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PAPER

Geometric Event-Based Quantum Mechanics

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Abstract

We propose a special relativistic framework for quantum mechanics. It is based on introducing a Hilbert space for events. Events are taken as primitive notions (as customary in relativity), whereas quantum systems (e.g. fields and particles) are emergent in the form of joint probability amplitudes for position *and* time of events. Textbook relativistic quantum mechanics and quantum field theory can be recovered by dividing the event Hilbert spaces into space and time (a foliation) and then conditioning the event states onto the time part. Our theory satisfies the full Lorentz symmetry as a 'geometric' unitary transformation, and possesses relativistic observables for space (location of an event) and time (position in time of an event).

Quantum mechanics (QM) relies on time-conditioned quantities: observables conditioned on time in the Heisenberg picture (e.g. $X(t)$ is the position operator at time t) or states conditioned on time in the Schrödinger one (e.g. $|\psi(t)\rangle$ is the state at time t). As such, it is inherently incompatible with the Poincaré symmetry of special relativity. Indeed QM can be formulated in a relativistic covariant fashion only in very specific circumstances, such as quantum field theory (QFT) when using Heisenberg picture operators acting on a relativistic invariant state such as the field vacuum. In this paper we depart from this approach and introduce a quantum mechanical theory of events, the geometric event-based QM (GEB), which is based on a modification of the Born rule which leads to unconditioned spacetime quantities and hence is intrinsically covariant. In QFT one starts from the dynamical equation of motion (either in the Hamiltonian formulation or from a Lagrangian [1]) and quantizes the dynamical solutions imposing equal time commutation relations. We take the opposite track: we start by defining an (unconditioned) purely kinematic Hilbert space $\mathcal{H}_\mathrm{E}$ which is well suited to account for the symmetries of a relativistic theory [2–4]. A formal correspondence with QM and QFT is then established by showing that the quantum evolutions defined by these theories can be identified as special subset $\mathcal{H}_\mathrm{QM}$ of the *distributions* of $\mathcal{H}_\mathrm{E}$ which is determined not via dynamical equations, but through purely geometrical constraints [5, 6].

The Hilbert space $\mathcal{H}_\mathrm{E}$ of GEB, rather than *systems*, can be thought to describe *events* [7]. In our approach the *event* is taken as a *primitive* notion, i.e. not something that is derived from a pre-existing notion of (say) a pre-existing particle that has been detected (as happens in QM). Indeed, in GEB the detection of particle at a particular location in space and time by an inertial measuring device [8] is a way to *identify* the event itself, and the particle (or, more generically, any quantum system) is a derived notion. A quantum system is then interpreted as a probability amplitude for an event out of a sequence of events, which takes the place of the 'sequence of events'[5], which is the customary definition of 'physical system' in relativity where events are often [9] taken as a primitive notion. Schrödinger had a similar observation: 'it is better to regard a particle not as a permanent entity but as an instantaneous event' [10]. Jorge Luis Borges is, obviously, more captivating: 'the world is not a concurrence of objects in space, but a heterogeneous series of independent

[5] In quantum theory, in the absence of trajectories, the concept of 'same system' (i.e. of a system that persists and is re-identifiable in time) is highly problematic [68]: the GEB formalism reflects this fully, as the system persistence and re-identifiability is only enforced as a probability amplitude.

acts' [11]. Examples of events (discussed below) are 'a fermion with spin σ is detected at a time t and position $\vec{x}$ in spacetime', or 'a boson is detected with energy E and momentum $\vec{p}$'. While we will be using some of the formalism developed in [2, 3], our interpretation of the formulas and the conceptual framework of GEB is fundamentally different. Relativistic versions of constrained QM [12–21] are also explored in [2–4, 22–34]. A different claim of 'covariant quantum mechanics' is in [35], where the Hilbert space is expressed through position eigenstates of a time-evolved (Heisenberg picture) position operator. While there is some similarity in the notation used, our approach is completely different: we do not use *any* dynamical assertion (hence, no pictures) in our treatment.

The outline follows. In section 1 we introduce the Hilbert space $\mathcal{H}_\mathbf{E}$ and the Born rule for a single-event, defining its relativistic observables, giving interpretations of the system 4D wave-function as unconditioned properties of events and the Lorentz transformations. In section 2 we consider multiple events in a first quantization and then in a second quantization Fock formalism. In section 3 we provide a formal connection between GEB and QM by introducing a correspondence rule that enables one to map quantum trajectories of the latter into distributions of the former. Conclusions and future perspectives are given in section 4. Technical details are in appendices.

1. The Hilbert space of GEB and its canonical observables

In this section, the Hilbert space $\mathcal{H}_\mathbf{E}$ which provides the mathematical setting for describing spacetime events is introduced. We start in section 1.1 by considering the simplest non trivial scenario, i.e. a Universe characterized by a single (spinless) event, which is the building block for the general case presented in section 2. In section 1.2 we discuss the covariance properties of the theory under Lorentz transformations, while in section 1.3 we show how to generalize the analysis to include spinor degrees of freedom.

1.1. A Universe with a single event

In the formulation of GEB we are guided by the fundamental observation that spacetime is physically meaningful only insofar as it is mapped by clocks and rods (clicks and ticks) which are events [36, 37]. From this we can infer that there is no localization in time of an event without any energy and energy spread (nothing can happen) and there is no localization in space of an event without momentum and momentum spread (any event would be delocalized over the whole space): so, in addition to being characterized by spacetime coordinates, any given event must also be connected to energy and momentum degrees of freedom. Accordingly, the structure of the Hilbert space $\mathcal{H}_\mathbf{E}$ associated with a single event can be identified by declaring that among the linear operators that acts on such space, there must exist (at least) a four component vectorial observable $\overline{X} := (X^0, X^1, X^2, X^3)$ that determines the 4-position $\overline{x} := (x^0 = t, \vec{x})$ in spacetime of the event, and an associated 4-momentum operator $\overline{P} := (P^0, P^1, P^2, P^3)$ that instead defines the corresponding energy-momentum values $\overline{p} := (p^0 = E, \vec{p})$ of the event[6]. (We use the overbar $\overline{x}$ to denote contravariant 4-vectors, the underbar $\underline{x}$ for covariant ones, and the arrow $\vec{x}$ for spatial 3-vectors—see appendix A) The existence of $\overline{X}$ and $\overline{P}$ is a minimal assumption of the theory: other observables can in fact be introduced that describe *extra* (not kinematic) degrees of freedom of the event, something that for instance will be revealed by the internal degree of freedom of an event-defined particle (say its spin) (see section 1.3).

We now impose the canonical commutation rules

$$[X^\mu, P^\nu] = -i\eta^{\mu\nu} \text{ and } [X^\mu, X^\nu] = [P^\mu, P^\nu] = 0\,, \tag{1}$$

with $\mu, \nu \in \{0, 1, 2, 3\}$ and η the metric diag$(1, -1, -1, -1)$ (note the minus sign in the 00 commutator). The rationale of this choice is that, on one hand, it allows us to satisfy Poincaré algebra [5]:

$$[M^{\mu\nu}, P^\rho] = -i(\eta^{\mu\rho}P^\nu - \eta^{\nu\rho}P^\mu), \tag{2}$$

$$[M^{\mu\nu}, M^{\rho\sigma}] = i(\eta^{\nu\rho}M^{\mu\sigma} - \eta^{\mu\rho}M^{\nu\sigma} - \eta^{\mu\sigma}M^{\rho\nu} + \eta^{\nu\sigma}M^{\rho\mu}), \tag{3}$$

[6] Notice that at this stage, as it happens in the axiomatic definition of QM, we do not provide any physical realization of $\overline{X}$ or $\overline{P}$, i.e. a way to detect the spacetime location and momentum of the event. It is intuitively clear however that these observables are related with the conventional definitions of position and momentum of particles in QM: e.g. on one hand if one conditions on the time t of an external clock, $\overline{X}$ conditioned on t is just the position measurement of a particle happening at some time t, the conventional position operator of QM that describes a screen that is turned on at a certain (externally controlled) time; on the other hand, if one conditions on the position $\vec{x}$ of a screen, then $\overline{X}$ conditioned on $\vec{x}$ is the time of arrival measurement of the particle at an active screen at position $\vec{x}$ [69, 70]. In the non-relativistic case, time observables were studied in, e.g. [15, 70, 71].

where $M^{\mu\nu} := X^\mu P^\nu - X^\nu P^\mu = (X \wedge P)^{\mu\nu}$ [38, 39] is the relativistic angular momentum tensor (the spatial part M^{ij} with $i, j \in \{1,2,3\}$, containing the angular momentum tensor, the generator of rotations, and the temporal part M^{0j} with $j \in \{1,2,3\}$ containing the generator of boosts)[7]. On the other hand, the condition (1) automatically imposes $\mathcal{H}_{\mathbf{E}}$ to be infinite dimensional and leads to the following spectral decompositions

$$X^\mu = \int d^4x x^\mu |\overline{x}\rangle\langle\overline{x}| \; , \qquad P^\mu = \int d^4p \, p^\mu |\overline{p}\rangle\langle\overline{p}| \; , \tag{4}$$

with $|\overline{x}\rangle$ and $|\overline{p}\rangle$ forming sets of generalized eigenstates each individually fulfilling delta-like orthogonality conditions, i.e. $\langle\overline{x}'|\overline{x}\rangle = \delta^{(4)}(\overline{x}' - \overline{x})$ and $\langle\overline{p}'|\overline{p}\rangle = \delta^{(4)}(\overline{p}' - \overline{p})$, while being related by a 4D Fourier transform

$$|\overline{p}\rangle = \int \tfrac{d^4x}{4\pi^2} e^{-i\overline{x}\cdot\underline{p}} |\overline{x}\rangle \; . \tag{5}$$

(where $\overline{a} = (a^0, \vec{a})$, $\underline{a} = \eta\overline{a} = (a^0, -\vec{a})$ so that $\overline{a} \cdot \underline{b} = \overline{a}\eta\overline{b} = a^\mu b_\mu$, appendix A)[8]. The states $|\overline{x}\rangle$ and $|\overline{p}\rangle$ are 4D extensions of the usual position and momentum eigenstates $|\vec{x}\rangle$, $|\vec{p}\rangle$ of QM: just as $|\vec{x}\rangle$ and $|\vec{p}\rangle$, both $|\overline{x}\rangle$ and $|\overline{p}\rangle$ are not elements of $\mathcal{H}_{\mathbf{E}}$ but objects that in functional analysis would correspond to *distributions*. Indeed they belong to the rigged-extended version $\mathcal{H}_{\mathbf{E}}^+$ of $\mathcal{H}_{\mathbf{E}}$ that, together with the dense nuclear subspace $\mathcal{H}_{\mathbf{E}}^-$ formed by the intersection of the (dense) domains of $\underline{X}$ and $\underline{P}$, defines the Gelfand triple $\mathcal{H}_{\mathbf{E}}^- \subset \mathcal{H}_{\mathbf{E}} \subset \mathcal{H}_{\mathbf{E}}^+$ of the model. (The Gelfand triple [40] is a collection of three objects: the Hilbert space itself $\mathcal{H}_{\mathbf{E}}$, the set of distributions on the space $\mathcal{H}_{\mathbf{E}}^+$, such as Dirac deltas identifying eigenvectors of continuous-variable eigenvalue observables, and the nuclear elements of the Hilbert space, i.e. a dense subset $\mathcal{H}_{\mathbf{E}}^-$ on which one can apply the distributions.) As such, similarly to $|\vec{x}\rangle$ and $|\vec{p}\rangle$ in QM, we shall not assign to them a precise physical interpretation. The best we can say is that $|\overline{x}\rangle$ ($|\overline{p}\rangle$) represent an (unphysical) absolute localization of the event in spacetime (resp. energy-momentum space) which can be used to characterize the statistical distributions of the elements of $\mathcal{H}_{\mathbf{E}}$ via decompositions of the form

$$|\Phi\rangle = \int d^4x \, \Phi(\overline{x}) \, |\overline{x}\rangle = \int d^4p \, \tilde{\Phi}(\overline{p}) \, |\overline{p}\rangle \; , \tag{6}$$

with the amplitudes

$$\Phi(\overline{x}) := \langle\overline{x}|\Phi\rangle \; , \; \tilde{\Phi}(\overline{p}) := \langle\overline{p}|\Phi\rangle = \int \tfrac{d^4x}{4\pi^2} e^{i\overline{x}\cdot\underline{p}} \, \Phi(\overline{x}) \; , \tag{7}$$

being square integrable functions (4D wave-functions)[9]. Quantum probabilities follow then directly from the Born rule. In particular the probability that the measurement of the observable X^μ has result $\overline{x}$ on a event state associated with the normalized vector $|\Phi\rangle \in \mathcal{H}_{\mathbf{E}}$ is

$$P(\overline{x}|\Phi) = |\Phi(\overline{x})|^2 = |\langle\overline{x}|\Phi\rangle|^2 \; . \tag{8}$$

We interpret this as the probability distribution that the event happens at spacetime position $\overline{x}$: this clarifies that in our approach the vectors $|\Phi\rangle$ are spacetime states, representing (in the position representation) the probability amplitudes of the spacetime locations of the event. This is a *different* version of the Born rule with respect to the QM-Born rule $p(\vec{x}|\psi, t) = |\langle\vec{x}_S|\psi_S(t)\rangle|^2 = |\langle\vec{x}_H(t)|\psi_H\rangle|^2$ (written in the Schrödinger S or Heisenberg H pictures), which gives the *conditional* probability that a particle in state ψ is found at position $\vec{x}$ *given* that the time is t. Instead, in GEB, the probability (8) is unconditioned: it is a *joint* probability that the event happens at position $\vec{x}$ *and* that the time is t for $\overline{x} = (t, \vec{x})$. Analogously, in GEB we can also interpret $P(\overline{p}|\Phi) = |\tilde{\Phi}(\overline{p})|^2 = |\langle\overline{p}|\Phi\rangle|^2$ as the probability that the event will carry an energy-momentum $\overline{p}$, and

$$p(\Phi_1|\Phi_2) = |\langle\Phi_1|\Phi_2\rangle|^2 \; , \tag{9}$$

[7] This is the simplest choice to satisfy the Poincaré algebra, but it is not unique: one can still satisfy the commutation relations with appropriate redefinitions of P_μ and $M_{\mu\nu}$. For example, the instant form [5] arises from the requirement that the position and momentum are referred to some instant of time, as in the conventional (conditioned) formulation of quantum mechanics [57]: in the Schrödinger picture the states are conditioned to time being t and the operators to $t = 0$, *viceversa* in the Heisenberg picture.

[8] Notice that one can divide the four degrees of freedom (1 temporal and 3 spatial) of $|\underline{x}\rangle$ using tensor products $|\underline{x}\rangle = |t\rangle|\vec{x}\rangle$, as is done in the nonrelativistic Page and Wootters formalism [12, 15] (and equivalent ones [22]) but, this tensor product structure is not absolute to preserve Poincaré invariance: it is observable-induced [72–74], since observers in different reference frames will tensorize it differently.

[9] Formally speaking the decomposition (6) holds for the dense subset $\mathcal{H}_{\mathbf{E}}^-$ of $\mathcal{H}_{\mathbf{E}}$ which together with $\mathcal{H}_{\mathbf{E}}^+$ define the Gelfand triple.

as the probability to confuse the event $|\Phi_2\rangle$ as the event $|\Phi_1\rangle \in \mathcal{H}_{\mathbf{E}}$. (Again, these are both different from the conventional QM Born rule: the first is again conditioned on time, $P(E|\psi(t)) = |\langle E|\psi(t)\rangle|^2$ is the probability of finding the energy E with eigenstate $|E\rangle$, given that the state is ψ *and the time is* t; the second is $P(\psi_1(t_1)|\psi_2(t_2)) = |\langle\psi_1(t_1)|\psi_2(t_2)\rangle|^2$ is the probability to confuse the state $|\psi_1\rangle$ at time t_1 with the state $|\psi_2\rangle$ at time t_2.)

A direct consequence of equation (1) are the Heisenberg–Robertson [41] uncertainty relations

$$\Delta X^\mu \Delta P^\nu \geqslant 1/2\ \delta_{\mu\nu}, \tag{10}$$

with δ the Kronecker delta, which for $\mu, \nu = 0, 1, 2, 3$ have to be fulfilled by the statistical distributions $P(\overline{x}|\Phi)$ and $P(\overline{p}|\Phi)$ for all choices of the normalized vector $|\Phi\rangle \in \mathcal{H}_{\mathbf{E}}$. These equations effectively translate in strict mathematical form our initial observation that no event can be spacetime localized without energy and momentum spread. While for $\mu = 1, 2, 3$, equation (10) is the usual Heisenberg uncertainty relation [42], for $\mu = 0$ it takes a special role [43, 44]: as a matter of fact it cannot [45] be derived in QM, where time is a parameter and not an observable. (Indeed, in the Mandelstamm–Tamm uncertainty relation [46] Δt takes the role of a time interval (the minimum time interval it takes for a system to evolve to an orthogonal configuration), not of a statistical uncertainty due to quantum stochasticity.) Here, instead, time X^0 *is* a quantum observable, and we can assign to $\Delta X^0 \Delta P^0 \geqslant \hbar/2$ the Heisenberg–Robertson interpretation. (The acute objections [45, 47] against this type of interpretation of the time-energy uncertainty were formulated in the framework of QM. In GEB, these objections do not apply.) This interpretation is a purely kinematical statement, not a dynamical one: P^0 is not some system's Hamiltonian, but it is the energy devoted to the event. The connection between the energy and a Hamiltonian (dynamics) is here only enforced as a constraint on the physical states, see below. We use constraints that select the dynamics instead of dynamical equations. This type of approach is quite traditional in the relativistic literature [2–6, 22–34]. Indeed, even the Einstein equations can be seen as constraints on the spacetime and on the stress-energy tensor rather than dynamical equations in the traditional sense.

1.2. Lorentz transforms

Relativistic QM, e.g. QFT, can be written in a covariant fashion only in very specific situations. Indeed, as discussed above, the Born rule probability when calculated in the Schrödinger, Heisenberg or interaction picture, is a conditional probability, conditioned on time. As such, in contrast to what it is claimed sometimes (e.g. [48]), QM probabilities are *not* covariant: they refer to a specific spacetime foliation, and a Lorentz transform cannot be simply applied to something defined on a single foliation slice. In QFT one can recover covariance if one writes the observables in the interaction picture in terms of creation and annihilation operators that have a spacetime dependence of the form $e^{\pm i\overline{x}\cdot\underline{p}}$. But observables by themselves are not sufficient to obtain quantum probabilities or expectation values: they must be applied to states. QM states explicitly depend on a foliation (at time t in the Schrödinger picture, at time $t = 0$ in the Heisenberg picture, and at a foliation determined by the state-evolution operator in the interaction picture). For this reason, one cannot in general apply a Lorentz transform to a quantum state. There is an important exception, customarily employed in QFT, where one uses states that are eigenstates of the Hamiltonian, e.g. the vacuum state. They are invariant for the dynamics, and hence can be easily Lorentz-transformed as they are left invariant.

