity

$$M_3(g) := \sup_{x \ne y} \frac{\|\mathrm{Hess}\, g(x) - \mathrm{Hess}\, g(y)\|_{op}}{\|x - y\|}.$$

The following lemma gives a solution to the multidimensional Stein's equation which comes in handy in the proof of Theorem 4.2. We refer the reader to Peccati and Zheng (2010) (see Lemma 2.17) and the references cited therein for a proof. We note that the bounds (3.4) and (3.5) improve the corresponding estimates in Peccati and Zheng (2010); the proofs of these improved bounds are provided in Gaunt (2016) (see Proposition 2.1).

Throughout this paper we denote by $\langle \cdot, \cdot \rangle$ the inner product on $\mathbb{R}^d$.

Lemma 3.5. *Fix $d \in \mathbb{N}$ and let $\Sigma = (\Sigma_{ij})_{1 \le i,j \le d}$ be a $d \times d$ positive definite symmetric real matrix. Then, for $g \in C^2(\mathbb{R}^d)$ with bounded first and second derivatives, we have that the function*

$$f_g(x) := \frac{1}{2} \int_0^1 \frac{1}{1-s} \mathbb{E}[g(\sqrt{s}\mathbf{N} + \sqrt{1-s}x) - g(\mathbf{N})] \, \mathrm{d}s$$

is solution to the following partial differential equation (with unknown f):

$$g(x) - \mathbb{E}[g(\mathbf{N})] = \langle x, \nabla f(x) \rangle - \langle \Sigma, \mathrm{Hess}\, f(x) \rangle_{H.S.}, \quad x \in \mathbb{R}^d. \tag{3.3}$$

Moreover, one has that

$$M_2(f_g) \le \sqrt{\frac{2}{\pi}} \|\Sigma^{-1/2}\|_{op} \|g\|_{\mathrm{Lip}} \tag{3.4}$$

and

$$M_3(f_g) \le \frac{\sqrt{2\pi}}{4} \|\Sigma^{-1/2}\|_{op} M_2(g). \tag{3.5}$$

Since it appears unclear how to deal with solutions f_g to the Stein equation (3.3) when g is non-smooth (such as the test functions involved in the definition of d_c) in order to prove Theorem 4.4 we need a slightly different version of the multidimentional Stein's method. The idea is to provide a solution to (3.3) considering in place of a non-smooth g an its regular version, say g_t, which depends on a smoothing parameter $t \in (0,1)$. Of course one makes some error by replacing the test functions defining d_c by their regular versions, but a smoothing lemma (see Lemma 3.9) allows us to bound this error by some constant multiple of $\sqrt{t}/(1-t)$.

Lemma 3.6. *Fix $d \in \mathbb{N}$ and let $\Sigma = (\Sigma_{ij})_{1 \le i,j \le d}$ be a $d \times d$ positive definite symmetric real matrix. Given $g : \mathbb{R}^d \to \mathbb{R}$ measurable and bounded and $t \in (0,1)$, we introduce the smoothed function*

$$g_t(x) := \mathbb{E}[g(\sqrt{t}\mathbf{N} + \sqrt{1-t}x)], \quad x \in \mathbb{R}^d. \tag{3.6}$$

Then:
(i) The function

$$f_{t,g}(x) := \frac{1}{2} \int_t^1 \frac{1}{1-s} \mathbb{E}[g(\sqrt{s}\mathbf{N} + \sqrt{1-s}x) - g(\mathbf{N})] \, \mathrm{d}s, \quad x \in \mathbb{R}^d$$

is solution to the partial differential equation (3.3) with g_t in place of g (and with unknown f).
(ii)

$$\sup_{x \in \mathbb{R}^d} \left| \frac{\partial f_{t,g}(x)}{\partial x_i} \right| \le \|g\|_\infty \frac{\sqrt{1-t}}{\sqrt{t}} \sum_{\ell,j=1}^d |(\Sigma^{-1/2})_{\ell j} (\Sigma^{-1/2})_{\ell i}| \sqrt{\Sigma_{jj}}, \quad \text{for any } i \in [d] \text{ and } t \in (0,1).$$

(iii) For any d-dimensional random vector $\mathbf{X}$ it holds

$$\sup_{g \in \mathcal{I}} \mathbb{E} \left[\sum_{i,j=1}^d \left(\frac{\partial^2 f_{t,g}(\mathbf{X})}{\partial x_i \partial x_j} \right)^2 \right] \le \|\Sigma^{-1}\|_{op}^2 (d^2 (\log t)^2 d_c(\mathbf{X}, \mathbf{N}) + 530 d^{17/6}), \quad \text{for any } t \in (0,1).$$

(iv) For any $\ell, k, j \in [d]$ and $x \in \mathbb{R}^d$ it holds

$$\sup_{x \in \mathbb{R}^d} \left| \frac{\partial^3 f_{t,g}(x)}{\partial x_\ell \partial x_k \partial x_j} \right| \le 6 d^3 \|\Sigma^{-1}\|_{op}^{3/2} \|g\|_\infty \frac{1}{\sqrt{t}}, \quad \text{for any } t \in (0,1).$$

The proof of this lemma is given in the Appendix.

3.4. *Preliminary lemmas.* We conclude this section by stating some further lemmas. We start with the multivariate Gaussian integration by parts formula. The proof is elementary and can be found e.g. in Talagrand (2003) (see Eq. (A.41) therein).

Lemma 3.7. *Let $g \in C^1(\mathbb{R}^d)$ with bounded derivatives and such that $\mathbb{E}[|N_i g(\mathbf{N})|] < \infty$, for any $i \in [d]$. Then*

$$\mathbb{E}[N_i g(\mathbf{N})] = \sum_{j=1}^{d} \Sigma_{ij} \mathbb{E}\left[\frac{\partial g(\mathbf{N})}{\partial x_j}\right], \quad \text{for any } i \in [d].$$

The next lemma (whose proof is given in the Appendix) provides a "corrected" integration by parts formula for functionals of $\mathbf{I}(\eta, \mathbf{h})$, which plays a crucial role in the proofs of the Quantitative Multidimensional Central Limit Theorems in the d_3 and the d_2 distances.

Lemma 3.8. *Suppose that the kernels $h_i \in L^2(\mathbb{X}, \sigma)$, $i \in [d] \cup \{0\}$, satisfy (1.2). Then, for any $g \in C^2(\mathbb{R}^d)$ with bounded first and second derivatives, we have*

$$\mathbb{E}[g(\mathbf{I}(\eta, \mathbf{h})) I(\eta, h_0)] = \frac{1}{\beta(\beta+1)} \mathbb{E}\left[\sum_{i=1}^{d} \frac{\partial}{\partial x_i} g(\mathbf{I}(\eta, \mathbf{h})) \int_{\mathbb{X}} h_0(x) h_i(x) \sigma(\mathrm{d}x)\right] + c,$$

where c is a constant such that

$$|c| \leq \frac{d \|g^{(2)}\|_\infty}{\beta(\beta+1)(\beta+2)} \left[\sum_{i=1}^{d} \int_{\mathbb{X}} |h_0(x)| |h_i(x)|^2 \sigma(\mathrm{d}x) \right.$$
$$\left. + \frac{1}{\beta(\beta+1)} \int_{\mathbb{X}} |h_0(x)| \sigma(\mathrm{d}x) \sum_{i=1}^{d} \int_{\mathbb{X}} |h_i(x)|^2 \sigma(\mathrm{d}x)\right].$$

Finally we state a remarkable smoothing lemma for the convex distance proved in Schulte and Yukich (2019), see Lemma 2.2 therein. It plays a crucial role in the proof of the Quantitative Multidimensional Central Limit Theorem in the metric d_c.

Lemma 3.9. *Let $\mathbf{X}$ be a d-dimensional random vector, $t \in (0, 1)$ and Σ a $d \times d$ real symmetric and positive definite matrix. Then*

$$d_c(\mathbf{X}, \mathbf{N}) \leq \frac{4}{3} \sup_{g \in \mathcal{I}} |\mathbb{E}[g_t(\mathbf{X}) - g_t(\mathbf{N})]| + \frac{20}{\sqrt{2}} d \frac{\sqrt{t}}{1-t},$$

where g_t is defined by (3.6).

4. Quantitative Multidimensional Central Limit Theorems

In this section we prove the main results of the paper, i.e., three different Quantitative Multidimensional Central Limit Theorems for the random vector $\mathbf{I}(\eta, \mathbf{h})$. The first gives an explicit upper bound on $d_3(\mathbf{I}(\eta, \mathbf{h}), \mathbf{N})$ and it is proved by exploiting the so-called "smart path" method. We refer the reader to Theorem 4.2 in Peccati and Zheng (2010) for a similar result on the Poisson space. The second provides an explicit upper bound on $d_2(\mathbf{I}(\eta, \mathbf{h}), \mathbf{N})$ and it is proved exploiting the Malliavin-Stein method. We refer the reader to Theorem 3.3 in Peccati and Zheng (2010) for a similar result for functionals of the Poisson measure. The third furnishes an explicit upper bound on $d_c(\mathbf{I}(\eta, \mathbf{h}), \mathbf{N})$. We refer the reader to Theorem 1.2 in Schulte and Yukich (2019) for a related result on the Poisson space.

Hereon, we consider the quantities

$$\gamma_1 := \sum_{i,j=1}^{d} \Big|\Sigma_{ij} - \frac{1}{\beta(\beta+1)} \int_{\mathbb{X}} h_i(x)h_j(x)\sigma(\mathrm{d}x)\Big|,$$

$$\gamma_2 := \frac{d}{\beta(\beta+1)(\beta+2)}\Bigg[\sum_{i,j=1}^{d} \int_{\mathbb{X}} |h_i(x)||h_j(x)|^2\sigma(\mathrm{d}x) + \frac{1}{\beta(\beta+1)} \sum_{i,j=1}^{d} \int_{\mathbb{X}} |h_i(x)|\sigma(\mathrm{d}x) \int_{\mathbb{X}} |h_j(x)|^2\sigma(\mathrm{d}x)\Bigg],$$

$$\gamma_3 := \frac{d^3}{\beta(\beta+1)^2} \max_{1\le i\le d} \int_{\mathbb{X}} |h_i(x)|^3\sigma(\mathrm{d}x),$$

$$\gamma_4 := \frac{d^3}{\beta(\beta+1)^2}\sqrt{\max_{1\le i\le d} \int_{\mathbb{X}} |h_i(x)|^2\sigma(\mathrm{d}x) \max_{1\le i\le d} \int_{\mathbb{X}} |h_i(x)|^4\sigma(\mathrm{d}x)},$$

$$\gamma_5 := \max\{\gamma_1, \sqrt{\gamma_7}\}, \quad \gamma_6 := \max\{\gamma_1, \sqrt{\gamma_8}\},$$

where

$$\gamma_7 := \frac{2}{3\beta(\beta+1)(\beta+2)}\Bigg[\sum_{j=1}^{d} \int_{\mathbb{X}} |h_j(x)|^3\sigma(\mathrm{d}x) + 2\sqrt{\frac{\beta+2}{\beta+1}\sum_{k=1}^{d}\int_{\mathbb{X}} h_k(x)^2\sigma(\mathrm{d}x)\sum_{k=1}^{d}\left(2\int_{\mathbb{X}} h_k(x)^4\sigma(\mathrm{d}x) + \frac{6}{\beta(\beta+1)}\left(\int_{\mathbb{X}} h_k(x)^2\sigma(\mathrm{d}x)\right)^2\right)}\Bigg]$$

and

$$\gamma_8 := \frac{2}{3\beta(\beta+1)(\beta+2)}\Bigg[\sqrt{\sum_{i=1}^{d} \int_{\mathbb{X}} |h_i(x)|^2\sigma(\mathrm{d}x)\sum_{i=1}^{d} \int_{\mathbb{X}} |h_i(x)|^4\sigma(\mathrm{d}x)} + 2\sqrt{\frac{\beta+2}{\beta+1}\sum_{k=1}^{d}\int_{\mathbb{X}} h_k(x)^2\sigma(\mathrm{d}x)\sum_{k=1}^{d}\left(2\int_{\mathbb{X}} h_k(x)^4\sigma(\mathrm{d}x) + \frac{6}{\beta(\beta+1)}\left(\int_{\mathbb{X}} h_k(x)^2\sigma(\mathrm{d}x)\right)^2\right)}\Bigg].$$

Hereafter, we denote by $\gamma_{i,M}$, $i \in [8]$, $M \in \mathbb{N}$, the quantity obtained replacing h_k, $k \in [d]$, σ and β in the definition of γ_i with a sequence of kernels $h_{k,M}$, a sequence of measures σ_M and a sequence of total masses $\beta_M := \sigma_M(\mathbb{X})$, respectively. We also set $\mathbf{h}_M := (h_{1,M}, \dots, h_{d,M})$.

Hereon, if not otherwise stated, the covariance matrix Σ is assumed to be positive semi-definite.

4.1. *A Quantitative Multidimensional Central Limit Theorem in the d_3 distance.*

Theorem 4.1. *Let $d \in \mathbb{N}$ be fixed and suppose that*

For any $i \in [d]$, $h_i : \mathbb{X} \to \mathbb{R}$ is a measurable function which satisfies (1.2) *with h_i in place of h.* (4.1)

Then:

(i)

$$d_3(\mathbf{I}(\eta, \mathbf{h}), \mathbf{N}) \le \frac{1}{2}(\gamma_1 + \gamma_2) \le \frac{1}{2}(\gamma_1 + \gamma_3) \le \frac{1}{2}(\gamma_1 + \gamma_4). \tag{4.2}$$

(ii) Let $\{\eta_M\}_{M\in\mathbb{N}}$ be a sequence of Dirichlet-Ferguson measures such that, for any $M \in \mathbb{N}$, η_M has parameter σ_M and $\beta_M := \sigma_M(\mathbb{X}) \to +\infty$, as $M \to +\infty$. For each $M \in \mathbb{N}$, let $\{h_{i,M}\}_{i\in[d]}$ be a

family of kernels in $L^3(\mathbb{X},\sigma_M)$ *which satisfy* (1.2), *with* $h_{i,M}$ *in place of* h *and* σ_M *in place of* σ, *and are such that*

$$\text{For any } i,j\in[d],\quad \frac{1}{\beta_M^2}\int_{\mathbb{X}} h_{i,M}(x)h_{j,M}(x)\sigma_M(\mathrm{d}x)\to\Sigma_{ij},\quad \text{as } M\to+\infty \tag{4.3}$$

and

$$\text{For any } i\in[d],\quad \frac{1}{\beta_M^3}\int_{\mathbb{X}} |h_{i,M}(x)|^3\sigma_M(\mathrm{d}x)\to 0,\quad \text{as } M\to+\infty. \tag{4.4}$$

Then, for each $M\in\mathbb{N}$, *the relation* (4.2) *is satisfied, with* η_M *in place of* η, $\mathbf{h}_M$ *in place of* $\mathbf{h}$ *and* $\gamma_{i,M}$ *in place of* γ_i, $i=1,2,3,4$, *and* $\gamma_{1,M},\gamma_{3,M}\to 0$, *as* $M\to+\infty$. *Therefore*

$$d_3(\mathbf{I}(\eta_M,\mathbf{h}_M),\mathbf{N})\to 0,\quad \text{as } M\to+\infty \tag{4.5}$$

and

$$\mathbf{I}(\eta_M,\mathbf{h}_M)\to\mathbf{N}\quad \text{in law, as } M\to+\infty. \tag{4.6}$$

(*iii*) *Let the sequence* $\{\eta_M\}_{M\in\mathbb{N}}$ *be as in* (*ii*). *For any* $M\in\mathbb{N}$, *let* $\{h_{i,M}\}_{i\in[d]}$ *be a family of kernels in* $L^4(\mathbb{X},\sigma_M)$ *which satisfy* (1.2), *with* $h_{i,M}$ *in place of* h *and* σ_M *in place of* σ, (4.3) *and*

$$\text{For any } i\in[d],\quad \frac{1}{\beta_M^4}\int_{\mathbb{X}} |h_{i,M}(x)|^4\sigma_M(\mathrm{d}x)\to 0,\quad \text{as } M\to+\infty. \tag{4.7}$$

Then, for each $M\in\mathbb{N}$, *the relation* (4.2) *is satisfied, with* η_M *in place of* η, $\mathbf{h}_M$ *in place of* $\mathbf{h}$ *and* $\gamma_{i,M}$ *in place of* γ_i, $i=1,2,3,4$, *and* $\gamma_{1,M},\gamma_{4,M}\to 0$, *as* $M\to+\infty$. *Therefore the relations* (4.5) *and* (4.6) *hold.*

Note that, under the foregoing assumptions, if Σ *is a diagonal matrix, then the* (*dependent*) *random variables* $I(\eta_M,h_{i,M})$, $i\in[d]$, *are, as* M *grows large, asymptotically independent with* (*one-dimensional*) *Gaussian law with mean zero and variance* Σ_{ii}, $i\in[d]$.

