

Article

The Spinor-Tensor Gravity of the Classical Dirac Field

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Abstract: In this work, with the help of the quantum hydrodynamic formalism, the gravitational equation associated with the classical Dirac field is derived. The hydrodynamic representation of the Dirac equation described by the evolution of four mass densities, subject to the theory-defined quantum potential, has been generalized to the curved space-time in the covariant form. Thence, the metric of space-time has been defined by imposing the minimum action principle. The derived gravity shows the spontaneous emergence of the “cosmological” gravity tensor (CGT), a generalization of the classical cosmological constant (CC), as a part of the energy-impulse tensor density (EITD). Even if the classical cosmological constant is set to zero, the CGT is non-zero, allowing a stable quantum vacuum (out of the collapsed branched polymer phase). The theory shows that in the classical macroscopic limit, the general relativity equation is recovered. In the perturbative approach, the CGT leads to a second-order correction to Newtonian gravity that takes contribution from the space where the mass is localized (and the space-time is curvilinear), while it tends to zero as the space-time approaches the flat vacuum, leading, as a means, to an overall cosmological constant that may possibly be compatible with the astronomical observations. The Dirac field gravity shows analogies with the modified Brans–Dicke gravity, where each spinor term brings an effective gravity constant G divided by its field squared. The work shows that in order to obtain the classical minimum action principle and the general relativity limit of the macroscopic classical scale, quantum decoherence is necessary.

Keywords: einstein gravity of charged fermion field; quantum potential-induced gravity; quantum potential of charged spin-half particles; cosmological gravity tensor of the quantum Dirac field; cosmological gravity tensor expectation value of fermions; spinor-tensor gravity; Brans–Dicke gravity

1. Introduction

One of the physics problems nowadays is to describe how the quantum mechanical properties of space-time (ST) and the second quantization affect gravity.

Even if general relativity (GR) has opened up some understanding about cosmological dynamics [1–6], the complete explanation of the generation of matter and its distribution in the universe need the integration of cosmological physics with quantum physics. To this end, quantum gravity (QG) represents the goal of the century for theoretical research [7–9]. Nevertheless, difficulties arise when one attempts to apply the quantum field approach to the force of gravity [10,11].

The difficulties of the integration of quantum field theory (QFT) and the GR become really evident in the so-called cosmological constant (CC) that Einstein introduced into its equation to give stability to the solution of universe evolution, but he then refused [6].

In fact, the renewed interest in the CC is basically semi-empirical arguments to explain the motion of the galaxies [10,11]. The great difficulty of the QFT to give a correct value to the CC relies on the fact that the energy-impulse tensor density (EITD) in the Einstein equation [12] (for classical bodies) owns a point-dependence by mass density, but does not have any analytical connection to the fields of matter described by the Dirac or Klein-Gordon equation (KGE).

If we place matter into a classic ST, each infinitesimal element of mass can be put freely in an infinitesimal volume without any interference with the neighboring ones. In quantum mechanical ST, things are different, and it is increasingly difficult to add more and more matter to the same place (e.g., forbidding punctual mass distribution).

The difference is the so-called quantum potential, an additional energy source that, in the hydrodynamic description of quantum mechanics, is added to the classical equation of motion. Therefore, the curvature of the space-times induced by the mass-energy density distribution is different whether or not we are considering a classical vacuum or a quantum mechanical vacuum that leads to distinct types of gravity. The incompleteness of general relativity in describing the gravity of the physical ST is recognized by all, and modified gravity models (on a semi-empirical basis) have been proposed. One example is scalar-tensor gravity, about which Brans–Dicke model [13–15] is the most known example. If, on the one hand, the conceptual generalization about the possible sources of gravity has been fruitful, showing how problems such as the cosmological constant, the universe inflation and the dark matter can be solved [16–18], on the other hand, it has brought the appearance of a scalar field of obscure physical reading.

The derivation of the quantum-mechanical gravity of mass distribution, obeying the Klein–Gordon equation [19] by using a classical-like hydrodynamic quantum representation that leads to terms that are identical to those contained in the Brans–Dicke model, shows that the scalar field of the BD theory, by fact, reproduces the effects of the quantum-mechanical property of the ST upon gravity.

Taking in account the contribution of the quantum potential energy (i.e., the energy connected to the pilot wave) in the definition of space-time curvature, the author has shown that the quantization of a scalar-uncharged field may lead to a mean cosmological constant value (on the universe scale) compatible with the astronomical observation, with the energy cut-off defined by the Planck wavelength [20].

This work generalizes the gravitational equation (GE) of the scalar KGE field [17] to the field of half-integer spin-charged particles obeying the Dirac-Fock-Weyl equation (DFWE).

In Section 2, the work defines the Lagrangian motion equation of the hydrodynamic representation of the Dirac field in curved space-time, describing the evolution of four mass densities related to the bispinor fields and subject to the nonlocal interaction of the theory-defined quantum potential (QP) [20–26]. This procedure allows us to include the QP energy in the definition of space-time curvature.

In Section 3, the gravity generated by the bispinor fields is derived using the minimum action principle [21] by postulating the covariance of the motion equations in curved space-time.

In Section 4, in order to obtain the weak gravity limit, the perturbative approach to the gravity-DFWE system of coupled equations is developed.

In Section 5, the expectation value of the cosmological constant generated by the quantum Dirac field is derived for quasi-Minkowskian space-time.

In Section 6, some aspects of the gravity of the Dirac field are discussed:

- i. The Dirac field gravity and the cosmological constant in pure quantum gravity;
- ii. Analogies with the Brans-Dicke model;
- iii. Quantum mechanical gravity and the foundations of quantum mechanics;
- iv. Possible experimental tests.

2. The Hydrodynamic Dirac Equation in Curved Space-Time

In order to derive the gravitational equation for the (classical) Dirac field, we make use of the hydrodynamic representation of its bispinor field, $\Psi = \begin{pmatrix} \psi_{1i} \\ \psi_{2i} \end{pmatrix} i = 1, 2$, that is equivalent to four mass densities

$$|\psi_{\pm i}|^2 = \left| \frac{\psi_{1i} \pm \psi_{2i}}{\sqrt{2}} \right|^2 \text{ (see Appendix A) moving with momenta } p_{\pm\mu} = \begin{pmatrix} p_{\pm 1\mu} \\ p_{\pm 2\mu} \end{pmatrix} = -\partial_\mu S_\pm = -\partial_\mu \begin{pmatrix} S_{\pm 1} \\ S_{\pm 2} \end{pmatrix}$$

(being $\psi_{\pm i} = |\psi_{\pm i}| \exp\left[i \frac{S_{\pm i}}{\hbar}\right]$) under the action of the quantum potential (A11). The gravitational curvature of space-time is obtained by imposing the assumption that the overall action comprehending the

gravitational field [19] is stationary. To this end, we generalize Lagrangian Euclidean formalism (given in Appendix A) to the curved space-time.

This method is supported by the fact that the hydrodynamic approach leads to the derivation of the Einstein equation with the correct electromagnetic energy-impulse tensor when applied to the boson field of the photon [19] (see Appendix B).

On the other hand, the experimental demonstration of the physical existence of quantum potential energy by the Bohm-Aharonov effect supports this approach.

One possible generalization of the Dirac equation to the curved space-time can be obtained by assuming the physics covariance of the equation. In reference [27], it has been shown that the physics covariance condition plays the same theoretical role as the inertial to gravitational mass equivalence postulate in classical general relativity.

The covariant Dirac equation in curved space-time, with respect to affine and spin connections, reads [28]

$$\left(i\hbar\gamma^\mu\left(\partial_\mu - \frac{i}{4}\sigma^{ab}\omega_{ab\mu} + \frac{ie}{\hbar}A_\mu\right) - mc\right)\Psi = 0 \tag{1}$$

where

$$\sigma^{ab} = \frac{i}{2}[\gamma^a, \gamma^b] \tag{2}$$

and

$$\omega_{ab\mu} = f_b^\alpha e_{a\beta} \Gamma_{\mu\alpha}^\beta - f_b^\alpha \partial_\mu e_{a\alpha} \tag{3}$$

where e_b^α , and f_b^α , are the vielbein and the inverse vielbein, respectively, and $\Gamma_{\mu\alpha}^\beta$ is the Christoffel matrix that reads

$$\Gamma_{\mu\alpha}^\beta = \frac{1}{2}g^{\beta\gamma}(\partial_\alpha g_{\gamma\mu} + \partial_\mu g_{\gamma\alpha} - \partial_\gamma g_{\mu\alpha}) \tag{4}$$

By assuming the covariant derivative for affine and spinor connections

$$D_\mu = \partial_\mu - \frac{i}{4}\sigma^{ab}\omega_{ab\mu} \tag{5}$$

Equation (1) more simply reads

$$\left(i\hbar\gamma^\mu\left(D_\mu + \frac{ie}{\hbar}A_\mu\right) - mc\right)\Psi = 0 \tag{6}$$

leading, by following the procedure in Appendix A, to

$$\frac{i\hbar}{mc}\sigma^\mu\left(D_\mu + \frac{ie}{\hbar}A_\mu\right)\psi_+ = \psi_- \tag{7}$$

$$\frac{i\hbar}{mc}\tilde{\sigma}^\mu\left(D_\mu + \frac{ie}{\hbar}A_\mu\right)\psi_- = \psi_+ \tag{8}$$

to the covariant equation

$$\left(g^{\mu\nu} + \alpha_\pm^{\mu\nu}\right)\left(\partial_\mu - \frac{i}{4}\sigma^{ab}\omega_{ab\mu} + \frac{ie}{\hbar}A_\mu\right)\left(\partial_\nu - \frac{i}{4}\sigma^{ab}\omega_{ab\nu} + \frac{ie}{\hbar}A_\nu\right)\psi_\pm = -\frac{m^2c^2}{\hbar^2}\psi_\pm \tag{9}$$

and to Equation

$$\left(\begin{array}{l} \left(\partial_\mu - \frac{i}{4}\sigma^{ab}\omega_{ab\mu} + \frac{ie}{\hbar}A_\mu\right)\left(g^{\mu\nu}\left(\partial_\nu - \frac{i}{4}\sigma^{ab}\omega_{ab\nu} + \frac{ie}{\hbar}A_\nu\right)\right) \\ + g^{\mu\beta}g^{\nu\alpha}\alpha_{\pm\beta\alpha}\left(\frac{ie}{2\hbar}F_{\mu\nu} - \frac{i}{4}\sigma^{ab}\left(\omega_{ab\mu}\partial_\nu + \partial_\mu\omega_{ab\nu}\right)\right) \end{array}\right)\psi_\pm = -\left(\frac{m^2c^2}{\hbar^2} + \right)\psi_\pm \tag{10}$$

where, due to the antisymmetry of $\alpha_\pm^{\mu\nu}$, the properties

$$\alpha_\pm^{\mu\nu}\sigma^{ab}\omega_{ab\mu}\sigma^{ab}\omega_{ab\nu} = 0 \tag{11}$$

$$\alpha_{\pm}^{\mu\nu} \sigma^{ab} \omega_{ab\mu} A_{\nu} + \alpha_{\pm}^{\mu\nu} A_{\mu} \sigma^{ab} \omega_{ab\nu} = 0 \quad (12)$$

have been used. Moreover, by using Equation (9) and equating the real part of Equation (10), the HHJE in curved space-time reads (see Appendix C)

$$\begin{aligned} & (\partial_{\mu} S_{\pm} - (\operatorname{Re}\{\frac{\hbar}{4} \sigma^{ab} \omega_{ab\mu}\} - eA_{\mu})) g^{\mu\nu} (\partial_{\nu} S_{\pm} - (\operatorname{Re}\{\frac{\hbar}{4} \sigma^{ab} \omega_{ab\nu}\} - eA_{\nu})) \\ & = (m^2 c^2 - mV_{qu\pm}) \end{aligned} \quad (13)$$

where (see Appendix C) the quantum potential (introducing the quantum non-local properties (see Appendix D)) reads

$$\begin{aligned} V_{qu} = & -\frac{\hbar^2}{m} \left(\left(\frac{\partial_{\mu} (g^{\mu\nu} \partial_{\nu} |\psi_{\pm 1}|)}{|\psi_{\pm 1}|} \right) + \frac{\hbar^2}{2} \operatorname{Im}\{\sigma^{ab} \omega_{ab\mu}\} g^{\mu\nu} \left(\frac{\partial_{\nu} |\psi_{\pm 1}|}{|\psi_{\pm 1}|} \right) \right. \\ & \left. + \partial_{\mu} g^{\mu\nu} \left(\frac{\operatorname{Im}\{\sigma^{ab} \omega_{ab\nu}\}}{4} \right) + g^{\mu\nu} \operatorname{Im}\left\{ \frac{1}{4} \sigma^{ab} \omega_{ab\mu} \right\} \operatorname{Im}\left\{ \frac{1}{4} \sigma^{ab} \omega_{ab\nu} \right\} \right) \\ & + \operatorname{Re} \left(g^{\mu\beta} g^{\nu\alpha} \left(\frac{ie}{2\hbar} \begin{pmatrix} \alpha_{\pm\beta\alpha 1j} \frac{|\psi_{\pm 1}|}{|\psi_{\pm 1}|} \\ \alpha_{\pm\beta\alpha 2j} \frac{|\psi_{\pm 2}|}{|\psi_{\pm 2}|} \end{pmatrix} F_{\mu\nu} \right. \right. \\ & \left. \left. - \frac{i}{4} \alpha_{\pm\beta\alpha} \sigma^{ab} \left(\omega_{ab\mu} \begin{pmatrix} \frac{\partial_{\nu} |\psi_{\pm 1}|}{|\psi_{\pm 1}|} \\ \frac{\partial_{\nu} |\psi_{\pm 2}|}{|\psi_{\pm 2}|} \end{pmatrix} + \begin{pmatrix} \frac{\partial_{\mu} \omega_{ab\nu} |\psi_{\pm 1}|}{|\psi_{\pm 1}|} \\ \frac{\partial_{\mu} \omega_{ab\nu} |\psi_{\pm 2}|}{|\psi_{\pm 2}|} \end{pmatrix} \right) \right) \right) \end{aligned} \quad (14)$$

The EITD and the Hydrodynamic Lagrangian Function for Eigenstates in Curved Space-Time

By assuming the covariance of the motion Equations (A44)–(A46) and (A48) in Appendix A, we need to define both the corresponding EITD and the Lagrangian function in order to give explicit meaning to them. By using Equation (3) for the k -th eigenstates, it follows that

$$p_{\pm i(k)\mu} = \left(m\gamma \dot{q}_{\pm i(k)\mu} \sqrt{1 - \frac{V_{qu\pm(k)}}{mc^2}} - \operatorname{Re}\left\{ \frac{\hbar}{4} \sigma^{ab} \omega_{ab\mu} \right\} + eA_{\mu} \right) \quad (15)$$

and that

$$\begin{aligned} L_{\pm i(k)} &= -\frac{mc^2}{\gamma} \sqrt{1 - \frac{V_{qu\pm(k)}}{mc^2}} - \left(\operatorname{Re}\left\{ \frac{\hbar}{4} \sigma^{ab} \omega_{ab\mu} \right\} - eA_{\mu} \right) \dot{q}_{\pm i(k)}^{\mu} \\ &= L_{\pm i(k)class} + L_{\pm i(k)Q} \end{aligned} \quad (16)$$

where

$$L_{\pm i(k)class} = -\frac{mc^2}{\gamma} - \left(\operatorname{Re}\left\{ \frac{\hbar}{4} \sigma^{ab} \omega_{ab\mu} \right\} - eA_{\mu} \right) \dot{q}_{\pm i(k)}^{\mu} \quad (17)$$

$$\begin{aligned} L_{\pm i(k)Q} &= a_{(V_{qu\pm i(k)})} \frac{mc^2}{\gamma} \\ &= -\left(1 - \sqrt{1 - \frac{V_{qu\pm i(k)}}{mc^2}} \right) \left(L_{\pm i(k)class} + \left(\operatorname{Re}\left\{ \frac{\hbar}{4} \sigma^{ab} \omega_{ab\mu} \right\} - eA_{\mu} \right) \dot{q}_{\pm i(k)}^{\mu} \right) \end{aligned} \quad (18)$$

$$\begin{aligned} p_{\pm i(k)class}^{\mu} &= \left(m\gamma_{\pm i} \dot{q}_{\pm i(k)}^{\mu} - \left(\operatorname{Re}\left\{ \frac{\hbar}{4} \sigma^{ab} \omega_{ab\mu} \right\} - eA_{\mu} \right) \right) \\ &= \frac{p_{\pm i(k)}^{\mu} + \left(\operatorname{Re}\left\{ \frac{\hbar}{4} \sigma^{ab} \omega_{ab\mu} \right\} - eA_{\mu} \right) \left(1 - \sqrt{1 - \frac{V_{qu\pm i(k)}}{mc^2}} \right)}{\sqrt{1 - \frac{V_{qu\pm i(k)}}{mc^2}}} \\ &= \frac{\partial_{\mu} S_{\pm i(k)} - \left(\operatorname{Re}\left\{ \frac{\hbar}{4} \sigma^{ab} \omega_{ab\mu} \right\} - eA_{\mu} \right) a_{(V_{qu\pm i(k)})}}{1 - a_{(V_{qu\pm i(k)})}} \end{aligned} \quad (19)$$

$$\begin{aligned} p_{\pm i(k)Q}^{\mu} &= -a_{(V_{qu\pm i(k)})} m\gamma_{\pm i} \dot{q}_{\pm i(k)}^{\mu} \\ &= -a_{(V_{qu\pm i(k)})} \left(p_{\pm i(k)class}^{\mu} + \left(\operatorname{Re}\left\{ \frac{\hbar}{4} \sigma^{ab} \omega_{ab\mu} \right\} - eA_{\mu} \right) \right) \end{aligned} \quad (20)$$

As far as it concerns the EITD $T_{\pm i(k)\mu}{}^{\nu} = |\psi_{\pm i(k)}|^2 T_{\pm i(k)\mu}{}^{\nu}$, we obtain that

$$T_{\pm i(k)\mu}{}^{\nu} = T_{\pm i(k)Class\mu}{}^{\nu} + L_{\pm i(k)Class}\delta_{\mu}{}^{\nu} + T_{\pm i(k)Q\mu}{}^{\nu} = m\gamma \begin{pmatrix} \dot{q}_{\pm i(k)\mu} \left(\sqrt{1 - \frac{V_{qu\pm(\pm k)}}{mc^2}} \dot{q}_{\pm i(k)}{}^{\nu} - \left(\text{Re} \left\{ \frac{\hbar}{4} \sigma^{ab} \omega_{ab}{}^{\nu} \right\} - eA^{\nu} \right) \right) \\ -\dot{q}_{\pm i(k)\alpha} \left(\sqrt{1 - \frac{V_{qu\pm(\pm k)}}{mc^2}} \dot{q}_{\pm i(k)}{}^{\alpha} - \left(\text{Re} \left\{ \frac{\hbar}{4} \sigma^{ab} \omega_{ab}{}^{\alpha} \right\} - eA^{\alpha} \right) \right) \delta_{\mu}{}^{\nu} \end{pmatrix} \quad (21)$$

where

$$T_{\pm i(k)Class\mu}{}^{\nu} = \frac{mc^2}{\gamma} u_{\pm i(k)\mu} \left(u_{\pm i(k)}{}^{\nu} - \frac{1}{mc} \left(\text{Re} \left\{ \frac{\hbar}{4} \sigma^{ab} \omega_{ab}{}^{\nu} \right\} - eA^{\nu} \right) \right) \quad (22)$$

where

$$T_{\pm i(k)Q\mu}{}^{\nu} = -a_{(V_{qu\pm i(k)})} \frac{mc^2}{\gamma_{\pm i(k)}} \left(u_{\pm i(k)\mu} u_{\pm i(k)}{}^{\nu} - u_{\pm i(k)\alpha} u_{\pm i(k)}{}^{\alpha} \delta_{\mu}{}^{\nu} \right) \quad (23)$$

Finally, by using the identities Equation (1), the identity Equation (7) can be rearranged as follows:

$$T_{\pm i(k)\mu\nu} = g_{\nu\kappa} T_{\pm i(k)\mu}{}^{\kappa} = - \left(-g_{\nu\kappa} \dot{q}_{\mu\pm i(k)} p^{\kappa}{}_{\pm i(k)} + g_{\alpha\kappa} \dot{q}_{\pm i(k)} p^{\alpha}{}_{\pm i(k)} g_{\mu\nu} \right) = c^2 \left[\frac{\partial S_{\pm i(k)}}{\partial t} - c \left(\text{Re} \left\{ \frac{\hbar}{4} \sigma^{ab} \omega_{ab0} \right\} - eA_0 \right) \right]^{-1} \begin{pmatrix} g_{\nu\kappa} \left(p_{\mu\pm i(k)} + \left(\text{Re} \left\{ \frac{\hbar}{4} \sigma^{ab} \omega_{ab\mu} \right\} - eA_{\mu} \right) \right) p^{\kappa}{}_{\pm i(k)} \\ -g_{\alpha\kappa} \left(p^{\kappa}{}_{\pm i(k)} + \left(\text{Re} \left\{ \frac{\hbar}{4} \sigma^{ab} \omega_{ab}{}^{\kappa} \right\} - eA^{\kappa} \right) \right) p^{\alpha}{}_{\pm i(k)} g_{\mu\nu} \end{pmatrix} \quad (24)$$

that, as a function of the DFW field, reads

$$T_{\pm i(k)\mu}{}^{\nu} = -\frac{\hbar c^2}{2i} \left(\frac{\partial}{\partial t} \ln \left[\frac{\psi_{\pm i(k)}}{\psi_{\pm i(k)}^*} \right] - \frac{2i}{\hbar} \left(\text{Re} \left\{ \frac{\hbar}{4} \sigma^{ab} \omega_{ab0} \right\} - eA_0 \right) \right)^{-1} \begin{pmatrix} \left(\frac{\partial \ln \left[\frac{\psi_{\pm i(k)}}{\psi_{\pm i(k)}^*} \right]}{\partial q^{\mu}} + \frac{2i}{\hbar} \left(\text{Re} \left\{ \frac{\hbar}{4} \sigma^{ab} \omega_{ab\mu} \right\} - eA_{\mu} \right) \right) \frac{\partial \ln \left[\frac{\psi_{\pm i(k)}}{\psi_{\pm i(k)}^*} \right]}{\partial q^{\nu}} \\ -g_{\alpha\kappa} \left(\frac{\partial \ln \left[\frac{\psi_{\pm i(k)}}{\psi_{\pm i(k)}^*} \right]}{\partial q^{\kappa}} + \frac{2i}{\hbar} \left(\text{Re} \left\{ \frac{\hbar}{4} \sigma^{ab} \omega_{ab}{}^{\kappa} \right\} - eA^{\kappa} \right) \right) \frac{\partial \ln \left[\frac{\psi_{\pm i(k)}}{\psi_{\pm i(k)}^*} \right]}{\partial q^{\alpha}} \delta_{\mu}{}^{\nu} \end{pmatrix} \quad (25)$$

where $\psi_{\pm i(k)}$ represents the eigenstates of the DFWE in curved space-time that for positive $\psi_{\pm i(k)}^{(+)}$ and negative $\psi_{\pm i(k)}^{(-)}$ solutions reads, respectively

$$\psi_{\pm i(k)}^{(+)} = \alpha_{\pm i(k)} \exp \left(\frac{iS_{\pm i(k)}}{\hbar} \right) \quad (26)$$

$$\psi_{\pm i(k)}^{(-)} = \beta_{\pm i(k)}^* \exp \left(\frac{iS_{\pm i(k)}(-k)}{\hbar} \right) \quad (27)$$

3. The Minimum Action in Curved Space-TIME and the Gravity Equation for the DFWE

In this section, we derive the gravity equation by applying the minimum action principle to the quantum hydrodynamic evolution associated with the fields $\psi_{\pm i}$.