In contrast, the GEB formalism we introduce here is fully covariant as the canonical observables of the theory are trivially transformed using a unitary representation of the Lorentz group. To clarify this fact let us consider two inertial observers O and O' sitting on different reference frames R and R' connected via the transformation matrix Λ of the Lorentz group so that, given $\overline{x} = (t, \vec{x})$ and $\overline{x}' = (t', \vec{x}')$ the coordinates that they assign to the same event one has

$$\overline{x}' = \Lambda\overline{x}\,, \tag{11}$$

(e.g. if O' has velocity v along the x-axis with respect to O', then $t' = \gamma(t - vx)$, $x' = \gamma(x - vt)$, $y' = y$, and $z' = z$). Under these conditions it follows that given a state event S, they will describe it as two different elements $|\Phi\rangle = \int d^4x \Phi(\overline{x})|\overline{x}\rangle$ and $|\Phi'\rangle = \int d^4x \Phi'(\overline{x})|\overline{x}\rangle$ of the space $\mathcal{H}_{\mathbf{E}}$ whose 4D wave-functions amplitudes are related by the identity

$$\Phi'(\overline{x}) = \Phi(\Lambda^{-1}\overline{x})\,, \tag{12}$$

see appendix B. Accordingly, we can relate the vectors $|\Phi\rangle$ and $|\Phi'\rangle$ as

$$|\Phi'\rangle = U_\Lambda|\Phi\rangle\,, \tag{13}$$

where U_Λ is the unitary mapping that represents Λ in $\mathcal{H}_\mathrm{E}$, i.e. the transformation associated with the generators $M^{\mu\nu}$ of equation (2). Equation (13) ensures that both observers will assign the same scalar products to any two couples of states, i.e.

$$\langle \Phi_1' | \Phi_2' \rangle = \langle \Phi_1 | \Phi_2 \rangle \,, \tag{14}$$

which implies that the probabilities (9) are invariant under change of reference frames. This is in line with the fact that the Born rule probabilities in GEB refer to the occurrence of events in spacetime (rather than the conditional probabilities on a foliation at a specific time t): they are unconditioned and hence invariant. As a direct consequence of (12), the spacetime probability distributions (8) that O' and O associate with S, i.e. the functions $P(\overline{x}|\Phi') = |\Phi'(\overline{x})|^2$ and $P(\overline{x}|\Phi) = |\Phi(\overline{x})|^2$ respectively, are transformed as any scalar field

$$P(\overline{x}|\Phi') = P(\Lambda^{-1}\overline{x}|\Phi) \,. \tag{15}$$

[Actually, the P are probability *densities*, but they transform as scalar fields for Lorentz transformations, since the 4-volume element is invariant.] Similarly for the energy-momentum distributions we get $\tilde{\Phi}'(\overline{p}) = \tilde{\Phi}(\Lambda\overline{p})$ and hence $P(\overline{p}|\Phi') = P(\Lambda^{-1}\overline{p}|\Phi)$, (these last follow directly from the identity (B7) of appendix B). A direct consequence of these relations is that any statement O and O' make on the expectation values of these observables on the states of $\mathcal{H}_\mathrm{E}$ (or on their higher momenta like those appearing in (10)) are automatically covariant under Lorentz (and more generally Poincaré) transformations. This can be made more explicit (appendix B) using the 4D 'Heisenberg picture' where the mapping (13) results in the following connection between the canonical observables of O and O'

$$\overline{X}' = U_\Lambda^\dagger \overline{X} U_\Lambda = \Lambda \overline{X} \,, \tag{16}$$

$$\overline{P}' = U_\Lambda^\dagger \overline{P} U_\Lambda = \Lambda \overline{P} \,. \tag{17}$$

1.3. Spinor

As discussed above, extra (non-kinematic) degrees of freedom can be included into the theory. This can be done for instance by promoting the generalized eigenvectors $|\overline{x}\rangle$, $|\overline{p}\rangle$ to spinor vectors $|\overline{x},\sigma\rangle$, $|\overline{p},\sigma\rangle$ with σ a spinor index (e.g. taking values 1,2,3,4) that fulfill generalized orthogonality conditions

$$\begin{aligned}\langle \overline{x}',\sigma' | \overline{x},\sigma\rangle &= \delta_{\sigma,\sigma'}\ \delta^{(4)}(\overline{x}'-\overline{x}) \,,\\ \langle \overline{p}',\sigma' | \overline{p},\sigma\rangle &= \delta_{\sigma,\sigma'}\ \delta^{(4)}(\overline{p}'-\overline{p}) \,.\end{aligned} \tag{18}$$

Accordingly, the decomposition (6) of the event state is

$$|\Phi\rangle = \sum_\sigma \int d^4x\, \Phi(\overline{x},\sigma)\, |\overline{x},\sigma\rangle = \sum_\sigma \int d^4p\, \tilde{\Phi}(\overline{p},\sigma)\, |\overline{p},\sigma\rangle, \tag{19}$$

where now the amplitudes $\Phi(\overline{x},\sigma) := \langle \overline{x},\sigma|\Phi\rangle$ (resp. $\tilde{\Phi}(\overline{p},\sigma) := \langle \overline{p},\sigma|\Phi\rangle$) when properly normalized define the joint probability $P(\overline{x},\sigma|\Phi) = |\Phi(\overline{x},\sigma)|^2$ (resp. $P(\overline{p},\sigma|\Phi) = |\tilde{\Phi}(\overline{p},\sigma)|^2$) of finding $|\Phi\rangle$ in location $\overline{x}$ (with momentum $\overline{p}$) and spin value σ. Under these conditions the identity (13) which relates the description of the observer O and O' still holds by updating (12) with

$$\Phi'(\overline{x},\sigma) = \sum_{\sigma'} S^{-1}_{\sigma',\sigma}(\Lambda)\Phi(\Lambda^{-1}\overline{x},\sigma') \,, \tag{20}$$

where now $S_{\sigma,\sigma'}(\Lambda)$ is the unitary matrix representation of the Lorentz transform in the spinor space [38].

2. Multiple events

The Hilbert space $\mathcal{H}_\mathrm{E}$ spanned by the vectors (19) describes a single spacetime event. The extension to the case of multiple events is obtained by considering tensor products of such space, possibly equipped with proper symmetrisation rules that account for the statistical properties of the particles that are defined through them (see below). An important advantage of this construction is that the transformations under the Lorentz group follow directly from those established for those of the single-event model (13). Indeed given $|\Phi^{[n]}\rangle, |\Phi^{[n]\prime}\rangle \in \mathcal{H}_\mathrm{E}^{\otimes n}$ the vectors used by the observers O and O' to describe the same state of n events, they will be connected via the mapping

$$|\Phi^{[n]\prime}\rangle = U_\Lambda^{\otimes n} |\Phi^{[n]}\rangle \,, \tag{21}$$

with U_Λ being the unitary operator that represents the matrix Λ of equation (11) for a single event. Of course with this choice, each individual event is connected to its own time of occurrence, whereas interpretations of the covariant formalism in terms of particles are problematic [2, 49] because the 'time of a particle' is a meaningless concept [45]. In the following paragraphs we discuss the different types of multi-event models that arise from the above formalization and show how this analysis can be lifted to a higher level of complexity by constructing an effective QFT version of GEB, through Fock space (second quantization).

2.1. Distinguishable vs Indistinguishable events

The simplest example of a multi-event model is represented by an Universe of *n distinguishable* spacetime events, i.e. events that define n distinguishable particles. In this case any normalized vector $|\Phi^{[n]}\rangle$ of $\mathcal{H}_{\mathbf{E}}^{\otimes n}$ qualifies for a proper GEB state of the model, with the decomposition (19) being replaced by

$$|\Phi^{[n]}\rangle = \sum_{\sigma_1,\ldots,\sigma_n} \int d^4x_1 \cdots d^4x_n \; \Phi^{[n]}(\bar{x}_1,\sigma_1;\ldots;\bar{x}_n,\sigma_n)|\bar{x}_1,\sigma_1;\ldots;\bar{x}_n,\sigma_n\rangle \tag{22}$$

$$= \sum_{\sigma_1,\ldots,\sigma_n} \int d^4p_1 \cdots d^4p_n \; \tilde{\Phi}^{[n]}(\bar{p}_1,\sigma_1;\ldots;\bar{p}_n,\sigma_n)|\bar{p}_1,\sigma_1;\ldots;\bar{p}_n,\sigma_n\rangle \;, \tag{23}$$

where $|\bar{x}_1,\sigma_1;\ldots;\bar{x}_n,\sigma_n\rangle$ stands for $\bigotimes_j |\bar{x}_j,\sigma_j\rangle$ with $|\bar{x}_j,\sigma_j\rangle_j$ the 4-position and spin eigenstate of the j-th event so that $\Phi^{[n]}(\bar{x}_1,\sigma_1;\ldots;\bar{x}_n,\sigma_n)$ is the wave-function which yields the *joint* probability

$$P^{[n]}(\bar{x}_1,\sigma_1;\ldots;\bar{x}_n,\sigma_n) = |\Phi^{[n]}(\bar{x}_1,\sigma_1;\ldots;\bar{x}_n,\sigma_n)|^2 \;, \tag{24}$$

of revealing the jth event in spacetime position $\bar{x}_j$ with spin σ_j, for all j. — similar definitions apply also to $|\bar{p}_1,\sigma_1;\ldots;\bar{p}_n,\sigma_n\rangle$ and $\tilde{\Phi}^{[n]}(\bar{p}_1,\sigma_1;\ldots;\bar{p}_n,\sigma_n)$ of equation (23).

Consider next the scenario of *n indistinguishable* spacetime events, i.e. events that are used to define n identical particles. We describe the states of such models via vectors (22) which induce the Bosonic or Fermionic character of the derived particles through the property of being either completely symmetric or completely anti-symmetric under exchange of the key indexes. Specifically a Bosonic n-event GEB model is described by the completely symmetric linear subset $\mathcal{H}_{\mathbf{E}}^{(n,\mathbf{S})} \subset \mathcal{H}_{\mathbf{E}}^{\otimes n}$ spanned by the vectors (22) with amplitudes probabilities $\Phi^{[n]}(\bar{x}_1,\sigma_1;\ldots;\bar{x}_n,\sigma_n)$ that are invariant under an arbitrary permutation $\mathbf{p}$ of the n systems labels, i.e.

$$\Phi^{[n]}(\bar{x}_{\mathbf{p}(1)},\sigma_{\mathbf{p}(1)};\ldots;\bar{x}_{\mathbf{p}(n)},\sigma_{\mathbf{p}(n)}) = \Phi^{[n]}(\bar{x}_1,\sigma_1;\ldots;\bar{x}_n,\sigma_n) \;, \quad \forall \mathbf{p} \;, \tag{25}$$

(a condition that automatically carries over to the 4D momentum wave-function). A Fermionic n-event GEB model, instead, will be described by the completely anti-symmetric linear subset $\mathcal{H}_{\mathbf{E}}^{(n,\mathbf{A})} \subset \mathcal{H}_{\mathbf{E}}^{\otimes n}$ formed by vectors with amplitudes that fulfill the identity

$$\Phi^{[n]}(\bar{x}_{\mathbf{p}(1)},\sigma_{\mathbf{p}(1)};\ldots;\bar{x}_{\mathbf{p}(n)},\sigma_{\mathbf{p}(n)}) = \mathrm{sign}[\mathbf{p}]\Phi^{[n]}(\bar{x}_1,\sigma_1;\ldots;\bar{x}_n,\sigma_n) \;, \quad \forall \mathbf{p} \;, \tag{26}$$

with sign[$\mathbf{p}$] being the sign of the permutation $\mathbf{p}$.

2.2. Fock space representation

The above construction can only deal with a fixed, predetermined number n of events. To analyze situations where n is, itself, a quantum degree of freedom, we need to escalate to a Fock space. The starting point of this construction is to introduce a '4D-vacuum' state vector $|0\rangle_4$ which represents the state of a Universe where there are no events *at any spacetime location*, and by defining raising and lowering operators [2, 4] $a^\dagger_{\bar{x},\sigma}$, $a_{\bar{x},\sigma}$ that act as creators/annihilators of spacetime events in the theory. Properly symmetrized versions of the generalized 4-position eigenstates will now be expressed as reiterated applications of the $a^\dagger_{\bar{x},\sigma}$'s on $|0\rangle_4$, i.e.

$$|\bar{x}_1,\sigma_1;\ldots;\bar{x}_n,\sigma_n\rangle \quad \mapsto \quad \frac{1}{\sqrt{n!}}\, a^\dagger_{\bar{x}_1,\sigma_1}\cdots a^\dagger_{\bar{x}_n,\sigma_n}|0\rangle_4 \;, \tag{27}$$

while the generalized 4-momentum eigenstates $|\bar{p}_1,\sigma_1;\ldots;\bar{p}_n,\sigma_n\rangle$ as

$$|\bar{p}_1,\sigma_1;\ldots;\bar{p}_n,\sigma_n\rangle \quad \mapsto \quad \frac{1}{\sqrt{n!}}\, a^\dagger_{\bar{p}_1,\sigma_1}\cdots a^\dagger_{\bar{p}_n,\sigma_n}|0\rangle_4 \;, \tag{28}$$

with $a^\dagger_{\bar{p},\sigma}$ and $a_{\bar{p},\sigma}$ connected with $a^\dagger_{\bar{x},\sigma}$, i.e. the operators

$$a^\dagger_{\bar{p},\sigma} := \int \frac{d^4x}{4\pi^2} e^{-i\bar{p}\cdot\bar{x}} a^\dagger_{\bar{x},\sigma} \;, \qquad a_{\bar{p},\sigma} := \int \frac{d^4x}{4\pi^2} e^{i\bar{p}\cdot\bar{x}} a_{\bar{x},\sigma} \;. \tag{29}$$

The consistency of the representation is enforced by assigning proper commutation relations to the raising and lowering operators (appendix C). Specifically, the Bosonic/Fermionic character follows by requiring

$$\begin{aligned}&\text{Bose: } [a_{\bar{x},\sigma}, a^\dagger_{\bar{x}',\sigma'}] = \delta_{\sigma,\sigma'}\delta^{(4)}(\bar{x}-\bar{x}'),\ [a_{\bar{x},\sigma}, a_{\bar{x}',\sigma'}] = 0,\\ &\text{Fermi: } \{a_{\bar{x},\sigma}, a^\dagger_{\bar{x}',\sigma'}\} = \delta_{\sigma,\sigma'}\delta^{(4)}(\bar{x}-\bar{x}'),\ \{a_{\bar{x},\sigma}, a_{\bar{x}',\sigma'}\} = 0,\end{aligned} \tag{30}$$

which automatically translate into analogous relations for the $a^\dagger_{\bar{p},\sigma}$, $a_{\bar{p},\sigma}$, i.e.

$$\begin{aligned}&\text{Bose: } [a_{\bar{p},\sigma}, a^\dagger_{\bar{p}',\sigma'}] = \delta_{\sigma,\sigma'}\delta^{(4)}(\bar{p}-\bar{p}'),\ [a_{\bar{p},\sigma}, a_{\bar{p}',\sigma'}] = 0,\\ &\text{Fermi: } \{a_{\bar{p},\sigma}, a^\dagger_{\bar{p}',\sigma'}\} = \delta_{\sigma,\sigma'}\delta^{(4)}(\bar{p}-\bar{p}'),\ \{a_{\bar{p},\sigma}, a_{\bar{p}',\sigma'}\} = 0.\end{aligned} \tag{31}$$

Accordingly, in the Fock state representation equations (22) and (23) can be expressed as

$$|\Phi^{[n]}\rangle = \frac{1}{\sqrt{n!}} \sum_{\sigma_1,\ldots,\sigma_n} \int d^4x_1 \cdots d^4x_n\, \Phi^{[n]}(\bar{x}_1,\sigma_1;\ldots;\bar{x}_n,\sigma_n)\, a^\dagger_{\bar{x}_1,\sigma_1}\cdots a^\dagger_{\bar{x}_n,\sigma_n}|0\rangle_4 \tag{32}$$

$$= \frac{1}{\sqrt{n!}} \sum_{\sigma_1,\ldots,\sigma_n} \int d^4p_1 \cdots d^4p_n\, \tilde{\Phi}^{[n]}(\bar{p}_1,\sigma_1;\ldots;\bar{p}_n,\sigma_n)\, a^\dagger_{\bar{p}_1,\sigma_1}\cdots a^\dagger_{\bar{p}_n,\sigma_n}|0\rangle_4\ , \tag{33}$$

with $\Phi^{[n]}(\bar{x}_1,\sigma_1;\ldots;\bar{x}_n,\sigma_n)$ and $\tilde{\Phi}^{[n]}(\bar{p}_1,\sigma_1;\ldots;\bar{p}_n,\sigma_n)$ retaining the same probabilistic meaning given in equation (24).

It is important to stress that the 4D-vacuum $|0\rangle_4$ is a distinct state from the 3D-vacuum $|0\rangle_3$ used in QFT. Indeed $|0\rangle_3$ represents a configuration in which there are no particles *at a specific time* (time t in the Schrödinger picture or $t=0$ in the Heisenberg one), whereas $|0\rangle_4$ has no events at any time. The QFT vacuum $|0\rangle_3$ is the ground state of a field. As such, it is the spatial part (in some foliation) of the GEB state relative to a zero 4-momentum event: $|0\rangle_3 =$foliate$(a^\dagger_{\bar{p}=0}|0\rangle_4)$. It is *not* the spatial part of $|0\rangle_4$. The state $|0\rangle_4$ represents no events, whereas $a^\dagger_{\bar{p}=0}|0\rangle_4$ represents a zero 4-momentum event which corresponds to a uniform distribution of zero-energy events in all spacetime $a^\dagger_{\bar{p}=0}|0\rangle_4 = \int d^4x\, a^\dagger_{\bar{x}}|0\rangle_4$: in other words, there is a difference between saying 'nothing happens everywhere and everywhen' (i.e. the QFT vacuum $|0\rangle_3$ at all times), and saying 'nothing happens anywhere and at any time', the Aristotelian void $|0\rangle_4$.

Also, the GEB raising and lowering operators are completely different from the QFT ones, and it is not just a matter of adding the temporal degree of freedom: $a_{\bar{p}} \neq a_{p^0}\otimes a_{\vec{p}}$ (the right-hand-side would give incorrect commutators). The QFT ones are obtained from the canonical quantization of the harmonic oscillator, namely starting from the *dynamics*. Here, instead, we are introducing the raising and lowering operator from the *kinematics*, namely the ones that applied to the 4-vacuum create the position (or momentum) eigenstates. The QFT operators lose their meaning when one changes the dynamics (e.g. by adding interactions), whereas ours do not. However, as in QFT, also in GEB the raising and lowering operators can be used to define a number operator.