Proof: *Proof of Part* (*i*). We start noticing that if $h_i\notin L^3(\mathbb{X},\sigma)$ for some $i\in[d]$ then $\gamma_2=\gamma_3=\gamma_4=+\infty$ and the bounds are trivially true. Therefore, we suppose that the functions $|h_i|^3$, $i\in[d]$, are all integrable with respect to σ. Moreover, without loss of generality, we may assume that η and $\mathbf{N}$ are independent. Throughout this proof for ease of notation we put $\mathbf{I}:=\mathbf{I}(\eta,\mathbf{h})$. Take $g\in\mathcal{H}_3$ and set

$$G(t):=\mathbb{E}[g(\sqrt{1-t}\mathbf{I}+\sqrt{t}\mathbf{N})],\quad t\in[0,1].$$

We have

$$|\mathbb{E}[g(\mathbf{N})]-\mathbb{E}[g(\mathbf{I})]|=|G(1)-G(0)|=\Big|\int_0^1 G'(t)\mathrm{d}t\Big|\le\int_0^1|G'(t)|\mathrm{d}t. \tag{4.8}$$

By the properties of g we have that G is differentiable on $(0,1)$ with derivative

$$\begin{aligned} G'(t)&=\sum_{i=1}^d\mathbb{E}\left[\frac{\partial}{\partial x_i}g(\sqrt{1-t}\mathbf{I}+\sqrt{t}\mathbf{N})\left(\frac{1}{2\sqrt{t}}N_i-\frac{1}{2\sqrt{1-t}}I(\eta,h_i)\right)\right]\\ &=\frac{1}{2\sqrt{t}}\mathfrak{N}-\frac{1}{2\sqrt{t-1}}\mathfrak{I}, \end{aligned}\tag{4.9}$$

where

$$\mathfrak{N}:=\sum_{i=1}^d\mathbb{E}\left[\frac{\partial}{\partial x_i}g(\sqrt{1-t}\mathbf{I}+\sqrt{t}\mathbf{N})N_i\right]\quad\text{and}\quad \mathfrak{I}:=\sum_{i=1}^d\mathbb{E}\left[\frac{\partial}{\partial x_i}g(\sqrt{1-t}\mathbf{I}+\sqrt{t}\mathbf{N})I(\eta,h_i)\right].$$

In what follows we denote by $\mathcal{L}_Z$ the law of a random variable Z, and integrals with respect to $\mathcal{L}_Z$ are implicitly realized over the support of Z. For fixed $i\in[d]$, $t\in[0,1]$ and $z\in\mathbb{R}^d$, set

$$\gamma_i^{t,z}(x):=\frac{\partial}{\partial x_i}g(\sqrt{1-t}z+\sqrt{t}x),\quad x\in\mathbb{R}^d$$

and note that, for any $j\in[d]$,

$$\frac{\partial}{\partial x_j}\gamma_i^{t,z}(x)=\sqrt{t}\frac{\partial^2}{\partial x_i\partial x_j}g(\sqrt{1-t}z+\sqrt{t}x),\quad x\in\mathbb{R}^d. \tag{4.10}$$

So, due to the fact that $g\in\mathcal{H}_3$, we have that $\gamma_i^{t,z}\in C^1(\mathbb{R}^d)$ with bounded derivatives. Since η is independent of $\mathbf{N}$, we have

$$\mathbb{E}\left[\frac{\partial}{\partial x_i}g(\sqrt{1-t}\mathbf{I}+\sqrt{t}\mathbf{N})N_i\right]=\int\mathbb{E}[\gamma_i^{t,z}(\mathbf{N})N_i]\mathcal{L}_{\mathbf{I}}(\mathrm{d}z). \tag{4.11}$$

We shall check later on that, for any $i\in[d]$, $t\in[0,1]$ and $z\in\mathbb{R}^d$,

$$\mathbb{E}[|\gamma_i^{t,z}(\mathbf{N})N_i|]<\infty. \tag{4.12}$$

Therefore by Lemma 3.7, for any $i\in[d]$, $t\in[0,1]$ and $z\in\mathbb{R}^d$, we have

$$\mathbb{E}[\gamma_i^{t,z}(\mathbf{N})N_i]=\sqrt{t}\sum_{j=1}^d\Sigma_{ij}\mathbb{E}\left[\frac{\partial^2}{\partial x_i\partial x_j}g(\sqrt{1-t}z+\sqrt{t}\mathbf{N})\right].$$

Combining this relation with (4.11) and using again the independence between η and $\mathbf{N}$, we have

$$\mathbb{E}\left[\frac{\partial}{\partial x_i}g(\sqrt{1-t}\mathbf{I}+\sqrt{t}\mathbf{N})N_i\right]=\sqrt{t}\sum_{j=1}^d\Sigma_{ij}\mathbb{E}\left[\frac{\partial^2}{\partial x_i\partial x_j}g(\sqrt{1-t}\mathbf{I}+\sqrt{t}\mathbf{N})\right].$$

Therefore

$$\mathfrak{N}=\sqrt{t}\sum_{i,j=1}^d\Sigma_{ij}\mathbb{E}\left[\frac{\partial^2}{\partial x_i\partial x_j}g(\sqrt{1-t}\mathbf{I}+\sqrt{t}\mathbf{N})\right]. \tag{4.13}$$

For fixed $i\in[d]$, $t\in[0,1]$ and $z\in\mathbb{R}^d$, define

$$\widetilde{\gamma}_i^{t,z}(x):=\frac{\partial}{\partial x_i}g(\sqrt{1-t}x+\sqrt{t}z)=\gamma_i^{t,x}(z).$$

Since $g\in\mathcal{H}_3$, we have that $\widetilde{\gamma}_i^{t,z}\in C^2(\mathbb{R}^d)$ with bounded first and second derivatives. Indeed,

$$\frac{\partial}{\partial x_j}\widetilde{\gamma}_i^{t,z}(x)=\sqrt{1-t}\frac{\partial^2}{\partial x_i\partial x_j}g(\sqrt{1-t}x+\sqrt{t}z), \tag{4.14}$$

$$\frac{\partial^2}{\partial x_k\partial x_j}\widetilde{\gamma}_i^{t,z}(x)=(1-t)\frac{\partial^3}{\partial x_i\partial x_j\partial x_k}g(\sqrt{1-t}x+\sqrt{t}z),$$

$$\|(\widetilde{\gamma}_i^{t,z})^{(1)}\|_\infty\le\sqrt{1-t}\|g^{(2)}\|_\infty\quad\text{and}\quad\|(\widetilde{\gamma}_i^{t,z})^{(2)}\|_\infty\le(1-t)\|g^{(3)}\|_\infty.$$

Using the independence between η and $\mathbf{N}$, Lemma 3.8 and relation (4.14), we have

$$\begin{aligned}\mathbb{E}\left[\frac{\partial}{\partial x_i}g(\sqrt{1-t}\mathbf{I}+\sqrt{t}\mathbf{N})I(\eta,h_i)\right]&=\int\mathbb{E}\left[\frac{\partial}{\partial x_i}g(\sqrt{1-t}\mathbf{I}+\sqrt{t}z)I(\eta,h_i)\right]\mathcal{L}_{\mathbf{N}}(\mathrm{d}z)\\&=\int\mathbb{E}[\widetilde{\gamma}_i^{t,z}(\mathbf{I})I(\eta,h_i)]\mathcal{L}_{\mathbf{N}}(\mathrm{d}z)\\&=\int\left\{\frac{1}{\beta(\beta+1)}\mathbb{E}\left[\sum_{j=1}^d\frac{\partial}{\partial x_j}\widetilde{\gamma}_i^{t,z}(\mathbf{I})\int_{\mathbb{X}}h_i(x)h_j(x)\sigma(\mathrm{d}x)\right]+c(i,t,z)\right\}\mathcal{L}_{\mathbf{N}}(\mathrm{d}z)\\&=\frac{\sqrt{1-t}}{\beta(\beta+1)}\mathbb{E}\left[\sum_{j=1}^d\frac{\partial^2}{\partial x_i\partial x_j}g(\sqrt{1-t}\mathbf{I}+\sqrt{t}\mathbf{N})\int_{\mathbb{X}}h_i(x)h_j(x)\sigma(\mathrm{d}x)\right]+\widetilde{c}(i,t),\end{aligned}$$

where $\widetilde{c}(i,t) := \int c(i,t,z)\mathcal{L}_{\mathbf{N}}(\mathrm{d}z)$. Therefore

$$\mathfrak{I} = \frac{\sqrt{1-t}}{\beta(\beta+1)}\mathbb{E}\left[\sum_{i,j=1}^{d} \frac{\partial^2}{\partial x_i \partial x_j} g(\sqrt{1-t}\mathbf{I}+\sqrt{t}\mathbf{N}) \int_{\mathbb{X}} h_i(x)h_j(x)\sigma(\mathrm{d}x)\right] + \sum_{i=1}^{d} \widetilde{c}(i,t). \tag{4.15}$$

By (4.9), (4.13) and (4.15) we have

$$G'(t) = \frac{1}{2}\sum_{i,j=1}^{d} \mathbb{E}\left[\frac{\partial^2}{\partial x_i \partial x_j} g(\sqrt{1-t}\mathbf{I}+\sqrt{t}\mathbf{N})\left(\Sigma_{ij} - \frac{1}{\beta(\beta+1)}\int_{\mathbb{X}} h_i(x)h_j(x)\sigma(\mathrm{d}x)\right)\right] - \frac{1}{2\sqrt{1-t}}\sum_{i=1}^{d}\widetilde{c}(i,t). \tag{4.16}$$

By Lemma 3.8 we also have that, for any $i \in [d]$, $t \in [0,1]$ and $z \in \mathbb{R}^d$,

$$\begin{aligned}|c(i,t,z)| &\leq \frac{d\|(\widetilde{\gamma}_i^{t,z})^{(2)}\|_\infty}{\beta(\beta+1)(\beta+2)}\left[\sum_{j=1}^{d}\int_{\mathbb{X}}|h_i(x)||h_j(x)|^2\sigma(\mathrm{d}x)\right.\\ &\qquad\left. + \frac{1}{\beta(\beta+1)}\int_{\mathbb{X}}|h_i(x)|\sigma(\mathrm{d}x)\sum_{j=1}^{d}\int_{\mathbb{X}}|h_j(x)|^2\sigma(\mathrm{d}x)\right]\\ &\leq \frac{(1-t)d\|g^{(3)}\|_\infty}{\beta(\beta+1)(\beta+2)}\left[\sum_{j=1}^{d}\int_{\mathbb{X}}|h_i(x)||h_j(x)|^2\sigma(\mathrm{d}x)\right.\\ &\qquad\left. + \frac{1}{\beta(\beta+1)}\int_{\mathbb{X}}|h_i(x)|\sigma(\mathrm{d}x)\sum_{j=1}^{d}\int_{\mathbb{X}}|h_j(x)|^2\sigma(\mathrm{d}x)\right] =: c(i,t)\end{aligned}$$

and so $|\widetilde{c}(i,t)| \leq c(i,t)$. The first inequality in (4.2) follows by this latter inequality, (4.8), (4.16) and the fact that $\|g^{(2)}\|_\infty, \|g^{(3)}\|_\infty \leq 1$. The second inequality in (4.2) follows from noticing that $\gamma_2 \leq \gamma_3$. Indeed, by Hölder's inequality we have

$$\begin{aligned}\gamma_2 &\leq \frac{d}{\beta(\beta+1)(\beta+2)}\left[\sum_{i,j=1}^{d}\left(\int_{\mathbb{X}}|h_i(x)|^3\sigma(\mathrm{d}x)\right)^{1/3}\left(\int_{\mathbb{X}}|h_j(x)|^3\sigma(\mathrm{d}x)\right)^{2/3}\right.\\ &\qquad\left. + \frac{1}{\beta(\beta+1)}\beta\sum_{i,j=1}^{d}\left(\int_{\mathbb{X}}|h_i(x)|^3\sigma(\mathrm{d}x)\right)^{1/3}\left(\int_{\mathbb{X}}|h_j(x)|^3\sigma(\mathrm{d}x)\right)^{2/3}\right]\\ &\leq \frac{d}{\beta(\beta+1)(\beta+2)}\left[\sum_{i,j=1}^{d}\max_{1\leq i\leq d}\int_{\mathbb{X}}|h_i(x)|^3\sigma(\mathrm{d}x) + \frac{1}{\beta+1}\sum_{i,j=1}^{d}\max_{1\leq i\leq d}\int_{\mathbb{X}}|h_i(x)|^3\sigma(\mathrm{d}x)\right] = \gamma_3.\end{aligned}$$

As far as the third upper bound in (4.2) is concerned, it suffices to note that the Cauchy-Schwarz inequality yields

$$\int_{\mathbb{X}}|h_i(x)|^3\sigma(\mathrm{d}x) \leq \sqrt{\int_{\mathbb{X}}|h_i(x)|^2\sigma(\mathrm{d}x)\int_{\mathbb{X}}|h_i(x)|^4\sigma(\mathrm{d}x)} \tag{4.17}$$

and therefore $\gamma_3 \leq \gamma_4$. It remains to check (4.12). By the Cauchy-Schwarz inequality it suffices to check

$$\text{For any } t \in (0,1),\ z \in \mathbb{R}^d \text{ and } i \in [d], \quad \mathbb{E}[|\gamma_i^{t,z}(\mathbf{N})|^2] < \infty.$$

By the Multivariate Mean Value Theorem and (4.10), we have

$$|\gamma_i^{t,z}(\mathbf{N})-\gamma_i^{t,z}(\mathbf{0})|\le\left(\sup_{y\in\mathbb{R}^d}\sqrt{\sum_{i=1}^d\Big|\frac{\partial}{\partial x_i}\gamma_i^{t,z}(y)\Big|^2}\right)\sqrt{\sum_{i=1}^d|N_i|^2}\le\sqrt{d}\|g^{(2)}\|_\infty\sqrt{\sum_{i=1}^d|N_i|^2},$$

and so

$$|\gamma_i^{t,z}(\mathbf{N})|\le\sqrt{d}\|g^{(2)}\|_\infty\sqrt{\sum_{i=1}^d|N_i|^2}+|\gamma_i^{t,z}(\mathbf{0})|.$$

The square integrability of $\gamma_i^{t,z}(\mathbf{N})$ then follows by the integrability properties of $\mathbf{N}$.
Proof of Part (ii) *and Part* (iii). The proofs of these parts of the claim are rather obvious (they directly follow by Part (i) and the assumptions), and therefore omitted. □

4.2. *A Quantitative Multidimensional Central Limit Theorem in the* d_2 *distance.* Denote by $\mathrm{Lip}_d(1)$ the set of Lipschitz functions $g:\mathbb{R}^d\to\mathbb{R}$, $d\in\mathbb{N}$, with Lipschitz constant at most 1, and by d_W the Wasserstein distance. Since $\mathcal{H}_2\subset\mathrm{Lip}_d(1)$ it immediately follows that

$$d_2(\mathbf{X},\mathbf{Y})\le d_W(\mathbf{X},\mathbf{Y}):=\sup_{g\in\mathrm{Lip}_d(1)}|\mathbb{E}[g(\mathbf{X})]-\mathbb{E}[g(\mathbf{Y})]|$$

for $\mathbb{R}^d$-valued random vectors $\mathbf{X}$ and $\mathbf{Y}$ such that $\mathbb{E}[\|\mathbf{X}\|],\mathbb{E}[\|\mathbf{Y}\|]<\infty$. Consequently, in the one-dimensional case, a Quantitative Central Limit Theorem in the d_2 distance is immediately given by Theorem 1.1. As already discussed in the introduction, in the multivariate case it appears unclear how to deal with solutions to the multidimensional Stein equation and the Wasserstein distance. Indeed, the degree of smoothness of the solution to the Stein equation is increased by one, over what is required in the one-dimensional case. As far as the d-dimensional case is concerned we have the following Quantitative Central Limit Theorem in the d_2 distance.