Given that the quantum hydrodynamic equations in the Minkowskian space-time satisfy the minimum action principle when we consider the covariant formulation in curved space-time, such variation takes a contribution from the variability of the metric tensor. When we consider the gravity and we assume that the geometry of the space-time is that one which makes null the overall variation of the action, we have a condition that leads to the definition of the GE.

By considering the variation of the action due both to the curvilinear coordinates and to the functional dependence by $\psi_{\pm i}$ [19] it follows that [27]

$$\delta S_D = \delta S_{\pm i} - \delta(\Delta S_{\pm i}) = \frac{1}{c} \int \iiint |\psi_{\pm i}|^2 \sum_k \left(\begin{array}{l} \frac{\partial \sqrt{-g} \bar{L}_{\pm i(k)}}{\partial g^{\mu\nu}} - \frac{\partial}{\partial q^\lambda} \frac{\partial \sqrt{-g} \bar{L}_{\pm i(k)}}{\partial \frac{\partial g^{\mu\nu}}{\partial q^\lambda}} \\ - \frac{\partial}{\partial |\psi_k|} \frac{\partial \sqrt{-g} \bar{L}_{\pm i(k)}}{\partial \frac{\partial g^{\mu\nu}}{\partial |\psi_{\pm ik}|}} \end{array} \right) \delta g^{\mu\nu} d\Omega \quad (28)$$

By imposing that the variation of action of the gravitational field [29]

$$\delta S_g = \frac{c^3}{16\pi G} \int \iiint \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) \delta g^{\mu\nu} \sqrt{-g} d\Omega \quad (29)$$

offsets that one produced both by the DFWE field, δS_D , and by the EM field [29]

$$\delta S_{em} = \frac{1}{c} \int \iiint (T_{\mu\nu em} \delta g^{\mu\nu} \sqrt{-g} d\Omega \quad (30)$$

where, since the equations for the EM field are the same both in the classical and quantum cases and the hydrodynamic description coincides with the classical one (i.e., the quantum potential is null), the EITD for the EM field $T_{\mu\nu em}$ equals the classical form [19,27] (see Appendix B). Therefore, it follows that

$$\begin{aligned} & \delta S_g + \sum_{\pm, i} (\delta S_{\pm i} - \delta(\Delta S_{\pm i})) + \delta S_{em} \\ & = \int \iiint \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \frac{8\pi G}{c^4} \left(\sum_{\pm, i} |\psi_{\pm i}|^2 \tau_{\pm i\mu\nu} + T_{\nu\mu em} \right) \right) \delta g^{\mu\nu} \sqrt{-g} d\Omega = 0 \end{aligned} \quad (31)$$

and we obtain the gravitational equation for the DFWE field

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} \left(\sum_{\pm, i} |\psi_{\pm i}|^2 \tau_{\pm i\mu\nu} + T_{\nu\mu em} \right) \quad (32)$$

where the fermion tensor density $\tau_{\pm i\mu\nu}$ can be rearranged in the form [27]

$$\tau_{\pm i\mu\nu} = \tau_{\pm i\mu\nu class} + \tau_{\pm i\mu\nu Q} + \tau_{\pm i\mu\nu mix} + \tau_{\pm i\mu\nu curv} \quad (33)$$

where

$$\tau_{\pm i(k)\mu\nu class} = 2 \frac{1}{\sqrt{-g}} \left(\begin{array}{l} \frac{\partial}{\partial g^{\mu\nu}} - \frac{\partial}{\partial q^\lambda} \frac{\partial}{\partial \frac{\partial g^{\mu\nu}}{\partial q^\lambda}} \\ - \frac{\partial}{\partial |\psi_{\pm i(k)}|} \frac{\partial}{\partial \frac{\partial g^{\mu\nu}}{\partial |\psi_{\pm i(k)}|}} \end{array} \right) \sqrt{-g} p_{\pm i(k) class} \dot{q}^\mu_{\pm i} \quad (34)$$

$$\tau_{\pm i(k)\mu\nu Q} = 2 \frac{1}{\sqrt{-g}} \left(\begin{array}{l} \frac{\partial}{\partial g^{\mu\nu}} - \frac{\partial}{\partial q^\lambda} \frac{\partial}{\partial \frac{\partial g^{\mu\nu}}{\partial q^\lambda}} \\ - \frac{\partial}{\partial |\psi_{\pm i(k)}|} \frac{\partial}{\partial \frac{\partial g^{\mu\nu}}{\partial |\psi_{\pm i(k)}|}} \end{array} \right) \sqrt{-g} p_{\pm i(k) Q} \dot{q}^\mu_{\pm i} \quad (35)$$

$$\tau_{\pm i(k)\mu\nu mix} = 2 \frac{1}{\sqrt{-g}} \left(\begin{array}{l} \frac{\partial}{\partial g^{\mu\nu}} - \frac{\partial}{\partial q^\lambda} \frac{\partial}{\partial \frac{\partial g^{\mu\nu}}{\partial q^\lambda}} \\ - \frac{\partial}{\partial |\psi_{\pm i(k)}|} \frac{\partial}{\partial \frac{\partial g^{\mu\nu}}{\partial |\psi_{\pm i(k)}|}} \end{array} \right) \sqrt{-g} p_{\pm i(k) mix} \dot{q}^\mu_{\pm i} \quad (36)$$

where

$$p_{\pm i(k) tot}^\mu = p_{\pm i(k) class}^\mu + p_{\pm i(k) Q}^\mu + p_{\pm i(k) mix}^\mu \quad (37)$$

where

$$p_{\pm i(k) mix}^\mu = -\frac{\hbar}{i} \partial_\mu \ln \alpha_{\pm ik} \quad (38)$$

$$p_{\mu \pm i(-k)mix} = -\frac{\hbar}{i} \partial_{\mu} \ln \beta_{\pm ik}^* \tag{39}$$

and where $\tau_{\pm i \mu \nu curv} = \sum_k (\tau_{\pm i(k) \mu \nu curv} + \tau_{\pm i(-k) \mu \nu curv})$ is given in [27].

Furthermore, as shown in [27], the classical contribution $\tau_{\pm i(k) \mu}^{\nu}_{class}$ is calculated to be

$$\sum_{\pm i} |\psi_{\pm i(k)}|^2 \tau_{\pm i(k) \mu}^{\nu}_{class} = \sum_{\pm i} |\psi_{\pm i(k)}|^2 T_{\pm i(k) class \mu}^{\nu} - \Lambda \delta_{\mu}^{\nu} \tag{40}$$

where the classical CC Λ is independent by k .

3.1. The GE for the DFWE Eigenstates

By using Equation (19), (20), (34) and (35) for the k eigenstate, it follows that

$$\begin{aligned} \tau_{\pm i(k) \mu \nu Q} &= -a_{(V_{qu\pm i})} \left(\tau_{\pm i(k) \mu \nu class} + \Theta_{\mu \nu} + \begin{pmatrix} p_{\alpha \pm i(k) class} - eA_{\alpha} \\ + Re\{\frac{\hbar}{4} \sigma^{ab} \omega_{ab\alpha}\} \end{pmatrix} \cdot \dot{q}_{\pm i(k)} \Delta_{\pm i(k) \mu \nu} \right) \\ &= -a_{(V_{qu\pm i})} \left(\begin{pmatrix} T_{\pm i(k) class \mu \nu} - \frac{\Lambda}{|\psi_{\pm i(k)}|^2} g_{\mu \nu} + \Theta_{\mu \nu} \\ + \left(L_{\pm i(k) class} + \begin{pmatrix} Re\{\frac{\hbar}{4} \sigma^{ab} \omega_{ab\alpha}\} \\ -eA_{\alpha} \end{pmatrix} \cdot \dot{q}_{\pm i(k)} \right) \Delta_{\pm i(k) \mu \nu} \end{pmatrix} \right) \end{aligned} \tag{41}$$

where $\Theta_{\mu \nu}$ and $\Delta_{\pm i(k) \mu \nu}$ are curvature linked terms given in [27].

Thence, by using for the eigenstates the conditions $\tau_{\pm i \mu \nu mix} = \tau_{\pm i(k) \mu \nu mix} = 0$, $\tau_{\mu \nu curv} = 0$, the GE reads

$$R_{\mu \nu} - \frac{1}{2} R g_{\mu \nu} = \frac{8\pi G}{c^4} \left(\sum_{\pm i} \left(\begin{pmatrix} T_{\pm i(k) \mu \nu class} (1 - a_{(V_{qu\pm i(k)})}) \\ \Lambda g_{\mu \nu} + |\psi_{\pm i(k)}|^2 \Theta_{\mu \nu} \\ + a_{(V_{qu\pm i(k)})} \begin{pmatrix} L_{\pm i(k) class} \\ -|\psi_{\pm i(k)}|^2 \left(\begin{pmatrix} L_{\pm i(k) class} \\ + \begin{pmatrix} Re\{\frac{\hbar}{4} \sigma^{ab} \omega_{ab\alpha}\} \\ -eA_{\alpha} \end{pmatrix} \cdot \dot{q}_{\pm i(k)} \end{pmatrix} \right) \Delta_{\pm i(k) \mu \nu} \end{pmatrix} \right) \right) \right) \tag{42}$$

Finally, by separating both $\Delta_{\mu \nu}$ and $\Theta_{\mu \nu}$ in the isotropic and stress part as follows:

$$\Delta_{\pm i(k) \mu \nu} = \frac{\Delta_{\pm i(k) \lambda \lambda}}{4} g_{\mu \nu} + \Delta_{s \pm i(k) \mu \nu} \tag{43}$$

$$\Theta_{\mu \nu} = \frac{\Theta_{\lambda \lambda}}{4} + \Theta_{s \mu \nu} \tag{44}$$

Equation (42) reads

$$R_{\mu \nu} - \frac{1}{2} R g_{\mu \nu} = \frac{8\pi G}{c^4} \left(\sum_{\pm i} \left(\begin{pmatrix} T_{\pm i(k) \mu \nu class} (1 - a_{(V_{qu\pm i(k)})}) \\ \Lambda g_{\mu \nu} + |\psi_{\pm i(k)}|^2 \left(\frac{\Theta_{\lambda \lambda}}{4} g_{\mu \nu} + \Theta_{s \mu \nu} \right) \\ + a_{(V_{qu})} \begin{pmatrix} L_{\pm i(k) class} + \\ \left(\begin{pmatrix} Re\{\frac{\hbar}{4} \sigma^{ab} \omega_{ab\alpha}\} \\ -eA_{\alpha} \end{pmatrix} \cdot \dot{q}_{\pm i(k)} \right) \frac{\Delta_{\pm i(k) \lambda \lambda}}{4} g_{\mu \nu} \end{pmatrix} \\ -|\psi_{\pm i(k)}|^2 \begin{pmatrix} L_{\pm i(k) class} + \\ \left(\begin{pmatrix} Re\{\frac{\hbar}{4} \sigma^{ab} \omega_{ab\alpha}\} \\ -eA_{\alpha} \end{pmatrix} \cdot \dot{q}_{\pm i(k)} \right) \Delta_{s \pm i(k) \mu \nu} \end{pmatrix} \end{pmatrix} \right) \right) \right) \tag{45}$$

Equations (42) and (45) can be expressed as a function of the DFWE field Ψ by using the relation [27]

$$L_{\pm i(k)} = -c^2 \frac{i\hbar}{2} \left(\frac{\partial_t \ln[\frac{\psi_{\pm ik}}{\psi_{\pm ik}^*}] - e\phi}{+Re\{\frac{\hbar c}{4}\sigma^{ab}\omega_{ab0}\}} \right)^{-1} g_{\mu\alpha} \partial^\alpha \ln[\frac{\psi_{\pm ik}}{\psi_{\pm ik}^*}] \left(\frac{\partial^\mu \ln[\frac{\psi_{\pm ik}}{\psi_{\pm ik}^*}] - eA_\mu}{+Re\{\frac{\hbar c}{4}\sigma^{ab}\omega_{ab\mu}\}} \right) \tag{46}$$

where from Equation (15) the following identity has been used

$$\dot{q}_{\pm i(k)\mu} = -c^2 \frac{\partial_\mu \ln[\frac{\psi_{\pm ik}}{\psi_{\pm ik}^*}] + \frac{2i}{\hbar}(eA_\mu - Re\{\frac{\hbar c}{4}\sigma^{ab}\omega_{ab\mu}\})}{(\partial_t \ln[\frac{\psi_{\pm ik}}{\psi_{\pm ik}^*}] - \frac{2i}{\hbar}(e\phi - Re\{\frac{\hbar c}{4}\sigma^{ab}\omega_{ab0}\}))} \tag{47}$$

3.2. The GE of the DFWE Field

Finally, in the matrix form, the GE can be written as a function of the general DFWE field (A33) and can be arranged in the form

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4} (Tr(\tau_{\mu\nu class} - \Lambda g_{\mu\nu} - \Lambda_Q g_{\mu\nu} + \Delta\tau_{\mu\nu stress}) + T_{\nu\mu em}) \tag{48}$$

where

$$\Lambda = \Lambda\delta_{hk} \tag{49}$$

$$\tau_{\mu\nu class} = \sum_{\pm i} \left(\begin{array}{l} \varsigma_{\pm i(k)} \left(1 - a_{(V_{qu\pm i(k)})} \right) |\psi_{\pm i(k)}|^2 T_{\pm i(k)\mu\nu class} \\ + \varsigma_{\pm i(-k)} \left(1 - a_{(V_{qu\pm i(-k)})} \right) |\psi_{\pm i(-k)}|^2 T_{\pm i(-k)\mu\nu class} \end{array} \right) \delta_{hk} \tag{50}$$

where

$$\varsigma_{\pm i(k)} = \frac{1}{2} (\Xi_{\pm i(k)} + \Xi_{\pm i(k)}^*) \tag{51}$$

where $\Xi_{\pm i(k)}$ is given in ref. [27]

$$\Lambda_Q = \Lambda_{Q(k)} + \Lambda_{Q(-k)} \tag{52}$$

where

$$\Lambda_{Q(k)} = -\sum_{\pm i} \frac{1}{2} \left(\begin{array}{l} \Xi_{\pm i(k)} \left(\begin{array}{l} \Lambda + |\psi_{\pm i(k)}|^2 \frac{\Theta_{\Lambda\Lambda}}{4} \\ a_{\pm i(k)} \left(\begin{array}{l} p_{\alpha_{\pm i(k) class}} - eA_\alpha \\ + Re\{\frac{\hbar c}{4}\sigma^{ab}\omega_{ab\alpha}\} \end{array} \right) \dot{q}_{\pm i}^{\alpha \frac{\Delta_{\pm i(k)\Lambda\Lambda}}{4}} \end{array} \right) \\ + |\psi_{\pm i(k)}|^2 \tau_{\pm i(k)\kappa\kappa mix} \end{array} \right) \\ + \Xi_{\pm i(k)}^* \left(\begin{array}{l} \Lambda + |\psi_{\pm i(k)}|^2 \frac{\Theta_{\Lambda\Lambda}}{4} \\ a_{\pm i(k)} \left(\begin{array}{l} p_{\alpha_{\pm i(k) class}} - eA_\alpha \\ + Re\{\frac{\hbar c}{4}\sigma^{ab}\omega_{ab\alpha}\} \end{array} \right) \dot{q}_{\pm i}^{\alpha \frac{\Delta_{\pm i(k)\Lambda\Lambda}}{4}} \\ - |\psi_{\pm i(k)}|^2 \tau_{\pm i(k)\kappa\kappa mix} \end{array} \right) \\ + |\psi_{\pm i(k)}|^2 \frac{\tau_{\pm i(k)\Lambda curv}}{8} \end{array} \right) \delta_{hk} \tag{53}$$

and

$$\Delta\tau_{\mu\nu stress} = \Delta\tau_{(+k)\mu\nu stress} + \Delta\tau_{(-k)\mu\nu stress} \tag{54}$$

where

$$\Delta\tau_{(k)\mu\nu stress} = \sum_{\pm i} \frac{|\psi_{\pm i(k)}|^2}{2} \left(\begin{array}{c} \Xi_{\pm i(k)} \left(\begin{array}{c} a_{\pm i(k)} \left(\begin{array}{c} p\alpha_{\pm i(k) class} - eA_\alpha \\ + Re\{\frac{\hbar}{4}\sigma^{ab}\omega_{ab\alpha}\} \end{array} \right) \dot{q}^{\alpha}_{\pm i} \frac{\Delta_{s\pm i(k)\mu\nu}}{4} \\ - a_{\pm i(k)} \Theta_{s\mu\nu} - \tau_{s\pm i(k)\mu\nu mix} \end{array} \right) \\ + \Xi_{\pm i(k)}^* \left(\begin{array}{c} a_{\pm i(k)} \left(\begin{array}{c} p\alpha_{\pm i(k) class} - eA_\alpha \\ + Re\{\frac{\hbar}{4}\sigma^{ab}\omega_{ab\alpha}\} \end{array} \right) \dot{q}^{\alpha}_{\pm i} \frac{\Delta_{s\pm i(k)\mu\nu}}{4} \\ - a_{\pm i(k)} \Theta_{s\mu\nu} + \tau_{s\pm i(k)\mu\nu mix} \end{array} \right) \\ + \frac{\tau_{s\pm i\mu\nu curv}}{2} \end{array} \right) \delta_{hk} \quad (55)$$

where both $\tau_{\pm i(k)\mu\nu mix}$ and $\tau_{s\pm i\mu\nu curv}$ have been split into the isotropic and stress parts as follows:

$$\tau_{\pm i(k)\mu\nu mix} = \frac{\tau_{\pm i(k)\lambda\lambda mix}}{4} g_{\mu\nu} + \tau_{s\pm i(k)\mu\nu mix} \quad (56)$$

$$\tau_{s\pm i\mu\nu curv} = \frac{\tau_{\pm i\lambda\lambda curv}}{4} g_{\mu\nu} + \tau_{s\pm i\mu\nu curv} \quad (57)$$

Since Equation (45) holds for the eigenstates, it represents the decoherent macroscopic limit of Equation (48).

Equations (48)–(54) can be expressed as a function of the DFWE field Ψ by using the relations (A27, A35, A72) and

$$\dot{q}_{\pm i\mu} = -c^2 \frac{\partial_\mu \ln\left[\frac{\psi_{\pm i}}{\psi_{\pm i}^*}\right] + \frac{2i}{\hbar} \left(eA_\mu - Re\left\{\frac{\hbar c}{4}\sigma^{ab}\omega_{ab\mu}\right\} \right)}{\left(\partial_t \ln\left[\frac{\psi_{\pm i}}{\psi_{\pm i}^*}\right] - \frac{2i}{\hbar} \left(e\phi - Re\left\{\frac{\hbar c}{4}\sigma^{ab}\omega_{ab0}\right\} \right) \right)} \quad (58)$$

derived by (15) and (16).

The nonphysical hydrodynamic states are excluded by writing it as a function of the original bispinor field Ψ (corresponding just to the hydrodynamic irrotational states).

The main difference with the classical Einstein equation is given by the quantum-mechanical terms $a(V_{qu})$, $\Delta\tau_{\mu\nu stress}$, $\tau_{\pm i\mu\nu mix}$ and $\tau_{s\pm i\mu\nu curv}$ that in macroscopic classical scale problems, for weakly interacting potentials (i.e., $\lim_{\frac{\mathcal{L}}{\lambda_c} \rightarrow 0} V_{qu} = 0$ where \mathcal{L} is the physical length of the problem and λ_c is the De Broglie length), tends to zero $\lim_{V_{qu} \rightarrow 0} a(V_{qu}) = 0$, $\lim_{macro} \Delta\tau_{\mu\nu stress} = 0$, $\lim_{macro} \tau_{\pm i\mu\nu curv} = 0$, and $\lim_{macro} \tau_{\pm i\mu\nu mix} = 0$.