Thanks to Fock space, we can now have states with superpositions of different numbers of events, i.e.

$$|\Phi\rangle = \sum_{n\geqslant 0} \alpha_n |\Phi^{[n]}\rangle\ , \tag{34}$$

with $|\Phi^{[0]}\rangle := |0\rangle_4$ and α_n probability amplitudes, so that $|\alpha_n|^2|\Phi^{[n]}(\bar{x}_1,\sigma_1;\ldots;\bar{x}_n,\sigma_n)|^2$ (resp. $|\alpha_n|^2|\tilde{\Phi}^{[n]}(\bar{p}_1,\sigma_1;\ldots;\bar{p}_n,\sigma_n)|^2$) is the joint probability density of having n detection events *and* that they happen at spacetime locations $\bar{x}_1,\ldots,\bar{x}_n$ (resp. momenta $\bar{p}_1,\ldots,\bar{p}_n$) and with spins $\sigma_1,\ldots,\sigma_n$ ($|\alpha_0|^2$ being the probability of no event).

The Lorentz transformations are a straightforward extension of (21). Indeed, indicating with $\mathcal{U}_\Lambda$ the unitary mapping that represents Λ in Fock space, given $|\Phi\rangle$ and $|\Phi'\rangle$ the states two observers O and O' assign to same state event, we can write

$$|\Phi'\rangle = \mathcal{U}_\Lambda|\Phi\rangle\ , \tag{35}$$

by requiring that the vacuum state is left invariant by $\mathcal{U}_\Lambda$, $\mathcal{U}_\Lambda|0\rangle_4 = |0\rangle_4$, and by imposing

$$\mathcal{U}_\Lambda a^\dagger_{\bar{x},\sigma}\mathcal{U}^\dagger_\Lambda = \sum_{\sigma'} S^{-1}_{\sigma,\sigma'}(\Lambda)\, a^\dagger_{\Lambda\bar{x},\sigma'}\ , \tag{36}$$

or, equivalently,

$$\mathcal{U}_\Lambda a^\dagger_{\bar{p},\sigma}\mathcal{U}^\dagger_\Lambda = \sum_{\sigma'} S^{-1}_{\sigma,\sigma'}(\Lambda)\, a^\dagger_{\Lambda\bar{p},\sigma'}\ . \tag{37}$$

3. QM/GEB correspondence

QM is a physical theory of systems while GEB is a physical theory of events: in this section we shall see that these two different approaches can be connected by associating the dynamical quantum trajectories of QM to elements of the distributions set $\mathcal{H}_{\mathbf{E}}^{+}$ of GEB. For the sake of simplicity we start in section 3.1 by considering the special case of a single event universe showing that it can be put in correspondence with the QM description of a point-like single particle system. The generalization to multi-event scenario will instead be addressed in section 3.2 and in section 3.3 where we shall make use of the Fock space representation introduced in section 2.2.

3.1. Single-particle/single-event correspondence

In QM the temporal evolution of a single particle described by an observer O sitting in his reference frame R, is obtained by assigning a 3D+1 spinor wave-function $\Psi_{\rm QM}(\vec{x},\sigma|t)$ which extends both in time and in space. The unitary character of the dynamics ensures that 3D norm of this function is a constant of motion. Accordingly, setting

$$\sum_{\sigma}\int d^3x|\Psi_{\rm QM}(\vec{x},\sigma|t)|^2=1\ , \qquad \forall t\in\mathbb{R} \tag{38}$$

the function $\Psi_{\rm QM}(\vec{x},\sigma|t)$ can be interpreted as the conditional probability amplitude that the observer O will find the particle at position $\vec{x}$ with spin σ, given that time is t. A natural correspondence between the single particle states of QM and the single event states of the GEB formalism can hence be established by interpreting $\Psi_{\rm QM}(\vec{x},\sigma|t)$ as a 4D spinor wave-function

$$\Psi_{\rm QM}(\bar{x},\sigma):=\Psi_{\rm QM}(\vec{x},\sigma|t)\ , \tag{39}$$

and then using the following mapping

$$\Psi_{\rm QM}(\bar{x},\sigma)\in {\rm QM}\mapsto|\Psi_{\rm QM}\rangle:\overset{!}{=}\sum_{\sigma}\int d^4x\ \Psi_{\rm QM}(\bar{x},\sigma)\ |\bar{x},\sigma\rangle\ . \tag{40}$$

The exclamation mark is a reminder that, with the normalization (38), the vector $|\Psi_{\rm QM}\rangle$ has an infinite norm, in contrast to $|\Phi\rangle$ of (6). Indeed the square integrability of the GEB wave-functions $\Phi(\bar{x},\sigma)$ is incompatible with (38) obeyed by the QM wave-function $\Psi_{\rm QM}(\bar{x},\sigma)$: in general an element $|\Phi\rangle$ of $\mathcal{H}_{\mathbf{E}}$ will exhibit modulations with respect to t that in QM would be interpreted as unphysical *losses* and *gains* of probability during the temporal evolution of the particle but which are perfectly allowed at the kinematic level in the GEB formalism (and they can then be removed at the dynamical level, see below).

Because of their infinite norm, the vectors $|\Psi_{\rm QM}\rangle$ introduced above are not elements of $\mathcal{H}_{\mathbf{E}}$ and cannot be interpreted as proper event states of GEB. The mapping (40) associates the QM wave-functions $\Psi_{\rm QM}(\vec{x},\sigma|t)$ of O to distributions of GEB. This fact is explicitly shown in appendix D: here we point out that equation (40) identifies only a proper subset $\mathcal{H}_{\mathbf{QM}}$ of $\mathcal{H}_{\mathbf{E}}^{+}$. Examples of elements of $\mathcal{H}_{\mathbf{E}}^{+}$ which are not in $\mathcal{H}_{\mathbf{QM}}$ are provided for instance by the generalized position and momentum eigenvectors $|\bar{x}\rangle$ and $|\bar{p}\rangle$ which clearly cannot be expressed as in (40) with 3D normalized QM solutions $\Psi_{{\rm QM},\sigma}(\vec{x}|t)$. We also notice $\mathcal{H}_{\mathbf{QM}}$ can be identified via geometric constraints analogous to those adopted in [2, 3, 12, 22, 26, 39, 50–53]. Specifically one has

$$|\Psi_{\rm QM}\rangle\in\mathcal{H}_{\mathbf{QM}}\qquad\Longleftrightarrow\qquad\begin{cases}K|\Psi_{\rm QM}\rangle=0\ ,\\[2ex] |\Psi_{\rm QM}\rangle\neq 0\ ,\end{cases} \tag{41}$$

where K is a constraint operator that encodes the QM dynamics (as discussed in appendix E we can also add extra constrains that force $|\Psi_{\rm QM}\rangle$ to represent 3D+1 spinor wave-functions that fulfill assigned initial conditions for a given observer O). We stress that in contrast to previous literature [2, 3, 30, 39] where the solutions of constraint equations are interpreted as history states for *systems*, in GEB they are used to identify distributions which define *event* states. Another difference with previous approaches is that for the purpose to generalizing the analysis to the multi-event scenarios, in our construction we find it convenient to work with constraint operators which are explicitly self-adjoint and semidefinite-positive, i.e. $K\geqslant 0$. Of course this choice can be enforced without loss of generality since given J a generic operator fulfilling (41) we can always identify a positive semi-definite one that does exactly the same e.g. by taking $K=J^\dagger J$ exploiting the fact that

$$J|\Psi_{\rm QM}\rangle=0\qquad\Longleftrightarrow\qquad J^\dagger J|\Psi_{\rm QM}\rangle=0\ . \tag{42}$$

For non-relativistic models, the QM dynamics takes always the form of a Schrödinger equation which can be cast in the form (41) along the lines detailed e.g. in in reference [15]. Unfortunately, there does not appear to be a similarly general method to describe the relativistic dynamics[10], but one should use covariant constraints to avoid ruining the theory's covariance. Indicating with $\Box := \partial_t^2 - \nabla^2$ the D'Alembert operator, in the case of spinless particle of mass m this can be done for instance by invoking the Klein–Gordon (KG) equation

$$(\Box + m^2)\Psi_{\rm QM}(\vec{x}|t)\Big|_+ = 0\,, \tag{43}$$

filtering out its positive energy solutions (see appendix F). For the case of a massive spin 1/2 particle instead one can use the Dirac equation

$$\sum_{\sigma=1}^{4}(i\overline{\gamma}_{\sigma',\sigma}\cdot\underline{\partial} - m\delta_{\sigma',\sigma})\Psi_{\rm QM}(\vec{x},\sigma|t) = 0\,, \tag{44}$$

with $\overline{\partial} := (\partial/\partial t, -\vec{\nabla})$ and $\overline{\gamma} := (\gamma^0,\gamma^1,\gamma^2,\gamma^3)$ the Dirac matrices (see equation (A11)). Adopting the position representation $P^\mu \to i\partial^\mu$, both equations (43) and (44) can be turned into a constraint of the form (41) for the associated distributions $|\Psi_{\rm QM}\rangle$[11]. Specifically, in the case of the positive-energy KG equation (43) one can identify the constraint operator K of (41) with the self-adjoint operator

$$J_{\rm KG+} := \int d^4p\,(\Theta(p^0)\,\overline{p}\cdot\underline{p} - m^2)|\overline{p}\rangle\langle\overline{p}|\,, \tag{45}$$

where $\Theta(x)$ is the Heaviside step function (see appendix F) or with its positive definite counterpart

$$K_{\rm KG+} := J_{\rm KG+}^2 = \int d^4p\,\left(\Theta(p^0)\,\overline{p}\cdot\underline{p} - m^2\right)^2|\overline{p}\rangle\langle\overline{p}|\,. \tag{46}$$

Similarly, for the Dirac equation: we can directly translate (44) into (41) by identifying K with the operator

$$J_{\rm D} := \overline{\gamma}\cdot\underline{P} - m = \sum_{\sigma,\sigma'}\int d^4p\,(\overline{\gamma}_{\sigma,\sigma'}\cdot\underline{p} - m\,\delta_{\sigma,\sigma'})|\overline{p},\sigma\rangle\langle\overline{p},\sigma'|\,, \tag{47}$$

(which is not self-adjoint), or with its associated positive semi-definite counterpart

$$K_{\rm D} := J_{\rm D}^\dagger J_{\rm D} = \sum_{\sigma=1}^{4}\int d^4p\,\lambda_\sigma^2(\overline{p})\,|\phi_\sigma(\overline{p})\rangle\langle\phi_\sigma(\overline{p})|\,, \tag{48}$$

with $|\lambda_\sigma(\overline{p})|$ being the singular eigenvalues of $J_{\rm D}$ and the generalized vectors $|\phi_\sigma(\overline{p})\rangle$ forming an orthonormal set

$$\langle\phi_{\sigma'}(\overline{p}')|\phi_\sigma(\overline{p})\rangle = \delta_{\sigma,\sigma'}\,\delta^{(4)}(\overline{p}\,' - \overline{p})\,, \tag{49}$$

(see appendix G). One of the advantages of adopting the above definitions for the constraint operator K is that all of them are Lorentz invariant quantities (this is clearly evident for (47), while for (45) an explicitly proof for proper Lorentz transformations is given in appendix F). Accordingly, the elements of $\mathcal{H}_{\rm QM}$ identified by one observer O via equation (41) will be related with those assigned by the observer O' via the same unitary transformation (13) that links their state event descriptions, i.e.

$$|\Psi'_{\rm QM}\rangle = U_\Lambda|\Psi_{\rm QM}\rangle\,, \tag{50}$$

or equivalently

$$\Psi'_{\rm QM}(\overline{x},\sigma) = \sum_\sigma S^{-1}_{\sigma',\sigma}(\Lambda)\Psi_{\rm QM}(\Lambda^{-1}\overline{x},\sigma')\,, \tag{51}$$

(the spinless case being obtained by simply removing S and neglecting the σ terms), which via (39) properly describes how to relate the QM 3D+1 spinor wave-functions $\Psi'_{\rm QM}(\vec{x},\sigma|t)$ and $\Psi_{\rm QM}(\vec{x},\sigma|t)$ the observers assign to the *same* single-particle trajectory (see figure 1).

[10] A guideline, suggested by Wigner and Bargmann, is to consider as physical fields the ones that correspond to irreducible representations of the Poincaré group [38].

[11] An interaction with an external electromagnetic field can be described through the minimal coupling substitution of P^μ with $P^\mu + eA^\mu$, with e the particle charge and A^μ the em 4potential.).

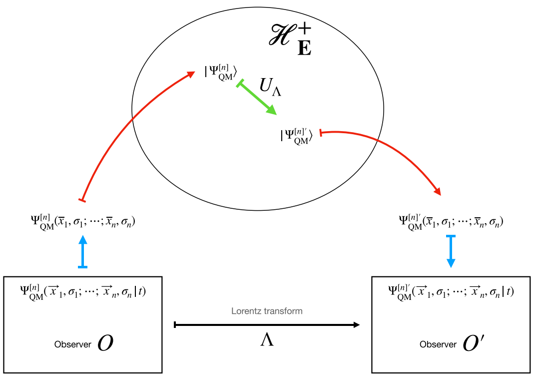

Figure 1. Schematic representation of the connection that links the 3D+1 spinor wave-functions $\Psi^{[n]}_{\rm QM}(\vec{x}_1,\sigma_1;\ldots;\vec{x}_n,\sigma_n|t)$ and $\Psi^{[n]\prime}_{\rm QM}(\vec{x}_1,\sigma_1;\ldots;\vec{x}_n,\sigma_n|t)$ of QM, that two observers O and O' assign to an n-particle state in their own reference frames: thanks to the QM/GEB correspondence (40) such connection can be expressed via the unitary mapping (50) (green arrow) that links the associated GEB distributions (54). The blue arrow elements represent the association of 3D+1 spinor wave-functions of QM with their 4D spinor GEB counterparts, given in (39) for $n=1$ and (52) for $n>1$; the red arrow elements instead represent the connection between 4D spinor wave-functions and the distributions of $\mathcal{H}_{\rm E}$.

3.2. Multi-event QM/GEB correspondence

To generalize the correspondence (40) to the multi-event case we need to address the problem that in QM the wave-function $\Psi^{[n]}_{\rm QM}(\vec{x}_1,\sigma_1;\ldots;\vec{x}_n,\sigma_n|t)$ of a n particle system is associated with n independent 3D spatial coordinates (plus possibly n spinor components) but with a single time-coordinate. It is hence not at all clear how to map such terms into elements (or distributions) of $\mathcal{H}_{\rm E}^{\otimes n}$ which instead possess n independent time coordinate values. In the case where the n particles are not interacting, we can use the fact that $\Psi^{[n]}_{\rm QM}(\vec{x}_1,\sigma_1;\ldots;\vec{x}_n,\sigma_n|t)$ can always be expressed as linear combinations of products of time-dependent single-particles, i.e.

$$\Psi^{[n]}_{\rm QM}(\vec{x}_1,\sigma_1;\ldots;\vec{x}_n,\sigma_n|t)=\sum_{\vec{\ell}}\alpha_{\vec{\ell}}\,\Psi^{(\ell_1)}_{\rm QM}(\vec{x}_1,\sigma_1|t)\cdots\Psi^{(\ell_n)}_{\rm QM}(\vec{x}_n,\sigma_n|t)\;, \tag{52}$$

where given $\vec{\ell}=(\ell_1,\ldots,\ell_n)$, $\alpha_{\vec{\ell}}$ are time-independent probability amplitudes, and where for $j=1,\ldots,n$, $\Psi^{(\ell_j)}_{\rm QM}(\vec{x}_j,\sigma_j|t)$ is the 3D+1 wave-function describing the evolution of the j-th particle of the system. Equation (52) is the key to generalize (40) as it allows us to formally associate $\Psi^{[n]}_{\rm QM}(\vec{x}_1,\sigma_1;\ldots;\vec{x}_n,\sigma_n|t)$ to a 4D spinor wave-function with n distinct time coordinates via the construction

$$\Psi^{[n]}_{\rm QM}(\bar{x}_1,\sigma_1;\ldots;\bar{x}_n,\sigma_n):=\sum_{\vec{\ell}}\alpha_{\vec{\ell}}\,\Psi^{(\ell_1)}_{\rm QM}(\vec{x}_1,\sigma_1|t_1)\cdots\Psi^{(\ell_n)}_{\rm QM}(\vec{x}_n,\sigma_n|t_n)\;, \tag{53}$$

and then using such term to identify the distribution $|\Psi_{\rm QM}\rangle$ via the identity

$$|\Psi^{[n]}_{\rm QM}\rangle:\overset{!}{=}\sum_{\sigma_1,\ldots,\sigma_2}\int d^4x_1\cdots\int d^4x_n\times\;\Psi^{[n]}_{\rm QM}(\bar{x}_1,\sigma_1;\ldots;\bar{x}_n,\sigma_n)\;|\bar{x}_1,\sigma_1;\ldots;\bar{x}_n,\sigma_n\rangle \tag{54}$$

(where, again, '!' is a reminder of the non-normalization). While the choice of the single-particles QM spinor wave-functions $\Psi^{(\ell_j)}_{\rm QM}(\vec{x}_j,\sigma_j|t)$ and of the coefficients $\alpha_{\vec{\ell}}$ entering in (52) are in general not unique, the vector (54) does not depend on such freedom ensuring that the connection between $\Psi^{[n]}_{\rm QM}(\vec{x}_1,\sigma_1;\ldots;\vec{x}_n,\sigma_n|t)$ and $|\Psi^{[n]}_{\rm QM}\rangle$ is one-to-one (see appendix H.1). Vice versa, given $|\Psi^{[n]}_{\rm QM}\rangle$ one can recover the QM spinor 3D wave-function $\Psi^{[n]}_{\rm QM}(\vec{x}_1,\sigma_1;\ldots;\vec{x}_n,\sigma_n|t)$ via the identity

$$\Psi^{[n]}_{\rm QM}(\vec{x}_1,\sigma_1;\ldots;\vec{x}_n,\sigma_n|t)=\langle\bar{x}_1,\sigma_1;\ldots;\bar{x}_n,\sigma_n|\Psi^{[n]}_{\rm QM}\rangle\Big|_{t_1=\cdots=t_n=t}\;. \tag{55}$$

As in the single-event case we can identify the distributions (54) by means of a geometric constraint (41) induced by an n-body operator $K^{[n]}$. To identify such a term we start form individual single particle constraint terms K_j that are explicitly positive semidefinite (i.e. $K_j\geqslant 0$), and take $K^{[n]}$ as their sum

$$K^{[n]}=\sum_{j=1}^{n}K_j\qquad\qquad(K_j\geqslant 0)\;. \tag{56}$$

The positivity requirement on the individual K_j is an important ingredient as it ensures that the kernel of $K^{[n]}$ coincides with the intersection of all the kernels of the single-particle constraints, i.e.