Theorem 4.2. *Let* $d\in\mathbb{N}$ *be fixed and suppose that the kernels* h_i, $i\in[d]$, *satisfy* (4.1) *and that the covariance matrix* Σ *is positive definite. Then:*
(i)

$$\begin{aligned}d_2(\mathbf{I}(\eta,\mathbf{h}),\mathbf{N})&\le\sqrt{\frac{2}{\pi}}\|\Sigma^{-1/2}\|_{op}\gamma_1+\frac{\sqrt{2\pi}\|\Sigma^{-1/2}\|_{op}}{4}\gamma_2\\&\le\sqrt{\frac{2}{\pi}}\|\Sigma^{-1/2}\|_{op}\gamma_1+\frac{\sqrt{2\pi}\|\Sigma^{-1/2}\|_{op}}{4}\gamma_3\\&\le\sqrt{\frac{2}{\pi}}\|\Sigma^{-1/2}\|_{op}\gamma_1+\frac{\sqrt{2\pi}\|\Sigma^{-1/2}\|_{op}}{4}\gamma_4.\end{aligned}\tag{4.18}$$

(ii) *Let* $\{\eta_M\}_{M\in\mathbb{N}}$ *be a sequence of Dirichlet-Ferguson measures such that, for any* $M\in\mathbb{N}$, η_M *has parameter* σ_M *and* $\beta_M:=\sigma_M(\mathbb{X})\to+\infty$, *as* $M\to+\infty$. *For each* $M\in\mathbb{N}$, *let* $\{h_{i,M}\}_{i\in[d]}$ *be a family of kernels in* $L^3(\mathbb{X},\sigma_M)$ *which satisfy* (1.2), *with* $h_{i,M}$ *in place of* h *and* σ_M *in place of* σ, (4.3) *and* (4.4). *Then, for each* $M\in\mathbb{N}$, *the relation* (4.18) *is satisfied, with* η_M *in place of* η, $\mathbf{h}_M$ *in place of* $\mathbf{h}$ *and* $\gamma_{i,M}$ *in place of* γ_i, $i=1,2,3,4$, *and* $\gamma_{1,M},\gamma_{3,M}\to0$, *as* $M\to+\infty$. *Therefore*

$$d_2(\mathbf{I}(\eta_M,\mathbf{h}_M),\mathbf{N})\to0,\quad \textit{as } M\to+\infty.\tag{4.19}$$

(iii) *Let the sequence* $\{\eta_M\}_{M\in\mathbb{N}}$ *be as in* (ii). *For any* $M\in\mathbb{N}$, *let* $\{h_{i,M}\}_{i\in[d]}$ *be a family of kernels in* $L^4(\mathbb{X},\sigma_M)$ *which satisfy* (1.2), *with* $h_{i,M}$ *in place of* h *and* σ_M *in place of* σ, (4.3) *and* (4.7). *Then, for each* $M\in\mathbb{N}$, *the relation* (4.18) *is satisfied, with* η_M *in place of* η, $\mathbf{h}_M$ *in place of* $\mathbf{h}$ *and* $\gamma_{i,M}$ *in place of* γ_i, $i=1,2,3,4$, *and* $\gamma_{1,M},\gamma_{4,M}\to0$, *as* $M\to+\infty$. *Therefore the relation* (4.19) *holds.*

Before proving Theorem 4.2, we briefly compare it with Theorem 4.1.

Remark 4.3. The bounds on $d_3(\mathbf{I}(\eta,\mathbf{h}),\mathbf{N})$ and $d_2(\mathbf{I}(\eta,\mathbf{h}),\mathbf{N})$ differ only for the multiplicative constants in front of γ_1, γ_2, γ_3 and γ_4. So the Parts (ii) and (iii) of Theorem 4.2 are an immediate consequence of the Part (i) of Theorem 4.2 and the Parts (ii) and (iii) of Theorem 4.1. The main difference between Theorems 4.1 and 4.2 is that, although the d_2 distance requires less regularity on the test functions than the d_3 distance, the bounds on the d_2 distance hold under the more restrictive assumption that the covariance matrix Σ is positive definite.

Proof of Theorem 4.2: Due to Remark 4.3, we only need to prove the Part (i). Note that if $h_i \notin L^3(\mathbb{X},\sigma)$ for some $i\in[d]$ then $\gamma_2=\gamma_3=\gamma_4=+\infty$ and the bounds are trivially true. Therefore, we suppose that the functions $|h_i|^3$, $i\in[d]$, are all integrable with respect to σ. Since $\gamma_2\le\gamma_3\le\gamma_4$ (see the proof of Theorem 4.1), we only need to show the first inequality in (4.18). Throughout this proof, for ease of notation, we set $\mathbf{I}:=\mathbf{I}(\eta,\mathbf{h})$. Let $g\in C^\infty(\mathbb{R}^d)$ with first and second bounded derivatives. Then it is easily seen that $\partial f_g/\partial x_j\in C^2(\mathbb{R}^d)$ with first and second bounded derivatives (see e.g. Lemma 3.3 in Nourdin et al. (2010)). By Lemma 3.5 and Lemma 3.8, we have

$$\begin{aligned}
&|\mathbb{E}[g(\mathbf{I})]-\mathbb{E}[g(\mathbf{N})]|\\
&\quad=\Big|\sum_{i,j=1}^d\mathbb{E}\left[\Sigma_{ij}\frac{\partial^2 f_g(\mathbf{I})}{\partial x_i\partial x_j}\right]-\sum_{j=1}^d\mathbb{E}\left[I(\eta,h_j)\frac{\partial f_g(\mathbf{I})}{\partial x_j}\right]\Big|\\
&\quad=\Big|\sum_{i,j=1}^d\mathbb{E}\left[\Sigma_{ij}\frac{\partial^2 f_g(\mathbf{I})}{\partial x_i\partial x_j}\right]-\frac{1}{\beta(\beta+1)}\sum_{i,j=1}^d\mathbb{E}\left[\frac{\partial^2 f_g(\mathbf{I})}{\partial x_i\partial x_j}\int_{\mathbb{X}}h_j(x)h_i(x)\sigma(\mathrm{d}x)\right]+\sum_{j=1}^d c_j\Big|\\
&\quad\le\sum_{i,j=1}^d\mathbb{E}\left[\Big|\frac{\partial^2 f_g(\mathbf{I})}{\partial x_i\partial x_j}\Big|\Big|\Sigma_{ij}-\frac{1}{\beta(\beta+1)}\int_{\mathbb{X}}h_j(x)h_i(x)\sigma(\mathrm{d}x)\Big|\right]+\sum_{j=1}^d|c_j|,
\end{aligned}\tag{4.20}$$

where c_j is a constant such that

$$\begin{aligned}
|c_j|\le\frac{d\|\left(\frac{\partial f_g}{\partial x_j}\right)^{(2)}\|_\infty}{\beta(\beta+1)(\beta+2)}&\Bigg[\sum_{i=1}^d\int_{\mathbb{X}}|h_j(x)||h_i(x)|^2\sigma(\mathrm{d}x)\\
&+\frac{1}{\beta(\beta+1)}\int_{\mathbb{X}}|h_j(x)|\sigma(\mathrm{d}x)\sum_{i=1}^d\int_{\mathbb{X}}|h_i(x)|^2\sigma(\mathrm{d}x)\Bigg].
\end{aligned}\tag{4.21}$$

By (3.4) we have that, for any $i,j\in[d]$, it holds

$$\Big|\frac{\partial^2 f_g(\mathbf{I})}{\partial x_i\partial x_j}\Big|\le\|\mathrm{Hess}\,f_g(\mathbf{I})\|_{H.S.}\le\sup_{x\in\mathbb{R}^d}\|\mathrm{Hess}f_g(x)\|_{H.S.}\le\sqrt{\frac{2}{\pi}}\|\Sigma^{-1/2}\|_{op}\|g\|_{\mathrm{Lip}},\quad\mathbb{P}\text{-a.s.}.\tag{4.22}$$

Moreover, a straightforward computation yields

$$\|\left(\frac{\partial f_g}{\partial x_j}\right)^{(2)}\|_\infty\le\|f_g^{(3)}\|_\infty\le M_3(f_g),\quad\text{for any } j\in[d]$$

and so by (4.21) and (3.5) we have

$$\sum_{j=1}^d|c_j|\le\frac{\sqrt{2\pi}\|\Sigma^{-1/2}\|_{op}}{4}M_2(g)\gamma_2.\tag{4.23}$$

Combining (4.20), (4.22) and (4.23), we have that for any $g\in C^\infty(\mathbb{R}^d)$ with first and second bounded derivatives,

$$|\mathbb{E}[g(\mathbf{I})]-\mathbb{E}[g(\mathbf{N})]|\le\sqrt{\frac{2}{\pi}}\|\Sigma^{-1/2}\|_{op}\|g\|_{\mathrm{Lip}}\gamma_1+\frac{\sqrt{2\pi}\|\Sigma^{-1/2}\|_{op}}{4}M_2(g)\gamma_2.\tag{4.24}$$

At this point, we note that for any $g \in \mathcal{H}_2$ there exists a family $\{g_\varepsilon\}_{\varepsilon>0}$ of functions in $C^\infty(\mathbb{R}^d)$ with bounded first and second derivatives such that $g_\varepsilon \to g$ uniformly, $\|g_\varepsilon\|_{\mathrm{Lip}} \leq \|g\|_{\mathrm{Lip}}$ and $M_2(g_\varepsilon) \leq M_2(g)$, for any $\varepsilon > 0$ (see the first lines of the proof of Theorem 2.3 in Chatterjee and Meckes (2008)). Then applying (4.24) to the approximating function g_ε, bounding $\|g_\varepsilon\|_{\mathrm{Lip}}$ and $M_2(g_\varepsilon)$ by 1, and finally taking the limit as $\varepsilon \to 0$, one has

$$|\mathbb{E}[g(\mathbf{I})] - \mathbb{E}[g(\mathbf{N})]| \leq \sqrt{\frac{2}{\pi}}\|\Sigma^{-1/2}\|_{op}\gamma_1 + \frac{\sqrt{2\pi}\|\Sigma^{-1/2}\|_{op}}{4}\gamma_2, \quad \text{for any } g \in \mathcal{H}_2.$$

The claim follows taking the supremum over $\mathcal{H}_2$ on this latter inequality. □

4.3. *A Quantitative Multidimensional Central Limit Theorem in the d_c distance.*

Theorem 4.4. *Let $d \in \mathbb{N}$ be a fixed integer and suppose that the kernels h_i, $i \in [d]$, satisfy (4.1) and that the covariance matrix Σ is positive definite. Then:*
(i)

$$\begin{aligned} d_c(\mathbf{I}(\eta,\mathbf{h}),\mathbf{N}) &\leq \left[\left(\frac{16d^{3/2}}{3}\frac{\sqrt{80}}{2^{1/4}} + 64d^{17/12} + 8d^6\right)\|\Sigma^{-1}\|_{op} + \frac{80d}{\sqrt{2}}\|\Sigma^{-1}\|_{op}^{1/2}\right]\gamma_5 \\ &\leq \left[\left(\frac{16d^{3/2}}{3}\frac{\sqrt{80}}{2^{1/4}} + 64d^{17/12} + 8d^6\right)\|\Sigma^{-1}\|_{op} + \frac{80d}{\sqrt{2}}\|\Sigma^{-1}\|_{op}^{1/2}\right]\gamma_6. \end{aligned} \tag{4.25}$$

(ii) Let $\{\eta_M\}_{M\in\mathbb{N}}$ be a sequence of Dirichlet-Ferguson measures such that, for any $M \in \mathbb{N}$, η_M has intensity σ_M and $\beta_M := \sigma_M(\mathbb{X}) \to +\infty$, as $M \to +\infty$. For each $M \in \mathbb{N}$, let $\{h_{i,M}\}_{i\in[d]}$ be a family of kernels in $L^4(\mathbb{X},\sigma_M)$ which satisfy (1.2), with $h_{i,M}$ in place of h and σ_M in place of σ, (4.3) and (4.7). Then, for each $M \in \mathbb{N}$, the relation (4.25) is satisfied, with η_M in place of η, $\mathbf{h}_M$ in place of $\mathbf{h}$ and $\gamma_{i,M}$ in place of γ_i, $i = 5,6$, and $\gamma_{6,M} \to 0$, as $M \to +\infty$. Therefore

$$d_c(\mathbf{I}(\eta_M,\mathbf{h}_M),\mathbf{N}) \to 0, \quad \textit{as } M \to +\infty. \tag{4.26}$$

Proof: *Proof of Part (i).* We start by noticing that if $h_i \notin L^4(\mathbb{X},\sigma)$ for some $i \in [d]$, then $\gamma_5 = \gamma_6 = +\infty$ (since $\gamma_7 = \gamma_8 = +\infty$) and the bounds are trivially true. Therefore, throughout this proof we suppose $h_i \in L^4(\mathbb{X},\sigma)$ for any $i \in [d]$. Note that the second inequality in (4.25) follows by the inequality $\gamma_7 \leq \gamma_8$, which, in turn, is a consequence of the Cauchy-Schwarz inequality (applied to the product measure $\sigma(\mathrm{d}x) \otimes \kappa(j)$, where $\kappa(j)$ is the counting measure on $[d]$). It remains to prove the first inequality in (4.25). Hereafter, for ease of notation we set $\mathbf{I} := \mathbf{I}(\eta,\mathbf{h})$. Let $t \in (0,1/2)$ and $g \in \mathcal{I}$ be arbitrarily fixed. The rough idea of the proof is first to bound the quantity $|\mathbb{E}[g_t(\mathbf{I})] - \mathbb{E}[g_t(\mathbf{N})]|$ and then to obtain a bound for $d_c(\mathbf{I},\mathbf{N})$ combining the smoothing Lemma 3.9 with a suitable choice of t. By Lemma 3.6(i)-(ii) and (3.2) (note that, for any $i \in [d]$ and $t \in (0,1/2)$, the function $\partial f_{t,g}/\partial x_i$ is bounded and therefore $\partial f_{t,g}(\mathbf{I})/\partial x_i \in L^2(\Omega,\mathbb{P})$), we have

$$\begin{aligned} |\mathbb{E}[g_t(\mathbf{I})] - \mathbb{E}[g_t(\mathbf{N})]| &= \Big|\sum_{i,j=1}^d \mathbb{E}\left[\Sigma_{ij}\frac{\partial^2 f_{t,g}(\mathbf{I})}{\partial x_i\partial x_j}\right] - \sum_{j=1}^d \mathbb{E}\left[I(\eta,h_j)\frac{\partial f_{t,g}(\mathbf{I})}{\partial x_j}\right]\Big| \\ &= \Big|\sum_{i,j=1}^d \mathbb{E}\left[\Sigma_{ij}\frac{\partial^2 f_{t,g}(\mathbf{I})}{\partial x_i\partial x_j}\right] - \sum_{j=1}^d \mathbb{E}\left[\int_{\mathbb{X}\times[0,1]} h_j(x)D_{(x,u)}\frac{\partial f_{t,g}(\mathbf{I})}{\partial x_j}\widehat{\sigma}(\mathrm{d}x,\mathrm{d}u)\right]\Big|. \end{aligned} \tag{4.27}$$

By the Fundamental Theorem of Calculus we have

$$\begin{aligned}D_{(x,u)}\frac{\partial}{\partial x_j}f_{t,g}(\mathbf{I}) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}z}\frac{\partial}{\partial x_j}f_{t,g}(\mathbf{I}+zD_{(x,u)}\mathbf{I})\,\mathrm{d}z\\ &= \int_0^1 \mathrm{d}z\sum_{k=1}^d \frac{\partial^2}{\partial x_k\partial x_j}f_{t,g}(\mathbf{I}+zD_{(x,u)}\mathbf{I})D_{(x,u)}I(\eta,h_k),\end{aligned}$$

and so

$$\begin{aligned}&\sum_{j=1}^d \mathbb{E}\left[\int_{\mathbb{X}\times[0,1]} h_j(x)D_{(x,u)}\frac{\partial f_{t,g}(\mathbf{I})}{\partial x_j}\widehat{\sigma}(\mathrm{d}x,\mathrm{d}u)\right]\\ &= \sum_{j,k=1}^d \mathbb{E}\left[\int_{\mathbb{X}\times[0,1]} h_j(x)\int_0^1 \frac{\partial^2}{\partial x_k\partial x_j}f_{t,g}(\mathbf{I}+zD_{(x,u)}\mathbf{I})D_{(x,u)}I(\eta,h_k)\mathrm{d}z\widehat{\sigma}(\mathrm{d}x,\mathrm{d}u)\right]\\ &= \sum_{j,k=1}^d \mathbb{E}\left[\int_{\mathbb{X}\times[0,1]} h_j(x)\frac{\partial^2 f_{t,g}(\mathbf{I})}{\partial x_k\partial x_j}D_{(x,u)}I(\eta,h_k)\widehat{\sigma}(\mathrm{d}x,\mathrm{d}u)\right]\\ &\quad+\sum_{j,k=1}^d \mathbb{E}\Bigg[\int_{\mathbb{X}\times[0,1]} h_j(x)\int_0^1\Bigg(\frac{\partial^2}{\partial x_k\partial x_j}f_{t,g}(\mathbf{I}+zD_{(x,u)}\mathbf{I})\\ &\qquad\qquad-\frac{\partial^2 f_{t,g}(\mathbf{I})}{\partial x_k\partial x_j}\Bigg)D_{(x,u)}I(\eta,h_k)\mathrm{d}z\widehat{\sigma}(\mathrm{d}x,\mathrm{d}u)\Bigg] =: \tau_1+\tau_2.\end{aligned}\tag{4.28}$$

Defining

$$\tau := \tau_1 - \sum_{j,k=1}^d \Sigma_{jk}\mathbb{E}\left[\frac{\partial^2 f_{t,g}(\mathbf{I})}{\partial x_k\partial x_j}\right],$$

by (4.27) and (4.28) we have

$$|\mathbb{E}[g_t(\mathbf{I})]-\mathbb{E}[g_t(\mathbf{N})]| \le |\tau|+|\tau_2|.\tag{4.29}$$

We continue dividing the proof of Part (i) in three steps. In the first step we upper bound $|\tau|$, in the second step we upper bound $|\tau_2|$, in the third step we conclude the proof.
Step 1: Bounding $|\tau|$.
We start by noticing that by the definition of the gradient D and the relation

$$I((1-t)\eta+t\varepsilon_x,h) = th(x)+(1-t)I(\eta,h),\tag{4.30}$$

we have

$$D_{(x,t)}I(\eta,h_i) = t(h_i(x)-I(\eta,h_i)),\quad (x,t)\in\mathbb{X}\times[0,1],\ i\in[d].\tag{4.31}$$