4. The GE-DFWE-EM System of Quantum Evolution

Once the system of evolutionary equations for the charged classical fermion field is defined in curved space-time

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4} \left(Tr(\tau_{\mu\nu class} - \Lambda g_{\mu\nu} - \Lambda_Q g_{\mu\nu} + \Delta\tau_{\mu\nu stress}) + T_{\nu\mu em} \right) \quad (59)$$

$$\left(i\hbar\gamma^\mu \left(\partial_\mu - \frac{i}{4}\sigma^{ab}\omega_{ab\mu} + \frac{ie}{\hbar}A_\mu \right) - mc \right) \Psi = 0 \quad (60)$$

$$F^{\mu\nu}{}_{;\nu} = -4\pi J^\mu \quad (61)$$

where the cosmological gravity tensor (CGT)

$$\Lambda g_{\mu\nu} + \Lambda_Q g_{\mu\nu} - \Delta\tau_{\mu\nu stress} \quad (62)$$

is given by the classical cosmological constant $\Lambda g_{\mu\nu}$ plus the quantum CGT $\Lambda_Q g_{\mu\nu} - \Delta\tau_{\mu\nu stress}$, it is possible to proceed with the quantization of the fermion field.

From the general point of view, we observe that the quantization of the Dirac field in curved space-time, with the metric defined by the GE (59), owns the following properties:

1. In the limit of flat space-time, the standard quantum electrodynamics (QED) is recovered (see Section 5).
2. Since all ordinary elementary particles (low energy vacuum states) own far mass densities very much smaller than the Planck mass in a sphere of Planck length, the first order of the approximation of the gravitational evolutionary system of equations can lead to precise results by using the perturbative approach.
3. For fermions in very high curved space-time or of Planckian mass, the superposition of the Dirac field eigenstates is quite different by the Fourier superposition of the Minkowskian case and the full treatment (e.g., high energy QFT) strongly diverges from the traditional results.

The Perturbative Approach to the GE-DFWE-EM Evolution

For particles very far from the Planckian mass density $\frac{m_p}{l_p^3} = \frac{c^5}{\hbar G^2}$, it is possible to solve the system of the GE Equations (59)–(61) by a perturbative iteration

$$R_{\mu\nu(\varepsilon_{\mu\nu})} \cong R^{(0)}_{\mu\nu(\varepsilon_{\mu\nu})} + R^{(1)}_{\mu\nu(\varepsilon_{\mu\nu})} + R^{(2)}_{\mu\nu(\varepsilon_{\mu\nu})} + \dots + R^{(n)}_{\mu\nu(\varepsilon_{\mu\nu})} \quad (63)$$

$$\Psi = \Psi^{(0)} + \Psi^{(1)} + \Psi^{(2)} + \dots + \Psi^{(n)} \quad (64)$$

for a small change of the space-time geometry ε^a_μ given by the vielbein e^a_μ transformation

$$e^a_\mu = \delta^a_\mu + \frac{\varepsilon^a_\mu}{2} \quad (65)$$

where ($\varepsilon_{\mu\nu}\varepsilon^{\mu\nu} \ll 1$), that leads (see Appendix E) to the metric tensor variation

$$g_{\mu\nu} \cong \eta_{\mu\nu} + \frac{\eta_{\mu a}\varepsilon^a_\nu + \eta_{\nu a}\varepsilon^a_\mu}{2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} + \varepsilon_{sim\mu\nu} \quad (66)$$

where

$$\varepsilon_{sim\mu\nu} = \frac{\eta_{\mu a}\varepsilon^a_\nu + \eta_{\nu a}\varepsilon^a_\mu}{2} \quad (67)$$

and where

$$\eta_{av}\varepsilon^a_\mu = \begin{bmatrix} \varepsilon^0_0 & \varepsilon^0_1 & \varepsilon^0_2 & \varepsilon^0_3 \\ -\varepsilon^1_0 & -\varepsilon^1_1 & -\varepsilon^1_2 & -\varepsilon^1_3 \\ -\varepsilon^2_0 & -\varepsilon^2_1 & -\varepsilon^2_2 & -\varepsilon^2_3 \\ -\varepsilon^3_0 & -\varepsilon^3_1 & -\varepsilon^3_2 & -\varepsilon^3_3 \end{bmatrix} \quad (68)$$

that for the successive steps of approximation $h^{(n)a}_\mu$

$$\varepsilon^a_\mu = h^{(1)a}_\mu + h^{(2)a}_\mu + \dots + h^{(n)a}_\mu \quad (69)$$

leads to

$$\begin{aligned} \varepsilon_{sim\mu\nu} &= \frac{\eta_{\mu a}h^{(1)a}_\nu + \eta_{\nu a}h^{(1)a}_\mu}{2} + \frac{\eta_{\mu a}h^{(2)a}_\nu + \eta_{\nu a}h^{(2)a}_\mu}{2} \\ &= h^{(1)}_{sim\mu\nu} + h^{(2)}_{sim\mu\nu} + \dots + h^{(n)}_{sim\mu\nu} \end{aligned} \quad (70)$$

where $\eta_{\nu\mu}$ satisfies the flat static solution $R_{\mu\nu}^{(0)}(\eta_{\nu\mu}) = 0$ of the zero-order GE

$$R_{\mu\nu}^{(0)} - \frac{1}{2}R^{(0)} = 0 \tag{71}$$

(we consider here this simple case since dynamical fluctuations of small vacuum curvature would lead both to the dark matter and possibly the quantum decoherence (in progress) on a macroscopic scale). Moreover, $\Psi^{(0)}$ is the solution of the zero-order DFWE

$$\left(i\hbar\gamma^\mu\left(\partial_\mu + \frac{ie}{\hbar}A_\mu\right) - mc\right)\Psi^{(0)} = 0 \tag{72}$$

$\Psi^{(1)}$ is the solution of the first order DFWE (see Appendix E)

$$\left(i\hbar\gamma^\mu\left(\partial_\mu + \frac{ie}{\hbar}A_\mu\right) - mc\right)\Psi^{(1)} \cong \frac{\hbar}{8}\gamma^\mu\sigma^{ab}(\partial_\mu\varepsilon_{ab})\Psi^{(0)} \tag{73}$$

and $h_{sim\mu\nu}^{(1)}$ is the solution of the first order GE

$$R_{\mu\nu}^{(1)}(h_{sim\mu\nu}^{(1)}) - \frac{1}{2}R_{\mu\nu}^{(1)}(h_{sim\mu\nu}^{(1)})g_{\mu\nu} = \frac{8\pi G}{c^4}\left(\text{Tr}\left(\tau_{\mu\nu}^{(0)} - \Lambda g_{\mu\nu} - \Lambda_Q^{(0)}g_{\mu\nu} + \Delta\tau_{\mu\nu}^{(0)}\right) + T_{\nu\mu}em\right) \tag{74}$$

where the Ricci tensor [19] reads

$$R_{\mu\nu}(h_{\mu\nu}) = \left(\partial_l\Gamma_{\mu\nu}^l - \partial_\nu\Gamma_{\mu l}^l + \Gamma_{\mu\nu}^l\Gamma_{lm}^m - \Gamma_{\mu l}^m\Gamma_{\nu m}^l\right) \tag{75}$$

where the given Christoffel symbols read [19]

$$\begin{aligned} \Gamma_{\mu\alpha}^\beta &= \frac{1}{2}\eta^{\beta\gamma}(\partial_\alpha\varepsilon_{sim\gamma\mu} + \partial_\mu\varepsilon_{sim\gamma\alpha} - \partial_\gamma\varepsilon_{sim\mu\alpha}) \\ &\cong \frac{1}{2}\eta^{\beta\gamma}\left(\left(\partial_\alpha h_{sim\gamma\mu}^{(1)} + \partial_\mu h_{sim\gamma\alpha}^{(1)} - \partial_\gamma h_{sim\mu\alpha}^{(1)}\right) + \dots\right) \end{aligned} \tag{76}$$

Moreover, by using the zero-order relations $V_{qu_0(k)} = 0, \alpha_{(V_{qu_0(k)})} = 0, \tau_{(k)\mu\nu}^{(0)} = 0$, the components $\tau_{\mu\nu}^{(0)}$, $\Delta\tau_{\mu\nu}^{(0)}$ and $\Lambda_Q^{(0)}g_{\mu\nu}$ read, respectively,

$$\tau_{\mu\nu}^{(0)} = \sum_{\pm i} \left(\zeta_{\pm i(k)}^{(0)}|\psi_{\pm i(k)}^{(0)}|^2 T_{\pm i(k)\mu\nu}^{(0)} + \zeta_{\pm i(-k)}^{(0)}|\psi_{\pm i(-k)}^{(0)}|^2 T_{\pm i(-k)\mu\nu}^{(0)}\right)\delta_{hk} \tag{77}$$

where $T_{\pm i(k)\mu\nu}^{(0)} = T_{\pm i(k)\mu\nu}^{(0)}(\psi_{\pm i}^{(0)})$ and where

$$T_{\pm i(k)\mu\nu}^{(0)} = \frac{mc^2}{\gamma}u_{\mu\pm i(k)}\left(u_{\nu\pm i(k)} - \frac{1}{mc}\left(\text{Re}\left\{\frac{\hbar}{4}\sigma^{ab}\omega_{ab\nu}\right\} - eA_\nu\right)\right) \tag{78}$$

$$\begin{aligned} u_{\pm i(k)\mu} &= \frac{p_{\pm i(k)\mu} + \text{Re}\left\{\frac{\hbar}{4}\sigma^{ab}\omega_{ab\mu}\right\} - eA_\mu}{mc\sqrt{1 - \frac{V_{qu\pm(k)}}{mc^2}}} \\ &= \frac{\frac{i\hbar}{2}g_{\mu\alpha}\partial^\alpha \ln\left[\frac{\psi_{\pm i(k)}}{\psi_{\pm i(k)}^*}\right] + \text{Re}\left\{\frac{\hbar}{4}\sigma^{ab}\omega_{ab\mu}\right\} - eA_\mu}{mc\sqrt{1 - \frac{V_{qu\pm(k)}}{mc^2}}} \end{aligned} \tag{79}$$

where $\varsigma_{\pm i(k)}^{(0)} = \varsigma_{\pm i(k)(\psi_{\pm i}^{(0)})}$, and, being $\tau_{\mu\nu curv(\psi 0(k))} = 0$,

$$\Delta\tau_{\mu\nu stress}^{(0)} = 0 \tag{80}$$

$$\Lambda_Q^{(0)} = 0 \tag{81}$$

the first-order GE reads

$$R_{\mu\nu}^{(1)}(h_{sim\mu\nu}^{(1)}) - \frac{1}{2}R_{\mu\nu}^{(1)}(h_{sim\mu\nu}^{(1)})g_{\mu\nu} = \frac{8\pi G}{c^4} \left(Tr(\tau_{\mu\nu class}^{(0)} - \Lambda g_{\mu\nu}) + T_{\nu\mu em} \right) \tag{82}$$

leading to the decoherent GE limit

$$R_{\mu\nu}^{dec}(h_{sim\mu\nu}^{(1)}) - \frac{1}{2}R_{\mu\nu}^{dec}(h_{sim\mu\nu}^{(1)})g_{\mu\nu} = \frac{8\pi G}{c^4} \left(\sum_{\pm i} |\psi_{\pm i(k)}^{(0)}|^2 T_{\pm i(k)\mu\nu class}^{(0)} + T_{\nu\mu em} - \Lambda g_{\mu\nu} \right) \tag{83}$$

where the identity $\varsigma_{\pm i(k)} = 1$ for the eigenstates has been used.

By performing the limit for $\hbar \rightarrow 0$ (when the DFWE disembogues into the KGE (A69), Equation (82) leads to the classical limit that coincides with the Einstein equation (see [19]).

The first contribution of the quantum part of the CGT $\Lambda_{\mu\nu}^{(qu)} = (\Delta\tau_{\mu\nu stress} - \Lambda_Q g_{\mu\nu})$, in Equation (59), comes from the second order of approximation (i.e., $\varepsilon_{sim\mu\nu} = h_{sim\mu\nu}^{(1)} + h_{sim\mu\nu}^{(2)}$)

$$R_{\mu\nu}^{(2)}(\varepsilon_{sim\mu\nu}) - \frac{1}{2}R_{\mu\nu}^{(2)}(\varepsilon_{sim\mu\nu})g_{\mu\nu} = \frac{8\pi G}{c^4} \left(Tr(\tau_{\mu\nu class}^{(1)} - \Lambda g_{\mu\nu} - \Lambda_Q^{(1)} g_{\mu\nu} + \Delta\tau_{\mu\nu stress}^{(1)}) + T_{\nu\mu em} \right) \tag{84}$$

where the fermion EITD $\tau_{\mu\nu class}^{(1)}$ is calculated by using $\psi_{\pm ik} \cong \psi_{\pm ik}^{(0)} + \psi_{\pm ik}^{(1)}$ obtained by Equation (73), where the CPTD $\Lambda_Q^{(1)}$ reads

$$\Lambda_Q^{(1)} = \Lambda_{Q(\psi_{\pm ik}^{(0)} + \psi_{\pm ik}^{(1)})} \equiv \Lambda_{\pm i(k)Q}^{(1)} \delta_{jk} \tag{85}$$

and where

$$\Delta\tau_{\mu\nu stress}^{(1)} = \Delta\tau_{\mu\nu stress(\psi_{\pm ik}^{(0)} + \psi_{\pm ik}^{(1)})} \equiv \Delta\tau_{\mu\nu stress(k)}^{(1)} \delta_{jk} \tag{86}$$

Moreover, being $\lim_{dec} \tau_{\mu\nu curv} = 0$, the macroscopic limit of the GE (to be used on a cosmological scale for corrections to Newtonian gravity) reads

$$R_{\mu\nu}^{(2)dec}(\varepsilon_{sim\mu\nu}) - \frac{1}{2}R_{\mu\nu}^{(2)dec}(\varepsilon_{sim\mu\nu})g_{\mu\nu} = \frac{8\pi G}{c^4} \left(\sum_{\pm i} |\psi_{\pm i(\tilde{k})}^{(1)}|^2 \left[\begin{aligned} & \left(1 - a_{\pm i(\tilde{k})}^{(1)} \right) T_{\pm i(k)\mu\nu class}^{(1)} \\ & + \Lambda_{\pm i(\tilde{k})Q}^{(1)} g_{\mu\nu} + \Delta\tau_{\mu\nu stress(k)}^{(1)} \end{aligned} \right] - \Lambda_{\mu\nu} + T_{\mu\nu em} \right) \tag{87}$$

where k is the eigenstates to which the wave-function collapses due to the decoherence process, where

$$T_{\pm i(k)\mu\nu class}^{(1)} = T_{\pm i(k)\mu\nu class(\psi_{\pm ik}^{(0)} + \psi_{\pm ik}^{(1)})} \tag{88}$$

and where

$$\Lambda_{(k)Q}^{(1)} = \Lambda_{(k)Q}^{(1)} + \Lambda_{(-k)Q}^{(1)} \tag{89}$$

where, by using Equation (3), it follows that

$$\begin{aligned}
 \Lambda_{(k)Q}^{(1)} &= \Lambda_{(k)Q} \Big|_{(\psi_{\pm ik} = \psi_{\pm ik}^{(0)} + \psi_{\pm ik}^{(1)})} \\
 &= \lim_{\psi = \psi^{(0)} + \psi^{(1)}} \sum_{\pm i} |\psi_{\pm i(k)}|^2 a_{\pm i(k)} \frac{\Delta_{\pm i(\pm k)\lambda\lambda}}{4} \left(\begin{array}{c} L_{\pm i(k)class} + \\ \left(\begin{array}{c} Re\left\{\frac{\hbar}{4}\sigma^{ab}\omega_{ab\alpha}\right\} \\ -eA_\alpha \end{array} \right) \dot{q}_{\pm i(k)}^\alpha \end{array} \right) \\
 &= \lim_{\psi = \psi^{(0)} + \psi^{(1)}} \sum_{\pm i} |\psi_{\pm i(k)}|^2 a_{\pm i(k)} \frac{\Delta_{\pm i(\pm k)\lambda\lambda}}{4} \sqrt{g_{\alpha\beta} \dot{q}_{\pm i(k)}^\beta \dot{q}_{\pm i(k)}^\alpha}
 \end{aligned} \tag{90}$$

and

$$\Delta\tau_{\mu\nu stress}^{(1)} = \Delta\tau_{\mu\nu stress(k)}^{(1)} + \Delta\tau_{\mu\nu stress(-k)}^{(1)} \tag{91}$$

where

$$\begin{aligned}
 \Delta\tau_{\mu\nu stress(k)}^{(1)} &= \sum_{\pm i} |\psi_{\pm i(k)}|^2 a_{\pm i(k)} \Delta_{s\pm i(k)\mu\nu} \left(L_{\pm i(k)class} + \left(\begin{array}{c} Re\left\{\frac{\hbar}{4}\sigma^{ab}\omega_{ab\alpha}\right\} \\ -eA_\alpha \end{array} \right) \dot{q}_{\pm i(k)}^\alpha \right) \\
 &= \sum_{\pm i} |\psi_{\pm i(k)}|^2 a_{\pm i(k)} \Delta_{s\pm i(k)\mu\nu} \sqrt{g_{\alpha\beta} \dot{q}_{\pm i(k)}^\beta \dot{q}_{\pm i(k)}^\alpha}
 \end{aligned} \tag{92}$$

The second-order GE (87) contains the cosmological pressure tensor density (CPTD) $\Lambda_{(k)Q}^{(1)}$ and the stress gravity term $\Delta\tau_{\mu\nu stress}^{(1)}$ that go to zero if \hbar is set to zero or the quantum potential can be disregarded. Thence, $\Lambda_{(k)Q}^{(1)}$ and $\Delta\tau_{\mu\nu stress}^{(1)}$ are actually macroscopic quantum-mechanical corrections to Newtonian gravity.

Since both terms $\alpha_{(V_{qu_1(k)})}^{(1)}$ and $\Delta_{s\mu\nu(k)}^{(1)}$ become relevant in very high-curvature space-time (beyond the first-order Newtonian limit), these corrections to the gravity should become noticeable when approaching very massive objects, such as black holes, and in the rotation of twin neutron stars or black holes.

5. Cosmological Tensor Density of the Quantum Dirac Field

In the classical treatment, except for the photon, the Einstein equation is not coupled to the particle fields but just to the energy impulse tensor of classical bodies.

The GE (59) is analytically coupled to the classical fermion field, and it takes into account the energy of the (nonlocal) quantum potential for determining the geometry of space-time.

The most important effect of the quantum potential is the generation of the quantum cosmological tensor QCT $\Lambda_{\mu\nu}^{(qu)} = (\Delta\tau_{\mu\nu stress} - \Lambda_Q g_{\mu\nu})$.

If we put to zero the quantum potential in the decoherent GE (45) (as it happens in the classical limit (i.e., $\hbar = 0$ and $V_{qu} = 0$)), the CGT (62) reduces to the classical cosmological constant.

Thence, the gravitational effect of the quantum potential leads to a not-null CGT that allows the quantum vacuum to be in the stable physical phase [9] even if the classical cosmological constant is null.

Given that the contribution of the quantum potential modifies both the classical metric tensor of general relativity and the quantization rules (see Appendix F), it also affects the field quantization.

In this section, we will derive the CPTD of the quantum fermion field in the quasi Minkowskian space-time in order to evaluate its mean value on a cosmological scale.