$$K^{[n]}|\Psi^{[n]}_{\rm QM}\rangle = 0 \Leftrightarrow K_j|\Psi^{[n]}_{\rm QM}\rangle = 0\;, \quad \forall j = 1,\ldots,n\;, \tag{57}$$

which automatically implies that the only acceptable solutions to (57) must have each individual particle evolving according to its own dynamical constraint (the model being interaction free for now). Observe also that as equation (56) is symmetric under exchange of the particle indexes it has no problem to act as constraint operator also in the case the particles are indistinguishable (in particular it does not mix the complete symmetric part $\mathcal{H}_{\bf E}^{(n,{\bf S})}$ of $\mathcal{H}_{\bf E}^{\otimes n}$ with the complete anti-symmetric part $\mathcal{H}_{\bf E}^{(n,{\bf A})}$). For instance, in the case of a Bosonic model governed by the positive energy KG equation (43), the positivity requirement on the K_j forces us to select (46) (instead of (45)) as the proper single particles terms: accordingly, for this model the n-body constraint operator $K^{[n]}$ can be identified with

$$K^{[n]}_{\rm KG+} := \sum_{j=1}^{n} (K_{\rm KG+})_j = \int d^4p_1 \cdots \int d^4p_n \sum_{j=1}^{n}\left(\Theta(p_j^0)\,\overline{p}_j\cdot \underline{p}_j - m^2\right)^2 |\overline{p}_1;\ldots;\overline{p}_n\rangle\langle\overline{p}_1;\ldots;\overline{p}_n|\;. \tag{58}$$

Similarly for Fermionic models we should identify the single-particle terms K_j with the operator (48) instead of (47). Accordingly in this case n-body constraint operator $K^{[n]}$ becomes

$$K^{[n]}_{\rm D} := \sum_{j=1}^{n}(K_{\rm D})_j = \sum_{\sigma_1}\int d^4p_1\cdots\sum_{\sigma_n}\int d^4p_n \sum_{j=1}^{n}\lambda^2_{\sigma_j}(\overline{p}_j)|\phi_{\sigma_1}(\overline{p}_1);\ldots;\phi_{\sigma_n}(\overline{p}_n)\rangle\langle\phi_{\sigma_1}(\overline{p}_1);\ldots;\phi_{\sigma_n}(\overline{p}_n)|\;. \tag{59}$$

If the QM dynamical equations that rule the equation of motion of the particles are relativistically covariant as in the cases of equations (58) and (59), then the identities (50) and (51) that in the single particle case allows us to connect the distributions of the observers O and O', translate into

$$|\Psi^{[n]\,\prime}_{\rm QM}\rangle = U_\Lambda^{\otimes n}|\Psi^{[n]}_{\rm QM}\rangle\;, \tag{60}$$

and

$$\Psi^{[n]\,\prime}_{\rm QM}(\overline{x}_1,\sigma_1;\ldots;\overline{x}_n,\sigma_n) = \sum_{\sigma_1,\ldots,\sigma_n} S^{-1}_{\sigma_1',\sigma_1}(\Lambda)\cdots S^{-1}_{\sigma_n',\sigma_n}(\Lambda)\Psi^{[n]}_{\rm QM}(\Lambda^{-1}\overline{x}_1,\sigma_1';\ldots;\Lambda^{-1}\overline{x}_n,\sigma_n')\;, \tag{61}$$

respectively. Notice also that setting $t_1 = \cdots = t_n = t$ in the last one, invoking equations (52) and (53) we obtain the connection between the spinor 3D wave-functions of QM $\Psi^{[n]}_{\rm QM}(\vec{x}_1,\sigma_1;\ldots;\vec{x}_n,\sigma_n|t)$ and $\Psi^{[n]\,\prime}_{\rm QM}(\vec{x}_1,\sigma_1;\ldots;\vec{x}_n,\sigma_n|t)$, that O and O' would assign to the *same* quantum trajectory of the n particles—see figure 1 for a schematic representation of this identity and appendix H.2 for a technical discussion.

3.3. QM/GEB correspondence in Fock space

In this section we generalize the correspondence (40) to the Fock space representation of GEB discussed in section 2.2. At variance with what we did in sections 3.1 and 3.2, here we start by first introducing the constraint operator (41), and then show that the associated solutions can be directly connected to those of QFT.

3.3.1. Constraint operators

To connect GEB to QFT, start by considering the case of a Bosonic model where each individual particle evolves according to the positive energy KG equation (43). As we have seen in the previous section, in the first quantization formalism the constraint operator of the model is provided by (58). When the total number of particles is fixed to n, the first quantization version of the constraint operator (58) assigns a contribution $\left(\Theta(p^0)\,\overline{p}\cdot\underline{p} - m^2\right)^2$ to each particle with 4-momentum $\overline{p}$, specifically

$$K^{[n]}_{\rm KG+}|S(\overline{p}_1;\ldots;\overline{p}_n)\rangle = \sum_{j=1}^{n}\left(\Theta(p_j^0)\,\overline{p}_j\cdot\underline{p}_j - m^2\right)^2|S(\overline{p}_1;\ldots;\overline{p}_n)\rangle\;, \tag{62}$$

with $|S(\overline{p}_1;\ldots;\overline{p}_n)\rangle$ the symmetric version of $|\overline{p}_1;\ldots;\overline{p}_n\rangle$ (see appendix C). Exploiting the correspondence (C5), Equation (62) can now be turned into its second quantization form by identifying $K^{[n]}_{\mathrm{KG}+}$ with the Fock operator

$$K^{(\mathrm{Fock})}_{\mathrm{KG}^+} := \int d^4p \left[\Theta(p^0)\,\overline{p}\cdot\underline{p} - m^2\right]^2 a^\dagger_{\overline{p}} a_{\overline{p}} \,, \tag{63}$$

with $a^\dagger_{\overline{p}}$, $a_{\overline{p}}$ the creation and annihilation operators that obey the canonical commutation rules (30).

In the Fermionic case we proceed in similar fashion. In this case, from (59), we see we need to introduce a Fock number operator that counts how many particles of the system are in single particle states described by the vectors $|\phi_\sigma(\overline{p})\rangle$. To construct such a term we introduce a new collection of annihilation operators

$$a_{\phi_\sigma(\overline{p})} := \sum_{\sigma'=1}^{4} u^*_{\sigma',\sigma}(\vec{p})\, a_{\overline{p},\sigma'} \,, \tag{64}$$

with $u_{\sigma,\sigma'}(\overline{p})$ the unitary matrices that connects the vectors $|\phi_\sigma(\overline{p})\rangle$ with the vectors $|\overline{p},\sigma\rangle$ (see appendix G). By construction they fulfill the same anti-commutation rules of $a^\dagger_{\overline{p},\sigma}$ and $a_{\overline{p},\sigma}$, i.e.

$$\{a_{\phi_\sigma(\overline{p})}, a^\dagger_{\phi'_\sigma(\overline{p}')}\} = \delta_{\sigma,\sigma'}\,\delta^{(4)}(\overline{p}-\overline{p}')\,,\ \{a_{\phi_\sigma(\overline{p})}, a_{\phi'_\sigma(\overline{p}')}\} = 0\,, \tag{65}$$

so that $a^\dagger_{\phi_\sigma(\overline{p})} a_{\phi_\sigma(\overline{p})}$ is exactly the number operator we are looking for. Accordingly we can construct the Fock counterpart of (59) by taking

$$K^{(\mathrm{Fock})}_{\mathrm{D}} := \sum_{\sigma=1}^{4} \int d^4p\, \lambda^2_\sigma(\overline{p})\, a^\dagger_{\phi_\sigma(\overline{p})} a_{\phi_\sigma(\overline{p})} \,, \tag{66}$$

which, via equation (64), can also be expressed as

$$K^{(\mathrm{Fock})}_{\mathrm{D}} = \sum_{\sigma',\sigma''=1}^{4} \int d^4p\, D_{\sigma',\sigma''}(\overline{p})\, a^\dagger_{\overline{p},\sigma'} a_{\overline{p},\sigma''}(\overline{p}) \,, \tag{67}$$

with

$$\begin{aligned} D_{\sigma',\sigma''}(\overline{p}) &:= \sum_{\sigma=1}^{4} u_{\sigma',\sigma}(\vec{p})\lambda^2_\sigma(\overline{p}) u^*_{\sigma'',\sigma}(\overline{p}) = \sum_{\sigma=1}^{4} (\overline{\gamma}^\dagger_{\sigma',\sigma}\cdot\underline{p})(\overline{\gamma}_{\sigma,\sigma''}\cdot\underline{p}) + m^2\delta_{\sigma',\sigma''} \\ &\quad - m(\overline{\gamma}^\dagger_{\sigma',\sigma''} + \overline{\gamma}_{\sigma',\sigma''})\cdot\underline{p}\,, \end{aligned} \tag{68}$$

where we used (G4) and (G8).

3.3.2. Connection with the QFT solutions

Here we analyze the solutions of the geometric constraint (41) that follow from the definitions of $K^{(\mathrm{Fock})}_{\mathrm{KG}^+}$ and $K^{(\mathrm{Fock})}_{\mathrm{D}}$ given in the previous section, i.e.

$$\int d^4p \left[\Theta(p^0)\overline{p}\cdot\underline{p} - m^2\right]^2 a^\dagger_{\overline{p}} a_{\overline{p}}\, |\Psi_{\mathrm{QM}}\rangle = 0\,, \tag{69}$$

for the Bosonic model, and

$$\sum_{\sigma=1}^{4} \int d^4p\, \lambda^2_\sigma(\overline{p})\, a^\dagger_{\phi_\sigma(\overline{p})} a_{\phi_\sigma(\overline{p})}\, |\Psi_{\mathrm{QM}}\rangle = 0\,, \tag{70}$$

for the Dirac one.

It is clear that in both scenarios the no-event state, that in the theory is represented by 4D vacuum state $|\Psi_{\mathrm{QM}}\rangle = |0\rangle_4$ is an allowed solution. It corresponds to the trivial case of no particles (Bosons for (69) or Fermions for (70)) at all times. To discuss the other solutions in what follow we shall address first the Bosonic case that allows for some simplification due to the absence of spinor components.

3.3.2.1. Bosonic model

Express the vector $|\Psi_{\rm QM}\rangle$ that appears on the r.h.s. of equation (69) as the one given in (34) (with no spin), namely

$$|\Psi_{\rm QM}\rangle \stackrel{!}{:=} \sum_n \frac{\alpha_n}{\sqrt{n!}} \left[\prod_{j=1}^n \int d^4x_j\, a^\dagger_{\bar{x}_j}\right] |0\rangle_4 \Psi^{[n]}_{\rm QM}(\bar{x}_1,\ldots,\bar{x}_n) = \sum_n \frac{\alpha_n}{\sqrt{n!}} \left[\prod_{j=1}^n \int d^4p_j\, a^\dagger_{\bar{p}_j}\right] |0\rangle_4 \tilde{\Psi}^{[n]}_{\rm QM}(\bar{p}_1;\ldots;\bar{p}_n) \tag{71}$$

with $\Psi^{[n]}_{\rm QM}(\bar{x}_1,\ldots,\bar{x}_n)$ and $\tilde{\Psi}^{[n]}_{\rm QM}(\bar{p}_1;\ldots;\bar{p}_n)$ connected via 4D Fourier transform:

$$\Psi^{[n]}_{\rm QM}(\bar{x}_1,\cdots) = \left[\prod_{j=1}^n \int \frac{d^4p_j}{4\pi^2} e^{-i\bar{p}_j\cdot\underline{x}_j}\right] \tilde{\Psi}^{[n]}_{\rm QM}(\bar{p}_1,\cdots)\,. \tag{72}$$

The functional dependence of the operator $K^{\rm (Fock)}_{\rm KG+}$ upon the number operator $a^\dagger_{\bar{p}}a_{\bar{p}}$ suggests to analyze equation (69) in the 4-momentum representation (71) instead of the position representation (72). Indeed when acting on $a^\dagger_{\bar{p}_1}\cdots a^\dagger_{\bar{p}_n}|0\rangle_4$, $a^\dagger_{\bar{p}}a_{\bar{p}}$ generates a multiplicative factor $\sum_{j=1}^n \delta(\bar{p}_j-\bar{p})$ that allows us to translate the constraint (69) into a constraint on the momentum-representation (33) of the wavefunction as

$$\tilde{\Phi}^{[n]}_{\rm QM}(\bar{p}_1;\ldots;\bar{p}_n)\left(\sum_{j=1}^n \left(\Theta(p_j^0)\bar{p}_j\cdot\underline{p}_j - m^2\right)^2\right) = 0\,. \tag{73}$$

Such equation forces $\tilde{\Psi}^{[n]}_{\rm QM}(\bar{p}_1;\ldots;\bar{p}_n)$ to have support only for values of the $\bar{p}_j$ momenta that satisfy the on-shell condition $\bar{p}_j\cdot\underline{p}_j = m^2$ with $p_j^0 \geqslant 0$. Specifically using

$$\delta(\bar{p}\cdot\underline{p} - m^2) = [\delta(p^0+E_p)+\delta(p^0-E_p)]/(2E_p)\,, \tag{74}$$

$$E_p := +\sqrt{|\vec{p}|^2+m^2}\,, \tag{75}$$

we can express the most general solution of (73) as

$$\tilde{\Psi}^{[n]}_{\rm QM}(\bar{p}_1;\ldots;\bar{p}_n) = \left(\Pi_{j=1}^n \delta\left(\bar{p}_j\cdot\underline{p}_j - m^2\right)\right) f^{[n]}(\bar{p}_1;\ldots;\bar{p}_n) = \left(\Pi_{j=1}^n \delta(p_j^0 - E_{p_j})\right)\frac{f^{[n]}(\vec{p}_1;\ldots;\vec{p}_n)}{2E_{p_1}\cdots 2E_{p_n}}, \tag{76}$$

$$\text{with } f^{[n]}(\vec{p}_1;\ldots;\vec{p}_n) := f^{[n]}(\bar{p}_1;\ldots;\bar{p}_n)\Big|_{p_j^0=E_{p_j}} \tag{77}$$

and $f^{[n]}(\bar{p}_1;\ldots;\bar{p}_n)$ an arbitrary function which nullifies for $p_j^0<0$ and that, in virtue of the implicit symmetry of (72), can always be forced to be completely symmetric under exchange of the indexes. The position representation (71) of the solution can now be recovered replacing equation (76) into equation (72), i.e.

$$\Psi^{[n]}_{\rm QM}(\bar{x}_1,\ldots,\bar{x}_n) = \left[\prod_j \int \frac{d^3p_j}{(2\pi)^{3/2}} e^{i(\vec{p}_j\cdot\vec{x}_j - E_{p_j}t_j)}\right] \frac{f^{[n]}(\vec{p}_1,\ldots,\vec{p}_n)}{\sqrt{8\pi}E_{p_1}\cdots\sqrt{8\pi}E_{p_n}}\,. \tag{78}$$

To put these solutions in correspondence with the QFT solutions of the corresponding Bosonic KG field equation we observe that in the Schrödinger picture, the general QFT solutions of a (positive-energy) Bosonic KG field equation writes as [54]

$$|\psi_{\rm QM}(t)\rangle = \sum_n \frac{\beta_n}{\sqrt{n!}}\left[\prod_j \int d^3x_j\, c^\dagger_{\vec{x}_j}\right] |0\rangle_3 \Psi^{[n]}_{\rm QM}(\vec{x}_1,\ldots,\vec{x}_n|t)\,, \tag{79}$$

where β_n are normalized amplitude probabilities, $|0\rangle_3$ is the 3D vacuum state of the field (not to be confused with the 4-vacuum state $|0\rangle_4$ of GEB) and the $c^\dagger_{\vec{x}}$'s are Bosonic creation operators fulfilling the equal-time canonical commutation rules

$$[c_{\vec{x}}, c^\dagger_{\vec{x}\,'}] = \delta^{(3)}(\vec{x}-\vec{x}\,'),\ [c_{\vec{x}}, c_{\vec{x}\,'}] = 0\,. \tag{80}$$

In the above expression $\Psi^{[n]}_{\rm QM}(\vec{x}_1,\ldots,\vec{x}_n|t)$ are (observer dependent) 3D+1 wave-functions that (under proper normalization conditions) define the joint probabilities of finding at time t, n particles in $\vec{x}_1,\ldots,\vec{x}_n$: their temporal dependence is fixed by the single-particle dispersion relation defined in equation (75) and is computed in equation (H6). Our goal is to show that the GEB solutions (72) with $\tilde{\Psi}^{[n]}_{\rm QM}(\overline{p}_1;\ldots;\overline{p}_n)$ as in equation (76) can be put in correspondence with (79) by taking $\beta_n=\alpha_n$ and setting

$$\tilde{\psi}^{[n]}_{\rm QM}(\vec{p}_1,\ldots,\vec{p}_n):=\frac{f^{[n]}(\vec{p}_1,\ldots,\vec{p}_n)}{\sqrt{8\pi}E_{p_1}\cdots\sqrt{8\pi}E_{p_n}}\,, \tag{81}$$

in equation (H6). To verify this fact notice that for fixed value of $n\geqslant 1$ one can invoke equations (C3)–(C5) to map the 4D wave-function $\Psi^{[n]}_{\rm QM}(\overline{x}_1,\ldots,\overline{x}_n)$ of equation (78) onto a QM 3D+1 wave-function of n Bosonic particles via equation (55): this exactly reproduces the QFT solution $\Psi^{[n]}_{\rm QM}(\vec{x}_1,\ldots,\vec{x}_n|t)$ of equation (H6) when we impose (81).