Therefore, for any function $h_0\in L^2(\mathbb{X},\sigma)$ whose integral against σ is equal to 0, for any $i\in[d]$, we have

$$\begin{aligned}\langle h_0(\cdot),D_{(x,t)}I(\eta,h_i)\rangle_{L^2(\mathbb{X}\times[0,1],\widehat{\sigma})} &= \int_{\mathbb{X}\times[0,1]} th_0(x)(h_i(x)-I(\eta,h_i))\widehat{\sigma}(\mathrm{d}x,\mathrm{d}t)\\ &= \int_0^1 t(1-t)^{\beta-1}\,\mathrm{d}t\int_{\mathbb{X}}h_0(x)h_i(x)\sigma(\mathrm{d}x) - I(\eta,h_i)\int_0^1 t(1-t)^{\beta-1}\,\mathrm{d}t\int_{\mathbb{X}}h_0(x)\sigma(\mathrm{d}x)\\ &= \frac{1}{\beta(\beta+1)}\int_{\mathbb{X}}h_0(x)h_i(x)\sigma(\mathrm{d}x).\end{aligned}\tag{4.32}$$

So

$$
\begin{aligned}
|\tau| &\leq \sum_{j,k=1}^{d} \mathbb{E}\left[\left|\Sigma_{jk} - \int_{\mathbb{X}\times[0,1]} h_j(x) D_{(x,u)} I(\eta, h_k) \widehat{\sigma}(\mathrm{d}x, \mathrm{d}u)\right| \left|\frac{\partial^2 f_{t,g}(\mathbf{I})}{\partial x_k \partial x_j}\right|\right] \\
&= \sum_{j,k=1}^{d} \left|\Sigma_{jk} - \frac{1}{\beta(\beta+1)} \int_{\mathbb{X}} h_j(x) h_k(x) \sigma(\mathrm{d}x)\right| \mathbb{E}\left[\left|\frac{\partial^2 f_{t,g}(\mathbf{I})}{\partial x_k \partial x_j}\right|\right] \\
&\leq \sum_{j,k=1}^{d} \left|\Sigma_{jk} - \frac{1}{\beta(\beta+1)} \int_{\mathbb{X}} h_j(x) h_k(x) \sigma(\mathrm{d}x)\right| \sqrt{\sum_{j,k=1}^{d} \mathbb{E}\left[\left|\frac{\partial^2 f_{t,g}(\mathbf{I})}{\partial x_k \partial x_j}\right|^2\right]},
\end{aligned}
$$

where we used (4.32) and the Cauchy-Schwarz inequality. Then by Lemma 3.6(iii) we have

$$
|\tau| \leq \|\Sigma^{-1}\|_{op} (d |\log t| \sqrt{d_c(\mathbf{I}, \mathbf{N})} + 24 d^{17/12}) \gamma_1, \quad t \in (0, 1/2). \tag{4.33}
$$

Step 2: Bounding $|\tau_2|$.
By the Multivariate Mean Value Theorem and Lemma 3.6(iv) (note that $g(\cdot) = \mathbf{1}\{\cdot \in K\}$ for some measurable convex set $K \subseteq \mathbb{R}^d$ and so $\|g\|_\infty \leq 1$), for any $k, j \in [d]$ we have

$$
\begin{aligned}
\left|\frac{\partial^2}{\partial x_k \partial x_j} f_{t,g}(\mathbf{I} + z D_{(x,u)} \mathbf{I}) - \frac{\partial^2}{\partial x_k \partial x_j} f_{t,g}(\mathbf{I})\right| &\leq \sup_{v \in \mathbb{R}} \sqrt{\sum_{i=1}^{d} \left|\frac{\partial^3 f_{t,g}(v)}{\partial x_i \partial x_k \partial x_j}\right|^2} \|z D_{(x,u)} \mathbf{I}\| \\
&\leq 6 d^4 \|\Sigma^{-1}\|_{op}^{3/2} \frac{1}{\sqrt{t}} \|z D_{(x,u)} \mathbf{I}\|.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
|\tau_2| &\leq 6 d^4 \|\Sigma^{-1}\|_{op}^{3/2} \frac{1}{\sqrt{t}} \int_0^1 z \mathrm{d}z \sum_{j,k=1}^{d} \mathbb{E}\left[\int_{\mathbb{X}\times[0,1]} |h_j(x)| \|D_{(x,u)} \mathbf{I}\| |D_{(x,u)} I(\eta, h_k)| \widehat{\sigma}(\mathrm{d}x, \mathrm{d}u)\right] \\
&= 3 d^4 \|\Sigma^{-1}\|_{op}^{3/2} \frac{1}{\sqrt{t}} \sum_{j,k=1}^{d} \mathbb{E}\left[\int_{\mathbb{X}\times[0,1]} |h_j(x)| \sqrt{\sum_{\ell=1}^{d} |D_{(x,u)} I(\eta, h_\ell)|^2} |D_{(x,u)} I(\eta, h_k)| \widehat{\sigma}(\mathrm{d}x, \mathrm{d}u)\right] \\
&\leq 3 d^4 \|\Sigma^{-1}\|_{op}^{3/2} \frac{1}{\sqrt{t}} \sum_{j,k,\ell=1}^{d} \mathbb{E}\left[\int_{\mathbb{X}\times[0,1]} |h_j(x)| |D_{(x,u)} I(\eta, h_\ell)| |D_{(x,u)} I(\eta, h_k)| \widehat{\sigma}(\mathrm{d}x, \mathrm{d}u)\right],
\end{aligned} \tag{4.34}
$$

where we used the elementary inequality $\sqrt{\sum_{i=1}^{d} |a_i|^2} \leq \sum_{i=1}^{d} |a_i|$, $a_1, \ldots, a_d \in \mathbb{R}$. So

$$
\begin{aligned}
&\mathbb{E}\left[\int_{\mathbb{X}\times[0,1]} |h_j(x)| |D_{(x,u)} I(\eta, h_k) D_{(x,u)} I(\eta, h_\ell)| \widehat{\sigma}(\mathrm{d}x, \mathrm{d}u)\right] \\
&\qquad = \frac{2}{\beta(\beta+1)(\beta+2)} \mathbb{E}\left[\int_{\mathbb{X}} |h_j(x)| |h_k(x) - I(\eta, h_k)| |h_\ell(x) - I(\eta, h_\ell)| \sigma(\mathrm{d}x)\right].
\end{aligned}
$$

By the arithmetic-geometric mean inequality and the convexity of $x \mapsto |x|^3$, we have

$$\begin{aligned}
&\mathbb{E}\left[\int_{\mathbb{X}\times[0,1]} |h_j(x)||D_{(x,u)}I(\eta,h_k)D_{(x,u)}I(\eta,h_\ell)|\widehat{\sigma}(\mathrm{d}x,\mathrm{d}u)\right] \\
&\le \frac{2}{\beta(\beta+1)(\beta+2)}\mathbb{E}\left[\int_{\mathbb{X}}\left(\frac{|h_j(x)|}{3}+\frac{|h_k(x)-I(\eta,h_k)|}{3}+\frac{|h_\ell(x)-I(\eta,h_\ell)|}{3}\right)^3\sigma(\mathrm{d}x)\right] \\
&\le \frac{2}{3\beta(\beta+1)(\beta+2)}\left(\int_{\mathbb{X}}|h_j(x)|^3\sigma(\mathrm{d}x)+\int_{\mathbb{X}}\mathbb{E}[|h_k(x)-I(\eta,h_k)|^3]\sigma(\mathrm{d}x)\right. \\
&\qquad\qquad\left.+\int_{\mathbb{X}}\mathbb{E}[|h_\ell(x)-I(\eta,h_\ell)|^3]\sigma(\mathrm{d}x)\right).
\end{aligned}$$

By this relation and the Cauchy-Schwarz inequality it follows

$$\begin{aligned}
&\sum_{j,k,\ell=1}^{d}\mathbb{E}\left[\int_{\mathbb{X}\times[0,1]} |h_j(x)||D_{(x,u)}I(\eta,h_k)D_{(x,u)}I(\eta,h_\ell)|\widehat{\sigma}(\mathrm{d}x,\mathrm{d}u)\right] \\
&\le \frac{2d^2}{3\beta(\beta+1)(\beta+2)}\left[\sum_{j=1}^{d}\int_{\mathbb{X}}|h_j(x)|^3\sigma(\mathrm{d}x)\right. \\
&\qquad\left.+2\sqrt{\sum_{k=1}^{d}\int_{\mathbb{X}}\mathbb{E}[|h_k(x)-I(\eta,h_k)|^2]\sigma(\mathrm{d}x)\sum_{k=1}^{d}\int_{\mathbb{X}}\mathbb{E}[|h_k(x)-I(\eta,h_k)|^4]\sigma(\mathrm{d}x)}\right].
\end{aligned}\tag{4.35}$$

By (2.3) and (2.4) we easily have

$$\mathbb{E}[|h_k(x)-I(\eta,h_k)|^2]=h_k(x)^2+\frac{1}{\beta(\beta+1)}\int_{\mathbb{X}}h_k(x)^2\sigma(\mathrm{d}x)\tag{4.36}$$

and

$$\begin{aligned}
&\mathbb{E}[|h_k(x)-I(\eta,h_k)|^4] \\
&\quad=\mathbb{E}[I(\eta,h_k)^4]-4h_k(x)\mathbb{E}[I(\eta,h_k)^3]+\frac{6h_k(x)^2}{\beta(\beta+1)}\int_{\mathbb{X}}h_k(x)^2\sigma(\mathrm{d}x)+h_k(x)^4.
\end{aligned}\tag{4.37}$$

By Jensen's inequality ($\eta(\omega)$ is a probability measure and $x \mapsto x^4$ is convex) and Lemma 2.1(ii), we have

$$\begin{aligned}
\mathbb{E}[I(\eta,h_k)^4]&=\mathbb{E}\left[\left(\int_{\mathbf{X}}h_k(x)\eta(\mathrm{d}x)\right)^4\right]\le\mathbb{E}\left[\int_{\mathbb{X}}h_k(x)^4\eta(\mathrm{d}x)\right]=\mathbb{E}[\mathbb{E}[h_k(U_1)^4\,|\,\eta]] \\
&=\mathbb{E}[h_k(U_1)^4]=\frac{1}{\beta}\int_{\mathbb{X}}h_k(x)^4\sigma(\mathrm{d}x),
\end{aligned}\tag{4.38}$$

where the latter equality follows by (2.1). Combining (4.37) and (4.38) we have

$$\begin{aligned}
\mathbb{E}[|h_k(x)-I(\eta,h_k)|^4]&\le\frac{1}{\beta}\int_{\mathbb{X}}h_k(x)^4\sigma(\mathrm{d}x)-4h_k(x)\mathbb{E}[I(\eta,h_k)^3] \\
&\qquad+\frac{6}{\beta(\beta+1)}h_k(x)^2\int_{\mathbb{X}}h_k(x)^2\sigma(\mathrm{d}x)+h_k(x)^4,
\end{aligned}$$

and so, using again (2.3), we get

$$\int_{\mathbb{X}}\mathbb{E}[|h_k(x)-I(\eta,h_k)|^4]\sigma(\mathrm{d}x)\le2\int_{\mathbb{X}}h_k(x)^4\sigma(\mathrm{d}x)+\frac{6}{\beta(\beta+1)}\left(\int_{\mathbb{X}}h_k(x)^2\sigma(\mathrm{d}x)\right)^2.\tag{4.39}$$

Furthermore, (4.36) yields

$$\int_{\mathbb{X}} \mathbb{E}[|h_k(x) - I(\eta, h_k)|^2]\sigma(\mathrm{d}x) = \frac{\beta+2}{\beta+1}\int_{\mathbb{X}} h_k(x)^2\sigma(\mathrm{d}x). \tag{4.40}$$

Combining (4.34), (4.35), (4.39) and (4.40), we easily have

$$|\tau_2| \le 3d^6\|\Sigma^{-1}\|_{op}^{3/2}\frac{1}{\sqrt{t}}\gamma_7, \quad t \in (0,1/2). \tag{4.41}$$

Step 3: Conclusion of the proof of Part (i).
By (4.29), (4.33) and (4.41) we have

$$|\mathbb{E}[g_t(\mathbf{I})] - \mathbb{E}[g_t(\mathbf{N})]| \le \|\Sigma^{-1}\|_{op}(d|\log t|\sqrt{d_c(\mathbf{I},\mathbf{N})} + 24d^{17/12})\gamma_1 + 3d^6\|\Sigma^{-1}\|_{op}^{3/2}\frac{1}{\sqrt{t}}\gamma_7,$$

and so Lemma 3.9 yields

$$d_c(\mathbf{I},\mathbf{N}) \le \frac{4}{3}\|\Sigma^{-1}\|_{op}(d|\log t|\sqrt{d_c(\mathbf{I},\mathbf{N})} + 24d^{17/12})\gamma_1 + 4d^6\|\Sigma^{-1}\|_{op}^{3/2}\frac{1}{\sqrt{t}}\gamma_7 + \frac{20d}{\sqrt{2}}\frac{\sqrt{t}}{1-t}.$$

Recall that $t \in (0,1/2)$ is arbitrarily fixed. Therefore $t^{1/4}|\log t| \le 2$ and $1-t > 1/2$. Then

$$d_c(\mathbf{I},\mathbf{N}) \le \frac{8d}{3}\frac{\sqrt{d_c(\mathbf{I},\mathbf{N})}}{t^{1/4}}\|\Sigma^{-1}\|_{op}\gamma_5 + 32d^{17/12}\|\Sigma^{-1}\|_{op}\gamma_5 + 4d^6\|\Sigma^{-1}\|_{op}^{3/2}\frac{\gamma_5^2}{\sqrt{t}} + \frac{40d}{\sqrt{2}}\sqrt{t}. \tag{4.42}$$

Since $d_c(\mathbf{I},\mathbf{N}) \le 1$, the first inequality in (4.25) is obvious if $\gamma_5 \ge \|\Sigma^{-1}\|_{op}^{-1/2}/\sqrt{2}$. So we suppose $\gamma_5 < \|\Sigma^{-1}\|_{op}^{-1/2}/\sqrt{2}$ and take $t \in (0,1/2)$ so that

$$\sqrt{t} = \max\left\{\frac{\sqrt{2}}{80d}d_c(\mathbf{I},\mathbf{N}), \|\Sigma^{-1}\|_{op}^{1/2}\gamma_5\right\}.$$

By this choice of t and (4.42), we have

$$\begin{aligned} d_c(\mathbf{I},\mathbf{N}) \le{}& \frac{8d^{3/2}}{3}\frac{\sqrt{80}}{2^{1/4}}\|\Sigma^{-1}\|_{op}\gamma_5 + 32d^{17/12}\|\Sigma^{-1}\|_{op}\gamma_5 + 4d^6\|\Sigma^{-1}\|_{op}\gamma_5 \\ &+ \frac{40d}{\sqrt{2}}\|\Sigma^{-1}\|_{op}^{1/2}\gamma_5 + \frac{1}{2}d_c(\mathbf{I},\mathbf{N}), \end{aligned}$$

which is equivalent to the first inequality in (4.25).
Proof of Part (ii). The proof of this part of the claim is rather obvious (it directly follows by Part (i) and the assumptions), and therefore omitted. □

We conclude this subsection noticing that the claims of Theorem 4.4 immediately transfer to the Kolmogorov distance

$$d_K(\mathbf{X},\mathbf{Y}) := \sup_{x_1,\dots,x_d\in\mathbb{R}} |\mathbb{P}(X_1 \le x_1,\dots,X_d \le x_d) - \mathbb{P}(X_1 \le x_1,\dots,X_d \le x_d)|$$

where $\mathbf{X} = (X_1,\dots,X_d)$ and $\mathbf{Y} = (Y_1,\dots,Y_d)$ are $\mathbb{R}^d$-valued random vectors. Indeed we clearly have that $d_K(\cdot,\cdot) \le d_c(\cdot,\cdot)$.

5. Gaussian approximation of linear transformations of random vectors with the Dirichlet distribution

For $M, d \in \mathbb{N}$ and $i \in [d]$, let $K_{iM} \ge 2$ be integers and let

$$\mathbf{Y}_M := (Y_{11},\dots,Y_{1K_{1M}}, Y_{21},\dots,Y_{2K_{2M}},\dots,Y_{d1},\dots,Y_{dK_{dM}})$$

be random vectors distributed according to the Dirichlet law with parameters

$$\alpha^{(M)} := (\alpha_{11}^{(M)},\dots,\alpha_{1K_{1M}}^{(M)}, \alpha_{21}^{(M)},\dots,\alpha_{2K_{2M}}^{(M)},\dots,\alpha_{d1}^{(M)},\dots,\alpha_{dK_{dM}}^{(M)}).$$

The next results provide Quantitative Central Limit Theorems for the sequence of random vectors defined by

$$\mathbf{S}_M := \left(C_{1M} \sum_{m=1}^{K_{1M}} u_{1m}\big(Y_{1m} - \mathbb{E}[Y_{1m}]\big), \dots, C_{dM} \sum_{m=1}^{K_{dM}} u_{dm}\big(Y_{dm} - \mathbb{E}[Y_{dm}]\big) \right),$$

where C_{iM}, $i \in [d]$, are normalizing constants and $\{u_{im}\}_{i\in[d],\, m=1,\dots,K_{iM}} \subset \{-1,+1\}$.