Since the EM field owns a null CGT (the classical theory coincides with the quantum hydrodynamic description (i.e., $V_{qu} = 0$ for the photon) and the gravitational coupling is the same as of classical

general relativity), for the sake of simplicity, we derive the CPTD by considering the case of uncharged fermions in the quasi-Minkowskian limit, whose Fourier decomposition [30] reads

$$\Psi = \sum_{r=1}^2 \iint \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \left(b_{r(p)} u_{(p)}^{(r)} \exp[-i\frac{p_\alpha q^\alpha}{\hbar}] + d_{r(p)}^\dagger v_{(p)}^{(r)} \exp[i\frac{p_\alpha q^\alpha}{\hbar}] \right) \quad (93)$$

$$\bar{\Psi} = \sum_{r=1}^2 \iint \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \left(b_{r(p)}^\dagger \bar{u}_{(p)}^{(r)} \exp[i\frac{p_\alpha q^\alpha}{\hbar}] + d_{r(p)} \bar{v}_{(p)}^{(r)} \exp[-i\frac{p_\alpha q^\alpha}{\hbar}] \right) \quad (94)$$

where $r = 1, 2$ and where

$$u_{(p)}^{(r)} = \begin{pmatrix} u_{1(p)}^{(r)} \\ u_{2(p)}^{(r)} \end{pmatrix} = \frac{1}{\sqrt{E_p + m}} \begin{pmatrix} (E_p + m)\chi^{(r)} \\ (\sigma \cdot p)\chi^{(r)} \end{pmatrix} \quad (95)$$

$$v_{(p)}^{(r)} = \begin{pmatrix} v_{1(p)}^{(r)} \\ v_{2(p)}^{(r)} \end{pmatrix} = \frac{1}{\sqrt{E_p + m}} \begin{pmatrix} (\sigma \cdot p)\chi^{(r)} \\ (E_p + m)\chi^{(r)} \end{pmatrix} \quad (96)$$

where

$$\chi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (97)$$

$$\chi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (98)$$

Moreover, by using the discrete form of field Fourier decomposition (i.e., $\iint \frac{d^3k}{(2\pi)^3} \rightarrow \frac{1}{\sqrt{V}} \sum_{k=0}$)

$$\begin{aligned} \Psi &= \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \frac{1}{\sqrt{V}} \sum_{p=0}^{\infty} \frac{1}{2\hbar\omega_p} \sum_{r=1}^2 \begin{pmatrix} b_{r(p)} u_{(p)}^{(r)} \exp[-i\frac{p_\alpha q^\alpha}{\hbar}] \\ + d_{r(p)}^\dagger v_{(p)}^{(r)} \exp[i\frac{p_\alpha q^\alpha}{\hbar}] \end{pmatrix} \\ &= \frac{1}{\sqrt{V}} \sum_{p=0}^{\infty} \frac{1}{2\hbar\omega_p} \sum_{r=1}^2 \begin{pmatrix} b_{r(p)} \begin{pmatrix} u_{1(p)}^{(r)} \\ u_{2(p)}^{(r)} \end{pmatrix} \exp[-i\frac{p_\alpha q^\alpha}{\hbar}] \\ + d_{r(p)}^\dagger \begin{pmatrix} v_{1(p)}^{(r)} \\ v_{2(p)}^{(r)} \end{pmatrix} \exp[i\frac{p_\alpha q^\alpha}{\hbar}] \end{pmatrix} \end{aligned} \quad (99)$$

it follows that

$$\psi_{\pm} = \frac{\psi_1 \pm \psi_2}{\sqrt{2}} = \frac{1}{\sqrt{V}} \sum_{p=0}^{\infty} \frac{1}{2\hbar\omega_p} \sum_{r=1}^2 \begin{pmatrix} b_{r(p)} \frac{u_{1(p)}^{(r)} \pm u_{2(p)}^{(r)}}{\sqrt{2}} \exp[-i\frac{p_\alpha q^\alpha}{\hbar}] \\ + d_{r(p)}^\dagger \frac{v_{1(p)}^{(r)} \pm v_{2(p)}^{(r)}}{\sqrt{2}} \exp[i\frac{p_\alpha q^\alpha}{\hbar}] \end{pmatrix} \quad (100)$$

that compared to the Minkowskian Equation (A33),

$$\psi_{\pm} = \sum_{k=0} \alpha_{\pm k} \exp[-i\frac{p_\alpha q^\alpha}{\hbar}] + \beta_{\pm k}^\dagger \exp[i\frac{p_\alpha q^\alpha}{\hbar}] \quad (101)$$

leads to

$$\alpha_{\pm p} = \frac{1}{2\hbar\omega_p} \sum_r b_{r(p)} \frac{u_{1(p)}^{(r)} \pm u_{2(p)}^{(r)}}{\sqrt{2}} \quad (102)$$

$$\beta_{\pm p}^\dagger = \frac{1}{2\hbar\omega_p} \sum_r d_{r(p)}^\dagger \frac{v_{1(p)}^{(r)} \pm v_{2(p)}^{(r)}}{\sqrt{2}} \quad (103)$$

Once $\alpha_{\pm p}$ and $\beta_{\pm p}$ are defined as a function of the creation and annihilation operators of the quantum Dirac field, we can obtain the expression of the expectation value of the CPTD that reads

$$\langle 0_p | \Lambda_{(k)Q} | 0_p \rangle = \sum_{\pm i} \int^{\omega_{k\max}} \frac{\Lambda_{\pm i(k)Q}}{(2\pi)^2 c^2} \sqrt{\frac{(\hbar\omega_p - e\phi)^2}{c^2} - m^2 c^2 \left(1 - \frac{V_{qu\pm i(p)}}{mc^2}\right)} d\omega_{\pm i(p)} \quad (104)$$

where, from Equation (45), the decoherent macroscopic CPTD $\Lambda_{\pm i(k)Q}$ reads

$$\Lambda_{\pm i(k)Q} = -|\psi_{\pm i(k)}|^2 a_{\pm i(k)} \left(\frac{\Theta_{\lambda\lambda}}{4} - \frac{\Delta_{\pm i(k)\lambda\lambda}}{4} \left(\begin{array}{l} L_{\pm i(k)class} - eA_\alpha \dot{q}_{\pm i(k)}^\alpha \\ + \text{Re} \left\{ \frac{\hbar}{4} \sigma^{ab} \omega_{ab\alpha} \right\} \dot{q}_{\pm i(k)}^\alpha \end{array} \right) \right) \quad (105)$$

and where (for uncharged fermion $e = 0$)

$$\omega_{\pm i(p)} = \frac{e\phi}{\hbar} + \sqrt{\frac{c^2 p_{\pm i}^2}{\hbar^2} + \frac{m^2 c^4}{\hbar^2} \left(1 - \frac{V_{qu\pm i(p)}}{mc^2}\right)} \quad (106)$$

obtained from the HHJE (27) and the identities (A16) $p_{\pm i\mu} = (p_{\pm i0}, -p_{\pm ij})$, $p_{\pm i}^2 = p_{\pm ij} p_{\pm ij}$, and $\frac{\hbar}{c} \omega_{\pm i} = p_{\pm i0}$. In order to have the explicit form of the quantum operator CPTD as a function of the creation and annihilation operators, the ordering problem in the expressions $|\psi_{\pm i(\pm k)}|^2$ and $V_{qu\pm i(\pm p)}$ must be defined. The criterion of ordering the creation and annihilation operators is defined by the necessity to avoid divergences [30]. Then, the normal ordering is assumed here for the CPTD, which reads

$$\begin{aligned} \langle 0_p | \Lambda_{(k)Q} | 0_p \rangle &= \\ &= \sum_{\pm i} \int^{\omega_{k\max}} \frac{:\Lambda_{\pm i(k)Q}:}{(2\pi)^2 c^2} \sqrt{\frac{(\hbar:\omega_{\pm i(p)}:-e\phi)^2}{c^2} - m^2 c^2 \left(1 - \frac{:V_{qu\pm i(p)}:}{mc^2}\right)} d : \omega_{\pm i(p)} : \end{aligned} \quad (107)$$

where

$$:\omega_{\pm i(p)}: = \frac{e\phi}{\hbar} + \sqrt{\frac{c^2 p_{\pm i}^2}{\hbar^2} + \frac{m^2 c^4}{\hbar^2} \left(1 - \frac{:V_{qu\pm i(p)}:}{mc^2}\right)} \quad (108)$$

Moreover, since at zero order of the Minkowskian limit, the mode k of the quantum potential (A11) for both positive and negative states (for uncharged fermions (i.e., $e = 0$)) reads

$$\begin{aligned} :V_{qu\pm i(+k)}: &= -\frac{\hbar^2}{m} \frac{1}{:|\psi_{\pm i(+k)}: \sqrt{-g}} \partial_\mu \sqrt{-g} g^{\mu\nu} \partial_\nu |(\alpha_{\pm k} \exp[ik_\alpha q^\alpha])| \\ &= -\frac{\hbar^2}{m} \frac{1}{:|\psi_{\pm i(+k)}: \sqrt{-g}} \partial_\mu \sqrt{-g} g^{\mu\nu} \partial_\nu :(\sqrt{\alpha_{\pm k} \alpha_{\pm k}^\dagger}): = 0 \end{aligned} \quad (109)$$

and

$$\begin{aligned} :V_{qu\pm i(-k)}: &= -\frac{\hbar^2}{m} \frac{1}{:|\psi_{\pm i(-k)}: \sqrt{-g}} \partial_\mu \sqrt{-g} g^{\mu\nu} \partial_\nu |(\beta_{\pm k}^\dagger \exp[-ik_\alpha q^\alpha])| \\ &= -\frac{\hbar^2}{m} \frac{1}{:|\psi_{\pm i(-k)}: \sqrt{-g}} \partial_\mu \sqrt{-g} g^{\mu\nu} \partial_\nu :(\sqrt{\beta_{\pm k}^\dagger \beta_{\pm k}}): = 0 \end{aligned} \quad (110)$$

it follows that

$$\left(: \omega_{\pm i(p)} : - \frac{e\phi}{\hbar} \right)^2 = \left(\omega_{\pm i(p)} - \frac{e\phi}{\hbar} \right)^2 = \frac{c^2 p_{\pm i}^2}{\hbar^2} + \frac{m^2 c^4}{\hbar^2} \quad (111)$$

(showing that, at zero order, the standard QFT identity is recovered) and that

$$: a_{\pm i(k)} (\hat{V}_{qu}) : = 0 \quad (112)$$

$$: \Delta_{\kappa\kappa}(k) : = 0 \quad (113)$$

$$: \Lambda_{(k)Q} : = 0 \quad (114)$$

$$\langle 0_p | : \Lambda_{(k)Q} : | 0_p \rangle = 0 \quad (115)$$

where (115) shows that the Minkowskian ST (very far from matter) does not contribute to the cosmological constant.

Weak Gravity Perturbation Hamiltonian of the Quantum Dirac Field

Given the weak-gravity coupling (see Appendix E)

$$J = \frac{\hbar}{8} \gamma^{\mu} \sigma^{ab} (\partial_{\mu} \varepsilon_{ab}) \quad (116)$$

where ε_{ab} is obtained by solving the gravitational Equation (59) at the zero-order of field approximation Ψ_0 , the interaction Lagrangian \mathcal{L}_I reads

$$\mathcal{L}_I = \bar{\Psi} J \Psi = \frac{\hbar}{8} (\bar{\Psi} \gamma^{\mu} (\partial_{\mu} \varepsilon_{ab}) \Psi) \quad (117)$$

leading to the correction to the Minkowskian zero order

$$: H_0 : = \frac{1}{2} \sum_{r=1}^2 \iint \frac{d^3 p}{(2\pi)^3} (b_{r(p)}^{\dagger} b_{r(p)} + d_{r(p)}^{\dagger} d_{r(p)}) \quad (118)$$

given by the perturbative interaction Hamiltonian [30]

$$\begin{aligned} H_1 &= - \int d^3 q \bar{\Psi} J \Psi \\ &= - \int \frac{d^3 k}{(2\pi)^3 2\omega_k} \left(\begin{aligned} &(b_{r(p)}^{\dagger} \bar{u}_{(p)}^{(r)} + d_{r(p)}^{\dagger} \bar{v}_{(p)}^{(r)}) \bar{J}(t, k) \exp[i \frac{p_{\alpha} q^{\alpha}}{\hbar}] \\ &+ (b_{r(p)} u_{(p)}^{(r)} + d_{r(p)} \bar{v}_{(p)}^{(r)}) J(t, k) \exp[-i \frac{p_{\alpha} q^{\alpha}}{\hbar}] \end{aligned} \right) \end{aligned} \quad (119)$$

where

$$J(t, k) = \int d^3 q J(t, q) e^{-ikq} = \frac{\hbar}{8} \gamma^{\mu} \sigma^{ab} \int d^3 q e^{-ikq} \partial_{\mu} \varepsilon_{ab} \quad (120)$$

$$\bar{J}(t, k) = \int d^3 q J(t, q) e^{+ikq} \quad (121)$$

In Appendix G, it is shown that Newtonian gravity leads to a null gravitational perturbation of the quantum Dirac field. This fact agrees with the experimental outputs of the Minkowskian QED that leads to the prediction of the anomalous magnetic moment with six significant digits [30]. The gravitational corrections coming out of higher-order terms can lead to non-zero contributions for predicting the anomalous magnetic moment with higher precision and for testing the theory.

The Equation (119) may make an important contribution to very heavy fermion particles.

6. Discussion

The hydrodynamic representation of the DFWE field makes it equivalent to the motion of four mass distributions $|\psi_{\pm i}|^2$ submitted to the theory-defined quantum potential in a curved space-time defined by the GE (59).

In agreement with the basic principle of general relativity, the theory includes the energy of the quantum potential (expressing the quantum-mechanical properties of the vacuum) in the contributions to the curvature of space-time.

On this basis, the quantum-mechanical nonlocal effects (e.g., the uncertainty principle) come into the gravity, leading, for instance, to the removal of the point singularity of black holes of general relativity [31] and to the theoretical appearance of the CGT in the GE.

In the classical treatment of general relativity, the Einstein equation for massive particles is not coupled to any matter field, but just to the energy impulse tensor of the classical bodies and does not contain any information about how it couples with their quantum-mechanical fields (i.e., their phases).

On the other hand, the coupled system of GE-DFWE-EM Equations (59)–(61) field equations can be further submitted to the second quantization, leading to quantum gravity that includes the nonlocal quantum-mechanical properties of the vacuum brought by the energy of the quantum potential.

6.1. The GE and the Quantum Gravity

Even if the quantization of the DFWE field in the curved ST is not treated in this work, the inspection of some features of the quantum gravity to the light of the GE (59) deserves mention.

Since the action of the GE (59) is basically given by the standard Einstein–Hilbert action plus terms stemming by the energy of the nonlocal quantum potential of DFWE for a massive bispionor field, the outputs of the canonical “pure quantum gravity” practically remains valid for the GE presented in this work.

As shown in [9], one interesting aspect of the quantum pure gravity is that the vacuum does not make a transition to the *collapsed branched polymer phase*, if and only if there exists a vanishing small cosmological constant (i.e., $\Lambda \neq 0$). (The branched polymer phase is the state of some polymer solutions where there is a phase separation between the fluid solution and the polymer molecules, with the appearance of separate domains of fluid alone and grouped polymeric molecules that exclude the fluid. This phase transition, treated by renormalization group theory, is used as an analogy to describe the quantum gravity of pure vacuum and its metrics. Out of the stable physical phase (with a continuous limit), the vacuum metrics collapse in the so-called branched polymer phase that has no sensible continuous limit (with domains of collapsed metrics).

Therefore the presence of the term Λ_Q in the GE (59), in principle, allows us to pose $\Lambda = 0$ (as strongly supported by Einstein [6]) with the GE that reads.

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \frac{8\pi G}{c^4}Tr(\Lambda_Q)g_{\mu\nu} = \frac{8\pi G}{c^4}(Tr(\tau_{\mu\nu}^{class} + \Delta\tau_{\mu\nu}^{stress}) + T_{\nu\mu}^{em}) \quad (122)$$

Moreover, since the matter makes the space-time curved and, hence,

$$a_{\pm i(\pm k)} \neq 0 \quad (123)$$

$$\Delta_{\kappa\kappa}(k) \neq 0 \quad (124)$$

$$\Delta\tau_{\mu\nu}^{stress} \neq 0 \quad (125)$$

and

$$\Lambda_Q \neq 0 \quad (126)$$

it follows that the matter itself stabilizes the vacuum in the physical strong gravity phase [9].

On the other hand, a perfect Minkowskian vacuum (i.e., without matter) will make the transition to the nonphysical *collapsed branched polymer phase* with no sensible continuum limit [9], leading to no-space and no-time, as is known.

6.2. Analogy with Modified Gravity Theories

The output of the work highlights an interesting analogy with Brans–Dicke [13–15] gravity that solves the problem of the cosmological constant [16] as well as those of inflation [17] and dark energy [18].

If we look in detail at the EITD $|\psi_{\pm i}|^2 \tau_{\pm i \mu \nu class}$ of the macroscopic decoherent limit (see Equation (40))

$$\lim_{dec} \sum_{\pm i} |\psi_{\pm i}|^2 \tau_{\pm i \mu \nu class} = \sum_{\pm i} T_{\pm i(k) \mu \nu class} \tag{127}$$

we can see that by using (A63), it leads to the expression

$$\begin{aligned} \frac{8\pi G}{c^4} T_{\pm i \mu \nu class} &= \frac{8\pi G}{c^4} \sum_{\pm i} |\psi_{\pm i}|^2 c^2 \left(\frac{\partial S_{\pm i}}{\partial t} \right)^{-1} \left(g^{\mu\beta} \frac{\partial S_{\pm i}}{\partial q^\beta} \frac{\partial S_{\pm i}}{\partial q^\nu} \right) \\ &= -\frac{8\pi G}{c^4} \sum_{\pm i} |\psi_{\pm i}|^2 \frac{\hbar^2 c^2}{4E} \left(g^{\mu\beta} \left(\frac{\partial_\beta \psi_{\pm i}}{\psi_{\pm i}} - \frac{\partial_\beta \psi_{\pm i}^*}{\psi_{\pm i}^*} \right) \left(\frac{\partial_\nu \psi_{\pm i}}{\psi_{\pm i}} - \frac{\partial_\nu \psi_{\pm i}^*}{\psi_{\pm i}^*} \right) \right) \\ &= -\frac{8\pi G}{c^4} \sum_{\pm i} \frac{G}{\psi_{\pm i}^2} |\psi_{\pm i}|^2 \frac{\hbar^2 c^2}{4E} \left(g^{\mu\beta} \partial_\beta \psi_{\pm i} \partial_\nu \psi_{\pm i} + \dots \right. \\ &\quad \left. \dots + \left(\frac{\psi_{\pm i}}{\psi_{\pm i}^*} \right)^2 g^{\mu\beta} \partial_\beta \psi_{\pm i}^* \partial_\nu \psi_{\pm i}^* \right) \end{aligned} \tag{128}$$

that is the summation over the spinor terms, each one owing to the form of scalar Brans-Dicke gravity [13] in the absence of external potentials (for the sake of simplicity, we have considered the case of uncharged fermions).

It is worth noting that the DWFE gravity endorses the concept of the effective gravitational constant of the Brans–Dicke model that in the spinor-tensor form Equation (123) leads to effective gravitational constants for each spinor $G_{eff \pm i} = \frac{G}{\psi_{\pm i}^2}$.

Moreover, as shown in The Perturbative Approach to the GE-DFWE-EM Evolution, the cosmological constant term is a second-order gravitational contribution to weak (Newtonian) gravity. This output has a similarity to the modified Newtonian dynamics (MOND) theories [32,33] that assume that Newtonian gravity, for very low accelerations, is slightly in order to fit the experimental data of the motion of galaxies (instead of hypothesizing the presence of dark matter). In the present model, the correction to Newtonian gravity comes from the energy of the quantum potential that is originated by a huge concentration of matter in a highly massive body. This is an important feature in order to compare the two theories. If MOND produces corrections of the gravity interaction on a distance of kiloparsec, the cosmological constant (CC) produces effects on megaparsec. On the other hand, this is a result that comes out by assuming the CC is evenly distributed in the space. In this work, it is shown that the CC comes out by a very massive body, such as the supermassive black holes (SMBHs) at the center of galaxies. Therefore, this localization of the CC (as a source of Newtonian corrections) produces a higher value of the CC in the region of space between SMBHs and hence between galaxies. Thence, this localized CC gravity source can lead to effects on distances of kiloparsec, substantially agreeing with MOND.

6.3. Quantum-Mechanical Gravity and the Foundations of the Quantum Theory

The output (A84) shows that in order to have macroscopic behavior in addition to the smallness of the Planck constants, the decoherence process induced by uncorrelated fluctuations is necessary. The decoherence process of the quantum states is necessary to obtain the classical extremal action principle and the axiomatic description of macroscopic classical physics.

The quantum hydrodynamic model (the Madelung–De Broglie-Bohm pilot wave approach) leads to the classic equation of motion by just posing tout court Planck’s constant to zero [22]. Actually, in the classical limit for, there is the change of the “mathematical nature” of the quantum hydrodynamic motion equations produced by the disappearance of the quantum potential that generates the eigenstates and the coherent evolution of their superposition [22–24]. In the presence of a noisy environment (when the quantum potential energy is much smaller than the energy amplitude of fluctuations) the coherent superposition of states cannot be maintained in time and undergo relaxation (decoherence) to the stationary eigenstate configurations, the only ones that are stable [24,34].

Actually, in the deterministic quantum hydrodynamic model used here, the decoherence is absent since it shows itself in the open-system [22–26].

The approach proposed here, on the minimum action condition (A83), shows that the decoherence process is definitely needed for the axiomatic derivation of classical mechanics by the minimum action principle.

6.4. Experimental Tests of the Theory

The result Equation (115) basically shows that in the Minkowskian space-time (i.e., the vacuum very far from particles), the CPTD is vanishing.

The CPTD appears at second-order quantum-mechanical corrections in the perturbative development of the GE.

By observing that, due to the high density of mass distribution in black holes, $V_{qu\pm i}$ acquires a maximum value comparable to the mass energy mc^2 (e.g., $\frac{V_{qu\pm i(\pm p)}}{mc^2} \rightarrow 1$ as well $a_{\pm i(\pm k)} \rightarrow 1$ [31]), it turns out that the largest contribution to the measured cosmological constant of the universe comes from the high-gravity regions of space (e.g., black holes, neutron stars, and supermassive BHs (SMBHs) at the center of the galaxies).

On the other hand, from the ordinary baryonic mass, the contribution to the CPTD is very small, given that the gravity is accurately described by Newtonian law.

Thence, the quantum-mechanical properties of the vacuum lead to a great lowering of the CPTD on a cosmological scale, with an order of magnitude that can be in agreement with the astronomical observations [20].

From the experimental point of view, the *quantum-mechanical corrections* to Newtonian gravity, $a_{(\vec{k})} \frac{mc^2}{\gamma} g_{\mu\alpha} u_{class}^\alpha u_{class}^\nu$, $\Lambda_{(k)Q} g_{\mu\nu}$ and $\Delta\tau_{\mu\nu stress}$, must be primarily detectable in the rotation of twin black holes or neutron stars (by means of the interferometric detection of the angular and frequency-dependent response functions [35] of gravitational waves), in the motion of stars in the bulge around the SMBHs at the center of the galaxies and in intergalactic interaction.

Moreover, since Newtonian gravity leads to a null first-order gravitational perturbation of the quantum Dirac field (see Appendix G), the prediction of the anomalous magnetic moment with six significant digits [30] in the Minkowskian QED is confirmed and explained by the theory. The gravitational corrections coming from higher-order terms (that can be relevant in high gravity and for very heavy fermions) can be a test for the theory that should also allow the prediction of the anomalous magnetic moment with higher precision.

7. Conclusions

The quantum hydrodynamic representation of the Dirac–Fock–Weil equation, describing the bispinor fields as the motion of four mass densities subject to the quantum potential, has been used to derive the correspondent gravity equation by using the minimum action principle. The gravity equation associated to the DFWE field takes into account the gravitational effects of the energy of the nonlocal quantum potential.

The macroscopic CPTD $\Lambda_{(k)Q}$ is not null if, and only if, massive particles are present but it tends to zero in the flat vacuum so that its spatial mean, on the cosmological scale, has a much smaller order of magnitude than that one derived tout court by accounting for the zero-point vacuum energy density, and possibly agrees with the measured cosmological constant.

The self-generation of the CPTD Λ_Q leads to the attractive hypothesis that matter itself generates the physical stable vacuum phase in which it is embedded and that outside the universe, there is no space and no time.