3.3.2.2. Fermionic model

Similar considerations apply to the Fermionic case. Here the functional dependence of the constraint operator $K^{(\rm Fock)}_{\rm D}$ upon the number operator $a^\dagger_{\phi_\sigma(\overline{p})}a_{\phi_\sigma(\overline{p})}$ suggests to expand the general solution (34), namely

$$|\Psi_{\rm QM}\rangle:\overset{!}{=}\sum_n\frac{\alpha_n}{\sqrt{n!}}\left[\prod_j\sum_{\sigma_j=1}^4\int d^4x_j\,a^\dagger_{\overline{x}_j,\sigma_j}\right]\Psi^{[n]}_{\rm QM}(\overline{x}_1,\sigma_1;\ldots;\overline{x}_n,\sigma_n)|0\rangle_4\,, \tag{82}$$

in terms of the creation operators $a^\dagger_{\phi_\sigma(\overline{p})}$, i.e.

$$|\Psi_{\rm QM}\rangle=\sum_n\frac{\alpha_n}{\sqrt{n!}}\left[\prod_j\sum_{\sigma_j=1}^4\int d^4p_j\,a^\dagger_{\phi_{\sigma_j}(\overline{p}_j)}\right]\tilde{\Psi}^{[n]}_{\rm QM}(\phi_{\sigma_1}(\overline{p}_1);\ldots;\phi_{\sigma_n}(\overline{p}_n))|0\rangle_4\,, \tag{83}$$

with the spinor wave-functions $\Psi^{[n]}_{\rm QM}(\overline{p}_1,\sigma_1;\ldots;\overline{p}_n,\sigma_n)$ that are connected with those of the 4-position representation via the identity

$$\Psi^{[n]}_{\rm QM}(\overline{x}_1,\sigma_1;\ldots;\overline{x}_n,\sigma_n)=\left[\prod_j\int\frac{d^4p_j}{4\pi^2}e^{-i\overline{p}_j\cdot\overline{x}_j}\sum_{\sigma_j'=1}^4u_{\sigma_j',\sigma_j}(\vec{p}_j)\right]\tilde{\Psi}^{[n]}_{\rm QM}(\phi_{\sigma_1'}(\overline{p}_1);\ldots;\phi_{\sigma_n'}(\overline{p}_n))\,. \tag{84}$$

Replacing (83) into (70) we get

$$\tilde{\Psi}^{[n]}_{\rm QM}(\phi_{\sigma_1}(\overline{p}_1);\ldots;\phi_{\sigma_n}(\overline{p}_n))\left(\sum_{j=1}^n\lambda^2_{\sigma_j}(\overline{p}_j)\right)=0\,, \tag{85}$$

which due to the positivity of the terms $\lambda^2_{\sigma_j}(\overline{p}_j)$ has solutions of the form

$$\begin{aligned}\tilde{\Psi}^{[n]}_{\rm QM}(\phi_{\sigma_1}(\overline{p}_1);\ldots;\phi_{\sigma_n}(\overline{p}_n))&=\left(\Pi_{j=1}^n\delta(\lambda_{\sigma_j}(\overline{p}_j))\right)f^{[n]}(\overline{p}_1,\sigma_1;\ldots;\overline{p}_n,\sigma_n)\\&=\left(\Pi_{j=1}^n\delta(\overline{p}^0_j-E^{(\sigma_j)}_{p_j})\right)f^{[n]}(\vec{p}_1,\sigma_1;\ldots;\vec{p}_n,\sigma_n)\,,\end{aligned}$$

where $E^{(\sigma)}_p=-E_p$ for $\sigma=1,3$ and $E^{(\sigma)}_p=E_p$ for $\sigma=2,4$ with $E_p=\sqrt{|\vec{p}|^2+m^2}$, see equation (G7), and where $f^{[n]}(\overline{p}_1,\sigma_1;\ldots;\overline{p}_n,\sigma_n)$ are arbitrary functions that can always be assumed to completely anti-symmetric under particle indexes exchange, and where finally

$$f^{[n]}(\vec{p}_1,\sigma_1;\ldots;\vec{p}_n,\sigma_n):=f^{[n]}(\overline{p}_1,\sigma_1;\ldots;\overline{p}_n,\sigma_n)\Big|_{p^0_j=E^{(\sigma_j)}_{p_j}}\,. \tag{86}$$

Substituting this into (84) we hence obtain

$$\Psi^{[n]}_{\rm QM}(\overline{x}_1,\sigma_1;\ldots;\overline{x}_n,\sigma_n)=\left[\prod_j\int\frac{d^3p_j}{(2\pi)^{3/2}}\sum_{\sigma_j'=1}^4\frac{u_{\sigma_j',\sigma_j}(\vec{p}_j)}{\sqrt{2\pi}}\times e^{i(\vec{p}_j\cdot\vec{x}_j-E^{(\sigma_j')}_{p_j}t_j)}\right]f^{[n]}(\vec{p}_1,\sigma_1';\ldots;\vec{p}_n,\sigma_n')\,. \tag{87}$$

To establish a formal correspondence between (82) and the solutions of QFT we observe that the 3D+1 spinor wave-function of n Fermionic particles that obey the Dirac equation is given by vectors of the form

$$|\psi_{\rm QM}(t)\rangle=\sum_n \frac{\beta_n}{\sqrt{n!}}\left[\prod_j \sum_{\sigma_j=1}^{4}\int d^3x_j c^\dagger_{\vec{x}_j,\sigma_j}\right]\Psi^{[n]}_{\rm QM}(\vec{x}_1,\sigma_1;\ldots;\vec{x}_n,\sigma_n|t)|0\rangle_3, \tag{88}$$

with creation operators that obey equal-time anti-commutation rules, i.e.

$$\{c_{\vec{x},\sigma},c^\dagger_{\vec{x}\,',\sigma'}\}=\delta^{(3)}_{\sigma,\sigma'}\,\delta(\vec{x}-\vec{x}\,'),\ \{c_{\vec{x},\sigma},c_{\vec{x}\,',\sigma'}\}=0\,. \tag{89}$$

and 3D+1 spinor wave-functions $\Psi^{[n]}_{\rm QM}(\vec{x}_1,\sigma_1;\ldots;\vec{x}_n,\sigma_n|t)$ defined in equation (H9). Invoking once more equations (C3)–(C5) we can hence conclude that equation (84) corresponds to (88) by setting $\beta_n=\alpha_n$ and $\phi^{[n]}(\vec{p}_1,\sigma_1';\ldots;\vec{p}_n,\sigma_n')$ of equation (H9) equal to $f^{[n]}(\vec{p}_1,\sigma_1';\ldots;\vec{p}_n,\sigma_n')$. Note that here we also consider possible entanglement between different spinor components, whence the n integrals in (88), which are not usually included in QFT treatments. Superpositions of a particle and an antiparticle are typically considered unphysical because one supposes that superselection rules will prevent them. However, such states have been proposed [55, 56], suggesting that superselection rules are never fundamental, but only practical limitations.

3.3.3. Lorentz transform

We conclude the section stressing that also in the Fock formalization of the model, the constrained operators are Lorentz invariant quantities, allowing us to extend the identity (35) also to the elements $|\Psi_{\rm QM}\rangle$ of $\mathcal{H}_{\rm QM}$, i.e.

$$|\Psi'_{\rm QM}\rangle=\mathcal{U}_\Lambda|\Psi_{\rm QM}\rangle\,, \tag{90}$$

indicating that in GEB Lorentz transforms can be done entirely using unitary representations of the Lorentz group as described above, according to Wigner's prescription for symmetry transformations, entirely at the kinematic level. This is clearly different to what happens in QFT where we quantize 'on shell', namely, the quantization procedure contains the dynamics. This implies for instance that the state $c^\dagger_{\vec{p}}|0\rangle_3$ lives in a $\mathcal{L}^2(\mathbb{R}^3)$ space of on-shell states, namely states whose energy is E_p. In order to Lorentz transform such state, one must first derive the new hyperboloid that satisfies $E_p'^2-p'^2=m^2$ in the new frame and *then* quantize in the new frame obtaining $c^\dagger_{\vec{p}\,'}|0\rangle_3$ in the new frame (the vacuum being Lorentz invariant).

4. Conclusions

In conclusion we presented an alternative framework (GEB) for special relativistic QM. The full axiomatic structure of QM (e.g. its statistical interpretation through the Born rule) is applied covariantly. The quantization is performed axiomatically in GEB, constructing a Hilbert space for events, rather than the customary QFT approach of quantizing the solutions of the dynamical equations. The usual textbook relativistic QM and QFT are obtained by conditioning over the temporal degrees of freedom of the GEB event states.

We have not considered interactions here: as in relativistic QM and QFT, interactions pose significant additional challenges (understatement!) that will be tackled in future work. Other covariant approaches that derive from Dirac forms [5] typically work only for free particles (since the Hamiltonian ends up in the boost generators): the 'no-interaction theorem' [57–59]. Our approach might, instead, be able to consider interactions, since we impose the dynamics only through a constraint, which is a procedure known to bypass the no-interaction theorem [24, 25, 39, 51, 60]. Moreover, GEB does not employ a quantization on the free-field dynamical equation solutions, so it might perhaps be able to describe interacting fields without the usual perturbative approach, if we will ever be able to devise appropriate, solvable, constraint equations. GEB replies affirmatively to a question raised by Kuchař [61] on whether the constraint formalism is able to describe localized relativistic particles (a completely different solution, based on the Newton–Wigner mechanism, is in [62]). Finally, it can treat situations that do not admit a Hamiltonian formulation [19] (such as solutions to the KG equation without positive-energy restriction, appendix F, or generic solutions of Einstein's field equations [6, 50, 63, 64]), since, as shown above, the constraint procedure does not require Hamiltonians to describe the dynamics.

Of course, we do not claim that QFT is inadequate: the formulation provided here is, as shown above, a (slight) extension of it and in all situations considered in this paper an equivalent QFT description exists (*mutatis mutandis*). It may perhaps be used to clarify some longstanding problems, such as Haag's theorem [65] or particle localization [65, 66] by recognizing that a localized particle (that stays localized for a period of time) is not a physical state (it does not satisfy the constraints), but it can be connected to a kinematic state that can be used as an eigenstate of an observable.

We believe that GEB opens new exciting avenues.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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Appendix A. Notation and conventions used

A.1. Physical Units

We use natural units setting $\hbar = 1$ and $c = 1$.

A.2. Spacetime coordinates

To represent 4D real vectors we use the notation

$$\overline{a} := (a^0, a^1, a^2, a^3) = (a^0, \vec{a}) \,, \tag{A1}$$

with a^0 the time-like component and $\vec{a}$ the associated space-like 3D vector

$$\vec{a} := (a^1, a^2, a^3) \,. \tag{A2}$$

Greeks labels are employed to indicate the four components of $\overline{a}$, and roman labels to indicate the three components of $\vec{a}$; e.g. a^μ with $\mu = \{0, 1, 2, 3\}$ indicates the μ-th term of $\overline{a}$, while a^i with $i = \{1, 2, 3\}$ indicates the i-th term of $\vec{a}$. Lower indexes 4D vectors are defined as

$$\underline{a} := (a_0, a_1, a_2, a_3) = (a^0, -\vec{a}) \,, \tag{A3}$$

which are connected with their upper indexes counterpart via the transformations

$$\underline{a} = \eta\, \overline{a} \,, \qquad \overline{a} = \eta\, \underline{a} \,, \tag{A4}$$

with η the 4×4 diagonal matrix

$$\eta := \mathrm{diag}(1, -1, -1, -1) \,, \tag{A5}$$

defining the metric tensor of the theory whose elements are represented with the symbol $\eta^{\mu\nu} = \eta_{\mu\nu}$. Recall next that given Λ a 4×4 real matrix associated to a generic Lorentz transformation we have

$$\Lambda^T \eta \Lambda = \eta \,, \tag{A6}$$

from which it follows that given the 4-vectors $\overline{a}$ and $\overline{b}$ the product

$$\overline{a} \cdot \underline{b} := \overline{a} \eta \overline{b} = \sum_{\mu=0}^{3} a^\mu b_\mu \,, \tag{A7}$$

is an invariant quantity, i.e. $\overline{a} \cdot \underline{b} = \overline{a}' \cdot \underline{b}'$ with $\overline{a}' = \Lambda \overline{a}$ and $\overline{b}' = \Lambda \overline{b}$ (notice that the same term can also be computed as $\underline{a} \eta \underline{b}$ or as $\underline{a} \cdot \overline{b}$).

Special examples of 4-vectors are provided by the 4-position and 4-momentum

$$\overline{x} := (t, \vec{x}) \ , \qquad \overline{p} := (p^0, \vec{p}) \ , \tag{A8}$$

by the associated differential term $\partial^\mu = \frac{\partial}{\partial x_\mu}$, i.e.

$$\overline{\partial} = (\partial^0, \partial^1, \partial^2, \partial^3) = (\partial/\partial t, -\vec{\nabla}) \ . \tag{A9}$$

Then,

$$\begin{aligned}
X^\mu &= \int d^4x\, x^\mu |\underline{x}\rangle\langle\underline{x}|, \ P^\mu = \int d^4p\, p^\mu |\underline{p}\rangle\langle\underline{p}| \Rightarrow \\
\langle\underline{x}|\underline{p}\rangle &= e^{-ip^\mu x_\mu}/(4\pi^2), \ \text{namely} \\
\langle x^0|p^0\rangle &= \langle t|E\rangle = \frac{e^{-iEt}}{\sqrt{2\pi}}, \ \langle x^1|p^1\rangle = \langle x|p_x\rangle = \frac{e^{+ip_x x}}{\sqrt{2\pi}}, \cdots \\
|\underline{x}\rangle \equiv c_x^\dagger|0\rangle_4 &= \int \frac{d^4p}{4\pi^2} e^{i\underline{px}}|p\rangle = \int \frac{d^4p}{4\pi^2} e^{i\underline{px}} c_p^\dagger|0\rangle_4 \\
c_x^\dagger &= \int \frac{d^4p}{4\pi^2} e^{i\underline{px}}\, c_p^\dagger, \ c_x = \int \frac{d^4p}{4\pi^2} e^{-i\underline{px}}\, c_p \\
c_{\vec{x}}^\dagger &= \int \frac{d^3p}{(2\pi)^{3/2}} e^{-i\vec{p}\cdot\vec{x}}\, c_{\vec{p}}^\dagger, \ c_{\vec{x}} = \int \frac{d^3p}{(2\pi)^{3/2}} e^{i\vec{p}\cdot\vec{x}}\, c_{\vec{p}} \ .
\end{aligned} \tag{A10}$$

The γ^μ matrices are given by

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \ , \tag{A11}$$

with σ^i being the Pauli operators.

Appendix B. Connecting different reference frames

Consider first the simple case of a single spin-less event space. Let O and O' be two inertial observer whose coordinates are linked as in equation (11) of the main text with the 4×4 matrix Λ representing an element of the Lorentz group. Let $\Phi(\overline{x})$ the wave-function of a state event S as described by O. To show that the observer O' in his reference frame will describe it as the function $\Phi'(\overline{x})$ of (12) assume that $\Phi(\overline{x})$ gets its maximum value $\Phi_{\max}$ for $\overline{x} = \overline{x}_0$, i.e. $\Phi_{\max} = \Phi(\overline{x}_0)$. The observer O' will assign to such point the coordinate $\overline{x}_0' = \Lambda\overline{x}_0$ that represents the value at which $\Phi'(\overline{x})$ reaches its maximum, i.e.

$$\Phi'(\overline{x}_0') = \Phi'(\Lambda\overline{x}_0) = \Phi_{\max} = \Phi(\overline{x}_0) \Longrightarrow \Phi'(\Lambda\overline{x}_0) = \Phi(\overline{x}_0),$$

which leads exactly to (12).

Let us now introduce the vectors $|\Phi\rangle$ and $|\Phi'\rangle$ of $\mathcal{H}_{\mathrm{E}}$ that O and O' will assign to the state S, i.e.

$$|\Phi\rangle = \int d^4x\, \Phi(\overline{x})|\overline{x}\rangle \ , \tag{B1}$$

$$|\Phi'\rangle = \int d^4x\, \Phi'(\overline{x})|\overline{x}\rangle = \int d^4x\, \Phi(\Lambda^{-1}\overline{x})|\overline{x}\rangle = \int d^4x\, \Phi(\overline{x})|\Lambda\overline{x}\rangle \ , \tag{B2}$$

(Figure 2). By direct inspection one can be easily verify that these vectors fulfill the identity (13) of the main text by identifying the unitary transformation U_Λ with the operator

$$U_\Lambda = \int d^4x |\Lambda\overline{x}\rangle\langle\overline{x}| = \int d^4x |\overline{x}\rangle\langle\Lambda^{-1}\overline{x}| \ , \tag{B3}$$

$$U_\Lambda^\dagger = \int d^4x |\overline{x}\rangle\langle\Lambda\overline{x}| = \int d^4x |\Lambda^{-1}\overline{x}\rangle\langle\overline{x}| \ , \tag{B4}$$

where the second identity in the first line follows by a simple chance of integration variables, while the second line is obtained by taking the adjoint of the first. Notice that U_Λ and $U_\Lambda^\dagger$ verify the conditions

$$U_\Lambda|\overline{x}\rangle = |\Lambda\overline{x}\rangle \ , \qquad \langle\overline{x}|U_\Lambda = \langle\Lambda^{-1}\overline{x}| \ , \tag{B5}$$

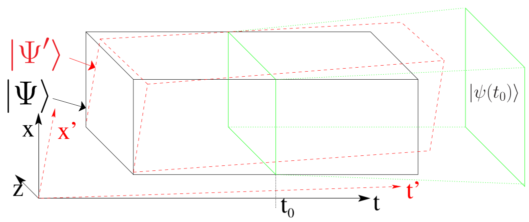

Figure 2. Figurative representation of the state $|\Psi\rangle$, its time-conditioning at time t_0 that gives the state $|\psi(t_0)\rangle$, and its relativistic boost $|\Psi'\rangle$ to the reference R' (dashed lines). The boosted reference (x', t') is obtained through a hyperbolic transformation (Lorentz transform) from the (x, t) reference, pictorially represented with the dashed lines. The foliation in the (x, t) reference gives the usual (conditioned) state $|\psi(t_0)\rangle$ of textbook quantum mechanics. A similar foliation in the (x', t') reference (not pictured) is required for the quantum state at time t' in the new reference.

$$U_\Lambda^\dagger|\overline{x}\rangle = |\Lambda^{-1}\overline{x}\rangle\,, \qquad \langle\overline{x}|U_\Lambda^\dagger = \langle\Lambda\overline{x}|\,, \tag{B6}$$

which represent the counterparts of (11) at the level of the generalized eigenstates of the position operator $\overline{X}$. Analogously for the generalized eigenvectors of the momentum operator $\overline{P}$ we get

$$U_\Lambda|\overline{p}\rangle = |\Lambda\overline{p}\rangle\,, \qquad \langle\overline{p}|U_\Lambda = \langle\Lambda^{-1}\overline{p}|\,, \tag{B7}$$

$$U_\Lambda^\dagger|\overline{p}\rangle = |\Lambda^{-1}\overline{p}\rangle\,, \qquad \langle\overline{p}|U_\Lambda^\dagger = \langle\Lambda\overline{p}|\,. \tag{B8}$$

The first for instance can be derived recalling equation (4) and observing that

$$U_\Lambda|\overline{p}\rangle = \int \tfrac{d^4x}{4\pi^2}\, e^{-i\overline{x}\cdot\underline{p}}|\Lambda\overline{x}\rangle = \int \tfrac{d^4x}{4\pi^2}\, e^{-i(\Lambda^{-1}\overline{x})\cdot\underline{p}}|\overline{x}\rangle = |\Lambda\overline{p}\rangle\,, \tag{B9}$$

where in the last identity we exploit the invariance of the product (A7) under Lorentz transform, i.e. $\overline{a}\cdot\underline{b} = \overline{a}'\cdot\underline{b}'$, for $\overline{a}' = \Lambda\overline{a}$ and $\overline{b}' = \Lambda\overline{b}$.