Set

$$\widetilde{\alpha}_M := \sum_{i=1}^{d} \sum_{m=1}^{K_{iM}} \alpha_{im}^{(M)}, \quad \widetilde{\alpha}_{iM} := \frac{1}{C_{iM}^2} \sum_{m=1}^{K_{iM}} \alpha_{im}^{(M)} \quad \text{and} \quad \mu_{iM} := \frac{1}{C_{iM}^2} \sum_{m=1}^{K_{iM}} u_{im} \alpha_{im}^{(M)}, \quad i \in [d].$$

In this section we consider the quantities

$$\gamma_{1,M} := \sum_{i=1}^{d} \left| \Sigma_{ii} - \frac{C_{iM}^4}{\widetilde{\alpha}_M(\widetilde{\alpha}_M+1)} \left(\widetilde{\alpha}_{iM} - \frac{C_{iM}^2 \mu_{iM}^2}{\widetilde{\alpha}_M} \right) \right| + \mathbf{1}\{d \geq 2\} \sum_{i,j\in[d]}^{i\neq j} \left| \Sigma_{ij} + \frac{C_{iM}^3 C_{jM}^3}{\widetilde{\alpha}_M^2(\widetilde{\alpha}_M+1)} \mu_{iM}\mu_{jM} \right|,$$

$$\begin{aligned}\gamma_{4,M} := {} & \frac{d^3}{\widetilde{\alpha}_M(\widetilde{\alpha}_M+1)^2} \left\{ \max_{1\leq i\leq d} C_{iM}^4 \left(\widetilde{\alpha}_{iM} - \frac{C_{iM}^2 \mu_{iM}^2}{\widetilde{\alpha}_M} \right) \right\}^{1/2} \\ & \times \left\{ \max_{1\leq i\leq d} C_{iM}^4 \left(\frac{C_{iM}^{10}\mu_{iM}^4 \widetilde{\alpha}_{iM}}{(\widetilde{\alpha}_M)^4} - 4\frac{C_{iM}^8 \mu_{iM}^4}{(\widetilde{\alpha}_M)^3} + 6\frac{C_{iM}^6 \mu_{iM}^2 \widetilde{\alpha}_{iM}}{(\widetilde{\alpha}_M)^2} - 4\frac{C_{iM}^4 \mu_{iM}^2}{\widetilde{\alpha}_M} \right.\right. \\ & \left.\left. + C_{iM}^2 \widetilde{\alpha}_{iM} + \mathbf{1}\{d\geq 2\} \frac{C_{iM}^8 \mu_{iM}^4}{(\widetilde{\alpha}_M)^4} \sum_{j=1,\, j\neq i}^{d} C_{jM}^2 \widetilde{\alpha}_{jM} \right) \right\}^{1/2},\end{aligned}$$

$$\gamma_{6,M} := \max\{\gamma_{1,M}, \sqrt{\gamma_{8,M}}\},$$

and

$$\begin{aligned}\gamma_{8,M} := {} & \frac{2}{3\widetilde{\alpha}_M(\widetilde{\alpha}_M+1)(\widetilde{\alpha}_M+2)} \Bigg\{ \left[\sum_{i=1}^{d} C_{iM}^4 \left(\widetilde{\alpha}_{iM} - \frac{C_{iM}^2 \mu_{iM}^2}{\widetilde{\alpha}_M} \right) \right]^{1/2} \\ & \times \left[\sum_{i=1}^{d} C_{iM}^4 \left(\frac{C_{iM}^{10}\mu_{iM}^4 \widetilde{\alpha}_{iM}}{(\widetilde{\alpha}_M)^4} - 4\frac{C_{iM}^8 \mu_{iM}^4}{(\widetilde{\alpha}_M)^3} + 6\frac{C_{iM}^6 \mu_{iM}^2 \widetilde{\alpha}_{iM}}{(\widetilde{\alpha}_M)^2} - 4\frac{C_{iM}^4 \mu_{iM}^2}{\widetilde{\alpha}_M} + C_{iM}^2 \widetilde{\alpha}_{iM} \right.\right. \\ & \left.\left. + \mathbf{1}\{d\geq 2\} \frac{C_{iM}^8 \mu_{iM}^4}{(\widetilde{\alpha}_M)^4} \sum_{j=1,\, j\neq i}^{d} C_{jM}^2 \widetilde{\alpha}_{jM} \right) \right]^{1/2} \\ & + 2 \left[\frac{\widetilde{\alpha}_M+2}{\widetilde{\alpha}_M+1} \sum_{i=1}^{d} C_{iM}^4 \left(\widetilde{\alpha}_{iM} - \frac{C_{iM}^2 \mu_{iM}^2}{\widetilde{\alpha}_M} \right) \right. \\ & \times \left[2 \sum_{i=1}^{d} C_{iM}^4 \left(\frac{C_{iM}^{10}\mu_{iM}^4 \widetilde{\alpha}_{iM}}{(\widetilde{\alpha}_M)^4} - 4\frac{C_{iM}^8 \mu_{iM}^4}{(\widetilde{\alpha}_M)^3} + 6\frac{C_{iM}^6 \mu_{iM}^2 \widetilde{\alpha}_{iM}}{(\widetilde{\alpha}_M)^2} - 4\frac{C_{iM}^4 \mu_{iM}^2}{\widetilde{\alpha}_M} + C_{iM}^2 \widetilde{\alpha}_{iM} \right.\right. \\ & \left. + \mathbf{1}\{d\geq 2\} \frac{C_{iM}^8 \mu_{iM}^4}{(\widetilde{\alpha}_M)^4} \sum_{j=1,\, j\neq i}^{d} C_{jM}^2 \widetilde{\alpha}_{jM} \right) \\ & \left.\left. + \frac{6}{\widetilde{\alpha}_M(\widetilde{\alpha}_M+1)} \sum_{i=1}^{d} \left(C_{iM}^4 \left(\widetilde{\alpha}_{iM} - \frac{C_{iM}^2 \mu_{iM}^2}{\widetilde{\alpha}_M} \right) \right)^2 \right] \right]^{1/2} \Bigg\}. \end{aligned} \tag{5.1}$$

The following theorem holds.

Theorem 5.1. *Let $d \in \mathbb{N}$ be fixed. Then:*
(i) If Σ is positive semi-definite, then

$$d_3(\mathbf{S}_M, \mathbf{N}) \le \frac{1}{2}(\gamma_{1,M} + \gamma_{4,M}). \tag{5.2}$$

(ii) If Σ is positive definite, then

$$d_2(\mathbf{S}_M, \mathbf{N}) \le \sqrt{\frac{2}{\pi}} \|\Sigma^{-1/2}\|_{op}\gamma_{1,M} + \frac{\sqrt{2\pi}\|\Sigma^{-1/2}\|_{op}}{4}\gamma_{4,M} \tag{5.3}$$

and

$$d_c(\mathbf{S}_M, \mathbf{N}) \le \left[\left(\frac{16d^{3/2}}{3}\frac{\sqrt{80}}{2^{1/4}} + 64d^{17/12} + 8d^6\right)\|\Sigma^{-1}\|_{op} + \frac{80d}{\sqrt{2}}\|\Sigma^{-1}\|_{op}^{1/2}\right]\gamma_{6,M}. \tag{5.4}$$

Hereafter, we write $a_n \sim b_n$ for two positive sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ such that $a_n/b_n \to 1$ as $n \to +\infty$. The next two theorems and the next remark refer to the d-dimensional case with $d \ge 2$. The one-dimensional case is provided by Theorem 5.5.

Theorem 5.2. *Let $d \ge 2$ be fixed, let the notation of Theorem 5.1 prevail and suppose:*
$(H1)$: The sequences $\{C_{iM}\}_{M\in\mathbb{N}}$, $i \in [d]$, diverge to $+\infty$, as $M \to +\infty$ and are such that, for any $j \in \{2, \dots, K\}$, there exists a positive constant $\kappa_j > 0$ such that $C_{jM} \sim \kappa_j C_{1M}$, as $M \to +\infty$.
$(H2)$: For any $j \in [d]$, $\widetilde{\alpha}_{jM} \to \alpha_j$, as $M \to +\infty$. Here $\alpha_1, \dots, \alpha_d$ are non-negative constants such that $(\alpha_1, \dots, \alpha_d) \neq \mathbf{0}$.
$(H3)$: For any $j \in [d]$, $\mu_{jM} \to \mu_j \in \mathbb{R}$, as $M \to +\infty$.
Let $\Sigma = (\Sigma_{ij})_{1\le i,j\le d}$ be the symmetric matrix with entries

$$\Sigma_{jj} := \frac{\kappa_j^4}{\kappa^2}\left(\alpha_j - \frac{\kappa_j^2\mu_j^2}{\kappa}\right), \quad j \in [d], \quad \Sigma_{jk} := -\frac{\kappa_j^3\kappa_k^3}{\kappa^3}\mu_j\mu_k, \quad j \neq k, j, k \in [d],$$

where

$$\kappa := \sum_{j=1}^{d} \kappa_j^2\alpha_j > 0 \quad \text{and} \quad \kappa_1 := 1.$$

Then:
(i) If Σ is positive semi-definite, then the upper bound (5.2) holds, $\gamma_{1,M}, \gamma_{4,M} \to 0$, as $M \to +\infty$, and so $d_3(\mathbf{S}_M, \mathbf{N}) \to 0$, as $M \to +\infty$.
(ii) If Σ is positive definite, then the upper bounds (5.3) and (5.4) hold, $\gamma_{1,M}, \gamma_{4,M}, \gamma_{8,M} \to 0$, as $M \to +\infty$, and so $d_j(\mathbf{S}_M, \mathbf{N}) \to 0$, as $M \to +\infty$, $j = 2, c$.

Theorem 5.3. *Let $d \ge 2$ be fixed, let the notation of Theorem 5.1 prevail, and suppose:*
$(H1)'$: The sequences $\{C_{iM}\}_{M\in\mathbb{N}}$, $i \in [d]$, diverge to $+\infty$, as $M \to +\infty$, and are such that, for any $j \in \{2, \dots, d\}$, there exists a positive constant $\kappa_j > 0$ such that $C_{jM}/C_{1M} = \kappa_j + O(C_{1M}^{-1})$, as $M \to +\infty$.
$(H2)'$: For any $j \in [d]$, $\widetilde{\alpha}_{jM} = \alpha_j + O(C_{1M}^{-2})$, as $M \to +\infty$. Here $\alpha_1, \dots, \alpha_d$ are non-negative constants such that $(\alpha_1, \dots, \alpha_d) \neq \mathbf{0}$.
$(H3)'$: For any $j \in [d]$, $\mu_{jM} = \mu_j + O(C_{1M}^{-1})$, as $M \to +\infty$. Here $\mu_{i_1}, \dots, \mu_{i_K}$ are real constants.
Then, letting Σ denote the matrix defined in the statement of Theorem 5.2, we have:
(i) If Σ is positive semi-definite, then $d_3(\mathbf{S}_M, \mathbf{N}) = O(C_{1M}^{-1})$, as $M \to +\infty$.
(ii) If Σ is positive definite, then $d_2(\mathbf{S}_M, \mathbf{N}) = O(C_{1M}^{-1})$ and $d_c(\mathbf{S}_M, \mathbf{N}) = O(C_{1M}^{-1/2})$, as $M \to +\infty$.

Remark 5.4. (i) Note that the assumptions $(H1)'$, $(H2)'$ and $(H3)'$ of Theorem 5.3 are stronger than the hypotheses $(H1)$, $(H2)$ and $(H3)$ of Theorem 5.2, respectively, and guarantee a rate for the various distances. (ii) There are various criteria which guarantee that a real symmetric square matrix is either positive semi-definite or positive definite. For instance, if a real symmetric square matrix has non-negative (positive, respectively) diagonal entries and it is diagonally dominant (strictly diagonally dominant, respectively) then it is positive semi-definite (positive definite, respectively). Therefore, the matrix Σ defined in the statement of Theorem 5.2 is positive semi-definite if

$$\frac{\kappa_j^4}{\kappa^2}\left(\alpha_j - \frac{\kappa_j^2\mu_j^2}{\kappa}\right) \geq \sum_{k\in[d]} \frac{\kappa_j^3\kappa_k^3}{\kappa^3}|\mu_j\mu_k|, \quad \forall\, j \in [d]$$

and it is positive definite if

$$\frac{\kappa_j^4}{\kappa^2}\left(\alpha_j - \frac{\kappa_j^2\mu_j^2}{\kappa}\right) > \sum_{k\in[d]} \frac{\kappa_j^3\kappa_k^3}{\kappa^3}|\mu_j\mu_k|, \quad \forall\, j \in [d].$$

The next theorem concerns the one-dimensional case.

Theorem 5.5. *Let $d = 1$, let the notation of Theorem 5.1 prevail and set*

$$S_M := C_{1M}\sum_{m=1}^{K_{1M}} u_{1m}\big(Y_{1m} - \mathbb{E}[Y_{1m}]\big).$$

(i) If $C_{1M} \to +\infty$, $\widetilde{\alpha}_{1M} \to \alpha > 0$, $\mu_{1M} \to \mu \in \mathbb{R}$, as $M \to +\infty$, and $\nu^2 := \alpha^{-2}(\alpha - \alpha^{-1}\mu^2) > 0$, then relations (5.2), (5.3) and (5.4) hold, with S_M in place of $\mathbf{S}_M$, $N(0, \nu^2)$ in place of $\mathbf{N}$, $d = 1$, ν^2 in place of Σ, and the upper bounds go to zero as $M \to +\infty$.
(ii) If, as $M \to \infty$, $C_{1M} \to +\infty$, $\widetilde{\alpha}_{1M} = \alpha + O(C_{1M}^{-2})$, for some $\alpha > 0$, $\mu_{1M} = \mu + O(C_{1M}^{-1})$, for some $\mu \in \mathbb{R}$, and $\nu^2 := \alpha^{-2}(\alpha - \alpha^{-1}\mu^2) > 0$, then

$$d_3(S_M, N(0,\nu^2)) = O(C_{1M}^{-1}), \quad d_2(S_M, N(0,\nu^2)) = O(C_{1M}^{-1}), \quad \text{as } M \to +\infty$$

and

$$d_c(S_M, N(0,\nu^2)) = O(C_{1M}^{-1/2}), \quad \text{as } M \to +\infty.$$

We conclude with a couple of examples.

Example 1 For $M \in \mathbb{N}$ and $d \geq 2$, define $K_{iM} := C$, $i \in [d]$, where C is a fixed even integer, $u_{im} := (-1)^m$, $\alpha_{im}^{(M)} := \varphi(M)$, $i \in [d]$, $m \in [C]$, where $\varphi : \mathbb{N} \to (0, +\infty)$ is a function which diverges to infinity, and $C_{iM} := \sqrt{C\varphi(M)}$, $i \in [d]$. It is readily checked that $\widetilde{\alpha}_{iM} = 1$ and $\mu_{iM} = 0$, $i \in [d]$. Therefore, letting

$$\mathbf{Y}_M := (Y_{11}, \ldots, Y_{1C}, Y_{21}, \ldots, Y_{2C}, \ldots, Y_{d1}, \ldots, Y_{dC})$$

denote a random vector with the Dirichlet distribution with parameters all equal to $\varphi(M)$ and setting

$$\mathbf{S}_M := \sqrt{C\varphi(M)}\left(\sum_{m=1}^{C}(-1)^m Y_{1m}, \ldots, \sum_{m=1}^{C}(-1)^m Y_{dm}\right),$$

by Theorem 5.3 we have

$$d_3(\mathbf{S}_M, \mathbf{N}) = O(\varphi(M)^{-1/2}), \quad d_2(\mathbf{S}_M, \mathbf{N}) = O(\varphi(M)^{-1/2}) \quad \text{and} \quad d_c(\mathbf{S}_M, \mathbf{N}) = O(\varphi(M)^{-1/4}),$$

as $M \to +\infty$, where $\mathbf{N}$ is a centered d-dimensional Gaussian random vector with diagonal covariance matrix $\Sigma = \mathrm{diag}(d^{-2}, \ldots, d^{-2})$. Similar rates can be obtained in the one-dimensional case applying Theorem 5.5; we omit the details.