The GE of the fermion field shows that the measured cosmological constant is the effect of the quantum cosmological tensor density that emerges as a second-order quantum-mechanical correction to Newton gravity. The GE of the classical DFWE field shows that the CPTD $\Lambda_{(k)Q}$, as well as the QCT, become increasingly relevant the higher the deviations from Newtonian gravity are. This output endorses the hypothesis that the gravity near supermassive black holes at the center

of galaxies (e.g., in the bulge), between twin black holes, or between galaxies can be influenced by the quantum-mechanical effects on gravity.

The spinor-tensor gravitational coupling of the fermion field owns (for each spinor) terms typical of Brans–Dicke modified gravity in the absence of external potentials.

The gravitational corrections to the quantum fermion field can be tested by predicting the anomalous magnetic moment with higher precision.

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Appendix A

The hydrodynamic form of the DFWE in the Minkowskian space-time.

In reference [21] the Dirac equation

$$\left(i\hbar\gamma^\mu\left(\partial_\mu + \frac{ie}{\hbar}A_\mu\right) - mc\right)\Psi = 0 \quad (\text{A1})$$

where

$$\gamma^\mu = (\gamma^0, \gamma^i) = \left(\left(\begin{array}{cc} I & 0 \\ 0 & -I \end{array}\right), \left(\begin{array}{cc} 0 & \sigma_i \\ -\sigma_i & 0 \end{array}\right)\right) \quad (\text{A2})$$

where $\sigma^\mu = (\sigma_0, \sigma_i)$ are the 4-D extended Pauli matrices [22], where

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \quad (\text{A3})$$

$$\bar{\Psi} = \Psi^\dagger \gamma^0 \quad (\text{A4})$$

and where $A_\mu = (\frac{\phi}{c}, -A_i)$ is the electromagnetic potential, has been expressed as a function of the components ψ_k of the bi-spinor $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$, in the form

$$\frac{i\hbar}{mc}\sigma^\mu\left(\partial_\mu + \frac{ie}{\hbar}A_\mu\right)\psi_+ = \psi_- \quad (\text{A5})$$

$$\frac{i\hbar}{mc}\tilde{\sigma}^\mu\left(\partial_\mu + \frac{ie}{\hbar}A_\mu\right)\psi_- = \psi_+ \quad (\text{A6})$$

where $\psi_\pm = \frac{\psi_1 \pm \psi_2}{\sqrt{2}}$. Moreover, by posing

$$\begin{aligned} \tilde{\sigma}^\mu\sigma^\nu &= g^{\mu\nu} + \alpha_{\pm}^{\mu\nu} \\ \sigma^\mu\tilde{\sigma}^\nu &= g^{\mu\nu} + \alpha_{\mp}^{\mu\nu} \end{aligned} \quad (\text{A7})$$

where $\tilde{\sigma}^\mu = (\tilde{\sigma}_0, \tilde{\sigma}_i) = (\sigma_0, -\sigma_i) = \sigma_\mu$, substituting (A5) in (A6) it follows that

$$\left[\left(\partial_\mu + \frac{ie}{\hbar}A_\mu\right)\left(\partial^\mu + \frac{ie}{\hbar}A^\mu\right) + \left(\frac{m^2c^2}{\hbar^2} + \frac{ie}{2\hbar}\alpha_{\pm}^{\mu\nu}F_{\mu\nu}\right)\right]\psi_\pm = 0 \quad (\text{A8})$$

where $F_{\mu\nu}$ is the electromagnetic (EM) tensor [22].

Furthermore, by using the hydrodynamic notation

$$\psi_\pm \equiv \psi_{\pm i} = \begin{pmatrix} \psi_{\pm 1} \\ \psi_{\pm 2} \end{pmatrix} = \begin{pmatrix} |\psi_{\pm 1}| \exp\left[i\frac{S_{\pm 1}}{\hbar}\right] \\ |\psi_{\pm 2}| \exp\left[i\frac{S_{\pm 2}}{\hbar}\right] \end{pmatrix} \quad (\text{A9})$$

and equating the real and imaginary part of Equation (A8), both the quantum hydrodynamic Hamilton-Jacobi equation (HHJE) of motion and the current conservation equation [21] of the spinors $\psi_{\pm i}$ follow.

The HHJE reads (see reference [21])

$$(\partial_{\mu} S_{\pm i} + eA_{\mu})(\partial^{\mu} S_{\pm i} + eA^{\mu}) = (m^2 c^2 - mV_{qu\pm i}) \quad (\text{A10})$$

where $S_{\pm i} = \begin{pmatrix} S_{\pm 1} \\ S_{\pm 2} \end{pmatrix}$, where $i = 1, 2$ is the spinor italic index and where the quantum potential reads

$$V_{qu\pm i} = \begin{pmatrix} V_{qu\pm 1} \\ V_{qu\pm 2} \end{pmatrix} = -\frac{\hbar^2}{m} \begin{pmatrix} \frac{\partial_{\mu} \partial^{\mu} |\psi_{\pm 1}|}{|\psi_{\pm 1}|} \\ \frac{\partial_{\mu} \partial^{\mu} |\psi_{\pm 2}|}{|\psi_{\pm 2}|} \end{pmatrix} - \text{Re} \left\{ \frac{ie\hbar}{2m} \begin{pmatrix} \alpha_{\pm}^{\lambda\kappa} \frac{|\psi_{\pm j}|}{|\psi_{\pm 1}|} \\ \lambda\kappa \frac{|\psi_{\pm j}|}{\pm 2j |\psi_{\pm 2}|} \end{pmatrix} F_{\lambda\kappa} \right\} \quad (\text{A11})$$

The current equation for the symmetric and antisymmetric combination of the spinors, $\psi_{\pm i}$, reads [21]

$$-(g^{\mu\nu} + \alpha_{\pm}^{\mu\nu}) \partial_{\mu} \left(\frac{|\psi_{\pm i}|^2}{m} (\partial_{\nu} S + eA_{\nu}) \right) = \partial_{\mu} J^{(\pm i)\mu} \quad (\text{A12})$$

that, being ψ_{+} and ψ_{-} coupled each other through the EM potential in Equation (A8), leads to the overall conservation of current J^{μ}

$$\frac{\partial((J^{(+1)\mu} + J^{(+2)\mu}) + (J^{(-1)\mu} + J^{(-2)\mu}))}{\partial q^{\mu}} = \frac{\partial J^{(+)\mu} + J^{(-)\mu}}{\partial q^{\mu}} = \frac{\partial J^{\mu}}{\partial q^{\mu}} = 0 \quad (\text{A13})$$

where the current J^{μ} reads [21]

$$\begin{aligned} J^{\mu} &= J^{(+)\mu} + J^{(-)\mu} \\ &= \frac{\hbar}{2im} \left[\psi_{+}^{\dagger} \tilde{\sigma}^{\mu} \sigma^{\nu} (\partial_{\nu} - \frac{e}{i\hbar} A_{\nu}) \psi_{+} - \psi_{+} \tilde{\sigma}^{\mu} \sigma^{\nu} (\partial_{\nu} + \frac{e}{i\hbar} A_{\nu}) \psi_{+} \right. \\ &\quad \left. + (\psi_{-}^{\dagger} \tilde{\sigma}^{\mu} \tilde{\sigma}^{\nu} (\partial_{\nu} - \frac{e}{i\hbar} A_{\nu}) \psi_{-} - \psi_{-} \sigma^{\mu} \tilde{\sigma}^{\nu} (\partial_{\nu} + \frac{e}{i\hbar} A_{\nu}) \psi_{-}) \right] \end{aligned} \quad (\text{A14})$$

that by using Equations (A5) and (A6), leads to the known relation [21]

$$\begin{aligned} J^{\mu} &= -\frac{\hbar}{2im} \left[\psi_{+}^{\dagger} \tilde{\sigma}^{\mu} \psi_{-} - \psi_{+} \tilde{\sigma}^{\mu} \psi_{-} \right. \\ &\quad \left. + \psi_{-}^{\dagger} \tilde{\sigma}^{\mu} \psi_{+} + \psi_{-} \tilde{\sigma}^{\mu} \psi_{+} \right] \\ &= -\frac{\hbar}{2im} [\bar{\Psi} \gamma^{\mu} \Psi + \bar{\Psi}^* \gamma^{\mu} \Psi^*] = J^{(+)\mu} + J^{(-)\mu} = -\frac{\hbar}{im} \bar{\Psi} \gamma^{\mu} \Psi \end{aligned} \quad (\text{A15})$$

A.1. The Solution of the Hydrodynamic Hamilton-Jacobi Equation for Eigenstates

Bu using the hydrodynamic notations

$$\partial_{\mu} S_{\pm i} = \partial_{\mu} \begin{pmatrix} S_{\pm 1} \\ S_{\pm 2} \end{pmatrix} = - \begin{pmatrix} p_{\pm 1\mu} \\ p_{\pm 2\mu} \end{pmatrix} = -p_{\pm i\mu} \quad (\text{A16})$$

the Hamilton-Jacobi Equation (A10), as a function of the spinors components,

$$\left(\partial_{\mu} \begin{pmatrix} S_{\pm 1} \\ S_{\pm 2} \end{pmatrix} + eA_{\mu} \right) \left(\partial^{\mu} \begin{pmatrix} S_{\pm 1} \\ S_{\pm 2} \end{pmatrix} + eA^{\mu} \right) = m^2 c^2 \left(\mathbf{1} - \frac{1}{mc^2} \begin{pmatrix} V_{qu\pm 1} \\ V_{qu\pm 2} \end{pmatrix} \right) \quad (\text{A17})$$

for the k -th eigenstate reads

$$- \left(\begin{pmatrix} p_{\pm 1(k)\mu} \\ p_{\pm 2(k)\mu} \end{pmatrix} - eA_{\mu} \right) \left(\begin{pmatrix} p_{\pm 1(k)\mu} \\ p_{\pm 2(k)\mu} \end{pmatrix} - eA^{\mu} \right) = m^2 c^2 \left(\mathbf{1} - \frac{1}{mc^2} \begin{pmatrix} V_{qu\pm 1(k)} \\ V_{qu\pm 2(k)} \end{pmatrix} \right) \quad (\text{A18})$$

Moreover, by defining the velocity, $\dot{q}_{\pm i}^v$ such as

$$\gamma_{\pm i} = \frac{1}{\sqrt{1 - \frac{\dot{q}_{\pm i}^2}{c^2}}} = \frac{1}{\sqrt{\frac{g_{\mu\nu}\dot{q}_{\pm i}^v\dot{q}_{\pm i}^\mu}{c^2}}} \quad (\text{A19})$$

by using the following identity

$$\begin{aligned} (\partial_\mu S_{\pm i} + eA_\mu)(\partial^\mu S_{\pm i} + eA^\mu) &= m^2\gamma_{\pm i}^2\dot{q}_{\mu\pm i}\dot{q}_{\pm i}^\mu\left(1 - \frac{V_{qu\pm i}}{mc^2}\right) \\ &= m^2\gamma_{\pm i}^2c^2\left(1 - \frac{V_{qu\pm i}}{mc^2}\right) - m^2\gamma_{\pm i}^2\dot{q}_{\pm i}^2 - \left(1 - \frac{V_{qu\pm i}}{mc^2}\right) \\ &= (p_{\pm i\mu} + eA_\mu)(p_{\pm i}^\mu + eA^\mu) = m^2\gamma_{\pm i}^2\dot{q}_{\mu\pm i}\dot{q}_{\pm i}^\mu\left(1 - \frac{V_{qu\pm i}}{mc^2}\right) \end{aligned} \quad (\text{A20})$$

and by using the notation

$$p_{\mu_{\pm i(k)}}^{(\pm)} = \pm p_{\mu_{\pm i(k)}} \quad (\text{A21})$$

where the superscript (\pm) stands for positive and negative solutions of (A30), respectively, for the k eigenstate it follows that the relation between the velocity and the momentum reads

$$p_{\mu_{\pm i(k)}} = \left(m\gamma_{\pm i}\dot{q}_{\mu_{\pm i(k)}} \sqrt{1 - \frac{V_{qu\pm i(k)}}{mc^2}} + eA_\mu \right) \quad (\text{A22})$$

$$E_{\pm i(k)} - e\phi = m\gamma_{\pm i}c^2 \sqrt{1 - \frac{V_{qu\pm i(k)}}{mc^2}} \quad (\text{A23})$$

Equation (A23) shows that the classical relativistic expression is recovered in the classical limit when

$$V_{qu\pm i(k)} \rightarrow 0$$

From (A16) to (A22) it follows that the hydrodynamic Lagrangian for the eigenstates reads

$$L_{\pm i(k)} = \frac{dS_{\pm i(k)}}{dt} = -p_{\pm i(k)\mu}\dot{q}_{\pm i(k)}^\mu = -\frac{mc^2}{\gamma} \sqrt{1 - \frac{V_{qu\pm i(k)}}{mc^2}} + eA_\mu\dot{q}_{\pm i(k)}^\mu \quad (\text{A24})$$

where $V_{qu\pm i(k)}$ is given by Equation (A11) and where the Lagrangian for negative-energy states reads

$$L_{\pm i(k)}^{(-)} = -p_{\pm i(k)\mu}^{(-)}\dot{q}_{\pm i(k)}^\mu = p_{\pm i(k)\mu}\dot{q}_{\pm i(k)}^\mu = -L_{\pm i(k)} \quad (\text{A25})$$

Moreover, it is useful to rearrange (A22) and (A24) as follows, respectively,

$$p_{\mu_{\pm i(k)}} = p_{\mu_{\pm i(k)class}} + p_{\mu_{\pm i(k)Q}} \quad (\text{A26})$$

where

$$\begin{aligned} p_{\mu_{\pm i(k)class}} &= \left(m\gamma_{\pm i}\dot{q}_{\mu_{\pm i(k)}} + eA_\mu \right) = \frac{p_{\mu_{\pm i(k)}} - eA_\mu \left(1 - \sqrt{1 - \frac{V_{qu\pm i(k)}}{mc^2}}\right)}{\sqrt{1 - \frac{V_{qu\pm i(k)}}{mc^2}}} \\ &= -\frac{\partial_\mu S_{\pm i(k)} + eA_\mu a(V_{qu\pm i(k)})}{1 - a(V_{qu\pm i(k)})} \end{aligned} \quad (\text{A27})$$

$$\begin{aligned} p_{\mu_{\pm i(k)Q}} &= -a(V_{qu\pm i(k)})m\gamma_{\pm i}\dot{q}_{\mu_{\pm i(k)}} \\ &= -a(V_{qu\pm i(k)})\left(p_{\mu_{\pm i(k)class}} - eA_\mu\right) \end{aligned} \quad (\text{A28})$$

and

$$L_{\pm i}(k) = -\frac{mc^2}{\gamma} \sqrt{1 - \frac{V_{qu\pm i}(k)}{mc^2}} + eA_{\mu} \dot{q}_{\pm i}(k)^{\mu} = L_{\pm i}(k)_{class} + L_{\pm i}(k)_Q \quad (A29)$$

where

$$L_{\pm i}(k)_{class} = -\frac{mc^2}{\gamma} + eA_{\mu} \dot{q}_{\pm i}(k)^{\mu} \quad (A30)$$

$$\begin{aligned} L_{\pm i}(k)_Q &= a(V_{qu\pm i}(k)) \frac{mc^2}{\gamma} \\ &= -\left(1 - \sqrt{1 - \frac{V_{qu\pm i}(k)}{mc^2}}\right) \left(L_{\pm i}(k)_{class} - eA_{\mu} \dot{q}_{\pm i}(k)^{\mu}\right) \end{aligned} \quad (A31)$$

where

$$a(V_{qu\pm i}(k)) = \left(1 - \sqrt{1 - \frac{V_{qu\pm i}(k)}{mc^2}}\right) \quad (A32)$$

A.2. The Lagrangian Motion Equations for the Dirac Field

Since the quantum hydrodynamic description submitted to irrotational condition is equivalent to the Dirac-Fock-Weil equation (DFWE) [22], given the generic fermion field

$$\begin{aligned} \psi_{\pm i} &= \begin{pmatrix} \psi_{\pm 1} \\ \psi_{\pm 2} \end{pmatrix} = \begin{pmatrix} |\psi_{\pm 1}| \exp\left[i\frac{S_{\pm 1}}{\hbar}\right] \\ |\psi_{\pm 2}| \exp\left[i\frac{S_{\pm 2}}{\hbar}\right] \end{pmatrix} = \begin{pmatrix} \sum_{k'=-\infty}^{\infty} a_{\pm 1k'} \exp\left[\frac{iS_{\pm 1}(k')}{\hbar}\right] \\ \sum_{k'=-\infty}^{\infty} a_{\pm 2k'} \exp\left[\frac{iS_{\pm 2}(k')}{\hbar}\right] \end{pmatrix} \\ &= \begin{pmatrix} \sum_{k=0}^{\infty} \alpha_{\pm 1k} \exp\left[\frac{iS_{\pm 1}(k)}{\hbar}\right] + \beta_{\pm 1k}^{\dagger} \exp\left[\frac{iS_{\pm 1}(-k)}{\hbar}\right] \\ \sum_{k=0}^{\infty} \alpha_{\pm 2k} \exp\left[\frac{iS_{\pm 2}(k)}{\hbar}\right] + \beta_{\pm 2k}^{\dagger} \exp\left[\frac{iS_{\pm 2}(-k)}{\hbar}\right] \end{pmatrix} \\ &\equiv \sum_{k=0}^{\infty} \alpha_{\pm ik} \exp\left[\frac{iS_{\pm i}(k)}{\hbar}\right] + \beta_{\pm ik}^{\dagger} \exp\left[\frac{iS_{\pm i}(-k)}{\hbar}\right] \end{aligned} \quad (A33)$$

the action reads

$$\begin{aligned} S_{\pm i} &= \frac{\hbar}{2i} \ln \left[\frac{\psi_{\pm i}}{\psi_{\pm i}^*} \right] = \frac{\hbar}{2i} \begin{pmatrix} \ln \left[\sum_{k'=-\infty}^{\infty} a_{\pm ik'} \exp\left(\frac{iS_{\pm i}(k')}{\hbar}\right) \right] \\ -\ln \left[\sum_{k'=-\infty}^{\infty} a_{\pm ik'}^* \exp\left(-\frac{iS_{\pm i}(k')}{\hbar}\right) \right] \end{pmatrix} \\ &= \frac{\hbar}{2i} \begin{pmatrix} \ln \left[\sum_{k=0}^{\infty} \alpha_{\pm ik} \exp\left(\frac{iS_{\pm i}(k)}{\hbar}\right) + \beta_{\pm ik}^* \exp\left(\frac{iS_{\pm i}(-k)}{\hbar}\right) \right] \\ -\ln \left[\sum_{k=0}^{\infty} \alpha_{\pm ik}^* \exp\left(-\frac{iS_{\pm i}(k)}{\hbar}\right) + \beta_{\pm ik} \exp\left(-\frac{iS_{\pm i}(-k)}{\hbar}\right) \right] \end{pmatrix} \end{aligned} \quad (A34)$$

where for the k -eigenstates $S_{\pm i}(k)$ reads

$$S_{\pm i}(k) = \begin{pmatrix} S_{\pm 1}(k) \\ S_{\pm 2}(k) \end{pmatrix} = \frac{\hbar}{2i} \begin{pmatrix} \ln \frac{\psi_{\pm 1}(k)}{\psi_{\pm 1}(k)^*} \\ \ln \frac{\psi_{\pm 2}(k)}{\psi_{\pm 2}(k)^*} \end{pmatrix} \quad (A35)$$