Consider next the expectation values of a generic operator Θ on S. The observer O will compute this as

$$\langle\Theta\rangle = \langle\Phi|\Theta|\Phi\rangle\,, \tag{B10}$$

while O' will see this as

$$\langle\Theta\rangle' = \langle\Phi'|\Theta|\Phi'\rangle = \langle\Phi|U_\Lambda^\dagger\Theta U_\Lambda|\Phi\rangle\,, \tag{B11}$$

which of course needs not to be the same as $\langle\Theta\rangle$. Notice that we can also rewrite $\langle\Theta\rangle' = \langle\Phi|\Theta'|\Phi\rangle$ where now

$$\Theta' = U_\Lambda^\dagger\Theta U_\Lambda\,, \tag{B12}$$

is a sort of 4D 'Heisenberg-picture' that allows us to transform the operators instead of the states in moving from the reference frame of O to the one by O'. In particular we shall say that Θ is invariant under Lorentz transformations if $\Theta' = \Theta$ for all choices of Λ, i.e.

$$U_\Lambda^\dagger\Theta U_\Lambda = \Theta\,, \qquad \forall\Lambda\,, \tag{B13}$$

while, given a collection of operators A^0, A^1, A^2, A^3 we shall call $\overline{A} = (A^0, A^1, A^2, A^3)$ a vectorial operator if

$$U_\Lambda^\dagger\,\overline{A}\,U_\Lambda = \Lambda\overline{A}\,, \qquad \forall\Lambda\,. \tag{B14}$$

From (A6) it then follows that given $\overline{A}$ and $\overline{B}$ arbitrary vectorial operators the operator $\overline{A}\cdot\underline{B}$ is invariant. Important examples of vectorial operators are provided by the canonical operators $\overline{X}$ and $\overline{P}$ of the theory, as anticipated in equations (16) and (17) of the main text. To see this explicit observe for instance that from (4) and (B12) we get

$$(X^\mu)' = U_\Lambda^\dagger X^\mu U_\Lambda = \int d^4x\, x^\mu|\Lambda^{-1}\overline{x}\rangle\langle\Lambda^{-1}\overline{x}| = \int d^4x\,(\Lambda\overline{x})^\mu|\overline{x}\rangle\langle\overline{x}| = (\Lambda\overline{X})^\mu\,, \tag{B15}$$

which leads to (16). We notice that the above expressions can be used to show that the U_Λ's admit as generators operators $M^{\mu,\nu}$ entering in (2): for example, a y-axis rotation by an angle θ is generated by $U_\Lambda = e^{-i\theta M^{13}}$ with $M^{13} = X^1P^3 - X^3P^1$ so that $U_\Lambda^\dagger X^3 U_\Lambda = X^3\cos\theta + X^1\sin\theta$; a x-axis directed boost by a rapidity v is generated by $U_\Lambda = e^{-ivM^{01}}$ with $M^{01} = X^1P^0 - X^0P^1$ so that $UX^1U^\dagger = X^1\cosh v + X^0\sinh v$ (the hyperbolic trigonometric functions appear because of the extra minus sign in $[X^\mu, P^\nu] = -i\eta^{\mu\nu}$). We stress also that (16) and (17) are fully consistent with the setting of the problem. In fact indicating with $\langle\overline{X}\rangle$ and $\langle\overline{X}\rangle'$ the mean position that O and O' assign to the same event state we notice that they are connected via the identity

$$\langle\overline{X}\rangle' := \langle\Phi|\overline{X'}|\Phi\rangle = \langle\Phi|\Lambda\overline{X}|\Phi\rangle = \Lambda\langle\Phi|\overline{X}|\Phi\rangle = \Lambda\langle\overline{X}\rangle\,, \tag{B16}$$

which is exactly what you would aspect from equation (11). Of course the same result can be obtained by working in the Schrödinger picture: in this case in fact we get

$$\langle\overline{X}\rangle' = \langle\Phi'|\overline{X}|\Phi'\rangle = \int d^4x\,\overline{x}\,|\Phi'(\overline{x})|^2 = \int d^4x\,\overline{x}\,|\Phi(\Lambda^{-1}\overline{x})|^2 = \int d^4x\,\Lambda\overline{x}\,|\Phi(\overline{x})|^2 = \Lambda\langle\overline{X}\rangle\,, \tag{B17}$$

where in the third identity we used (12).

B.1. Spinors

In the presence of spinorial degree of freedom, the 4D spinor wave-functions $\Phi'(\overline{x},\sigma')$ and $\Phi(\overline{x},\sigma)$ assigned by the observers O' and O', will be connected as in equation (20). This implies that equations (B3) and (B4) are replaced by

$$U_\Lambda = \sum_{\sigma,\sigma'} S^{-1}_{\sigma,\sigma'}(\Lambda)\int d^4x|\Lambda\overline{x},\sigma'\rangle\langle\overline{x},\sigma| = \sum_{\sigma,\sigma'} S^{-1}_{\sigma,\sigma'}(\Lambda)\int d^4x|\overline{x},\sigma'\rangle\langle\Lambda^{-1}\overline{x},\sigma|\,, \tag{B18}$$

$$U_\Lambda^\dagger = \sum_{\sigma,\sigma'} S_{\sigma,\sigma'}(\Lambda)\int d^4x|\overline{x},\sigma'\rangle\langle\Lambda\overline{x},\sigma| = \sum_{\sigma,\sigma'} S_{\sigma,\sigma'}(\Lambda)\int d^4x|\Lambda^{-1}\overline{x},\sigma'\rangle\langle\overline{x},\sigma|\,, \tag{B19}$$

so that

$$U_\Lambda|\overline{x},\sigma\rangle = \sum_{\sigma'} S^{-1}_{\sigma,\sigma'}(\Lambda)|\Lambda\overline{x},\sigma'\rangle\,, \tag{B20}$$

$$U_\Lambda^\dagger|\overline{x},\sigma\rangle = \sum_{\sigma'} S_{\sigma,\sigma'}(\Lambda)|\Lambda^{-1}\overline{x},\sigma'\rangle\,, \tag{B21}$$

and

$$U_\Lambda|\overline{p},\sigma\rangle = \sum_{\sigma'} S^{-1}_{\sigma,\sigma'}(\Lambda)|\Lambda\overline{p},\sigma'\rangle\,, \tag{B22}$$

$$U_\Lambda^\dagger|\overline{p},\sigma\rangle = \sum_{\sigma'} S_{\sigma,\sigma'}(\Lambda)|\Lambda^{-1}\overline{p},\sigma'\rangle\,. \tag{B23}$$

Appendix C. Multi-event tensor representation and Fock representation

Recall that the projectors $\Pi^{(n,\mathbf{S})}$ and $\Pi^{(n,\mathbf{A})}$ associated with the completely symmetric $\mathcal{H}_{\mathbf{E}}^{(n,\mathbf{S})}$ and the completely anti-symmetric $\mathcal{H}_{\mathbf{E}}^{(n,\mathbf{A})}$ subspaces of $\mathcal{H}_{\mathrm{E}}^{\otimes n}$, can be expressed as

$$\Pi^{(n,\mathbf{S})} = \frac{1}{n!}\sum_{\mathbf{p}} V_{\mathbf{p}}\,,\quad \Pi^{(n,\mathbf{A})} = \frac{1}{n!}\sum_{\mathbf{p}}\mathrm{sign}[\mathbf{p}]V_{\mathbf{p}}\,, \tag{C1}$$

where the sums over $\mathbf{p}$ run on the set of permutations of n elements, and $V_{\mathbf{p}}$ is the unitary operator which represents $\mathbf{p}$ on $\mathcal{H}_{\mathrm{E}}^{\otimes n}$.

As mentioned in the main text the n event states of Bosonic QM/GEB are by vectors $|\Phi^{[n]}\rangle$ of (22) with 4D spinor wave-functions $\Phi^{[n]}(\overline{x}_1,\sigma_1;\ldots;\overline{x}_n,\sigma_n)$ obeying the symmetry condition (25) and normalization condition

$$\sum_{\sigma_1,\ldots,\sigma_n}\int d^4x_1\cdots d^4x_n|\Phi^{[n]}(\overline{x}_1,\sigma_1;\ldots;\overline{x}_n,\sigma_n)|^2 = 1\,. \tag{C2}$$

Since these vectors belong to the completely symmetric $\mathcal{H}_{\mathbf{E}}^{(n,\mathbf{S})}$ subspace of $\mathcal{H}_{\mathbf{E}}^{\otimes n}$ we have $|\Phi\rangle = \Pi^{(n,\mathbf{S})}|\Phi\rangle$ which exploiting (C1) allows one to equivalently rewrite equation (22) as

$$|\Phi^{[n]}\rangle = \frac{1}{\sqrt{n!}} \sum_{\sigma_1,\ldots,\sigma_n} \int d^4x_1 \cdots d^4x_n \Phi^{[n]}(\bar{x}_1,\sigma_1;\ldots;\bar{x}_n,\sigma_n)|S(\bar{x}_1,\sigma_1;\ldots;\bar{x}_n,\sigma_n)\rangle \ , \tag{C3}$$

with

$$\begin{aligned}|S(\bar{x}_1,\sigma_1;\ldots;\bar{x}_n,\sigma_n)\rangle &:= \frac{1}{\sqrt{n!}}\sum_{\mathbf{p}} V_{\mathbf{p}}|\bar{x}_1,\sigma_1;\ldots;\bar{x}_n,\sigma_n\rangle \\ &= \frac{1}{\sqrt{n!}}\sum_{\mathbf{p}} |\bar{x}_{\mathbf{p}(1)},\sigma_{\mathbf{p}(1)};\ldots;\bar{x}_{\mathbf{p}(n)},\sigma_{\mathbf{p}(n)}\rangle\end{aligned} \tag{C4}$$

the completely symmetric counterpart of $|\bar{x}_1,\sigma_1;\ldots;\bar{x}_n,\sigma_n\rangle$. In the Fock space representation $|S(\bar{x}_1,\sigma_1;\ldots;\bar{x}_n,\sigma_n)\rangle$ is the vector (not $|\bar{x}_1,\sigma_1;\ldots;\bar{x}_n,\sigma_n\rangle$) that is formally expressed as the application of sequences of the Bosonic creation operators $a^{\dagger}_{\bar{x},\sigma}$'s to the 4-vacuum state, i.e.

$$a^{\dagger}_{\bar{x}_1,\sigma_1}\cdots a^{\dagger}_{\bar{x}_n,\sigma_n}|0\rangle_4\Big|_{\rm BOS} \equiv |S(\bar{x}_1,\sigma_1;\ldots;\bar{x}_n,\sigma_n)\rangle \ , \tag{C5}$$

which replaced into (C3) leads to (32) (to justify (C5) notice that due to the commutation rules (30) the two family of states on the l.h.s. and the r.h.s. of the above equation have the same symmetry under permutation of indexes and the same scalar products).

Similar considerations apply for the Fermionic case where the 4D spinor wave-function appearing in (22) fulfill the anti-symmetric relation (26). Invoking hence the fact that they are elements of the completely anti-symmetric $\mathcal{H}_{\mathbf{E}}^{(n,\mathbf{A})}$ subspace of $\mathcal{H}_{\mathbf{E}}^{\otimes n}$ we have now $|\Phi\rangle = \Pi^{(n,\mathbf{A})}|\Phi\rangle$, which allows one to replace equation (C3) with

$$|\Phi^{[n]}\rangle = \frac{1}{\sqrt{n!}} \sum_{\sigma_1,\ldots,\sigma_n} \int d^4x_1 \cdots d^4x_n \Phi^{[n]}(\bar{x}_1,\sigma_1;\ldots;\bar{x}_n,\sigma_n)|A(\bar{x}_1,\sigma_1;\ldots;\bar{x}_n,\sigma_n)\rangle \ , \tag{C6}$$

with

$$|A(\bar{x}_1,\sigma_1;\ldots;\bar{x}_n,\sigma_n)\rangle := \frac{1}{\sqrt{n!}}\sum_{\mathbf{p}} \mathrm{sign}[\mathbf{p}] V_{\mathbf{p}}|\bar{x}_1,\sigma_1;\ldots;\bar{x}_n,\sigma_n\rangle = \frac{1}{\sqrt{n!}}\sum_{\mathbf{p}} |\bar{x}_{\mathbf{p}(1)},\sigma_{\mathbf{p}(1)};\ldots;\bar{x}_{\mathbf{p}(n)},\sigma_{\mathbf{p}(n)}\rangle$$

the vector that is now identified by sequences of Fermionic creation operator $a^{\dagger}_{\bar{x},\sigma}$'s to the 4D-vacuum state, i.e.

$$a^{\dagger}_{\bar{x}_1,\sigma_1}\cdots a^{\dagger}_{\bar{x}_n,\sigma_n}|0\rangle_4\Big|_{\rm FER} \equiv |A(\bar{x}_1,\sigma_1;\ldots;\bar{x}_n,\sigma_n)\rangle \ , \tag{C7}$$

leading once more to (32).

Appendix D. More on the QM/GEB correspondence

Here we analyze in detail the technical aspects of the QM/GEB correspondence introduced in section 3. Specifically we shall show that the vectors $|\Psi_{\rm QM}\rangle$ introduced in equation (40), while not being elements of $\mathcal{H}_{\mathbf{E}}$, form a special subset $\mathcal{H}_{\mathbf{QM}}$ of the distributions set $\mathcal{H}_{\mathbf{E}}^{+}$ of the theory, i.e. the rigged-extended version of $\mathcal{H}_{\mathbf{E}}$ which we introduce when discussing the generalized position and momentum eigenvectors of GEB.

We have already commented the fact that the normalization condition (38) implies that the $|\Psi_{\rm QM}\rangle$'s of (40) have a divergent norm. This automatically excludes them from the Hilbert space $\mathcal{H}_{\mathbf{E}}$. To prove that they are distributions, we need to show that there exists a dense subset $\mathcal{D}$ of $\mathcal{H}_{\mathbf{E}}$ formed by (normalized) vectors $|\Phi\rangle$ such that the quantity $\langle\Psi_{\rm QM}|\Phi\rangle$ exists and is finite. To exhibit such subset let first introduce the spectral decomposition of the QM Hamiltonian H which is ruling the dynamical evolution of the single-particle of the problem (i.e. the generator which is responsible for the time evolution of the 3D wave-function $\Psi_{\rm QM}(\vec{x}|t)$). We will consider explicitly the case where H has a (possibly degenerate) continuous spectrum but the analysis can be easily applied to the cases of discrete spectra (or even mixed discrete/continuous spectra). Accordingly, we write

$$H := \int dE \sum_k E|E,k\rangle\langle E,k| \ , \tag{D1}$$

with the discrete variable k accounting for the degeneracy of the E-energy level, and where $\{|E,k\rangle\}_{E,k}$ are the generalized orthonormal eigenvectors that fulfill

$$\langle E',k'|E,k\rangle = \delta_{k,k'}\delta(E-E') \;, \tag{D2}$$

with $\delta_{k,k'}$ the Kronecker delta symbol. Similarly to [12, 15], we now adopt a spacetime foliation that separate the temporal coordinate of $\mathcal{H}_\mathbf{E}$ vs the spatial ones via a tensor product, writing

$$|\underline{x}\rangle = |t\rangle|\vec{x}\rangle \;, \tag{D3}$$

(notice that while this choice breaks the covariance of the theory, this is not a problem as in our case we shall compute scalar products between vectors which are explicitly invariant quantities). We then expand a generic normalized element of $\mathcal{H}_\mathbf{E}$ in the following form

$$|\Phi\rangle = \int dE \sum_{n,k} c_{n,k}(E)|n\rangle|E,k\rangle \;, \tag{D4}$$

where we introduced a discrete complete orthonormal set $\{|n\rangle\}_n$ for the temporal axis while we adopted the generalized eigenstates $\{|E,k\rangle\}_{E,k}$ of the QM Hamiltonian H to expand the spatial degree of freedom of the system. In the above equation $c_{n,k}(E)$ are probability amplitudes fulfilling the normalization condition

$$\int dE \sum_{n,k} |c_{n,k}(E)|^2 = \langle\Phi|\Phi\rangle = 1 \;, \tag{D5}$$

Now we define $\mathcal{D}$ to be set of vectors of $\mathcal{H}_\mathbf{E}$ which admits a decomposition (D4) with coefficients $c_{n,k}(E)$ that, besides (D5), fulfill also the extra constraint

$$\sum_n \sqrt{\int dE \sum_k |c_{n,k}(E)|^2} < \infty \;, \tag{D6}$$

(to see that $\mathcal{D}$ is dense observe that such space contains all the vectors $|\Phi\rangle$ with $c_{n,k}(E) \neq 0$ only for a finite set of values of n). Expressing now $|\Psi_{\rm QM}\rangle$ of equation (40) in terms of the same spacetime foliation used in (D4), i.e.

$$|\Psi_{\rm QM}\rangle = \int dt \int d^3x \Psi_{\rm QM}(\vec{x}|t)|t\rangle|\vec{x}\rangle = \int dt|t\rangle|\psi(t)\rangle \;, \tag{D7}$$

with

$$|\psi(t)\rangle = \int d^3x \Psi_{\rm QM}(\vec{x}|t)|\vec{x}\rangle \;, \tag{D8}$$

we notice that

$$\begin{aligned}\langle\Phi|\Psi_{\rm QM}\rangle &= \int dt \int dE \sum_{n,k} c^*_{n,k}(E)\langle n|t\rangle\langle E,k|\psi(t)\rangle = \int dt \int dE \sum_{n,k} c^*_{n,k}(E)\langle n|t\rangle\alpha_k(E)e^{-iEt} \\ &= \sqrt{2\pi}\int dE \sum_{n,k} c^*_n(E)\alpha_k(E)\langle n|\pi(E)\rangle,\end{aligned} \tag{D9}$$

where in the second identity we introduced the probability amplitudes

$$\alpha_k(E)e^{-iEt} := \langle E,k|\psi(t)\rangle \tag{D10}$$

of the state $|\psi(t)\rangle$ with e^{-iEt} being their associated dynamical phase (remember that $\{|E,k\rangle\}_{E,k}$ are eigenvectors of the system Hamiltonian), and where in the third identity we introduce the vectors

$$|\pi(E)\rangle := \frac{1}{\sqrt{2\pi}}\int dt e^{-iEt}|t\rangle \;. \tag{D11}$$

Observe that this last is a distribution for the temporal coordinate (indeed it is the Fourier transform of position coordinates), that fulfills the orthonormalization rule

$$\langle\pi(E')|\pi(E)\rangle = \delta(E-E') \;. \tag{D12}$$

As a matter of fact we can identify $|\pi(E)\rangle$ as a generalized eigenstate of the canonical momentum of the temporal position axis. Accordingly, we can interpret $\langle\pi(E)|n\rangle$ as the momentum amplitude probability distribution of $|n\rangle$ evaluated at momentum E. Remember next that $\{|n\rangle\}_n$ is a basis that we can choose freely. We now take such basis as the orthonormal set of the spectrum of the Harmonic oscillator which allows us to explicitly compute the value of $\langle\pi(E)|n\rangle$ as

$$\langle\pi(E)|n\rangle = \frac{\pi^{-1/4}}{\sqrt{2^n n!}}\exp[-E^2/2]H_n(E)\,, \tag{D13}$$

where for the sake of simplicity we are expressing here the function in renormalized units where all the physical constants are set equal to 1, and where $H_n(x)$ are the Hermite polynomials. Now the only fundamental aspect of the problem here is that we can put an upper bound on such terms, independently of the choice of n and E. In particular we can show that

$$\langle\pi(E)|n\rangle \leqslant \langle\pi(E)=0|n=0\rangle = \frac{1}{\pi^{1/4}}\,. \tag{D14}$$

Hence invoking the Cauchy–Schwarz inequality, we can now bound the term (D9) as follows:

$$\begin{aligned}|\langle\Phi|\Psi_{\rm QM}\rangle| &\leqslant \sqrt{2\pi}\sum_n\int dE\sum_k\left|c^*_{n,k}(E)\alpha_k(E)\langle n|\pi(E)\rangle\right| \leqslant \frac{\sqrt{2\pi}}{\pi^{1/4}}\sum_n\int dE\sum_k\left|c^*_{n,k}(E)\alpha_k(E)\right|\\ &\leqslant \frac{\sqrt{2\pi}}{\pi^{1/4}}\sum_n\sqrt{\int dE\sum_k|c^*_{n,k}(E)|^2}\,,\end{aligned} \tag{D15}$$

which is finite due to equation (D6).