Example 2 For $M \in \mathbb{N}$ and $d \geq 2$, define $K_{iM} := M$, $\alpha_{im}^{(M)} := \gamma > 0$ and $C_{iM} := \sqrt{\gamma M}$, $i \in [d]$, $m \in [M]$. Let $u_{im} := u_m$, $i \in [d]$, $m \in [M]$, where $\{u_m\}_{m\in[M]} \subset \{+1, -1\}$ is a sequence such that

$$\mathrm{Card}\big(\{m \in [M] \,:\, u_m = +1\}\big) = \frac{M}{2} + O(\sqrt{M}), \quad \text{as } M \text{ goes to infinity}$$

(here the symbol $\mathrm{Card}(S)$ denotes the cardinality of the set S). It is readily checked that, for any $i \in [d]$, $\widetilde{\alpha}_{iM} = 1$ and

$$\mu_{iM} = \frac{1}{M}\sum_{m=1}^{M} u_{im} = O(M^{-1/2}), \quad \text{as } M \to \infty.$$

Therefore, letting

$$\mathbf{Y}_M := (Y_{11}, \dots, Y_{1M}, Y_{21}, \dots, Y_{2M}, \dots, Y_{d1}, \dots, Y_{dM})$$

denote a random vector with the Dirichlet distribution with parameters all equal to γ and setting

$$\mathbf{S}_M := \sqrt{\gamma M}\left(\sum_{m=1}^{M} u_m\left(Y_{1m} - \frac{1}{dM}\right), \dots, \sum_{m=1}^{M} u_m\left(Y_{dm} - \frac{1}{dM}\right)\right),$$

by Theorem 5.3 we have

$$d_3(\mathbf{S}_M, \mathbf{N}) = O(M^{-1/2}), \quad d_2(\mathbf{S}_M, \mathbf{N}) = O(M^{-1/2}) \quad \text{and} \quad d_c(\mathbf{S}_M, \mathbf{N}) = O(M^{-1/4}),$$

as $M \to +\infty$, where $\mathbf{N}$ is a centered d-dimensional Gaussian random vector with diagonal covariance matrix $\Sigma = \mathrm{diag}(d^{-2}, \dots, d^{-2})$. Similar rates can be obtained in the one-dimensional case applying Theorem 5.5; we omit the details.

6. Appendix

6.1. *Proof of Lemma 3.6.* We refer the reader to p. 337 of Chen et al. (2011) and Lemma 3.3 in Nourdin et al. (2010) for a proof of Part (i), to Proposition 2.3 in Schulte and Yukich (2019) for a proof of Part (iii) and to p. 12 formula (2.5) in Schulte and Yukich (2019) for a proof of Part (iv). As far as Part (ii) is concerned, letting φ denote the density of $\mathbf{N}$, for any $t \in (0,1)$, $x \in \mathbb{R}^d$ and $i \in [d]$, a simple computation yields

$$\begin{aligned}\Big|\frac{\partial f_{t,g}(x)}{\partial x_i}\Big| &= \frac{1}{2}\Big|\int_t^1 \frac{1}{\sqrt{s}\sqrt{1-s}}\int_{\mathbb{R}^d} g(\sqrt{s}z + \sqrt{1-s}x)\frac{\partial \varphi(z)}{\partial x_i}\,\mathrm{d}z\mathrm{d}s\Big| \\ &\le \frac{\|g\|_\infty}{2\sqrt{t}}\int_t^1 \frac{1}{\sqrt{1-s}}\,\mathrm{d}s\int_{\mathbb{R}^d}\Big|\frac{\partial\varphi(z)}{\partial x_i}\Big|\,\mathrm{d}z = \|g\|_\infty\frac{\sqrt{1-t}}{\sqrt{t}}\int_{\mathbb{R}^d}\Big|\frac{\partial\varphi(z)}{\partial x_i}\Big|\,\mathrm{d}z. \end{aligned} \tag{6.1}$$

Setting $\zeta_\ell(\hat{\mathbf{z}}_i) := \sum_{j=1,\, j\neq i}^{d}(\Sigma^{-1/2})_{\ell j}z_j$, a straightforward calculation shows

$$\begin{aligned}\varphi(z) &= \frac{1}{\sqrt{(2\pi)^d \det\Sigma}}\exp\left(-\frac{1}{2}\sum_{\ell=1}^{d}\left(\sum_{j=1}^{d}(\Sigma^{-1/2})_{\ell j}z_j\right)^2\right) \\ &= \frac{1}{\sqrt{(2\pi)^d \det\Sigma}}\exp\left(-\frac{1}{2}\sum_{\ell=1}^{d}\left(\zeta_\ell(\hat{\mathbf{z}}_i) + (\Sigma^{-1/2})_{\ell i}z_i\right)^2\right) \\ &= \frac{1}{\sqrt{(2\pi)^d \det\Sigma}}\exp\left(-\frac{1}{2}\sum_{\ell=1}^{d}\zeta_\ell(\hat{\mathbf{z}}_i)^2 - \frac{z_i^2}{2}\sum_{\ell=1}^{d}[(\Sigma^{-1/2})_{\ell i}]^2 - z_i\sum_{\ell=1}^{d}\zeta_\ell(\hat{\mathbf{z}}_i)(\Sigma^{-1/2})_{\ell i}\right)\end{aligned}$$

and therefore

$$\begin{aligned}\frac{\partial\varphi(z)}{\partial x_i} &= -z_i\varphi(z)\sum_{\ell=1}^d[(\Sigma^{-1/2})_{\ell i}]^2-\varphi(z)\sum_{\ell=1}^d\zeta_\ell(\hat{\mathbf{z}}_i)(\Sigma^{-1/2})_{\ell i}\\ &= -z_i\varphi(z)\sum_{\ell=1}^d[(\Sigma^{-1/2})_{\ell i}]^2-\sum_{\ell=1}^d\sum_{j=1,\,j\neq i}^d(\Sigma^{-1/2})_{\ell j}(\Sigma^{-1/2})_{\ell i}z_j\varphi(z)\\ &= -\sum_{\ell,j=1}^d(\Sigma^{-1/2})_{\ell j}(\Sigma^{-1/2})_{\ell i}z_j\varphi(z).\end{aligned}$$

The claim easily follows by this expression, (6.1), the fact that N_j is Gaussian distributed with mean zero and variance Σ_{jj} and the Cauchy-Schwarz inequality.

6.2. *Proof of Lemma 3.8*. The proof of Lemma 3.8 exploits, in turn, the following lemma, which provides a “corrected" multivariate chain rule for the gradient D.

Lemma 6.1. *Let $\mathbf{F}=(F_1,\dots,F_d):\mathcal{P}(\mathbb{X})\to\mathbb{R}^d$ be a measurable mapping. Then, for all $f\in C^2(\mathbb{R}^d)$ with $\|f^{(2)}\|_\infty<\infty$, for $\mathbb{P}\otimes\widehat{\sigma}$-almost all (ω,x,t), we have*

$$D_{(x,t)}f(\mathbf{F}(\eta))=\sum_{i=1}^d\frac{\partial}{\partial x_i}f(\mathbf{F}(\eta))D_{(x,t)}F_i(\eta)+R(\mathbf{F}(\eta),D_{(x,t)}\mathbf{F}(\eta)),$$

where $R:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$ is a measurable mapping such that

$$|R(\mathbf{F}(\eta),D_{(x,t)}\mathbf{F}(\eta))|\leq\frac{1}{2}\|f^{(2)}\|_\infty\left(\sum_{i=1}^d|D_{(x,t)}F_i(\eta)|\right)^2,\quad \textit{for } \mathbb{P}\otimes\widehat{\sigma}\textit{-almost all the } (\omega,x,t).$$

The proof of this lemma is provided later on in this Appendix, and now we proceed by proving Lemma 3.8. Throughout the proof, for ease of notation, we put $\mathbf{I}:=\mathbf{I}(\eta,\mathbf{h})$. Let $g\in C^2(\mathbb{R}^d)$ with bounded first and second derivatives. Then $g(\mathbf{I})\in L^2(\Omega,\mathbb{P})$. Indeed, by a simple application of the Multivariate Mean Value Theorem we have

$$|g(\mathbf{I})-g(\mathbf{0})|\leq\left(\sup_{z\in\mathbb{R}^d}\sqrt{\sum_{i=1}^d\left|\frac{\partial g(z)}{\partial x_i}\right|^2}\right)\sqrt{\sum_{i=1}^d|I(\eta,h_i)|^2}\leq\sqrt{d}\|g^{(1)}\|_\infty\sqrt{\sum_{i=1}^d|I(\eta,h_i)|^2}.$$

By (2.4) we immediately have

$$\mathbb{E}\left[\sum_{i=1}^d I(\eta,h_i)^2\right]=\frac{1}{\beta(\beta+1)}\sum_{i=1}^d\int_{\mathbb{X}}|h_i(x)|^2\sigma(\mathrm{d}x)<\infty,$$

and so $g(\mathbf{I})-g(\mathbf{0})\in L^2(\Omega,\mathbb{P})$, i.e., $g(\mathbf{I})\in L^2(\Omega,\mathbb{P})$. Thus by Lemma 3.1 (with $G(\eta):=g(\mathbf{I})$) and Lemma 6.1 (with $f:=g$ and $\mathbf{F}(\eta):=\mathbf{I}$) we have

$$\begin{aligned}\mathbb{E}[g(\mathbf{I})I(\eta,h_0)] &= \mathbb{E}\Big[\langle h_0(\cdot),D.g(\mathbf{I})\rangle_{L^2(\mathbb{X}\times[0,1],\widehat{\sigma})}\Big]\\ &= \mathbb{E}\left[\sum_{i=1}^d\frac{\partial g(\mathbf{I})}{\partial x_i}\langle h_0(\cdot),D.I(\eta,h_i)\rangle_{L^2(\mathbb{X}\times[0,1],\widehat{\sigma})}\right]+\mathbb{E}[\langle h_0(\cdot),R(\mathbf{I},D.\mathbf{I})\rangle_{L^2(\mathbb{X}\times[0,1],\widehat{\sigma})}]\end{aligned}\tag{6.2}$$

with

$$
\begin{aligned}
&|\mathbb{E}[\langle h_0, R(\mathbf{I}, D.\mathbf{I})\rangle_{L^2(\mathbb{X}\times[0,1],\widehat{\sigma})}]| \\
&\quad \leq \frac{1}{2}\|g^{(2)}\|_\infty \int_{\mathbb{X}\times[0,1]} |h_0(x)|\mathbb{E}\left[\left(\sum_{i=1}^d |D_{(x,t)}I(\eta,h_i)|\right)^2\right]\widehat{\sigma}(\mathrm{d}x,\mathrm{d}t). \qquad (6.3)
\end{aligned}
$$

Furthermore, for any $(x,t)\in\mathbb{X}\times[0,1]$, by the convexity of $x\mapsto x^2$ and (2.3), (2.4), (4.31) we have

$$
\begin{aligned}
\mathbb{E}\left[\left(\sum_{i=1}^d |D_{(x,t)}I(\eta,h_i)|\right)^2\right] &= d^2\mathbb{E}\left[\left(\frac{1}{d}\sum_{i=1}^d |D_{(x,t)}I(\eta,h_i)|\right)^2\right] \\
&\leq d\sum_{i=1}^d \mathbb{E}[|D_{(x,t)}I(\eta,h_i)|^2] \\
&= t^2 d\left(\sum_{i=1}^d |h_i(x)|^2 + \frac{1}{\beta(\beta+1)}\sum_{i=1}^d \int_{\mathbb{X}} |h_i(x)|^2\sigma(\mathrm{d}x)\right).
\end{aligned}
$$

The claim follows on combining this latter relation with (6.2), (4.32) and (6.3).

6.3. *Proof of Lemma 6.1.* For $\mathbb{P}\otimes\widehat{\sigma}$-almost all the (ω,x,t), we have $D_{(x,t)}f(\mathbf{F}(\eta)) = f(D_{(x,t)}\mathbf{F}(\eta)+\mathbf{F}(\eta)) - f(\mathbf{F}(\eta))$, and so by the Multivariate Taylor's Formula with the integral remainder we have

$$
D_{(x,t)}f(\mathbf{F}(\eta)) = \sum_{i=1}^d \frac{\partial}{\partial x_i} f(\mathbf{F}(\eta))D_{(x,t)}F_i(\eta) + R(\mathbf{F}(\eta), D_{(x,t)}\mathbf{F}(\eta)),
$$

where

$$
\begin{aligned}
R(\mathbf{F}(\eta), D_{(x,t)}\mathbf{F}(\eta)) = 2 &\sum_{(n_1,\dots,n_d)\in\mathbb{N}\cup\{0\}:\ n_1+\dots+n_d=2} \frac{(D_{(x,t)}F_1(\eta))^{n_1}\dots(D_{(x,t)}F_d(\eta))^{n_d}}{n_1!\dots n_d!} \\
&\times \int_0^1 (1-t)\frac{\partial^{n_1+\dots+n_d} f(\mathbf{F}(\eta)+tD_{(x,t)}\mathbf{F}(\eta))}{\partial x_1^{n_1}\dots\partial x_d^{n_d}}\,\mathrm{d}t.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&|R(\mathbf{F}(\eta), D_{(x,t)}\mathbf{F}(\eta))| \\
&\quad \leq \int_0^1 (1-t)\,\mathrm{d}t\,\|f^{(2)}\|_\infty \sum_{(n_1,\dots,n_d)\in\mathbb{N}_0:\ n_1+\dots+n_d=2} \frac{2}{n_1!\dots n_d!}|D_{(x,t)}F_1(\eta)|^{n_1}\dots|D_{(x,t)}F_d(\eta)|^{n_d} \\
&\quad = \frac{1}{2}\|f^{(2)}\|_\infty\left(\sum_{i=1}^d |D_{(x,t)}F_i(\eta)|\right)^2, \quad \text{for } \mathbb{P}\otimes\widehat{\sigma}\text{-almost all the } (\omega,x,t)
\end{aligned}
$$

where the latter relation follows by the Multinomial Theorem.

6.4. *Proof of Theorem 5.1.* Let $M,d\in\mathbb{N}$, σ_M a positive and finite measure on $(\mathbb{X},\mathcal{B}(\mathbb{X}))$ and

$$
\{X_{11},\dots,X_{1K_{1M}},X_{21},\dots,X_{2K_{2M}},\dots,X_{d1},\dots,X_{dK_{dM}}\}\in\mathfrak{P}_{\sum_{i\in[d]}K_{iM}}(\mathbb{X})
$$

a partition of $\mathbb{X}$ such that $\sigma_M(X_{im}) = \alpha_{im}^{(M)} > 0$, $i\in[d]$, $m\in[K_{iM}]$. Let η_M be the Dirichlet-Ferguson measure with parameter σ_M. Then

$$
\beta_M := \sigma_M(\mathbb{X}) = \sum_{i=1}^d\sum_{m=1}^{K_{iM}} \alpha_{im}^{(M)} = \widetilde{\alpha}_M
$$

and, by the definition of η_M,

$$(\eta_M(X_{11}),\ldots,\eta_M(X_{1K_{1M}}),\eta_M(X_{21}),\ldots,\eta_M(X_{2K_{2M}}),\ldots,\eta_M(X_{d1}),\ldots,\eta_M(X_{dK_{dM}})) \stackrel{\mathcal{L}}{=} \mathbf{Y}_M. \tag{6.4}$$

For $i \in [d]$, define the function

$$h_{i,M}(x) := C_{iM} \sum_{m=1}^{K_{iM}} u_{im} \left(\mathbf{1}_{X_{im}}(x) - \frac{\alpha_{im}^{(M)}}{\beta_M} \right), \quad x \in \mathbb{X}$$

and note that

$$I(\eta_M, h_{i,M}) = C_{iM} \sum_{m=1}^{K_{iM}} u_{im} \big(\eta_M(X_{im}) - \mathbb{E}[\eta_M(X_{im})] \big),$$

indeed $\mathbb{E}[\eta_M(X_{im})] = \alpha_{im}^{(M)}/\beta_M$ since $\eta_M(X_{im})$ is the im-th marginal of $\mathrm{Dir}[\alpha^{(M)}]$. Therefore, by (6.4) we have that

$$\mathbf{I}(\eta_M, \mathbf{h}_M) = (I(\eta_M, h_{1,M}), \ldots, I(\eta_M, h_{d,M})) \stackrel{\mathcal{L}}{=} \mathbf{S}_M.$$

Since $\sigma_M(\mathbb{X}) = \beta_M$, we have

$$\int_{\mathbf{X}} h_{i,M}(x)\, \sigma_M(\mathrm{d}x) = 0.$$

Therefore, the kernels $h_{i,M}$ satisfy the condition (1.2) with $h_{i,M}$ in place of h and σ_M in place of σ. The inequalities (5.2) and (5.3), with

$$\gamma_{1,M} := \sum_{i,j=1}^{d} \left| \Sigma_{ij} - \frac{1}{\widetilde{\alpha}_M(\widetilde{\alpha}_M + 1)} \int_{\mathbb{X}} h_{i,M}(x) h_{j,M}(x) \sigma_M(\mathrm{d}x) \right| \tag{6.5}$$

and

$$\gamma_{4,M} := \frac{d^3}{\widetilde{\alpha}_M(\widetilde{\alpha}_M + 1)^2} \sqrt{\max_{1 \le i \le d} \int_{\mathbb{X}} |h_{i,M}(x)|^2 \sigma_M(\mathrm{d}x) \max_{1 \le i \le d} \int_{\mathbb{X}} |h_{i,M}(x)|^4 \sigma_M(\mathrm{d}x)}, \tag{6.6}$$

and the inequality (5.4), with $\gamma_{6,M} := \max\{\gamma_{1,M}, \sqrt{\gamma_{8,M}}\}$ and

$$\begin{aligned} \gamma_{8,M} := \frac{2}{3\widetilde{\alpha}_M(\widetilde{\alpha}_M + 1)(\widetilde{\alpha}_M + 2)} & \Bigg\{ \sqrt{\sum_{i=1}^{d} \int_{\mathbb{X}} |h_{i,M}(x)|^2 \sigma_M(\mathrm{d}x) \sum_{i=1}^{d} \int_{\mathbb{X}} |h_{i,M}(x)|^4 \sigma_M(\mathrm{d}x)} \\ + 2 \Bigg[\frac{\widetilde{\alpha}_M + 2}{\widetilde{\alpha}_M + 1} & \sum_{k=1}^{d} \int_{\mathbb{X}} h_{k,M}(x)^2 \sigma_M(\mathrm{d}x) \\ & \times \sum_{k=1}^{d} \left(2 \int_{\mathbb{X}} h_{k,M}(x)^4 \sigma_M(\mathrm{d}x) + \frac{6}{\widetilde{\alpha}_M(\widetilde{\alpha}_M + 1)} \left(\int_{\mathbb{X}} h_{k,M}(x)^2 \sigma_M(\mathrm{d}x) \right)^2 \right) \Bigg]^{1/2} \Bigg\} \end{aligned} \tag{6.7}$$

then follow by Theorems 4.1, 4.2 and 4.4, respectively.