Moreover, from Equation (A33), the hydrodynamic momentum $p_{\pm i\mu}$ and the Lagrangian function L_{\pm} read, respectively,

$$\begin{aligned} p_{\pm i\mu} &= -\partial_{\mu} S_{\pm i} = -\frac{1}{2} \frac{\sum_{k=0}^{\infty} \left(\alpha_{\pm ik} \exp\left(\frac{iS_{\pm i}(k)}{\hbar}\right) \left(\frac{\hbar}{i} \partial_{\mu} \ln \alpha_{\pm ik} - p_{\mu \pm i}(k)\right) + \beta_{\pm ik}^* \exp\left(\frac{iS_{\pm i}(-k)}{\hbar}\right) \left(\frac{\hbar}{i} \partial_{\mu} \ln \beta_{\pm ik}^* - p_{\mu \pm i}(-k)\right) \right)}{\sum_{k=0}^{\infty} \alpha_{\pm ik} \exp\left(\frac{iS_{\pm i}(k)}{\hbar}\right) + \beta_{\pm ik}^* \exp\left(\frac{iS_{\pm i}(-k)}{\hbar}\right)} \\ &\quad + \frac{1}{2} \frac{\sum_{k=0}^{\infty} \left(\alpha_{\pm ik}^* \exp\left(-\frac{iS_{\pm i}(k)}{\hbar}\right) \left(\frac{\hbar}{i} \partial_{\mu} \ln \alpha_{\pm ik}^* + p_{\mu \pm i}(k)\right) + \beta_{\pm ik} \exp\left(-\frac{iS_{\pm i}(-k)}{\hbar}\right) \left(\frac{\hbar}{i} \partial_{\mu} \ln \beta_{\pm ik} + p_{\mu \pm i}(-k)\right) \right)}{\sum_{k=0}^{\infty} \alpha_{\pm ik}^* \exp\left(-\frac{iS_{\pm i}(k)}{\hbar}\right) + \beta_{\pm ik} \exp\left(-\frac{iS_{\pm i}(-k)}{\hbar}\right)} = Tr(p_{\pm i\mu}) \end{aligned} \quad (A36)$$

where the matrix $p_{\pm i\mu}$ reads

$$p_{\pm i\mu} = \begin{pmatrix} \left(\frac{\alpha_{\pm ik} \exp\left(\frac{iS_{\pm i(k)}}{\hbar}\right) \left(\frac{\hbar}{i} \partial_{\mu} \ln \alpha_{\pm ik} - p_{\mu_{\pm i(k)}}\right) + \beta_{\pm ik}^* \exp\left(\frac{iS_{\pm i(-k)}}{\hbar}\right) \left(\frac{\hbar}{i} \partial_{\mu} \ln \beta_{\pm ik}^* - p_{\mu_{\pm i(-k)}}\right) \right)}{\sum_{k_j=0} \alpha_{\pm ik_j} \exp\left(\frac{iS_{\pm i(k_j)}}{\hbar}\right) + \beta_{\pm ik_j}^* \exp\left(\frac{iS_{\pm i(-k_j)}}{\hbar}\right)} \right) \\ + \frac{1}{2} \left(\frac{\alpha_{\pm ik}^* \exp\left(-\frac{iS_{\pm i(k)}}{\hbar}\right) \left(\frac{\hbar}{i} \partial_{\mu} \ln \alpha_{\pm ik}^* + p_{\mu_{\pm i(k)}}\right) + \beta_{\pm ik} \exp\left(-\frac{iS_{\pm i(-k)}}{\hbar}\right) \left(\frac{\hbar}{i} \partial_{\mu} \ln \beta_{\pm ik} + p_{\mu_{\pm i(-k)}}\right)}{\sum_{k_j=0} \alpha_{\pm ik_j}^* \exp\left(-\frac{iS_{\pm i(k_j)}}{\hbar}\right) + \beta_{\pm ik_j} \exp\left(-\frac{iS_{\pm i(-k_j)}}{\hbar}\right)} \right) \end{pmatrix} \delta_{mk} = \tilde{p}_{(k)\pm i\mu} \delta_{mk} \tag{A37}$$

$$L_{\pm i} = \frac{dS_{\pm i}}{dt} = \frac{\partial S_{\pm i}}{\partial t} + \frac{\partial S_{\pm i}}{\partial q_i} \dot{q}_i = -p_{\mu_{\pm i}} \dot{q}^{\mu}_{\pm i} = Tr(\mathbf{L}_{\pm i}) \\ = \frac{1}{2} \left(\frac{\sum_{k=0} \left(\alpha_{\pm ik} \exp\left(\frac{iS_{\pm i(k)}}{\hbar}\right) \left(\frac{\hbar}{i} \dot{q}^{\mu}_{\pm i} \partial_{\mu} \ln \alpha_{\pm ik} - p_{\mu_{\pm i(k)}} \dot{q}^{\mu}_{\pm i}\right) + \beta_{\pm ik}^* \exp\left(\frac{iS_{\pm i(-k)}}{\hbar}\right) \left(\frac{\hbar}{i} \dot{q}^{\mu}_{\pm i} \partial_{\mu} \ln \beta_{\pm ik}^* - p_{\mu_{\pm i(-k)}} \dot{q}^{\mu}_{\pm i}\right) \right)}{\sum_{k=0} \left(\alpha_{\pm ik} \exp\left(\frac{iS_{\pm i(k)}}{\hbar}\right) + \beta_{\pm ik}^* \exp\left(\frac{iS_{\pm i(-k)}}{\hbar}\right) \right)} \right) \\ - \frac{1}{2} \left(\frac{\sum_{k=0} \left(\alpha_{\pm ik}^* \exp\left(-\frac{iS_{\pm i(k)}}{\hbar}\right) \left(\frac{\hbar}{i} \dot{q}^{\mu}_{\pm i} \partial_{\mu} \ln \alpha_{\pm ik}^* + p_{\mu_{\pm i(k)}} \dot{q}^{\mu}_{\pm i}\right) + \beta_{\pm ik} \exp\left(-\frac{iS_{\pm i(-k)}}{\hbar}\right) \left(\frac{\hbar}{i} \dot{q}^{\mu}_{\pm i} \partial_{\mu} \ln \beta_{\pm ik} + p_{\mu_{\pm i(-k)}} \dot{q}^{\mu}_{\pm i}\right) \right)}{\sum_{k=0} \left(\alpha_{\pm ik}^* \exp\left(-\frac{iS_{\pm i(k)}}{\hbar}\right) + \beta_{\pm ik} \exp\left(-\frac{iS_{\pm i(-k)}}{\hbar}\right) \right)} \right) \tag{A38}$$

where the Lagrangian matrix reads

$$\mathbf{L}_{\pm i} = -p_{\mu_{\pm i}} \dot{q}^{\mu}_{\pm i} = -\tilde{p}_{\mu_{\pm i(k)}} \dot{q}^{\mu}_{\pm i} \delta_{m\bar{k}} = \tilde{L}_{\pm i(k)} \delta_{m\bar{k}} \tag{A39}$$

where

$$\dot{q}_{\mu_{\pm i}} = \frac{(p_{\mu_{\pm i}} + eA_{\mu})}{m\gamma_{\pm i} \sqrt{1 - \frac{V_{qu\pm i}}{mc^2}}} = c^2 \frac{(p_{\mu_{\pm i}} + eA_{\mu})}{E_{\pm i}} = -c^2 \frac{(p_{\mu_{\pm i}} + eA_{\mu})}{\frac{\partial S_{\pm i}}{\partial t} + e\phi} \tag{A40}$$

where

$$\tilde{L}_{\pm i(k)} = \frac{1}{2} \left(\frac{\left(\alpha_{\pm ik} \exp\left(\frac{iS_{\pm i(k)}}{\hbar}\right) \left(\frac{\hbar}{i} \dot{q}_{\pm i}^{\mu} \partial_{\mu} \ln \alpha_{\pm ik} - p_{\mu_{\pm i(k)}} \dot{q}_{\pm i}^{\mu}\right) + \beta_{\pm ik}^* \exp\left(\frac{iS_{\pm i(-k)}}{\hbar}\right) \left(\frac{\hbar}{i} \dot{q}_{\pm i}^{\mu} \partial_{\mu} \ln \beta_{\pm ik}^* - p_{\mu_{\pm i(-k)}} \dot{q}_{\pm i}^{\mu}\right) \right)}{\sum_{k_j=0} \left(\alpha_{\pm ik_j} \exp\left(\frac{iS_{\pm i(k_j)}}{\hbar}\right) + \beta_{\pm ik_j}^* \exp\left(\frac{iS_{\pm i(-k_j)}}{\hbar}\right) \right)} \right) \\ - \frac{1}{2} \left(\frac{\left(\alpha_{\pm ik}^* \exp\left(-\frac{iS_{\pm i(k)}}{\hbar}\right) \left(\frac{\hbar}{i} \dot{q}_{\pm i}^{\mu} \partial_{\mu} \ln \alpha_{\pm ik}^* + p_{\mu_{\pm i(k)}} \dot{q}_{\pm i}^{\mu}\right) + \beta_{\pm ik} \exp\left(-\frac{iS_{\pm i(-k)}}{\hbar}\right) \left(\frac{\hbar}{i} \dot{q}_{\pm i}^{\mu} \partial_{\mu} \ln \beta_{\pm ik} + p_{\mu_{\pm i(-k)}} \dot{q}_{\pm i}^{\mu}\right) \right)}{\sum_{k_j=0} \left(\alpha_{\pm ik_j}^* \exp\left(-\frac{iS_{\pm i(k_j)}}{\hbar}\right) + \beta_{\pm ik_j} \exp\left(-\frac{iS_{\pm i(-k_j)}}{\hbar}\right) \right)} \right) \tag{A41}$$

Thence, by using (A37) and (A41) it follows that [27]

$$-\frac{\partial \tilde{L}_{\pm i(k)}}{\partial \dot{q}^{\mu}} = \tilde{p}_{(k)\pm i\mu} \tag{A42}$$

Moreover, given that, for the \bar{k} -eigenstates

$$\tilde{L}_{(\bar{k})\pm i} = L_{(\bar{k})\pm i} \tag{A43}$$

$$\tilde{p}(\tilde{k})_{\pm i\mu} = p(\tilde{k})_{\pm i\mu} \quad (\text{A44})$$

and that the hydrodynamic Lagrangian function of the \tilde{k} -eigenstate $L_{\pm i(\tilde{k})}$ in Equation (A24) of time-independent systems does not explicitly depend by time, it follows that [27] $-\frac{\partial L_{\pm i(\tilde{k})}}{\partial q^\mu} = \dot{p}_{\pm i(\tilde{k})\mu}$, Equation (A43) and that, for the eigenstates, the Lagrangian hydrodynamic motion equations read

$$p(\tilde{k})_{\pm i\mu} = -\frac{\partial L_{\pm i(\tilde{k})}}{\partial \dot{q}^\mu_{\pm i(\tilde{k})}} \quad (\text{A45})$$

$$\dot{p}(\tilde{k})_{\pm i\mu} = -\frac{\partial L_{\pm i(\tilde{k})}}{\partial q^\mu} \quad (\text{A46})$$

leading to the motion equation

$$\frac{d}{dt} \left(-\frac{\partial L_{\pm i(\tilde{k})}}{\partial \dot{q}^\mu_{\pm i(\tilde{k})}} \right) = -\frac{\partial L_{\pm i(\tilde{k})}}{\partial q^\mu} \quad (\text{A47})$$

As shown in [23] the stationary states (i.e., the eigenstates), of Equation (A47) obey to the current conservation (A12) and are irrotational solutions of the DFWE.

Equation (A47) (whose non-relativistic limit (i.e., the Pauli equation) is given in ref. [22]). Moreover, generally speaking, by using the identities (A37) and (A41) the generalized quantum-hydrodynamic relativistic motion equation of the generic DFWE state (A33) [27] reads

$$\frac{d}{dt} \left(-\frac{\partial \tilde{L}_{(k)\pm i}}{\partial \dot{q}^\mu_{\pm i(k)}} \right) = -\frac{\partial \tilde{L}_{(k)\pm i}}{\partial q^\mu} - \tilde{p}_{(k)\pm i\nu} \frac{\partial \dot{q}_{(k)\pm i\nu}}{\partial q^\mu} \quad (\text{A48})$$

Moreover, by making the summation over k , Equation (A48) leads to

$$\text{Tr}(\dot{p}_{\pm i\mu}) = \frac{d}{dt} \text{Tr}(p_{\pm i\mu}) = -\frac{\partial \text{Tr}(L_{\pm i})}{\partial q^\mu} - \text{Tr}(p_{\pm i\nu} \partial_\mu \dot{q}_{\pm i\nu}) \quad (\text{A49})$$

or to

$$\dot{p}_{\pm i\mu} = -\frac{\partial L_{\pm i}}{\partial q^\mu} - \sum_k \tilde{p}_{(k)\pm i\nu} \frac{\partial \dot{q}_{(k)\pm i\nu}}{\partial q^\mu} \quad (\text{A50})$$

where, in the following, it is useful to pose

$$\text{Tr}(L_{\pm i}) = L_{\pm i} = L_{\pm i\text{Class}} + L_{\pm i\text{Q}} + L_{\pm i\text{mix}} \quad (\text{A51})$$

where

$$L_{\pm i\text{class}} = -\frac{1}{2} \frac{\sum_{k=0} \left(\alpha_{\pm ik} e^{\frac{iS_{\pm i(k)}}{\hbar}} p_{\pm i(k)\text{class}} \dot{q}^\mu_{\pm i} + \beta_{\pm ik}^* e^{\frac{iS_{\pm i(-k)}}{\hbar}} p_{\pm i(-k)\text{class}} \dot{q}^\mu_{\pm i} \right)}{\sum_{k=0} \left(\alpha_{\pm ik} e^{\frac{iS_{\pm i(k)}}{\hbar}} + \beta_{\pm ik}^* e^{\frac{iS_{\pm i(-k)}}{\hbar}} \right)} - \frac{1}{2} \frac{\sum_{k=0} \left(\alpha_{\pm ik}^* e^{-\frac{iS_{\pm i(k)}}{\hbar}} p_{\pm i(k)\text{class}} \dot{q}^\mu_{\pm i} + \beta_{\pm ik} e^{-\frac{iS_{\pm i(-k)}}{\hbar}} p_{\pm i(-k)\text{class}} \dot{q}^\mu_{\pm i} \right)}{\sum_{k=0} \left(\alpha_{\pm ik}^* e^{-\frac{iS_{\pm i(k)}}{\hbar}} + \beta_{\pm ik} e^{-\frac{iS_{\pm i(-k)}}{\hbar}} \right)} \quad (\text{A52})$$

$$\tilde{L}_{\pm i Q} = -\frac{1}{2} \frac{\sum_{k=0} \left(\alpha_{\pm ik} e^{\frac{iS_{\pm i(k)}}{\hbar}} a(V_{qu_{\pm i(k)}}) \left(p_{\mu_{\pm i(k)class}} \dot{q}_{\pm i}^{\mu} + eA_{\mu} \dot{q}_{\pm i}^{\mu} \right) + \beta_{\pm ik}^* e^{-\frac{iS_{\pm i(-k)}}{\hbar}} a(V_{qu_{\pm i(-k)}}) \left(p_{\mu_{\pm i(-k)class}} \dot{q}_{\pm i}^{\mu} + eA_{\mu} \dot{q}_{\pm i}^{\mu} \right) \right)}{\sum_{k=0} \left(\alpha_{\pm ik} e^{\frac{iS_{\pm i(k)}}{\hbar}} + \beta_{\pm ik}^* e^{-\frac{iS_{\pm i(-k)}}{\hbar}} \right)} \tag{A53}$$

$$-\frac{1}{2} \frac{\sum_{k=0} \left(\alpha_{\pm ik}^* e^{-\frac{iS_{\pm i(k)}}{\hbar}} a(V_{qu_{\pm i(k)}}) \left(p_{\mu_{\pm i(k)class}} \dot{q}_{\pm i}^{\mu} + eA_{\mu} \dot{q}_{\pm i}^{\mu} \right) + \beta_{\pm ik} e^{-\frac{iS_{\pm i(-k)}}{\hbar}} a(V_{qu_{\pm i(-k)}}) \left(p_{\mu_{\pm i(-k)class}} \dot{q}_{\pm i}^{\mu} + eA_{\mu} \dot{q}_{\pm i}^{\mu} \right) \right)}{\sum_{k=0} \left(\alpha_{\pm ik}^* e^{-\frac{iS_{\pm i(k)}}{\hbar}} + \beta_{\pm ik} e^{-\frac{iS_{\pm i(-k)}}{\hbar}} \right)}$$

$$\tilde{L}_{\pm i mix} = \frac{\hbar}{2i} \frac{\left(\alpha_{\pm ik} e^{\frac{iS_{\pm i(k)}}{\hbar}} \dot{q}_{\pm i}^{\mu} \partial_{\mu} \ln \alpha_{\pm ik} + \beta_{\pm ik}^* e^{-\frac{iS_{\pm i(-k)}}{\hbar}} \dot{q}_{\pm i}^{\mu} \partial_{\mu} \ln \beta_{\pm ik}^* \right)}{\sum_{k=0} \left(\alpha_{\pm ik} e^{\frac{iS_{\pm i(k)}}{\hbar}} + \beta_{\pm ik}^* e^{-\frac{iS_{\pm i(-k)}}{\hbar}} \right)} - \frac{\hbar}{2i} \frac{\left(\alpha_{\pm ik}^* e^{-\frac{iS_{\pm i(k)}}{\hbar}} \dot{q}_{\pm i}^{\mu} \partial_{\mu} \ln \alpha_{\pm ik}^* + \beta_{\pm ik} e^{-\frac{iS_{\pm i(-k)}}{\hbar}} \dot{q}_{\pm i}^{\mu} \partial_{\mu} \ln \beta_{\pm ik} \right)}{\sum_{k=0} \left(\alpha_{\pm ik}^* e^{-\frac{iS_{\pm i(k)}}{\hbar}} + \beta_{\pm ik} e^{-\frac{iS_{\pm i(-k)}}{\hbar}} \right)} \tag{A54}$$

A.3. The Hydrodynamic Energy-Impulse Tensor of the Dirac Field

In this section we derive the expression of the hydrodynamic energy-impulse tensor of the Dirac field. By posing the EITD $T_{\pm i(k)\mu}^{\nu} = |\psi_{\pm i(k)}|^2 T_{\pm i(\pm k)\mu}^{\nu}$ [21] we obtain [27]

$$T_{\pm i(k)\mu}^{\nu} = \frac{mc^2}{\gamma} \left(\left(u_{\mu_{\pm i(k)}} \sqrt{1 - \frac{V_{qu_{\pm i(k)}}}{mc^2}} u_{\pm i(k)}^{\nu} - u_{\alpha_{\pm i(k)}} \sqrt{1 - \frac{V_{qu_{\pm i(k)}}}{mc^2}} u_{\pm i(k)}^{\alpha} \right) + \frac{eA_{\mu}}{mc} u_{\pm i(k)}^{\nu} - \frac{eA_{\alpha}}{mc} u_{\pm i(k)}^{\alpha} \delta_{\mu}^{\nu} \right) \tag{A55}$$

that rearranged in the form

$$T_{\pm i(k)\mu}^{\nu} = T_{\pm i(k)Class} \mu^{\nu} + L_{\pm i(k)Class} \delta_{\mu}^{\nu} + T_{\pm i(k)Q_{\mu}}^{\nu} \tag{A56}$$

where

$$T_{\pm i(k)Class} \mu^{\nu} = \frac{mc^2}{\gamma} \left(u_{\mu_{\pm i(k)}} + \frac{eA^{\nu}}{mc} \right) u_{\pm i(k)}^{\nu} = -c^2 p_{\mu_{\pm i(k)class}} \frac{(p^{\nu}_{\pm i(k)} - eA^{\nu})}{E_{\pm i(k)} - e\phi} \tag{A57}$$

$$T_{\pm i(k)Q_{\mu}}^{\nu} = -a(V_{qu_{\pm i(k)}}) \frac{mc^2}{\gamma_{\pm i(k)}} \left(u_{\mu_{\pm i(k)}} u_{\pm i(k)}^{\nu} - u_{\alpha_{\pm i(k)}} u_{\pm i(k)}^{\alpha} \delta_{\mu}^{\nu} \right) \tag{A58}$$

with the help of both the identity

$$S_{\pm i} = \frac{\hbar}{2i} \ln \left[\frac{\psi_{\pm i}}{\psi_{\pm i}^*} \right] = \begin{pmatrix} S_{\pm 1} \\ S_{\pm 2} \end{pmatrix} = \frac{\hbar}{2i} \begin{pmatrix} \ln \left[\frac{\psi_{\pm 1}}{\psi_{\pm 1}^*} \right] \\ \ln \left[\frac{\psi_{\pm 2}}{\psi_{\pm 2}^*} \right] \end{pmatrix} \tag{A59}$$

and (A16), (A22) and (A33), leads to the expression of the EITD (A55) as a function of the Dirac field, independently by the hydrodynamic formalism, that reads

$$T_{\pm i(k)\mu}^{\nu} = |\psi_{\pm i(k)}|^2 \frac{i\hbar c^2}{2} \left(\frac{\partial}{\partial t} \ln \left[\frac{\psi_{\pm i(k)}}{\psi_{\pm i(k)}^*} \right] + \frac{2ie}{\hbar} \phi \right)^{-1} \begin{pmatrix} \frac{\partial \ln \left[\frac{\psi_{\pm i(k)}}{\psi_{\pm i(k)}^*} \right]}{\partial q^{\mu}} \left(\frac{\partial \ln \left[\frac{\psi_{\pm i(k)}}{\psi_{\pm i(k)}^*} \right]}{\partial q^{\nu}} - \frac{2ie}{\hbar} A^{\nu} \right) \\ - \frac{\partial \ln \left[\frac{\psi_{\pm i(k)}}{\psi_{\pm i(k)}^*} \right]}{\partial q^{\alpha}} \left(\frac{\partial \ln \left[\frac{\psi_{\pm i(k)}}{\psi_{\pm i(k)}^*} \right]}{\partial q^{\alpha}} - \frac{2ie}{\hbar} A^{\alpha} \right) \delta_{\mu}^{\nu} \end{pmatrix} \tag{A60}$$

It is useful to note that expression (A60) excludes the hydrodynamic solutions that do not satisfy the irrotational condition.

Moreover, since for

$$k' > 0, k = k', \beta_{\pm ik} = 0 \tag{A61}$$

$$k' < 0, k = -k', \alpha_{\pm ik} = 0 \tag{A62}$$

the positive ($k' > 0$, superscript $(+)$) and negative energy ($k' < 0$, superscript $(-)$) eigenstates, (A60) reads, respectively,

$$T_{\pm i(k)\mu}^{\nu} = T_{\pm i(k'>0)\mu}^{\nu} = |\alpha_{\pm i(k)}|^2 \frac{i\hbar c^2}{2} \left(\frac{\partial}{\partial t} \left(\ln \left[\frac{\alpha_{\pm i(k)}}{\alpha_{\pm i(k)}^*} \right] + \frac{2iS_{\pm 1(k)}}{\hbar} \right) + \frac{2ie}{\hbar} \phi \right)^{-1} \left(\begin{array}{l} \frac{\partial \ln \left[\frac{\alpha_{\pm i(k)}}{\alpha_{\pm i(k)}^*} \right] + \frac{2iS_{\pm 1(k)}}{\hbar}}{\partial q^\mu} \left(\frac{\partial \ln \left[\frac{\alpha_{\pm i(k)}}{\alpha_{\pm i(k)}^*} \right] + \frac{2iS_{\pm 1(k)}}{\hbar}}{\partial q_\nu} - \frac{2ie}{\hbar} A^\nu \right) \\ - \frac{\partial \ln \left[\frac{\alpha_{\pm i(k)}}{\alpha_{\pm i(k)}^*} \right] + \frac{2iS_{\pm 1(k)}}{\hbar}}{\partial q^\alpha} \left(\frac{\partial \ln \left[\frac{\alpha_{\pm i(k)}}{\alpha_{\pm i(k)}^*} \right] + \frac{2iS_{\pm 1(k)}}{\hbar}}{\partial q_\alpha} - \frac{2ie}{\hbar} A^\alpha \right) \delta_{\mu}^{\nu} \end{array} \right) \tag{A63}$$

$$T_{\pm i(k)\mu}^{\nu} = T_{\pm i(k'<0)\mu}^{\nu} = |\beta_{\pm i(k)}|^2 \frac{i\hbar c^2}{2} \left(\frac{\partial}{\partial t} \left(\ln \left[\frac{\beta_{\pm i(k)}}{\beta_{\pm i(k)}^*} \right] + \frac{2iS_{\pm 1(-k)}}{\hbar} \right) + \frac{2ie}{\hbar} \phi \right)^{-1} \left(\begin{array}{l} \frac{\partial \ln \left[\frac{\beta_{\pm i(k)}}{\beta_{\pm i(k)}^*} \right] + \frac{2iS_{\pm 1(-k)}}{\hbar}}{\partial q^\mu} \left(\frac{\partial \ln \left[\frac{\beta_{\pm i(k)}}{\beta_{\pm i(k)}^*} \right] + \frac{2iS_{\pm 1(-k)}}{\hbar}}{\partial q_\nu} - \frac{2ie}{\hbar} A^\nu \right) \\ - \frac{\partial \ln \left[\frac{\beta_{\pm i(k)}}{\beta_{\pm i(k)}^*} \right] + \frac{2iS_{\pm 1(-k)}}{\hbar}}{\partial q^\alpha} \left(\frac{\partial \ln \left[\frac{\beta_{\pm i(k)}}{\beta_{\pm i(k)}^*} \right] + \frac{2iS_{\pm 1(-k)}}{\hbar}}{\partial q_\alpha} - \frac{2ie}{\hbar} A^\alpha \right) \delta_{\mu}^{\nu} \end{array} \right) \tag{A64}$$

where the explicit forms of both $\alpha_{\pm i(k)}$ and $\beta_{\pm i(k)}$ are defined by the solution of the generalized theory into the curvilinear space-time.