Appendix E. Initial conditions

The constraint equation (41) merely selects all possible distributions which are compatible with an assigned QM dynamical law. One can add extra constraints that enforce possibly observer dependent 'initial' (rather, boundary) conditions or better specify the system evolution. For instance we can identify the element of $\mathcal{H}_{\mathbf{QM}}$ associated to the QM quantum trajectory of a spin-less single-particle which at time τ as measured for the observer O, corresponds to a certain target 3D spinor wave-function $\psi_0(\vec{x},\sigma)$, by looking for the $|\Psi_{\rm QM}\rangle$ fulfilling (41) which verifies the extra condition

$$\Pi_\tau|\Psi_{\rm QM}\rangle = \sum_\sigma\int d^3x\,\psi_0(\vec{x},\sigma)|\bar{x},\sigma\rangle\Big|_{t=\tau}\,, \tag{E1}$$

with $\Pi_\tau = \sum_\sigma\int d^4x\,\delta(t-\tau)|\bar{x},\sigma\rangle\langle\bar{x},\sigma|$ being a generalized projector on $\mathcal{H}_{\mathbf{E}}$.

Appendix F. Constraint operator for the KG model

In the absence of the energy constraint the general solution of the KG equation

$$(\Box + m^2)\Psi_{\rm KG}(\bar{x}) = 0\,, \tag{F1}$$

expressed in term of spacetime coordinates of an inertial observer O is given by the sum of two independent contributions

$$\Psi_{\rm KG}(\bar{x}) = \Psi^+_{\rm KG}(\bar{x}) + \Psi^-_{\rm KG}(\bar{x})\,, \tag{F2}$$

$$\Psi^\pm_{\rm KG}(\bar{x}) := \int\frac{d^3p}{(2\pi)^{3/2}}e^{\mp iE_pt+i\vec{p}\cdot\vec{x}}\psi^{(\pm)}(\vec{p})\,, \tag{F3}$$

with $E_p := \sqrt{|\vec{p}|^2+m^2}$ and with the functions $\psi^{(\pm)}(\vec{p})$ fixed by imposing boundary conditions. Without introducing extra structure on the problem, equation (F2) is not compatible with unitary evolutions predicted by QM since, as discussed below, the two parts can be seen as time evolutions according to two different Hamiltonian (e.g. given $\Psi_{{\rm KG},1}(\bar{x})$ and $\Psi_{{\rm KG},2}(\bar{x})$ solutions of (F1) we get $\int d^3x\Psi^*_{{\rm KG},1}(\bar{x})\Psi_{{\rm KG},2}(\bar{x})$ is an explicit function of t). Yet one still use equation (40) to associate to $\Psi_{\rm KG}(\bar{x})$ a distribution $|\Psi_{\rm KG}\rangle$ of GEB and observe that the resulting vector can be identified with the solutions of an eigenvalue equation (41)

$$J_{\rm KG}|\Psi_{\rm KG}\rangle = 0\,, \tag{F4}$$

with constraint operator

$$J_{\rm KG} := \overline{P}\cdot \underline{P} - m^2 = \int d^4p\, (\overline{p}\cdot\underline{p} - m^2)|\overline{p}\rangle\langle\overline{p}| \,, \tag{F5}$$

that is explicit Lorentz invariant. From the equation (F2) it follows that we can be written as $|\Psi_{\rm KG}\rangle$ the sum of two terms

$$|\Psi_{\rm KG}\rangle = \int d^4x\, \Psi_{\rm KG}(\overline{x})\, |\overline{x}\rangle = |\Psi^+_{\rm KG}\rangle + |\Psi^-_{\rm KG}\rangle \,, |\Psi^\pm_{\rm KG}\rangle := \int d^4x\, \Psi^\pm_{\rm KG}(\overline{x})\, |\overline{x}\rangle \,, \tag{F6}$$

which also satisfy (F4), i.e.

$$J_{\rm KG}|\Psi^\pm_{\rm KG}\rangle = 0 \,. \tag{F7}$$

Selecting the positive (negative) energy solutions of (F2) corresponds to identifying $\Psi_{\rm KG}(\overline{x})$ with just the component $\Psi^+_{\rm KG}(\overline{x})$ (resp. $\Psi^-_{\rm KG}(\overline{x})$), i.e. to imposing $\psi^{(-)}(\vec{p}) = 0$ (resp. $\psi^{(+)}(\vec{p}) = 0$) as boundary condition of the problem. By construction, these special functions can be seen as solutions of ordinary Schrödinger equations with single-particle Hamiltonian $H := \sqrt{m^2 - \nabla^2}$, i.e.

$$i\partial_t \Psi^+_{\rm KG}(\overline{x}) = H\Psi^+_{\rm KG}(\overline{x}) \,, \tag{F8}$$

(the same holds also for $\Psi^-_{\rm KG}(\overline{x})$, choosing $-H$ as Hamiltonian). Therefore, $\Psi^+_{\rm KG}(\overline{x})$ represents a proper unitary temporal evolution that preserves equal time, 3D scalar products.

A better insight on the properties of the distributions $|\Psi^\pm_{\rm KG}\rangle$ can be gained by rewriting (F6) as

$$|\Psi^\pm_{\rm KG}\rangle := \int d^4p\, \Psi^\pm_{\rm KG}(\overline{p})\, |\overline{p}\rangle \,, \tag{F9}$$

where $\Psi^\pm_{\rm KG}(\overline{p}) = \int \frac{d^4x}{4\pi^2} e^{i\overline{x}\cdot\underline{p}}\, \tilde{\Psi}^\pm_{\rm KG}(\overline{x})$ is the 4D Fourier transform of $\tilde{\Psi}^\pm_{\rm KG}(\overline{x})$ which, by explicit computation, is given by

$$\tilde{\Psi}^\pm_{\rm KG}(\overline{p}) := \sqrt{2\pi}\delta(p^0 \mp E_p)\psi^{(\pm)}(\vec{p}) \,. \tag{F10}$$

Introducing the orthogonal projectors

$$\Pi^+ := \int d^4p\, \Theta(p^0)|\overline{p}\rangle\langle\overline{p}| \,, \tag{F11}$$

$$\Pi^- := \mathbb{1}_{\rm E} - \Pi^+ = \int d^4p\, \Theta(-p^0)|\overline{p}\rangle\langle\overline{p}| \,, \tag{F12}$$

that identify the positive/negative energy subspaces of $\mathcal{H}$E, we note that they admit $|\Psi^\pm_{\rm KG}\rangle$ as eigenvectors that solve the identities

$$\Pi^+|\Psi^+_{\rm KG}\rangle = |\Psi^+_{\rm KG}\rangle \,, \quad \Pi^-|\Psi^-_{\rm KG}\rangle = |\Psi^-_{\rm KG}\rangle \,. \tag{F13}$$

or equivalently

$$\Pi^-|\Psi^+_{\rm KG}\rangle = 0 \,, \quad \Pi^+|\Psi^-_{\rm KG}\rangle = 0 \,, \tag{F14}$$

Thanks to (F7) this allows us to uniquely identify $|\Psi^+_{\rm KG}\rangle$ as the special vectors which are in the intersection of the kernels of $J_{\rm KG}$ and Π^-, i.e.

$$J_{\rm KG+}|\Psi^+_{\rm KG}\rangle = 0 \,, \tag{F15}$$

with the new constraint operator

$$J_{\rm KG+} := J_{\rm KG}\Pi^+ - m^2\Pi^- = \Pi^+ J_{\rm KG} - m^2\Pi^- = \int d^4p\, \left[\Theta(p^0)\,\overline{p}\cdot\underline{p} - m^2\right] |\overline{p}\rangle\langle\overline{p}| \,, \tag{F16}$$

(note the $-m^2\Pi^-$ term!) reported in equation (45) of the main text. Notice that such a term is explicitly self-adjoint ($J^\dagger_{\rm KG+} = J_{\rm KG+}$), but not positive semidefinite (indeed its generalized eigenvalues $\Theta(p^0)\,\overline{p}\cdot\underline{p} - m^2$

can take any real values for proper choices of the 4-momentum $\bar{p}$). Similarly the negative energy terms can be uniquely identified by writing $J_{\mathrm{KG}^-}|\Psi^-_{\mathrm{KG}}\rangle = 0$ with

$$J_{\mathrm{KG}^-} := J_{\mathrm{KG}}\Pi^- - m^2\Pi^+ = \Pi^- J_{\mathrm{KG}} - m^2\Pi^+ \tag{F17}$$

$$= \int d^4p \left[\Theta(-p^0)\,\bar{p}\cdot\underline{p} - m^2\right]|\bar{p}\rangle\langle\bar{p}|\,. \tag{F18}$$

Consider next what happens when we introduce a new observer O' sitting in a reference frame R' whose 4D coordinates $\bar{x}'$ are connected with those of O via the mapping (11). Due to the explicit covariant structure of (43), in the new reference frame the general solution $\Psi_{\mathrm{KG}}(\bar{x})$ is replaced by the new function

$$\Psi'_{\mathrm{KG}}(\bar{x}) = \Psi_{\mathrm{KG}}(\Lambda^{-1}\bar{x})\,, \tag{F19}$$

which corresponds to the identity (51) which at the level of the correspondence (40), leads to equation (50) of the main text. To verify that the same holds for the positive (negative) solutions as well, the important observation is that these functions do not mix under Lorentz transformations. Specifically one can verify that $\Psi'_{\mathrm{KG}}(\bar{x})$ still maintain the same structure of (F2),

$$\Psi'_{\mathrm{KG}}(\bar{x}) = \Psi'^{+}_{\mathrm{KG}}(\bar{x}) + \Psi'^{-}_{\mathrm{KG}}(\bar{x})\,, \tag{F20}$$

with new positive and negative energy terms

$$\Psi'^{\pm}_{\mathrm{KG}}(\bar{x}) = \int \frac{d^3p}{(2\pi)^{3/2}} e^{\mp iE_pt + i\vec{p}\cdot\vec{x}}\psi'^{(\pm)}(\vec{p})\,, \tag{F21}$$

that are associated with those of O via the same coordinate change of (F19), i.e.

$$\Psi'^{\pm}_{\mathrm{KG}}(\bar{x}) = \Psi^{\pm}_{\mathrm{KG}}(\Lambda^{-1}\bar{x})\,. \tag{F22}$$

For instance assuming Λ to represent a boost along the x direction (i.e. $t' = \gamma(t - vx)$, $x' = \gamma(x - vt)$, $y' = y$, and $z' = z$) we get $\psi'^{(\pm)}(\vec{p}) = \psi^{(\pm)}(\gamma(p^1 \pm vE_p), p^2, p^3))\frac{\gamma(E_p \pm vp^1)}{E_p}$ which shows the independence of $\Psi'^{+}_{\mathrm{KG}}(\bar{x})$ ($\Psi'^{-}_{\mathrm{KG}}(\bar{x})$) from $\Psi^-_{\mathrm{KG}}(\bar{x})$ (resp. $\Psi^+_{\mathrm{QM}}(\bar{x})$). An important consequence of the property (F22) is that it implies that we can drop the negative energy terms in equation (F2) without affecting the Lorentz invariance of equation (43) hence ensuring that also the non explicitly covariant equation (F8) yields Lorentz covariant solutions (this is exactly what we need to show that equation (50) also applies in the special case where we focus on the positive (negative) solutions of the KG equation (43)).

We now briefly comment on the physical significance of the negative energy solutions of the Klein–Gordon (KG) equation. Remember that the wave equation $(\Box - m^2)f(t,\vec{r}) = 0$ has solutions with spacetime dependence $f = g(\vec{r} - \vec{v}t) + h(\vec{r} + \vec{v}t)$, with $\vec{v}$ the propagation velocity (both signs of the velocity must appear in the general solution as the wave equation contains only v^2). One can expand g and h in terms of plane waves $e^{i\vec{k}\cdot(\vec{r}\pm\vec{v}t)} \equiv e^{i(\vec{k}\cdot\vec{r} - \omega t)}$, where the frequency $\omega \equiv \mp\vec{k}\cdot\vec{v}$ can be positive or negative depending on the propagation *direction* of the wave with respect to the wave vector $\vec{k}$. With an appropriate choice of sign in the definition of ω, one can consider a negative-frequency wave as an *advanced* solution to the wave equation and a positive-frequency wave as a *retarded* solution, since these solutions can be obtained from one another by time reversal. Usually the advanced solution is discarded (set to zero) appealing to some vague notion of causality, e.g. [75], but more careful analyses [76, 77] interpret the retarded solutions as a prediction based on past boundary conditions and the advanced solutions as a retrodiction based on future boundary conditions. Then the choice of which frequency sign to choose (or even a combination of the two [77]) is dictated purely by the available boundary conditions. Clearly, past boundary conditions are more useful in general. One can discard the negative frequency solutions by imposing, in addition to the KG equation of motion, an additional *physical* condition of positive-energy (as was done in the main text).

In closing we comment on the 'negative probability densities' that historically have plagued the acceptance of the KG equation (notoriously, it was discovered, but then discarded, by Schrödinger [78, 79]). This problem ensues from the observation that, if one defines a four-current for the KG wave-function ψ_1 as $j^\mu = \psi_1^*\partial^\mu\psi_1 - \psi_1\partial^\mu\psi_1^*$, it does satisfy a conservation equation $\partial_\mu j^\mu = 0$, but the density j^0 (representing a putative probability density) is not positive definite (and should be interpreted as a charge density). It is not such j^0 that should take the role of a probability density of the particle position *at a certain time*, but rather $|\psi_1(\bar{x})|^2$ that is the probability density of finding a particle-detection event at space*time* position $\bar{x} = (t, \vec{x})$: a joint probability for both the position and for time, rather than a conditioned probability for the position, given the time. As such, $|\psi_1(\bar{x})|^2$ is a *scalar* quantity, not the temporal component of a 4-current, and needs not satisfy any current conservation. Moreover, it is obviously always positive definite. In contrast, in the case of the Dirac field, one can build also a (conserved) probability current (see below).

Appendix G. Constraint operator for the Dirac model

The Dirac equation for the spinor wave-function $\Psi_{\rm QM}(\vec{x},\sigma|t)$ of single particle is a collection of the four differential equations reported in equation (44). By taking the 4D Fourier transform we can turn them into the equivalent form

$$\sum_{\sigma=1}^{4}(\overline{\gamma}_{\sigma',\sigma}\cdot \underline{p} - m\,\delta_{\sigma',\sigma})\tilde{\Psi}_{\rm QM}(\overline{p},\sigma) = 0\,, \tag{G1}$$

with

$$\tilde{\Psi}_{\rm QM}(\overline{p},\sigma) = \int \frac{d^4x}{4\pi^2}\exp[i\overline{p}\cdot\underline{x}]\Psi_{\rm QM}(\vec{x},\sigma|t)\,. \tag{G2}$$

Contracting the index σ' of (G1) with the matrix elements of the invertible matrix γ^0, we can further modify equation (44) into the identity

$$\sum_{\sigma=1}^{4} M_{\sigma',\sigma}(\overline{p})\tilde{\Psi}_{\rm QM}(\overline{p},\sigma) = 0\,, \tag{G3}$$

where

$$M_{\sigma',\sigma''}(\overline{p}) := \sum_{\sigma=1}^{4}\gamma^0_{\sigma',\sigma}(\overline{\gamma}_{\sigma,\sigma''}\cdot\underline{p} - m\,\delta_{\sigma,\sigma''})\,, \tag{G4}$$

are elements of the self-adjoint (yet not positive) 4×4 matrix

$$M(\overline{p}) := \begin{pmatrix} (p^0-m)\mathbb{1} & -\vec{\sigma}\cdot\vec{p} \\ -\vec{\sigma}\cdot\vec{p} & (p^0+m)\mathbb{1}\end{pmatrix}, \tag{G5}$$

with eigenvalues

$$\lambda_\sigma(\overline{p}) := p^0 - E_p^{(\sigma)}\,, \tag{G6}$$

where given $E_p = \sqrt{|\vec{p}|^2+m^2}$ we introduced the quantities

$$E_p^{(\sigma)} := \begin{cases} -E_p & \text{for} \quad \sigma = 1,3\,, \\ E_p & \text{for} \quad \sigma = 2,4\,, \end{cases} \tag{G7}$$

Equation (G1) can hence be interpreted as an eigenvector equation which, for any assigned $\overline{p}$, selects eigenvectors of $M(\overline{p})$ which are associated with null eigenvalues ($\lambda_\sigma(\overline{p}) = 0$). More precisely casting $M(\overline{p})$ in diagonal form

$$M_{\sigma',\sigma''}(\overline{p}) = \sum_{\sigma=1}^{4} u_{\sigma',\sigma}(\vec{p})\,\lambda_\sigma(\overline{p})\,u^*_{\sigma'',\sigma}(\vec{p})\,, \tag{G8}$$

with $u_{\sigma,\sigma'}(\vec{p})$ the elements of a 4×4 unitary matrix (see the end of the section for explicit expressions), it follows that the most generic solution of equation (G1) can be written as

$$\tilde{\Psi}_{\rm QM}(\overline{p},\sigma) = \sum_{\sigma'=1}^{4}\delta(p^0 - E_p^{(\sigma')})\alpha_{\sigma'}(\vec{p})\,u_{\sigma,\sigma'}(\vec{p})\,, \tag{G9}$$

with $\alpha_{\sigma'}(\vec{p})$ arbitrary functions, i.e.

$$\Psi_{\rm QM}(\vec{x},\sigma|t) = \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}}e^{i\vec{p}\cdot\vec{x}}\sum_{\sigma'=1}^{4}\frac{u_{\sigma,\sigma'}(\vec{p})}{\sqrt{2\pi}}\,e^{-iE_p^{(\sigma')}t}\,\alpha_{\sigma'}(\vec{p})\,, \tag{G10}$$

at the level of the 3D+1 spinor wave-function.