The rest of the proof consists in computing the integrals involved in the expressions of $\gamma_{1,M}$, $\gamma_{4,M}$ and $\gamma_{8,M}$.

Step 1: Computing $\gamma_{1,M}$.

For any $i, j \in [d]$ and $x \in \mathbb{X}$, we have

$$h_{i,M}(x)h_{j,M}(x) = C_{iM}C_{jM} \sum_{m=1}^{K_{iM}} \sum_{n=1}^{K_{jM}} u_{im}u_{jn} \left(\mathbf{1}_{X_{im}}(x) - \frac{\alpha_{im}^{(M)}}{\beta_M} \right) \left(\mathbf{1}_{X_{jn}}(x) - \frac{\alpha_{jn}^{(M)}}{\beta_M} \right)$$
$$= C_{iM}C_{jM} \sum_{m=1}^{K_{iM}} \sum_{n=1}^{K_{jM}} u_{im}u_{jn} \left(\mathbf{1}_{X_{im} \cap X_{jn}}(x) - \mathbf{1}_{X_{im}}(x) \frac{\alpha_{jn}^{(M)}}{\beta_M} - \mathbf{1}_{X_{jn}}(x) \frac{\alpha_{im}^{(M)}}{\beta_M} + \frac{\alpha_{im}^{(M)}\alpha_{jn}^{(M)}}{\beta_M^2} \right),$$

and so

$$\int_{\mathbb{X}} h_{i,M}(x)h_{j,M}(x)\sigma_M(\mathrm{d}x) = C_{iM}C_{jM} \sum_{m=1}^{K_{iM}} \sum_{n=1}^{K_{jM}} u_{im}u_{jn} \left(\sigma_M(X_{im} \cap X_{jn}) - \frac{\alpha_{im}^{(M)}\alpha_{jn}^{(M)}}{\beta_M} \right).$$

Hence

$$\int_{\mathbb{X}} h_{i,M}(x)h_{j,M}(x)\sigma_M(\mathrm{d}x) = -\frac{C_{iM}C_{jM}}{\widetilde{\alpha}_M} \sum_{m=1}^{K_{iM}} \sum_{n=1}^{K_{jM}} u_{im}u_{jn}\alpha_{im}^{(M)}\alpha_{jn}^{(M)}$$
$$= -\frac{C_{iM}^3 C_{jM}^3}{\widetilde{\alpha}_M} \mu_{iM}\mu_{jM}, \quad \text{for } i \neq j \text{ and } d \geq 2 \tag{6.8}$$

and

$$\int_{\mathbb{X}} h_{i,M}(x)^2 \sigma_M(\mathrm{d}x) = C_{iM}^2 \left[\sum_{m=1}^{K_{iM}} \alpha_{im}^{(M)} - \frac{1}{\beta_M} \sum_{m=1}^{K_{iM}} (\alpha_{im}^{(M)})^2 - \frac{1}{\beta_M} \sum_{n,m \in [K_{iM}]}^{n \neq m} u_{in}u_{im}\alpha_{im}^{(M)}\alpha_{in}^{(M)} \right]$$
$$= C_{iM}^2 \left[\sum_{m=1}^{K_{iM}} \alpha_{im}^{(M)} - \frac{1}{\beta_M} \left(\sum_{m=1}^{K_{iM}} u_{im}\alpha_{im}^{(M)} \right)^2 \right]$$
$$= C_{iM}^2 \left(\sum_{m=1}^{K_{iM}} \alpha_{im}^{(M)} - \frac{1}{\beta_M} C_{iM}^4 \mu_{iM}^2 \right)$$
$$= C_{iM}^4 \left(\widetilde{\alpha}_{iM} - \frac{C_{iM}^2 \mu_{iM}^2}{\widetilde{\alpha}_M} \right). \tag{6.9}$$

The expression of $\gamma_{1,M}$ in the statement of the theorem easily follows by (6.5), (6.8) and (6.9).
Step 2: Computing $\gamma_{4,M}$ and $\gamma_{8,M}$.

The claimed expressions of $\gamma_{4,M}$ and $\gamma_{8,M}$ follow by (6.6), (6.7), (6.9), and the following computation:

$$\begin{aligned}
\int_{\mathbb{X}} h_{i,M}(x)^4\,\sigma_M(\mathrm{d}x) &= C_{iM}^4 \int_{\mathbb{X}} \left(\sum_{m=1}^{K_{iM}} u_{im}\left(\mathbf{1}_{X_{im}}(x) - \frac{\alpha_{im}^{(M)}}{\beta_M}\right)\right)^4 \sigma_M(\mathrm{d}x)\\
&= C_{iM}^4 \sum_{j=1}^{d}\sum_{m=1}^{K_{jM}} \int_{X_{jm}} \left(\sum_{k=1}^{K_{iM}} u_{ik}\left(\mathbf{1}_{X_{ik}}(x) - \frac{\alpha_{ik}^{(M)}}{\beta_M}\right)\right)^4 \sigma_M(\mathrm{d}x)\\
&= C_{iM}^4 \sum_{m=1}^{K_{iM}} \int_{X_{im}} \left(\sum_{k=1}^{K_{iM}} u_{ik}\left(\mathbf{1}_{X_{ik}}(x) - \frac{\alpha_{ik}^{(M)}}{\beta_M}\right)\right)^4 \sigma_M(\mathrm{d}x)\\
&\quad + \mathbf{1}\{d\geq 2\} C_{iM}^4 \sum_{j=1,\, j\neq i}^{d}\sum_{m=1}^{K_{jM}} \int_{X_{jm}} \left(\sum_{k=1}^{K_{iM}} u_{ik}\left(\mathbf{1}_{X_{ik}}(x) - \frac{\alpha_{ik}^{(M)}}{\beta_M}\right)\right)^4 \sigma_M(\mathrm{d}x)\\
&= C_{iM}^4 \sum_{m=1}^{K_{iM}} \int_{X_{im}} \left(u_{im}\left(\mathbf{1}_{X_{im}}(x) - \frac{\alpha_{im}^{(M)}}{\beta_M}\right) - \frac{1}{\beta_M}\sum_{k=1,\, k\neq m}^{K_{iM}} u_{ik}\alpha_{ik}^{(M)}\right)^4 \sigma_M(\mathrm{d}x)\\
&\quad + \mathbf{1}\{d\geq 2\} C_{iM}^4 \sum_{j=1,\, j\neq i}^{d}\sum_{m=1}^{K_{jM}} \alpha_{jm}^{(M)} \left(\sum_{k=1}^{K_{iM}} u_{ik}\frac{\alpha_{ik}^{(M)}}{\beta_M}\right)^4\\
&= C_{iM}^4 \sum_{m=1}^{K_{iM}} \int_{X_{im}} \left(u_{im}\left(1 - \frac{\alpha_{im}^{(M)}}{\beta_M}\right) - \frac{1}{\beta_M}\sum_{k=1,\, k\neq m}^{K_{iM}} u_{ik}\alpha_{ik}^{(M)}\right)^4 \sigma_M(\mathrm{d}x)\\
&\quad + \mathbf{1}\{d\geq 2\} \frac{C_{iM}^{12}\mu_{iM}^4}{(\widetilde{\alpha}_M)^4} \sum_{j=1,\, j\neq i}^{d} C_{jM}^2\widetilde{\alpha}_{jM}\\
&= C_{iM}^4\left[\sum_{m=1}^{K_{iM}} \alpha_{im}^{(M)}\left(u_{im}\left(1 - \frac{\alpha_{im}^{(M)}}{\beta_M}\right) - \frac{1}{\beta_M}\sum_{k=1,\, k\neq m}^{K_{iM}} u_{ik}\alpha_{ik}^{(M)}\right)^4 + \mathbf{1}\{d\geq 2\}\frac{C_{iM}^8\mu_{iM}^4}{(\widetilde{\alpha}_M)^4}\sum_{j=1,\, j\neq i}^{d} C_{jM}^2\widetilde{\alpha}_{jM}\right]\\
&= C_{iM}^4\left[\sum_{m=1}^{K_{iM}} \alpha_{im}^{(M)}\left(u_{im} - \frac{C_{iM}^2\mu_{iM}}{\widetilde{\alpha}_M}\right)^4 + \mathbf{1}\{d\geq 2\}\frac{C_{iM}^8\mu_{iM}^4}{(\widetilde{\alpha}_M)^4}\sum_{j=1,\, j\neq i}^{d} C_{jM}^2\widetilde{\alpha}_{jM}\right]
\end{aligned}$$

and we also easily have

$$\sum_{m=1}^{K_{iM}} \alpha_{im}^{(M)}\left(u_{im} - \frac{C_{iM}^2\mu_{iM}}{\widetilde{\alpha}_M}\right)^4 = \frac{C_{iM}^{10}\mu_{iM}^4\widetilde{\alpha}_{iM}}{(\widetilde{\alpha}_M)^4} - 4\frac{C_{iM}^8\mu_{iM}^4}{(\widetilde{\alpha}_M)^3} + 6\frac{C_{iM}^6\mu_{iM}^2\widetilde{\alpha}_{iM}}{(\widetilde{\alpha}_M)^2} - 4\frac{C_{iM}^4\mu_{iM}^2}{\widetilde{\alpha}_M} + C_{iM}^2\widetilde{\alpha}_{iM}.$$

6.5. *Proof of Theorem 5.2.* We divide the proof in three steps where we show that $\gamma_{j,M} \to 0$, as $M \to +\infty$, for any $j = 1, 4, 8$. The claims (i) and (ii) then follow by Theorem 5.1.

Step 1: Proof of $\gamma_{1,M} \to 0$.

Note that $\widetilde{\alpha}_M = \sum_{i=1}^d C_{iM}^2\widetilde{\alpha}_{iM}$. Therefore by the assumptions $(H1)$ and $(H2)$ we have

$$\widetilde{\alpha}_M \sim \kappa C_{1M}^2, \quad \text{as } M \to +\infty. \tag{6.10}$$

By this relation and the assumptions $(H1)$, $(H2)$ and $(H3)$ we easily have, as $M \to +\infty$,

$$\frac{C_{jM}^4}{\widetilde{\alpha}_M(\widetilde{\alpha}_M + 1)}\left(\widetilde{\alpha}_{jM} - \frac{C_{jM}^2\mu_{jM}^2}{\widetilde{\alpha}_M}\right) \to \frac{\kappa_j^4}{\kappa^2}\left(\alpha_j - \frac{\kappa_j^2\mu_j^2}{\kappa}\right), \quad j \in [d] \tag{6.11}$$

and

$$\frac{C_{jM}^3 C_{kM}^3}{\widetilde{\alpha}_M^2(\widetilde{\alpha}_M+1)}\mu_{jM}\mu_{kM} \to \frac{\kappa_j^3\kappa_k^3}{\kappa^3}\mu_j\mu_k, \quad j\neq k,\ j,k\in[d].$$

On combining these relations with the definition of $\gamma_{1,M}$ and the definition of the matrix Σ in the statement of the theorem, we immediately have $\gamma_{1,M}\to 0$, as $M\to+\infty$.
Step 2: Proof of $\gamma_{4,M}\to 0$.
Note that

$$\begin{aligned}\gamma_{4,M} = d^3 &\left\{\frac{1}{(\widetilde{\alpha}_M+1)^2}\max_{1\le i\le d} C_{iM}^4\left(\widetilde{\alpha}_{iM}-\frac{C_{iM}^2\mu_{iM}^2}{\widetilde{\alpha}_M}\right)\right\}^{1/2}\\ &\times\left\{\frac{1}{(\widetilde{\alpha}_M)^2(\widetilde{\alpha}_M+1)^2}\max_{1\le i\le d} C_{iM}^4\left(\frac{C_{iM}^{10}\mu_{iM}^4\widetilde{\alpha}_{iM}}{(\widetilde{\alpha}_M)^4}-4\frac{C_{iM}^8\mu_{iM}^4}{(\widetilde{\alpha}_M)^3}+6\frac{C_{iM}^6\mu_{iM}^2\widetilde{\alpha}_{iM}}{(\widetilde{\alpha}_M)^2}-4\frac{C_{iM}^4\mu_{iM}^2}{\widetilde{\alpha}_M}\right.\right.\\ &\left.\left.+C_{iM}^2\widetilde{\alpha}_{iM}+\frac{C_{iM}^8\mu_{iM}^4}{(\widetilde{\alpha}_M)^4}\sum_{j=1,\,j\neq i}^{d}C_{jM}^2\widetilde{\alpha}_{jM}\right)\right\}^{1/2}.\end{aligned} \tag{6.12}$$

By (6.11) and the fact that $\widetilde{\alpha}_M\to+\infty$ (this follows by (6.10), $\kappa>0$ and $C_{1M}\to+\infty$), we have that, as $M\to+\infty$,

$$\frac{1}{(\widetilde{\alpha}_M)^2}\max_{1\le i\le d} C_{iM}^4\left(\widetilde{\alpha}_{iM}-\frac{C_{iM}^2\mu_{iM}^2}{\widetilde{\alpha}_M}\right)=O(1). \tag{6.13}$$

As far as the term inside the second square root in (6.12) is concerned, exploiting (6.10) and the assumptions $(H1)$, $(H2)$ and $(H3)$ easily follows that, as $M\to+\infty$,

$$\begin{aligned}\frac{1}{(\widetilde{\alpha}_M)^4}\max_{1\le i\le d} C_{iM}^4&\left(\frac{C_{iM}^{10}\mu_{iM}^4\widetilde{\alpha}_{iM}}{(\widetilde{\alpha}_M)^4}-4\frac{C_{iM}^8\mu_{iM}^4}{(\widetilde{\alpha}_M)^3}+6\frac{C_{iM}^6\mu_{iM}^2\widetilde{\alpha}_{iM}}{(\widetilde{\alpha}_M)^2}-4\frac{C_{iM}^4\mu_{iM}^2}{\widetilde{\alpha}_M}\right.\\ &\left.+C_{iM}^2\widetilde{\alpha}_{iM}+\frac{C_{iM}^8\mu_{iM}^4}{(\widetilde{\alpha}_M)^4}\sum_{j=1,\,j\neq i}^{d}C_{jM}^2\widetilde{\alpha}_{jM}\right)=o(1).\end{aligned} \tag{6.14}$$

On combining (6.12) with (6.13) and (6.14) it immediately follows that $\gamma_{4,M}\to 0$, as $M\to+\infty$.
Step 3: Proof of $\gamma_{8,M}\to 0$.