A.4. The Macroscopic Limit

Since in the macroscopic classical limit (i.e., when the action values are much bigger than the Planck constant and when the physical length is much bigger than the De Broglie length [24]) the superposition of states (A33) undergo collapse to an eigenstate (due to the quantum decoherence produced by fluctuations [23–26]), the macroscopic GE of the general relativity cannot be obtained just by applying the limiting condition $\lim_{\hbar \rightarrow 0}$ but also by imposing the decoherence condition (terming it \lim_{dec}) as follows

$$\lim_{macro} \equiv \lim_{dec} \lim_{\hbar \rightarrow 0} = \lim_{\hbar \rightarrow 0} \lim_{dec} \tag{A65}$$

where the *decoherence* effect on the quantum state reads

$$\lim_{dec} \psi_{\pm i} = \lim_{dec} \left(\begin{array}{l} \sum_{k=0} \alpha_{\pm 1k} \exp \left[\frac{iS_{\pm 1(k)}}{\hbar} \right] + \beta_{\pm 1k}^{\dagger} \exp \left[\frac{iS_{\pm 1(-k)}}{\hbar} \right] \\ \sum_{k=0} \alpha_{\pm 2k} \exp \left[\frac{iS_{\pm 2(k)}}{\hbar} \right] + \beta_{\pm 2k}^{\dagger} \exp \left[\frac{iS_{\pm 2(-k)}}{\hbar} \right] \end{array} \right) \begin{array}{l} \text{for } k' > 0 \\ \left(\begin{array}{l} \alpha_{\pm 1\bar{k}} \exp \left[\frac{iS_{\pm 1(k)}}{\hbar} \right] \\ \alpha_{\pm 2\bar{k}} \exp \left[\frac{iS_{\pm 2(k)}}{\hbar} \right] \end{array} \right) \\ = \\ \text{for } k' < 0 \\ \left(\begin{array}{l} \beta_{\pm 1\bar{k}}^{\dagger} \exp \left[\frac{iS_{\pm 1(-k)}}{\hbar} \right] \\ \beta_{\pm 2\bar{k}}^{\dagger} \exp \left[\frac{iS_{\pm 2(-k)}}{\hbar} \right] \end{array} \right) \end{array} \tag{A66}$$

leading to

$$\begin{aligned} \lim_{dec} \lim_{\hbar \rightarrow 0} \psi_{\pm i} &= \lim_{dec} \psi_0 = \lim_{dec} \sum_{k=0} \alpha_{0k} \exp\left[\frac{iS_{0(k)}}{\hbar}\right] + \beta_0^\dagger \exp\left[\frac{iS_{0(-k)}}{\hbar}\right] \\ &= \text{for } k' > 0 = \alpha_{0k} \exp\left[\frac{iS_{0(k)}}{\hbar}\right] \\ &= \text{for } k' < 0 = \beta_0^\dagger \exp\left[\frac{iS_{0(-k)}}{\hbar}\right] \end{aligned} \quad (A67)$$

where \tilde{k} is the eigenstates to which the wave-function collapses and where

$$\lim_{\hbar \rightarrow 0} \psi_{\pm i} = \psi_0 = \sum_{k=0} \alpha_{0k} \exp\left[\frac{iS_{0(k)}}{\hbar}\right] + \beta_0^\dagger \exp\left[\frac{iS_{0(-k)}}{\hbar}\right] \quad (A68)$$

is the solution of the Klein-Gordon (KG) Equation

$$\begin{aligned} \lim_{\hbar \rightarrow 0} \left[\hbar^2 \left(\partial_\mu + \frac{ie}{\hbar} A_\mu \right) \left(\partial^\mu + \frac{ie}{\hbar} A^\mu \right) + \left(m^2 c^2 + \frac{ie\hbar}{2} \alpha^{\mu\nu} F_{\mu\nu} \right) \right] \psi_{\pm} \\ \equiv \left[\left(\partial_\mu + \frac{ie}{\hbar} A_\mu \right) \left(\partial^\mu + \frac{ie}{\hbar} A^\mu \right) + m^2 c^2 \right] \psi_0 = 0 \end{aligned} \quad (A69)$$

and $\alpha_k, \beta_k^\dagger$ obey to the relations

$$\lim_{\hbar \rightarrow 0} \alpha_{\pm 1\tilde{k}} = \lim_{\hbar \rightarrow 0} \alpha_{\pm 2\tilde{k}} = \alpha_{0\tilde{k}} \quad (A70)$$

$$\lim_{\hbar \rightarrow 0} \beta_{\pm 1\tilde{k}}^\dagger = \lim_{\hbar \rightarrow 0} \beta_{\pm 2\tilde{k}}^\dagger = \beta_{0\tilde{k}}^\dagger \quad (A71)$$

Moreover, by introducing (A70, A71) into (A66), the exchangeability of the two limits

$$\lim_{dec} \lim_{\hbar \rightarrow 0} \psi_{\pm i} = \lim_{\hbar \rightarrow 0} \lim_{dec} \psi_{\pm i} \quad (A72)$$

follows.

The detailed stochastic hydrodynamic derivation of the macroscopic limit (A65) is given in ref. [24–26], but it is suffice to say here that the quantum non local interaction is maintained beyond the De Broglie length in the case of strong interactions [23–26] such as in the strong gravitational field of a black hole. Therefore, we assume the existence of the macroscopic limit in the curved space-time taking care that the macroscopic scale may go much beyond the De Broglie length in very high curvature space-time.

A.5. The Minimum Action Principle in the Hydrodynamic Formalism

Since the hydrodynamic Lagrangian depends also by the quantum potential and hence by $|\psi_{\pm i}| = f_1(|\psi_{\pm i(k)}|)$ and $\partial_\mu |\psi_{\pm i}| = f_2(|\psi_{\pm i(k)}|, \partial_\mu |\psi_{\pm i(k)}|)$, the problem of defining the equation of motion can be generally carried out by using the set of variables: $x_{(k)}^\mu = (q_\mu, \dot{q}_{\mu \pm i(k)}, |\psi_{\pm i(k)}|, \partial_\mu |\psi_{\pm i(k)}|)$.

Thence, the variation of the hydrodynamic action $S = \int \iiint \mathcal{L} dV dt$ (between the fixed starting and ending points, $q_{\mu a}, q_{\mu b}$, respectively) reads [19]

$$\begin{aligned} \delta S_{\pm i} &= \int \iiint |\psi_{\pm i}|^2 \left(\frac{\partial Tr L}{\partial x_{(k)}^\mu} \right) \delta x_{(k)}^\mu dV dt \\ &= \frac{1}{c} \int \iiint |\psi_{\pm i}|^2 \sum_k \left(\begin{aligned} &\left(\frac{\partial \tilde{L}_{\pm i(k)}}{\partial q^\mu} - \frac{d}{dt} \frac{\partial \tilde{L}_{\pm i(k)}}{\partial \dot{q}^\mu} \right) \delta q^\mu \\ &+ \left(\frac{\partial \tilde{L}_{\pm i(k)}}{\partial |\psi_{\pm i(k)}|} - \partial^\mu \frac{\partial \tilde{L}_{\pm i(k)}}{\partial \partial^\mu |\psi_{\pm i(k)}|} \right) \delta |\psi_{\pm i(k)}| \end{aligned} \right) d\Omega \end{aligned} \quad (A73)$$

Given that the quantum motion equations for eigenstates satisfy the condition

$$\left(\frac{\partial \tilde{L}_{\pm i(k)}}{\partial q^\mu} - \frac{d}{dt} \frac{\partial \tilde{L}_{\pm i(k)}}{\partial \dot{q}^\mu} \right) = \left(\frac{\partial L_{\pm i(k)}}{\partial q^\mu} - \frac{d}{dt} \frac{\partial L_{\pm i(k)}}{\partial \dot{q}^\mu} \right) = 0 \quad (A74)$$

the variation of the action $\delta S_{\pm i(k)}$ for the k -th eigenstates reads

$$\delta S_{\pm i(k)} = \frac{1}{c} \int \iiint |\psi_{\pm i(k)}|^2 \left(\frac{\partial L_{\pm i(k)}}{\partial |\psi_{\pm i(k)}|} - \partial^\mu \frac{\partial L_{\pm i(k)}}{\partial \partial_\mu |\psi_{\pm i(k)}|} \right) \delta |\psi_k| d\Omega = \delta \left(\Delta S_{\pm i(k)Q} \right) \quad (\text{A75})$$

that is not null for the dependence of the hydrodynamic Lagrangian $L_{\pm i(k)}$ by the quantum potential.

Thence, for the quantum hydrodynamic evolution, the variational principle reads

$$\delta S_{\pm i(k)} - \delta \left(\Delta S_{\pm i(k)Q} \right) = 0 \quad (\text{A76})$$

that in the classical limit (i.e., for $\hbar \rightarrow 0$, $V_{qu} \rightarrow 0$) so that

$$\frac{\partial (\lim_{\hbar \rightarrow 0} L_{\pm i(k)})}{\partial |\psi_{\pm i(k)}|} = 0 \quad (\text{A77})$$

$$\frac{\partial (\lim_{\hbar \rightarrow 0} L_{\pm i(k)})}{\partial^\mu |\psi_{\pm i(k)}|} = 0 \quad (\text{A78})$$

leads to the classical extremal principle

$$\lim_{\hbar \rightarrow 0} \delta S_{\pm i(k)} = \lim_{\hbar \rightarrow 0} \delta \left(\Delta S_{\pm i(k)Q} \right) = 0 \quad (\text{A79})$$

Moreover, generally speaking, by using (A48), for the general superposition of state (A33) the variation of the action reads

$$\begin{aligned} \delta S_{\pm i} &= -\frac{1}{c} \int \iiint |\psi_{\pm i}|^2 \sum_{\bar{k}} \left(\left(\tilde{p}_{\pm i(\bar{k})} \frac{\partial \dot{q}_{\pm i(\bar{k})\nu}}{\partial q^\mu} \right) \delta q^\mu \right. \\ &\quad \left. - \left(\frac{\partial \tilde{L}_{\pm i(\bar{k})}}{\partial |\psi_{\pm i(\bar{k})}|} - \partial^\mu \frac{\partial \tilde{L}_{\pm i(\bar{k})}}{\partial \partial^\mu |\psi_{\pm i(\bar{k})}|} \right) \delta |\psi_{\bar{k}}| \right) d\Omega \\ &= \delta \left(\Delta S_{\pm iQ_{mix}} + \Delta S_{\pm iQ} \right) = \delta \left(\Delta S_{\pm i} \right) \end{aligned} \quad (\text{A80})$$

where

$$\delta \left(\Delta S_{\pm iQ_{mix}} \right) = -\frac{1}{c} \int \iiint |\psi_{\pm i}|^2 \sum_{\bar{k}} \tilde{p}_{\pm i(\bar{k})} \frac{\partial \dot{q}_{\pm i(\bar{k})\nu}}{\partial q^\mu} \delta |\psi_{\pm i(k)}| d\Omega \quad (\text{A81})$$

is due to the mixing of the quantum superposition of states and where

$$\Delta S_{\pm iQ} = \frac{1}{c} \int \iiint |\psi_{\pm i}|^2 \sum_{\bar{k}} \left(\frac{\partial \tilde{L}_{\pm i(k)}}{\partial |\psi_{\pm i(\bar{k})}|} - \partial^\mu \frac{\partial \tilde{L}_{\pm i(k)}}{\partial \partial^\mu |\psi_{\pm i(\bar{k})}|} \right) \delta |\psi_k| d\Omega \quad (\text{A82})$$

Thence, by (A80), the minimum action principle, generally reads

$$\delta S_{\pm i} - \delta \left(\Delta S_{\pm i} \right) = 0 \quad (\text{A83})$$

that, for the decoherence (decay to an eigenstate) at the macroscopic scale

$$\begin{aligned} \lim_{macro} \delta S_{\pm i} &= \lim_{macro} \delta \left(\Delta S_{\pm i} \right) = \lim_{\hbar \rightarrow 0} \lim_{dec} \delta \left(\Delta S_{\pm iQ} + \Delta S_{\pm iQ_{mix}} \right) \\ &= \lim_{\hbar \rightarrow 0} \delta \left(\Delta S_{\pm iQ(\bar{k})} + \lim_{dec} \Delta S_{\pm iQ_{mix}} \right) \\ &= \lim_{\hbar \rightarrow 0} \delta \left(\Delta S_{\pm iQ(\bar{k})} \right) = 0 \end{aligned} \quad (\text{A84})$$

converges to the extremal classical principle.

Appendix B

The EITD of the boson field, given the KGE

$$\psi_{;\mu}^{;\mu} = (g^{\mu\nu}\partial_\nu\psi)_{;\mu} = \frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}(g^{\mu\nu}\partial_\nu\psi) = 0 \quad (\text{A85})$$

reads [19]

$$T_{\mu\nu} = |\psi|^2 T_{\mu\nu} = |\psi|^2 c^2 \left(\frac{\partial S}{\partial t} \right)^{-1} \left(\frac{\partial S}{\partial q^\mu} \frac{\partial S}{\partial q^\nu} - g_{\alpha\beta} \frac{\partial S}{\partial q^\beta} \frac{\partial S}{\partial q^\alpha} g_{\mu\nu} \right) \quad (\text{A86})$$

Given the wave function of a photon

$$\psi_{(q,t)} \propto \frac{A_{(q,t)}}{c} \sqrt{\frac{\omega}{\hbar}} \left[l^{-\frac{3}{2}} \right] \quad (\text{A87})$$

where $|\psi|^2$ represents the number of photons per volume and where A is the vector potential plane wave of the photon in the Minkowskian case

$$A = A_0 \exp[-ik_\mu q^\mu] = A_0 \exp\left[i\frac{S}{\hbar}\right] \quad (\text{A88})$$

from (A86) it follows that

$$T_{\mu\nu} \propto |A|^2 k_\mu k_\nu \quad (\text{A89})$$

that compared with the output of the EM theory for the photon [29]

$$T_{\nu\mu\text{em}} = \frac{|E|^2}{4\pi} \frac{c^2}{\omega^2} k_\mu k_\nu \propto \frac{|A|^2}{4\pi} k_\mu k_\nu \quad (\text{A90})$$

leads to

$$T_{\mu\nu} \propto 4\pi T_{\nu\mu\text{em}} \quad (\text{A91})$$

Moreover, since for the photon the hydrodynamic Lagrangian reads

$$L = L_{\text{class}} = -p_{(k)_\mu} \dot{q}^\mu_{(k)} = -\hbar g_{\mu\alpha} k^\alpha \dot{q}^\mu = -\frac{\hbar\omega}{|k|^2} g_{\mu\alpha} k^\alpha k^\mu = 0 \quad (\text{A92})$$

it follows that

$$\tau_{\mu\nu\text{class}} = T_{(\tilde{k})\mu\nu\text{class}} \quad (\text{A93})$$

$$\Lambda_{\mu\nu Q} = -\frac{|\psi_k|^2}{2} \left(\frac{\tau_{(k)\kappa\kappa\text{mix}}}{4} - \frac{\tau_{(k)\kappa\kappa\text{mix}}}{4} \right) g_{\mu\nu} \delta_{hk} = 0 \quad (\text{A94})$$

$$\Delta\tau_{\mu\nu\text{stress}} = -\frac{|\psi_k|^2}{2} \left(\tau_{s(k)\mu\nu\text{mix}} - \tau_{s(k)\mu\nu\text{mix}} \right) \delta_{hk} = 0 \quad (\text{A95})$$

From which we can see that the quantum part of the CGT, as well as the CPTD, for the photon is null.

Appendix C

By using the identity (A9), the Equation (10) reads

$$\begin{aligned} & \left(\begin{array}{l} \partial_\mu g^{\mu\nu} \partial_\nu + \frac{ie}{\hbar} \left(\begin{array}{l} -\frac{i}{4} \sigma^{ab} \omega_{ab\mu} + \frac{ie}{\hbar} A_\mu \\ + \partial_\mu g^{\mu\nu} \left(-\frac{i}{4} \sigma^{ab} \omega_{ab\nu} + \frac{ie}{\hbar} A_\nu \right) \end{array} \right) g^{\mu\nu} \partial_\nu \\ + \left(-\frac{i}{4} \sigma^{ab} \omega_{ab\mu} + \frac{ie}{\hbar} A_\mu \right) g^{\mu\nu} \left(-\frac{i}{4} \sigma^{ab} \omega_{ab\nu} + \frac{ie}{\hbar} A_\nu \right) \end{array} \right) \left(\begin{array}{l} |\psi_{\pm 1}| \exp\left[i\frac{S_{\pm 1}}{\hbar}\right] \\ |\psi_{\pm 2}| \exp\left[i\frac{S_{\pm 2}}{\hbar}\right] \end{array} \right) \\ & = -\left(\frac{m^2 c^2}{\hbar^2} + g^{\mu\beta} g^{\nu\alpha} \alpha_{\pm\beta\alpha} \left(\frac{ie}{2\hbar} F_{\mu\nu} - \frac{i}{4} \sigma^{ab} (\omega_{ab\mu} \partial_\nu + \partial_\mu \omega_{ab\nu}) \right) \right) \left(\begin{array}{l} |\psi_{\pm 1}| \exp\left[i\frac{S_{\pm 1}}{\hbar}\right] \\ |\psi_{\pm 2}| \exp\left[i\frac{S_{\pm 2}}{\hbar}\right] \end{array} \right) \end{aligned} \quad (\text{A96})$$

that, by developing the derivatives, leads to

$$\begin{aligned}
 & \left(\begin{aligned} & \left(\begin{aligned} & \exp\left[i\frac{S_{\pm 1}}{\hbar}\right]\partial_{\mu}(g^{\mu\nu}\partial_{\nu}|\psi_{\pm 1}| + \frac{i}{\hbar}g^{\mu\nu}\partial_{\nu}S_{\pm 1}) \\ & \exp\left[i\frac{S_{\pm 2}}{\hbar}\right]\partial_{\mu}(g^{\mu\nu}\partial_{\nu}|\psi_{\pm 2}| + \frac{i}{\hbar}g^{\mu\nu}\partial_{\nu}S_{\pm 2}) \end{aligned} \right) \\ & + \left(\begin{aligned} & \frac{i}{\hbar}\exp\left[i\frac{S_{\pm 1}}{\hbar}\right]\partial_{\mu}S_{\pm 1}(g^{\mu\nu}\partial_{\nu}|\psi_{\pm 1}| + \frac{i}{\hbar}|\psi_{\pm 1}|g^{\mu\nu}\partial_{\nu}S_{\pm 1}) \\ & \frac{i}{\hbar}\exp\left[i\frac{S_{\pm 2}}{\hbar}\right]\partial_{\mu}S_{\pm 2}(g^{\mu\nu}\partial_{\nu}|\psi_{\pm 2}| + \frac{i}{\hbar}|\psi_{\pm 2}|g^{\mu\nu}\partial_{\nu}S_{\pm 2}) \end{aligned} \right) \\ & + \left(\begin{aligned} & \left(-\frac{i}{4}\sigma^{ab}\omega_{ab\mu} + \frac{ie}{\hbar}A_{\mu} \right) g^{\mu\nu} \begin{pmatrix} \exp\left[i\frac{S_{\pm 1}}{\hbar}\right](\partial_{\nu}|\psi_{\pm 1}| + \frac{i}{\hbar}|\psi_{\pm 1}|\partial_{\nu}S_{\pm 1}) \\ \exp\left[i\frac{S_{\pm 2}}{\hbar}\right](\partial_{\nu}|\psi_{\pm 2}| + \frac{i}{\hbar}|\psi_{\pm 2}|\partial_{\nu}S_{\pm 2}) \end{pmatrix} \\ & + g^{\mu\nu}\left(-\frac{i}{4}\sigma^{ab}\omega_{ab\nu} + \frac{ie}{\hbar}A_{\nu}\right) \begin{pmatrix} \exp\left[i\frac{S_{\pm 1}}{\hbar}\right](\partial_{\mu}|\psi_{\pm 1}| + \frac{i}{\hbar}|\psi_{\pm 1}|\partial_{\mu}S_{\pm 1}) \\ \exp\left[i\frac{S_{\pm 2}}{\hbar}\right](\partial_{\mu}|\psi_{\pm 2}| + \frac{i}{\hbar}|\psi_{\pm 2}|\partial_{\mu}S_{\pm 2}) \end{pmatrix} \end{aligned} \right) \\ & + \left(\begin{aligned} & |\psi_{\pm 1}|\exp\left[i\frac{S_{\pm 1}}{\hbar}\right] \\ & |\psi_{\pm 2}|\exp\left[i\frac{S_{\pm 2}}{\hbar}\right] \end{aligned} \right) \partial_{\mu}g^{\mu\nu}\left(-\frac{i}{4}\sigma^{ab}\omega_{ab\nu} + \frac{ie}{\hbar}A_{\nu}\right) \\ & - g^{\mu\nu}\left(-\frac{1}{4}\sigma^{ab}\omega_{ab\mu} + \frac{e}{\hbar}A_{\mu}\right)\left(-\frac{1}{4}\sigma^{ab}\omega_{ab\nu} + \frac{e}{\hbar}A_{\nu}\right) \begin{pmatrix} |\psi_{\pm 1}|\exp\left[i\frac{S_{\pm 1}}{\hbar}\right] \\ |\psi_{\pm 2}|\exp\left[i\frac{S_{\pm 2}}{\hbar}\right] \end{pmatrix} \end{aligned} \right) \\ & = -\left(\frac{m^2c^2}{\hbar^2} + g^{\mu\beta}g^{\nu\alpha}\alpha_{\pm\beta\alpha}\left(\frac{ie}{2\hbar}F_{\mu\nu} - \frac{i}{4}\sigma^{ab}(\omega_{ab\mu}\partial_{\nu} + \partial_{\mu}\omega_{ab\nu})\right)\right) \begin{pmatrix} |\psi_{\pm 1}|\exp\left[i\frac{S_{\pm 1}}{\hbar}\right] \\ |\psi_{\pm 2}|\exp\left[i\frac{S_{\pm 2}}{\hbar}\right] \end{pmatrix}
 \end{aligned} \tag{A97}$$