Expressed as in equation (G1) it is easy to verify that, at the level of the GEB distribution $|\Psi_{\rm QM}\rangle = \sum_{\sigma=1}^{4}\int d^4x\, \Psi_{\rm QM}(\vec{x},\sigma|t)|\overline{x},\sigma\rangle$, the Dirac equation (44) corresponds to the identity $J_{\rm D}|\Psi_{\rm QM}\rangle = 0$ with $J_{\rm D}$ as in equation (47). Indeed, to show this, we need the fact that thanks to (4) $\tilde{\Psi}_{\rm QM}(\overline{p},\sigma)$ provides the 4D-momentum spinor wave-functions expansion of $|\Psi_{\rm QM}\rangle$, i.e. $|\Psi_{\rm QM}\rangle = \sum_{\sigma=1}^{4}\int d^4p\, \tilde{\Psi}_{\rm QM}(\overline{p},\sigma)|\overline{p},\sigma\rangle$. As mentioned in the main text the operator $J_{\rm D}$ is not a self-adjoint: this is a direct consequence of the fact that for all $i = 1,2,3$ the matrices γ^i are anti-Hermitian (indeed $(\gamma^i)^\dagger = -\gamma^i = \gamma_i$), while γ^0 is Hermitian, so that $J_{\rm D}^\dagger = \sum_{\mu=1}^{4}(\gamma^\mu)^\dagger P_\mu - m = \sum_{\mu=1}^{4}\gamma_\mu P_\mu - m \neq J_{\rm D}$. Notice however that exploiting the fact that $\gamma^0\gamma^0 = \mathbb{1}$, and $\gamma^0\gamma^i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0\end{pmatrix}$, we can write

$$J_{\rm D} = \gamma^0 J_{\rm D}^{(H)}\,, \tag{G11}$$

where given the matrix elements $M_{\sigma',\sigma''}(\overline{p})$ of equation (G4) $J_{\rm D}^{(H)}$ is the self-adjoint operator

$$J_{\rm D}^{(H)} := \sum_{\sigma',\sigma''}\int d^4p\, M_{\sigma',\sigma''}(\overline{p})|\overline{p},\sigma'\rangle\langle\overline{p},\sigma''|\,. \tag{G12}$$

Equation (48) finally follows by using equation (G8) observing that the vectors

$$|\phi_\sigma(\overline{p})\rangle := \sum_{\sigma'=1}^{4} u_{\sigma',\sigma}(\vec{p})|\overline{p},\sigma'\rangle\,, \tag{G13}$$

obey generalized orthonormal conditions (49) thanks to the unitary properties of the matrix elements $u_{\sigma',\sigma}(\vec{p})$: indeed with this choice equation (G12) becomes

$$J_{\rm D}^{(H)} = \sum_{\sigma=1}^{4}\int d^4p\, \lambda_\sigma(\overline{p})\, |\phi_\sigma(\overline{p})\rangle\langle\phi_\sigma(\overline{p})|\,, \tag{G14}$$

and hence

$$K_{\rm D} = J_{\rm D}^\dagger J_{\rm D} = \left(J_{\rm D}^{(H)}\right)^2 = \sum_{\sigma=1}^{4}\int d^4p\, \lambda_\sigma^2(\overline{p})\, |\phi_\sigma(\overline{p})\rangle\langle\phi_\sigma(\overline{p})|\,. \tag{G15}$$

We conclude reporting explicit expressions for the $|\phi_\sigma(\overline{p})\rangle$:

$$|\phi_1(\overline{p})\rangle := \frac{1}{\sqrt{2}}\left(\sqrt{1-\frac{m}{E_p}}|\overline{p},s_1(\hat{n})\rangle - \frac{|\vec{p}|}{\sqrt{1-\frac{m}{E_p}}}|\overline{p},s_2(\hat{n})\rangle\right)\,, \tag{G16}$$

$$|\phi_2(\overline{p})\rangle := \frac{1}{\sqrt{2}}\left(\sqrt{1+\frac{m}{E_p}}|\overline{p},s_1(\hat{n})\rangle + \frac{|\vec{p}|}{\sqrt{1+\frac{m}{E_p}}}|\overline{p},s_2(\hat{n})\rangle\right)\,, \tag{G17}$$

$$|\phi_3(\overline{p})\rangle := \frac{1}{\sqrt{2}}\left(\sqrt{1-\frac{m}{E_p}}|\overline{p},s_3(\hat{n})\rangle + \frac{|\vec{p}|}{\sqrt{1-\frac{m}{E_p}}}|\overline{p},s_4(\hat{n})\rangle\right)\,, \tag{G18}$$

$$|\phi_4(\overline{p})\rangle := \frac{1}{\sqrt{2}}\left(\sqrt{1+\frac{m}{E_p}}|\overline{p},s_3(\hat{n})\rangle - \frac{|\vec{p}|}{\sqrt{1+\frac{m}{E_p}}}|\overline{p},s_4(\hat{n})\rangle\right)\,, \tag{G19}$$

where for $\hat{n} := \vec{p}/|\vec{p}|$, $|\overline{p},s_\sigma(\hat{n})\rangle$ are the orthonormal vectors

$$|\overline{p},s_1(\hat{n})\rangle := \frac{1}{\sqrt{2}}\left(\sqrt{1+n^3}\,|\overline{p},1\rangle + \frac{n^1+in^2}{\sqrt{1+n^3}}\,|\overline{p},2\rangle\right)\,, \tag{G20}$$

$$|\overline{p},s_2(\hat{n})\rangle := \frac{1}{\sqrt{2}}\left(\sqrt{1+n^3}\,|\overline{p},3\rangle + \frac{n^1+in^2}{\sqrt{1+n^3}}\,|\overline{p},4\rangle\right)\,, \tag{G21}$$

$$|\bar{p}, s_3(\hat{n})\rangle := \frac{1}{\sqrt{2}}\left(\sqrt{1-n^3}\,|\bar{p},1\rangle - \frac{n^1+in^2}{\sqrt{1-n^3}}\,|\bar{p},2\rangle\right) , \tag{G22}$$

$$|\bar{p}, s_4(\hat{n})\rangle := \frac{1}{\sqrt{2}}\left(\sqrt{1-n^3}\,|\bar{p},3\rangle - \frac{n^1+in^2}{\sqrt{1-n^3}}\,|\bar{p},4\rangle\right) . \tag{G23}$$

Observe that via equation (G13) these identities implicitly define the matrix elements $u_{\sigma,\sigma'}(\vec{p})$: for instance we get

$$u_{1,1}(\vec{p}) = \frac{1}{2}\sqrt{1-\frac{m}{E_p}}\sqrt{1+n^3} , \qquad u_{2,1}(\vec{p}) = \frac{1}{2}\sqrt{1-\frac{m}{E_p}}\frac{n^1+in^2}{\sqrt{1+n^3}} , \tag{G24}$$

$$u_{3,1}(\vec{p}) = -\frac{1}{2}\frac{|\vec{p}|}{\sqrt{1-\frac{m}{E_p}}}\sqrt{1+n^3} , \qquad \cdots \tag{G25}$$

If one appropriately normalizes the state, one can recover, just as for Bosons, a *scalar* probability density for each spinor component, since the Dirac equation implies the KG one: $(\gamma^\mu p_\mu + m)(\gamma^\nu p_\nu - m) = (p^\mu p_\mu - m^2)\mathbb{1}_4$. In addition, one can, as usual, also introduce a conserved 4-current $j^\mu \equiv \Psi^\dagger(\bar{x})\gamma^0\gamma^\mu\Psi(\bar{x}) = \bar{\Psi}(\bar{x})\gamma^\mu\Psi(\bar{x})$, where Ψ is the column vector of the *conditioned* spinors in the position representation, namely the column of position-representation amplitudes. Since the zeroth component $j^0 = \Psi^\dagger(\bar{x})\Psi(\bar{x})$ is positive definite, it can be given a probabilistic interpretation as the *conditional* probability density of finding a particle at position $\vec{x}$, given that time is t, where $\bar{x} = (t, \vec{x})$.

Appendix H. Extra observations on the multi-event QM/GEB correspondence

This section is dedicated to making explicit some technical aspects of the QM/GEB correspondence in multi-event scenario discussed in section 3.2. We start by showing that the vector (54) is uniquely defined; then we verify that equation (61) gives the right prescription to compute the evolution of a QM 3D+1 spinor wave-function under Lorentz transformations.

H.1. Uniqueness of the the multi-event QM/GEB correspondence

Here we prove that the vector (54) is uniquely defined.

To begin with recall that, for all t, the QM spinor 3D wave-function $\Psi^{[n]}_{\rm QM}(\vec{x}_1,\sigma_1;\ldots;\vec{x}_n,\sigma_n|t)$ of n particles can be expressed as

$$\Psi^{[n]}_{\rm QM}(\vec{x}_1,\sigma_1;\ldots;\vec{x}_n,\sigma_n|t) = \langle\vec{x}_1,\sigma_1;\ldots;\vec{x}_n,\sigma_n|\psi^{[n]}_{\rm QM}(t)\rangle , \tag{H1}$$

where

$$|\psi^{[n]}_{\rm QM}(t)\rangle = \sum_{\sigma_1,\ldots,\sigma_n}\int d^3x_1\cdots\int d^3x_n n\Psi^{[n]}_{\rm QM}(\vec{x}_1,\sigma_1;\ldots;\vec{x}_n,\sigma_n|t)|\vec{x}_1,\sigma_1;\ldots;\vec{x}_n,\sigma_n\rangle , \tag{H2}$$

is the associated wave-vector, and $|\vec{x}_j,\sigma_j\rangle$ the generalized 3D position eigenvectors of the j-th particle. Recalling then that we are dealing with non-interacting systems, we can now write

$$|\psi^{[n]}_{\rm QM}(t)\rangle = U_1(t)\otimes\cdots\otimes U_n(t)|\psi^{[n]}_{\rm QM}(0)\rangle , \tag{H3}$$

where for $j = 1,\ldots,n$, $U_j(t)$ stands for the QM unitary transformation that rules the free evolution of the j-th particle. Equation (H3) leads to equation (52) by expressing $|\psi^{[n]}_{\rm QM}(0)\rangle$ in terms of an arbitrary local basis for the n particles, i.e.

$$|\psi^{[n]}_{\rm QM}(0)\rangle = \sum_{\vec{\ell}}\alpha_{\vec{\ell}}\,|\psi^{(\ell_1)}_{\rm QM}\rangle\otimes\cdots\otimes|\psi^{(\ell_n)}_{\rm QM}\rangle , \tag{H4}$$

and using the identities

$$\Psi^{(\ell_j)}_{\rm QM}(\vec{x}_j,\sigma_j|t) := \langle\vec{x}_j,\sigma_j|U_j(t)|\psi^{(\ell_j)}_{\rm QM}\rangle . \tag{H5}$$

Replacing this into (53) and (54) we finally arrive to

$$|\Psi^{[n]}_{\rm QM}\rangle = \sum_{\sigma_1,\ldots,\sigma_2}\int d^4x_1\cdots\int d^4x_n|\bar{x}_1,\sigma_1;\ldots;\bar{x}_n,\sigma_n\rangle\langle\vec{x}_1,\sigma_1;\ldots;\vec{x}_n,\sigma_n|U_1(t_1)\otimes\cdots\otimes U_n(t_n)|\psi^{[n]}_{\rm QM}(0)\rangle,$$

which explicitly shows that $|\Psi_{\rm QM}\rangle$ carries no functional dependence upon the specific choice of the local basis $\{|\psi^{(\ell)}_{\rm QM}\rangle\}_\ell$ used in (H4).

As an application of the above identities we report here the special cases of particles obeying to the positive energy KG equation and the Dirac equation. For the KG equation, setting $\tilde{\psi}^{[n]}_{\rm QM}(\vec{p}_1,\ldots,\vec{p}_n) := \sum_{\vec{\ell}}\alpha_{\vec{\ell}}\psi^{(\ell_1)}(\vec{p}_1)\cdots\psi^{(\ell_2)}(\vec{p}_n)$ and replacing (F3) into equation (52) we get

$$\Psi^{[n]}_{\rm QM}(\vec{x}_1,\ldots,\vec{x}_n|t) = \int \frac{d^3p_1}{(2\pi)^{3/2}}\cdots\frac{d^3p_n}{(2\pi)^{3/2}} e^{i(\vec{p}_1\cdot\vec{x}_1+\cdots+\vec{p}_n\cdot\vec{x}_n)}e^{-i(E_{p_1}+\cdots+E_{p_n})t}\,\tilde{\psi}^{[n]}_{\rm QM}(\vec{p}_1,\ldots,\vec{p}_n)\,, \tag{H6}$$

with associated 4D GEB spinor wave-function

$$\Psi^{[n]}_{\rm QM}(\bar{x}_1,\ldots,\bar{x}_n) = \int \frac{d^3p_1}{(2\pi)^{3/2}}\cdots\frac{d^3p_n}{(2\pi)^{3/2}} e^{i(\vec{p}_1\cdot\vec{x}_1+\cdots+\vec{p}_n\cdot\vec{x}_n)}e^{-i(E_{p_1}t_1+\cdots+E_{p_n}t_n)}\,\tilde{\psi}^{[n]}_{\rm QM}(\vec{p}_1,\ldots,\vec{p}_n)\,. \tag{H7}$$

Similarly for the Dirac equation from equation (G10), setting

$$\phi^{[n]}(\vec{p}_1,\sigma_1';\ldots;\vec{p}_n,\sigma_n') := \sum_{\vec{\ell}}\alpha_{\vec{\ell}}\,\alpha_{\sigma_1'}(\vec{p}_1)\cdots\alpha_{\sigma_n'}(\vec{p}_n)\,, \tag{H8}$$

we get

$$\begin{aligned}\Psi^{[n]}_{\rm QM}(\vec{x}_1,\sigma_1;\ldots;\vec{x}_n,\sigma_n|t) = \int \frac{d^3p_1}{(2\pi)^{3/2}}\cdots\frac{d^3p_n}{(2\pi)^{3/2}}\sum_{\sigma_1'=1}^{4}\frac{u_{\sigma_1',\sigma_1}(\vec{p}_1)}{\sqrt{2\pi}}\cdots\sum_{\sigma_n'=1}^{4}\frac{u_{\sigma_n',\sigma_n}(\vec{p}_n)}{\sqrt{2\pi}}\,e^{i(\vec{p}_1\cdot\vec{x}_1+\cdots+\vec{p}_n\cdot\vec{x}_n)}\\ \times e^{-i(E^{(\sigma_1')}_{p_1}+\cdots+E^{(\sigma_n')}_{p_n})t}\,\phi^{[n]}(\vec{p}_1,\sigma_1';\ldots;\vec{p}_n,\sigma_n')\,.\end{aligned} \tag{H9}$$

which at the level of GEB corresponds to

$$\begin{aligned}\Psi^{[n]}_{\rm QM}(\bar{x}_1,\sigma_1;\ldots;\bar{x}_n,\sigma_n) = \int \frac{d^3p_1}{(2\pi)^{3/2}}\cdots\frac{d^3p_n}{(2\pi)^{3/2}}\sum_{\sigma_1'=1}^{4}\frac{u_{\sigma_1',\sigma_1}(\vec{p}_1)}{\sqrt{2\pi}}\cdots\sum_{\sigma_n'=1}^{4}\frac{u_{\sigma_n',\sigma_n}(\vec{p}_n)}{\sqrt{2\pi}}\,e^{i(\vec{p}_1\cdot\vec{x}_1+\cdots+\vec{p}_n\cdot\vec{x}_n)}\\ \times e^{-i(E^{(\sigma_1')}_{p_1}t_1+\cdots+E^{(\sigma_n')}_{p_n}t_n)}\,\phi^{[n]}(\vec{p}_1,\sigma_1';\ldots;\vec{p}_n,\sigma_n')\,.\end{aligned} \tag{H10}$$

H.2. Lorentz transformations

Equation (H1) represents the (time-dependent) 3D+1 spinor wave-function that an observer O would assign to describe the state of the n particles on his reference frame R. Assuming the dynamical evolution is relativistic consistent (e.g. the particles obey KG or Dirac dynamical equations), we are now interested in determining the spinor 3D wave-function a second observer O' sitting in the reference frame R' with 4D coordinates $\bar{x}'$ that are linked with those of R as in equation (11) will assign to such a state. Since particles are independent (i.e. no interactions are present in the model), this can be done using the decomposition (52) for $\Psi^{[n]}_{\rm QM}(\vec{x}_1,\sigma_1;\ldots;\vec{x}_n,\sigma_n|t$ and applying the single-particle transformation (51) to each individual term $\Psi^{(\ell)}_{\rm QM}(\vec{x},\sigma|t)$. Accordingly, we can write

$$\begin{aligned}\Psi^{[n]\,\prime}_{\rm QM}(\vec{x}_1,\sigma_1;\ldots;\vec{x}_n,\sigma_n|t) = \sum_{\sigma_1,\ldots,\sigma_n} S^{-1}_{\sigma_1',\sigma_1}(\Lambda)\cdots S^{-1}_{\sigma_n',\sigma_n}(\Lambda)\\ \times\sum_{\vec{\ell}}\alpha_{\vec{\ell}}\,\Psi^{(\ell_1)}_{\rm QM}(\Lambda^{-1}\bar{x}_1,\sigma_1')|_{t_1=t}\cdots\Psi^{(\ell_n)}_{\rm QM}(\Lambda^{-1}\bar{x}_n,\sigma_n')|_{t_n=t}\,,\end{aligned} \tag{H11}$$

whose r.h.s. exactly matches with the one of equation (61) of the main text. It is important to stress that while not immediately evident from the resulting expression $\Psi^{[n]\,\prime}_{\rm QM}(\vec{x}_1,\sigma_1;\ldots;\vec{x}_n,\sigma_n|t)$ does not depends upon the specific choice of the local decomposition used in (H4). One easy way to verify this is e.g. to use the fact that (54) does not depends on such a choice (see previous section) and the fact that thanks to equation (61) we can write

$$\begin{aligned}\Psi^{[n]\,\prime}_{\rm QM}(\vec{x}_1,\sigma_1;\ldots;\vec{x}_n,\sigma_n|t) &= \langle\bar{x}_1,\sigma_1;\ldots;\bar{x}_n,\sigma_n|\Psi^{[n]\,\prime}_{\rm QM}\rangle\Big|_{t_1=\cdots=t_n=t}\\ &= \langle\bar{x}_1,\sigma_1;\ldots;\bar{x}_n,\sigma_n|U^{\otimes n}_{\Lambda}|\Psi^{[n]}_{\rm QM}\rangle\Big|_{t_1=\cdots=t_n=t}\,,\end{aligned} \tag{H12}$$

where we made use of the invertion formula (55) and of equation (60).

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