Note that

$$\gamma_{8,M} \le \frac{2}{3\widetilde{\alpha}_M(\widetilde{\alpha}_M+1)(\widetilde{\alpha}_M+2)}\Bigg\{ d\Bigg[\max_{1\le i\le d} C_{iM}^4\Bigg(\widetilde{\alpha}_{iM} - \frac{C_{iM}^2\mu_{iM}^2}{\widetilde{\alpha}_M}\Bigg)\Bigg]^{1/2}$$
$$\times\Bigg[\max_{1\le i\le d} C_{iM}^4\Bigg(\frac{C_{iM}^{10}\mu_{iM}^4\widetilde{\alpha}_{iM}}{(\widetilde{\alpha}_M)^4} - 4\frac{C_{iM}^8\mu_{iM}^4}{(\widetilde{\alpha}_M)^3} + 6\frac{C_{iM}^6\mu_{iM}^2\widetilde{\alpha}_{iM}}{(\widetilde{\alpha}_M)^2} - 4\frac{C_{iM}^4\mu_{iM}^2}{\widetilde{\alpha}_M} + C_{iM}^2\widetilde{\alpha}_{iM}$$
$$+ \mathbf{1}\{d\ge 2\}\frac{C_{iM}^8\mu_{iM}^4}{(\widetilde{\alpha}_M)^4}\sum_{j=1,\,j\neq i}^{d} C_{jM}^2\widetilde{\alpha}_{jM}\Bigg)\Bigg]^{1/2}$$
$$+2\Bigg[\frac{\widetilde{\alpha}_M+2}{\widetilde{\alpha}_M+1} d\max_{1\le i\le d} C_{iM}^4\Bigg(\widetilde{\alpha}_{iM} - \frac{C_{iM}^2\mu_{iM}^2}{\widetilde{\alpha}_M}\Bigg)$$
$$\times\Bigg[2d\max_{1\le i\le d} C_{iM}^4\Bigg(\frac{C_{iM}^{10}\mu_{iM}^4\widetilde{\alpha}_{iM}}{(\widetilde{\alpha}_M)^4} - 4\frac{C_{iM}^8\mu_{iM}^4}{(\widetilde{\alpha}_M)^3} + 6\frac{C_{iM}^6\mu_{iM}^2\widetilde{\alpha}_{iM}}{(\widetilde{\alpha}_M)^2} - 4\frac{C_{iM}^4\mu_{iM}^2}{\widetilde{\alpha}_M}$$
$$+ C_{iM}^2\widetilde{\alpha}_{iM} + \mathbf{1}\{d\ge 2\}\frac{C_{iM}^8\mu_{iM}^4}{(\widetilde{\alpha}_M)^4}\sum_{j=1,\,j\neq i}^{d} C_{jM}^2\widetilde{\alpha}_{jM}\Bigg)$$
$$+\frac{6}{\widetilde{\alpha}_M(\widetilde{\alpha}_M+1)} d\Bigg(\max_{1\le i\le d} C_{iM}^4\Bigg(\widetilde{\alpha}_{iM} - \frac{C_{iM}^2\mu_{iM}^2}{\widetilde{\alpha}_M}\Bigg)\Bigg)^2\Bigg]\Bigg]^{1/2}\Bigg\}. \tag{6.15}$$

The claim follows on noticing that this upper bound converges to zero as $M\to+\infty$ by (6.10), (6.13) and (6.14).

6.6. *Proof of Theorem 5.3.* We divide the proof in three steps. In the first step we prove that $\gamma_{1,M} = O(C_{1M}^{-1})$, in the second step we prove that $\gamma_{4,M} = O(C_{1M}^{-1})$ and in the third step we prove that $\gamma_{8,M} = O(C_{1M}^{-1})$. The claim then follows by the bounds on the distances d_3, d_2 and d_c of Theorem 5.1.

Step 1: Proof of $\gamma_{1,M} = O(C_{1M}^{-1})$.

We first show that the addend $\mathbf{1}\{d\ge 2\}\sum_{i,j\in[d]}^{i\neq j}|\cdots|$ in the expression of $\gamma_{1,M}$ is $O(C_{1M}^{-1})$, as $M\to+\infty$. By the definition of the matrix Σ and the assumption $(H3)'$, we have

$$C_{1M}\sum_{i,j\in[d]}^{i\neq j}\Bigg|\Sigma_{ij} + \frac{C_{iM}^3C_{jM}^3}{\widetilde{\alpha}_M^2(\widetilde{\alpha}_M+1)}\mu_{iM}\mu_{jM}\Bigg| = C_{1M}\sum_{j,k\in[d]}^{j\neq k}\Bigg| - \frac{\kappa_j^3\kappa_k^3}{\kappa^3}\mu_j\mu_k + \frac{C_{jM}^3C_{kM}^3}{\widetilde{\alpha}_M^2(\widetilde{\alpha}_M+1)}(\mu_j\mu_k + O(C_{1M}^{-1}))\Bigg|$$
$$=: \mathcal{T}_{1M}.$$

Note that

$$\frac{C_{jM}^3C_{kM}^3}{\widetilde{\alpha}_M^2(\widetilde{\alpha}_M+1)} = \frac{1}{\widetilde{\alpha}_M^3/(C_{jM}^3C_{kM}^3) + \widetilde{\alpha}_M^2/(C_{jM}^3C_{kM}^3)}$$
$$= \frac{1}{\widetilde{\alpha}_M^3/(C_{jM}^3C_{kM}^3) + O(C_{1M}^{-2})}, \quad j\neq k,\ j,k\in[d].$$

We shall show later on that

$$\widetilde{\alpha}_M^3/(C_{jM}^3C_{kM}^3) = \frac{\kappa^3/(\kappa_j^3\kappa_k^3) + O(C_{1M}^{-1})}{1+O(C_{1M}^{-1})}, \quad \text{for any } j\neq k,\ j,k\in[d]. \tag{6.16}$$

Therefore

$$\mathcal{T}_{1M} = C_{1M} \sum_{j,k\in[d]}^{j\neq k} \Big| - \frac{\kappa_j^3\kappa_k^3}{\kappa^3}\mu_j\mu_k + \frac{\mu_j\mu_k + O(C_{1M}^{-1})}{\kappa^3/(\kappa_j^3\kappa_k^3) + O(C_{1M}^{-1})}\Big|$$
$$= C_{1M} \sum_{j,k\in[d]}^{j\neq k} \Big|\frac{O(C_{1M}^{-1})}{\kappa^3/(\kappa_j^3\kappa_k^3) + O(C_{1M}^{-1})}\Big|,$$

whose lim sup as $M \to +\infty$ is finite. This shows that the addend $\mathbf{1}\{d \geq 2\} \sum_{i,j\in[d]}^{i\neq j} |\cdots|$ in the expression of $\gamma_{1,M}$ is $O(C_{1M}^{-1})$, as $M \to +\infty$.

Now we show that the addend $\sum_{i=1}^{d} |\cdots|$ in the expression of $\gamma_{1,M}$ is $O(C_{1M}^{-1})$, as $M \to +\infty$, which, combined with what we have previously proved, yields $\gamma_{1,M} = O(C_{1M}^{-1})$, as $M \to +\infty$. By the definition of the matrix Σ we have

$$C_{1M} \sum_{i=1}^{d} \Big|\Sigma_{ii} - \frac{C_{iM}^4}{\widetilde{\alpha}_M(\widetilde{\alpha}_M + 1)}\left(\widetilde{\alpha}_{iM} - \frac{C_{iM}^2\mu_{iM}^2}{\widetilde{\alpha}_M}\right)\Big|$$
$$= C_{1M} \sum_{j\in[d]} \Big|\frac{\kappa_j^4}{\kappa^2}\left(\alpha_j - \frac{\kappa_j^2\mu_j^2}{\kappa}\right) - \frac{C_{jM}^4}{\widetilde{\alpha}_M(\widetilde{\alpha}_M + 1)}\left(\widetilde{\alpha}_{jM} - \frac{C_{jM}^2\mu_{jM}^2}{\widetilde{\alpha}_M}\right)\Big|$$
$$=: \mathcal{T}_{2M}. \tag{6.17}$$

By the assumptions $(H2)'$ and $(H3)'$ we have

$$\frac{\kappa_j^4}{\kappa^2}\left(\alpha_j - \frac{\kappa_j^2\mu_j^2}{\kappa}\right) - \frac{C_{jM}^4}{\widetilde{\alpha}_M(\widetilde{\alpha}_M + 1)}\left(\widetilde{\alpha}_{jM} - \frac{C_{jM}^2\mu_{jM}^2}{\widetilde{\alpha}_M}\right)$$
$$= \frac{\kappa_j^4}{\kappa^2}\left(\alpha_j - \frac{\kappa_j^2\mu_j^2}{\kappa}\right) - \frac{C_{jM}^4}{\widetilde{\alpha}_M(\widetilde{\alpha}_M + 1)}(\alpha_j + O(C_{1M}^{-2})) + \frac{C_{jM}^6}{\widetilde{\alpha}_M^2(\widetilde{\alpha}_M + 1)}(\mu_j^2 + O(C_{1M}^{-1})), \quad j \in [d]. \tag{6.18}$$

We shall show later on that

$$\frac{C_{jM}^4}{\widetilde{\alpha}_M(\widetilde{\alpha}_M + 1)} = \frac{\kappa_j^4/\kappa^2 + O(C_{1M}^{-1})}{1 + O(C_{1M}^{-1})} \quad \text{and} \quad \frac{C_{jM}^6}{\widetilde{\alpha}_M^2(\widetilde{\alpha}_M + 1)} = \frac{\kappa_j^6/\kappa^3 + O(C_{1M}^{-1})}{1 + O(C_{1M}^{-1})}, \quad j \in [d]. \tag{6.19}$$

Combining these relations with (6.18) we have

$$\frac{\kappa_j^4}{\kappa^2}\left(\alpha_j - \frac{\kappa_j^2\mu_j^2}{\kappa}\right) - \frac{C_{jM}^4}{\widetilde{\alpha}_M(\widetilde{\alpha}_M + 1)}\left(\widetilde{\alpha}_{jM} - \frac{C_{jM}^2\mu_{jM}^2}{\widetilde{\alpha}_M}\right)$$
$$= \frac{\kappa_j^4}{\kappa^2}\left(\alpha_j - \frac{\kappa_j^2\mu_j^2}{\kappa}\right) - \frac{\kappa_j^4/\kappa^2 + O(C_{1M}^{-1})}{1 + O(C_{1M}^{-1})}(\alpha_{i_j} + O(C_{1M}^{-2})) + \frac{\kappa_j^6/\kappa^3 + O(C_{1M}^{-1})}{1 + O(C_{1M}^{-1})}(\mu_{jM}^2 + O(C_{1M}^{-1}))$$
$$= O(C_{1M}^{-1}), \quad \text{for any } j \in [d] \tag{6.20}$$

and so the lim sup as $M \to \infty$ of $\mathcal{T}_{2M}$ is finite, and therefore the addend $\sum_{i=1}^{d} |\cdots|$ in the expression of $\gamma_{1,M}$ is $O(C_{1M}^{-1})$, as $M \to +\infty$. To conclude the proof of Step 1 it remains to show the relations in (6.16) and (6.19). We only check (6.16). Indeed, the relations in (6.19) can be verified along similar computations. By the assumptions $(H1)'$ and $(H2)'$, we have

$$\widetilde{\alpha}_M^3/(C_{jM}^3 C_{kM}^3) = \frac{C_{1M}^{-6}\left(\sum_{i=1}^d C_{iM}^2 \widetilde{\alpha}_{iM}\right)^3}{C_{1M}^{-6} C_{jM}^3 C_{kM}^3} = \frac{\left(\sum_{j\in[d]}\left(\frac{C_{jM}}{C_{1M}}\right)^2 (\alpha_j + O(C_{1M}^{-2}))\right)^3}{(\kappa_j + O(C_{1M}^{-1}))^3 (\kappa_k + O(C_{1M}^{-1}))^3}$$

$$= \frac{\left(\sum_{j\in[d]}\left(\kappa_j + O(C_{1M}^{-1})\right)^2 (\alpha_j + O(C_{1M}^{-2}))\right)^3}{\kappa_j^3 \kappa_k^3 + O(C_{1M}^{-1})}$$

$$= \frac{(\kappa + O(C_{1M}^{-1}))^3}{\kappa_j^3\kappa_k^3 + O(C_{1M}^{-1})} = \frac{\kappa^3 + O(C_{1M}^{-1})}{\kappa_j^3\kappa_k^3 + O(C_{1M}^{-1})}, \quad \text{for any } j \neq k,\ j,k \in [d]$$

which gives the claim.
Step 2: Proof of $\gamma_{4,M} = O(C_{1M}^{-1})$.
Rewriting $\gamma_{4,M}$ as in (6.12), thanks to (6.13), it suffices to show that

$$\frac{C_{1M}^2}{(\widetilde{\alpha}_M)^4} \max_{1\le i\le d} C_{iM}^4 \Bigg(\frac{C_{iM}^{10}\mu_{iM}^4 \widetilde{\alpha}_{iM}}{(\widetilde{\alpha}_M)^4} - 4\frac{C_{iM}^8 \mu_{iM}^4}{(\widetilde{\alpha}_M)^3} + 6\frac{C_{iM}^6\mu_{iM}^2\widetilde{\alpha}_{iM}}{(\widetilde{\alpha}_M)^2} - 4\frac{C_{iM}^4\mu_{iM}^2}{\widetilde{\alpha}_M} + C_{1M}^2 \widetilde{\alpha}_{iM} + \frac{C_{iM}^8\mu_{iM}^4}{(\widetilde{\alpha}_M)^4} \sum_{j=1,\, j\neq i}^d C_{jM}^2 \widetilde{\alpha}_{jM} \Bigg) = O(1), \tag{6.21}$$

as $M \to +\infty$. This relation follows noticing that by (6.10) and the assumptions $(H1)'$, $(H2)'$ and $(H3)'$ we have

$$\frac{C_{1M}^2 C_{jM}^4}{(\widetilde{\alpha}_M)^4} = O(C_{1M}^{-2}), \quad \frac{C_{jM}^{10}\mu_{jM}^4\widetilde{\alpha}_{jM}}{(\widetilde{\alpha}_M)^4} = O(C_{1M}^2), \quad \text{for any } j \in [d]$$

$$\frac{C_{jM}^8\mu_{jM}^4}{(\widetilde{\alpha}_M)^3} = O(C_{1M}^2), \quad \frac{C_{jM}^4\mu_{jM}^2}{\widetilde{\alpha}_M} = O(C_{1M}^2), \quad \text{for any } j \in [d]$$

$$\frac{C_{jM}^6\mu_{jM}^2\widetilde{\alpha}_{jM}}{(\widetilde{\alpha}_M)^2} = O(C_{1M}^2), \quad C_{1M}^2\widetilde{\alpha}_{jM} = O(C_{1M}^2), \quad \text{for any } j \in [d]$$

$$\frac{C_{jM}^8\mu_{jM}^4}{(\widetilde{\alpha}_M)^4} = O(1), \quad \sum_{k=1,\, k\neq j}^d C_{kM}^2\widetilde{\alpha}_{kM} = O(C_{1M}^2), \quad \text{for any } j \in [d].$$

Step 3: Proof of $\gamma_{8,M} = O(C_{1M}^{-1})$.
Note that it suffices to show that the right-hand side of the inequality (6.15) is $O(C_{1M}^{-1})$, as $M \to +\infty$. Using (6.10) and that $\gamma_{4,M} = O(C_{1M}^{-1})$ as $M \to +\infty$, we easily see that it suffices to prove that

$$\frac{d}{(\widetilde{\alpha}_M)^6} \max_{1\le i\le d} C_{iM}^4 \left(\widetilde{\alpha}_{iM} - \frac{C_{iM}^2\mu_{iM}^2}{\widetilde{\alpha}_M} \right) \times \Bigg[2d \max_{1\le i\le d} C_{iM}^4 \Bigg(\frac{C_{iM}^{10}\mu_{iM}^4\widetilde{\alpha}_{iM}}{(\widetilde{\alpha}_M)^4} - 4\frac{C_{iM}^8\mu_{iM}^4}{(\widetilde{\alpha}_M)^3} + 6\frac{C_{iM}^6\mu_{iM}^2\widetilde{\alpha}_{iM}}{(\widetilde{\alpha}_M)^2} - 4\frac{C_{iM}^4\mu_{iM}^2}{\widetilde{\alpha}_M} + C_{1M}^2\widetilde{\alpha}_{iM} + \frac{C_{iM}^8\mu_{iM}^4}{(\widetilde{\alpha}_M)^4} \sum_{j\neq i}^{1,d} C_{jM}^2\widetilde{\alpha}_{jM} \Bigg) + \frac{6d}{\widetilde{\alpha}_M(\widetilde{\alpha}_M + 1)} \max_{1\le i\le d} \left(C_{iM}^4 \left(\widetilde{\alpha}_{iM} - \frac{C_{iM}^2\mu_{iM}^2}{\widetilde{\alpha}_M} \right) \right)^2 \Bigg] = O(C_{1M}^{-2}).$$

By (6.21) we rewrite the left-hand side of this relation as

$$\frac{d}{(\widetilde{\alpha}_M)^2}\max_{1\le i\le d} C_{iM}^4\left(\widetilde{\alpha}_{iM}-\frac{C_{iM}^2\mu_{iM}^2}{\widetilde{\alpha}_M}\right)\left[O(C_{1M}^{-2})+\frac{6d}{(\widetilde{\alpha}_M)^5(\widetilde{\alpha}_M+1)}\max_{1\le i\le d}\left(C_{iM}^4\left(\widetilde{\alpha}_{iM}-\frac{C_{iM}^2\mu_{iM}^2}{\widetilde{\alpha}_M}\right)\right)^2\right]. \tag{6.22}$$

By (6.10) and the assumptions $(H1)'$, $(H2)'$ and $(H3)'$ we have

$$\frac{C_{jM}^4}{(\widetilde{\alpha}_M)^2}\left(\widetilde{\alpha}_{jM}-\frac{C_{jM}^2\mu_{jM}^2}{\widetilde{\alpha}_M}\right)=O(1),\quad \text{for any } j\in[d].$$

Therefore the term in (6.22) is

$$O(1)(O(C_{1M}^{-2})+O(C_{1M}^{-4}))=O(C_{1M}^{-2})$$

and the proof is completed.

6.7. *Proof of Theorem 5.5.* The proof of Theorem 5.5 is similar to the proofs of Theorems 5.2 and 5.3, and therefore it is omitted.

Acknowledgements

The author would like to thank an anonymous referee for a careful reading of the paper and useful suggestions.

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