and, by dividing each component by $\exp\left[i\frac{S_{\pm i}}{\hbar}\right]$, where $i = 1, 2$, to

$$\begin{aligned}
 & \left(\begin{aligned} & \left(\begin{aligned} & \partial_{\mu}(g^{\mu\nu}\partial_{\nu}|\psi_{\pm 1}| + \frac{i}{\hbar}g^{\mu\nu}\partial_{\nu}S_{\pm 1}) \\ & \partial_{\mu}(g^{\mu\nu}\partial_{\nu}|\psi_{\pm 2}| + \frac{i}{\hbar}g^{\mu\nu}\partial_{\nu}S_{\pm 2}) \end{aligned} \right) + \left(\begin{aligned} & \frac{i}{\hbar}\partial_{\mu}S_{\pm 1}(g^{\mu\nu}\partial_{\nu}|\psi_{\pm 1}| + \frac{i}{\hbar}|\psi_{\pm 1}|g^{\mu\nu}\partial_{\nu}S_{\pm 1}) \\ & \frac{i}{\hbar}\partial_{\mu}S_{\pm 2}(g^{\mu\nu}\partial_{\nu}|\psi_{\pm 2}| + \frac{i}{\hbar}|\psi_{\pm 2}|g^{\mu\nu}\partial_{\nu}S_{\pm 2}) \end{aligned} \right) \\ & + \left(\begin{aligned} & \left(-\frac{i}{4}\sigma^{ab}\omega_{ab\mu} + \frac{ie}{\hbar}A_{\mu} \right) g^{\mu\nu} \begin{pmatrix} (\partial_{\nu}|\psi_{\pm 1}| + \frac{i}{\hbar}|\psi_{\pm 1}|\partial_{\nu}S_{\pm 1}) \\ (\partial_{\nu}|\psi_{\pm 2}| + \frac{i}{\hbar}|\psi_{\pm 2}|\partial_{\nu}S_{\pm 2}) \end{pmatrix} \\ & + g^{\mu\nu}\left(-\frac{i}{4}\sigma^{ab}\omega_{ab\nu} + \frac{ie}{\hbar}A_{\nu}\right) \begin{pmatrix} (\partial_{\mu}|\psi_{\pm 1}| + \frac{i}{\hbar}|\psi_{\pm 1}|\partial_{\mu}S_{\pm 1}) \\ (\partial_{\mu}|\psi_{\pm 2}| + \frac{i}{\hbar}|\psi_{\pm 2}|\partial_{\mu}S_{\pm 2}) \end{pmatrix} \end{aligned} \right) \\ & + \left(\begin{aligned} & |\psi_{\pm 1}| \\ & |\psi_{\pm 2}| \end{aligned} \right) \partial_{\mu}g^{\mu\nu}\left(-\frac{i}{4}\sigma^{ab}\omega_{ab\nu} + \frac{ie}{\hbar}A_{\nu}\right) \\ & - g^{\mu\nu}\left(-\frac{1}{4}\sigma^{ab}\omega_{ab\mu} + \frac{e}{\hbar}A_{\mu}\right)\left(-\frac{1}{4}\sigma^{ab}\omega_{ab\nu} + \frac{e}{\hbar}A_{\nu}\right) \begin{pmatrix} |\psi_{\pm 1}| \\ |\psi_{\pm 2}| \end{pmatrix} \end{aligned} \right) \\ & = -\left(\frac{m^2c^2}{\hbar^2} + g^{\mu\beta}g^{\nu\alpha}\alpha_{\pm\beta\alpha}\left(\frac{ie}{2\hbar}F_{\mu\nu} - \frac{i}{4}\sigma^{ab}(\omega_{ab\mu}\partial_{\nu} + \partial_{\mu}\omega_{ab\nu})\right)\right) \begin{pmatrix} |\psi_{\pm 1}| \\ |\psi_{\pm 2}| \end{pmatrix}
 \end{aligned} \tag{A98}$$

Moreover, by grouping the terms and equating the real part of the equation, it follows that

$$\begin{aligned}
 & \left(\begin{aligned} & \left(\frac{\partial_\mu (g^{\mu\nu} \partial_\nu |\psi_{\pm 1}|)}{|\psi_{\pm 1}|} \right) + \left(\frac{1}{\hbar^2} |\psi_{\pm 1}| \partial_\mu S_{\pm 1} (g^{\mu\nu} \partial_\nu S_{\pm 1}) \right) \\ & \left(\frac{\partial_\mu (g^{\mu\nu} \partial_\nu |\psi_{\pm 2}|)}{|\psi_{\pm 2}|} \right) + \left(\frac{1}{\hbar^2} |\psi_{\pm 2}| \partial_\mu S_{\pm 2} (g^{\mu\nu} \partial_\nu S_{\pm 2}) \right) \\ & \frac{1}{\hbar} \left(\operatorname{Re} \left\{ \frac{1}{4} \sigma^{ab} \omega_{ab\mu} \right\} - \frac{e}{\hbar} A_\mu \right) g^{\mu\nu} \begin{pmatrix} |\psi_{\pm 1}| \partial_\nu S_{\pm 1} \\ |\psi_{\pm 2}| \partial_\nu S_{\pm 2} \end{pmatrix} \\ & + \frac{1}{\hbar} g^{\mu\nu} \left(\operatorname{Re} \left\{ \frac{1}{4} \sigma^{ab} \omega_{ab\nu} \right\} - \frac{e}{\hbar} A_\nu \right) \begin{pmatrix} |\psi_{\pm 1}| \partial_\mu S_{\pm 1} \\ |\psi_{\pm 2}| \partial_\mu S_{\pm 2} \end{pmatrix} \\ & + \begin{pmatrix} |\psi_{\pm 1}| \\ |\psi_{\pm 2}| \end{pmatrix} \partial_\mu g^{\mu\nu} \left(-\frac{\operatorname{Im} \{ \sigma^{ab} \omega_{ab\nu} \}}{4} \right) \\ & - \operatorname{Im} \left\{ \frac{1}{4} \sigma^{ab} \omega_{ab\mu} \right\} g^{\mu\nu} \begin{pmatrix} \partial_\nu |\psi_{\pm 1}| \\ \partial_\nu |\psi_{\pm 2}| \end{pmatrix} \\ & - g^{\mu\nu} \operatorname{Im} \left\{ \frac{1}{4} \sigma^{ab} \omega_{ab\nu} \right\} \begin{pmatrix} \partial_\mu |\psi_{\pm 1}| \\ \partial_\mu |\psi_{\pm 2}| \end{pmatrix} \\ & - g^{\mu\nu} \left(\operatorname{Re} \left\{ \frac{1}{4} \sigma^{ab} \omega_{ab\mu} \right\} - \frac{e}{\hbar} A_{\mu\nu} \right) \left(\operatorname{Re} \left\{ \frac{1}{4} \sigma^{ab} \omega_{ab\nu} \right\} - \frac{e}{\hbar} A \right) \begin{pmatrix} |\psi_{\pm 1}| \\ |\psi_{\pm 2}| \end{pmatrix} \\ & - g^{\mu\nu} \operatorname{Im} \left\{ \frac{1}{4} \sigma^{ab} \omega_{ab\mu} \right\} \operatorname{Im} \left\{ \frac{1}{4} \sigma^{ab} \omega_{ab\nu} \right\} \begin{pmatrix} |\psi_{\pm 1}| \\ |\psi_{\pm 2}| \end{pmatrix} \end{aligned} \right) \\
 & = - \left(\frac{m^2 c^2}{\hbar^2} + \operatorname{Re} \left\{ g^{\mu\beta} g^{\nu\alpha} \alpha_{\pm\beta\alpha} \left(\frac{ie}{2\hbar} F_{\mu\nu} - \frac{i}{4} \sigma^{ab} (\omega_{ab\mu} \partial_\nu + \partial_\mu \omega_{ab\nu}) \right) \right\} \right) \begin{pmatrix} |\psi_{\pm 1}| \\ |\psi_{\pm 2}| \end{pmatrix}
 \end{aligned} \tag{A99}$$

and, by dividing each component by $|\psi_{\pm i}|$, where $i = 1, 2$, that

$$\begin{aligned}
 & \left(\begin{aligned} & \left(\frac{\partial_\mu (g^{\mu\nu} \partial_\nu |\psi_{\pm 1}|)}{|\psi_{\pm 1}|} \right) + \left(\frac{1}{\hbar^2} \partial_\mu S_{\pm 1} (g^{\mu\nu} \partial_\nu S_{\pm 1}) \right) \\ & \left(\frac{\partial_\mu (g^{\mu\nu} \partial_\nu |\psi_{\pm 2}|)}{|\psi_{\pm 2}|} \right) + \left(\frac{1}{\hbar^2} \partial_\mu S_{\pm 2} (g^{\mu\nu} \partial_\nu S_{\pm 2}) \right) \\ & \frac{1}{\hbar} \left(\operatorname{Re} \left\{ \frac{1}{4} \sigma^{ab} \omega_{ab\mu} \right\} - \frac{e}{\hbar} A_\mu \right) g^{\mu\nu} \begin{pmatrix} \partial_\nu S_{\pm 1} \\ \partial_\nu S_{\pm 2} \end{pmatrix} \\ & + \frac{1}{\hbar} g^{\mu\nu} \left(\operatorname{Re} \left\{ \frac{1}{4} \sigma^{ab} \omega_{ab\nu} \right\} - \frac{e}{\hbar} A_\nu \right) \begin{pmatrix} \partial_\mu S_{\pm 1} \\ \partial_\mu S_{\pm 2} \end{pmatrix} \\ & + \partial_\mu g^{\mu\nu} \left(-\frac{\operatorname{Im} \{ \sigma^{ab} \omega_{ab\nu} \}}{4} \right) \\ & - \frac{1}{2} \operatorname{Im} \left\{ \sigma^{ab} \omega_{ab\mu} \right\} g^{\mu\nu} \begin{pmatrix} \frac{\partial_\nu |\psi_{\pm 1}|}{|\psi_{\pm 1}|} \\ \frac{\partial_\nu |\psi_{\pm 2}|}{|\psi_{\pm 2}|} \end{pmatrix} \\ & - g^{\mu\nu} \left(\operatorname{Re} \left\{ \frac{1}{4} \sigma^{ab} \omega_{ab\mu} \right\} - \frac{e}{\hbar} A_\mu \right) \left(\operatorname{Re} \left\{ \frac{1}{4} \sigma^{ab} \omega_{ab\nu} \right\} - \frac{e}{\hbar} A_\nu \right) \\ & - g^{\mu\nu} \operatorname{Im} \left\{ \frac{1}{4} \sigma^{ab} \omega_{ab\mu} \right\} \operatorname{Im} \left\{ \frac{1}{4} \sigma^{ab} \omega_{ab\nu} \right\} \end{aligned} \right) \\
 & = - \left[\frac{m^2 c^2}{\hbar^2} + \operatorname{Re} \left\{ g^{\mu\beta} g^{\nu\alpha} \left(\begin{aligned} & \frac{ie}{2\hbar} \begin{pmatrix} \alpha_{\beta\alpha 1j} \frac{|\psi_{\pm j}|}{|\psi_{\pm 1}|} \\ \alpha_{\beta\alpha 2j} \frac{|\psi_{\pm j}|}{|\psi_{\pm 2}|} \end{pmatrix} F_{\mu\nu} \\ & - \frac{i}{4} \alpha_{\beta\alpha} \sigma^{ab} \begin{pmatrix} \omega_{ab\mu} \left(\frac{\partial_\nu |\psi_{\pm 1}|}{|\psi_{\pm 1}|} \right) \\ \omega_{ab\mu} \left(\frac{\partial_\nu |\psi_{\pm 2}|}{|\psi_{\pm 2}|} \right) \end{pmatrix} + \begin{pmatrix} \frac{\partial_\mu \omega_{ab\nu} |\psi_{\pm 1}|}{|\psi_{\pm 1}|} \\ \frac{\partial_\mu \omega_{ab\nu} |\psi_{\pm 2}|}{|\psi_{\pm 2}|} \end{pmatrix} \end{aligned} \right) \right\} \right]
 \end{aligned} \tag{A100}$$

$$\begin{aligned}
& \left(\begin{array}{l} \left(\begin{array}{l} \partial_\mu S_{\pm 1} (g^{\mu\nu} \partial_\nu S_{\pm 1}) \\ \partial_\mu S_{\pm 2} (g^{\mu\nu} \partial_\nu S_{\pm 2}) \end{array} \right) \\ + 2\hbar \left(\operatorname{Re} \left\{ \frac{1}{4} \sigma^{ab} \omega_{ab\mu} \right\} - \frac{e}{\hbar} A_\mu \right) g^{\mu\nu} \begin{pmatrix} \partial_\nu S_{\pm 1} \\ \partial_\nu S_{\pm 2} \end{pmatrix} \\ - \hbar^2 g^{\mu\nu} \left(\operatorname{Re} \left\{ \frac{1}{4} \sigma^{ab} \omega_{ab\mu} \right\} - \frac{e}{\hbar} A_\mu \right) \left(\operatorname{Re} \left\{ \frac{1}{4} \sigma^{ab} \omega_{ab\nu} \right\} - \frac{e}{\hbar} A_\nu \right) \\ - \hbar^2 \partial_\mu g^{\mu\nu} \left(\frac{\operatorname{Im} \{ \sigma^{ab} \omega_{ab\nu} \}}{4} \right) \\ - \hbar^2 g^{\mu\nu} \operatorname{Im} \left\{ \frac{1}{4} \sigma^{ab} \omega_{ab\mu} \right\} \operatorname{Im} \left\{ \frac{1}{4} \sigma^{ab} \omega_{ab\nu} \right\} \end{array} \right) \\
= & - \left(\begin{array}{l} m^2 c^2 + \operatorname{Re} \left\{ g^{\mu\beta} g^{\nu\alpha} \begin{pmatrix} \frac{ie}{2\hbar} \begin{pmatrix} \alpha_{\pm\beta\alpha 1j} \frac{|\psi_{\pm j}|}{|\psi_{\pm 1}|} \\ \alpha_{\pm\beta\alpha 2j} \frac{|\psi_{\pm j}|}{|\psi_{\pm 2}|} \end{pmatrix} F_{\mu\nu} \\ - \frac{i}{4} \alpha_{\pm\beta\alpha} \sigma^{ab} \left(\omega_{ab\mu} \begin{pmatrix} \frac{\partial_\nu |\psi_{\pm 1}|}{|\psi_{\pm 1}|} \\ \frac{\partial_\nu |\psi_{\pm 2}|}{|\psi_{\pm 2}|} \end{pmatrix} + \begin{pmatrix} \frac{\partial_\mu \omega_{ab\nu} |\psi_{\pm 1}|}{|\psi_{\pm 1}|} \\ \frac{\partial_\mu \omega_{ab\nu} |\psi_{\pm 2}|}{|\psi_{\pm 2}|} \end{pmatrix} \right) \end{pmatrix} \right) \\ - \hbar^2 \left(\frac{\partial_\mu (g^{\mu\nu} \partial_\nu |\psi_{\pm 1}|)}{|\psi_{\pm 1}|} \right) \left(\frac{\partial_\mu (g^{\mu\nu} \partial_\nu |\psi_{\pm 2}|)}{|\psi_{\pm 2}|} \right) + \frac{\hbar^2}{2} \operatorname{Im} \{ \sigma^{ab} \omega_{ab\mu} \} g^{\mu\nu} \begin{pmatrix} \frac{\partial_\nu |\psi_{\pm 1}|}{|\psi_{\pm 1}|} \\ \frac{\partial_\nu |\psi_{\pm 2}|}{|\psi_{\pm 2}|} \end{pmatrix} \end{array} \right) \\ \left(\begin{array}{l} \partial_\mu \left(S_{\pm 1} + \left(\operatorname{Re} \left\{ \frac{\hbar}{4} \sigma^{ab} \omega_{ab\mu} \right\} - e A_\mu \right) \right) (g^{\mu\nu} \partial_\nu (S_{\pm 1} + \left(\operatorname{Re} \left\{ \frac{\hbar}{4} \sigma^{ab} \omega_{ab\mu} \right\} - e A_\mu \right))) \\ \partial_\mu \left(S_{\pm 2} + \left(\operatorname{Re} \left\{ \frac{\hbar}{4} \sigma^{ab} \omega_{ab\mu} \right\} - e A_\mu \right) \right) (g^{\mu\nu} \partial_\nu (S_{\pm 2} + \left(\operatorname{Re} \left\{ \frac{\hbar}{4} \sigma^{ab} \omega_{ab\mu} \right\} - e A_\mu \right))) \end{array} \right) \\ = & - \left(\begin{array}{l} m^2 c^2 + \operatorname{Re} \left\{ g^{\mu\beta} g^{\nu\alpha} \begin{pmatrix} \frac{ie}{2\hbar} \begin{pmatrix} \alpha_{\pm\beta\alpha 1j} \frac{|\psi_{\pm j}|}{|\psi_{\pm 1}|} \\ \alpha_{\pm\beta\alpha 2j} \frac{|\psi_{\pm j}|}{|\psi_{\pm 2}|} \end{pmatrix} F_{\mu\nu} \\ - \frac{i}{4} \alpha_{\pm\beta\alpha} \sigma^{ab} \left(\omega_{ab\mu} \begin{pmatrix} \frac{\partial_\nu |\psi_{\pm 1}|}{|\psi_{\pm 1}|} \\ \frac{\partial_\nu |\psi_{\pm 2}|}{|\psi_{\pm 2}|} \end{pmatrix} + \begin{pmatrix} \frac{\partial_\mu \omega_{ab\nu} |\psi_{\pm 1}|}{|\psi_{\pm 1}|} \\ \frac{\partial_\mu \omega_{ab\nu} |\psi_{\pm 2}|}{|\psi_{\pm 2}|} \end{pmatrix} \right) \end{pmatrix} \right) \\ - \hbar^2 \left(\frac{\partial_\mu (g^{\mu\nu} \partial_\nu |\psi_{\pm 1}|)}{|\psi_{\pm 1}|} \right) \left(\frac{\partial_\mu (g^{\mu\nu} \partial_\nu |\psi_{\pm 2}|)}{|\psi_{\pm 2}|} \right) + \frac{\hbar^2}{2} \operatorname{Im} \{ \sigma^{ab} \omega_{ab\mu} \} g^{\mu\nu} \begin{pmatrix} \frac{\partial_\nu |\psi_{\pm 1}|}{|\psi_{\pm 1}|} \\ \frac{\partial_\nu |\psi_{\pm 2}|}{|\psi_{\pm 2}|} \end{pmatrix} \\ + \hbar^2 \partial_\mu g^{\mu\nu} \left(\frac{\operatorname{Im} \{ \sigma^{ab} \omega_{ab\nu} \}}{4} \right) \\ + \hbar^2 g^{\mu\nu} \operatorname{Im} \left\{ \frac{1}{4} \sigma^{ab} \omega_{ab\mu} \right\} \operatorname{Im} \left\{ \frac{1}{4} \sigma^{ab} \omega_{ab\nu} \right\} \end{array} \right) \end{aligned} \tag{A101}
\end{aligned}$$

leading to the final expression

$$\begin{aligned}
& g^{\mu\nu} \left(\partial_\mu \begin{pmatrix} S_{\pm 1} \\ S_{\pm 2} \end{pmatrix} + \left(\operatorname{Re} \left\{ \frac{\hbar}{4} \sigma^{ab} \omega_{ab\mu} \right\} - e A_\mu \right) \right) \left(\partial_\nu \begin{pmatrix} S_{\pm 1} \\ S_{\pm 2} \end{pmatrix} + \left(\operatorname{Re} \left\{ \frac{\hbar}{4} \sigma^{ab} \omega_{ab\nu} \right\} - e A_\nu \right) \right) \\
& = - \left(m^2 c^2 + m V_{qu} \right) \end{aligned} \tag{A102}$$

that, more synthetically can read

$$\begin{aligned}
& g^{\mu\nu} \left(\partial_\mu S_\pm + \left(\operatorname{Re} \left\{ \frac{\hbar}{4} \sigma^{ab} \omega_{ab\mu} \right\} - e A_\mu \right) \right) \left(\partial_\nu S_\pm + \left(\operatorname{Re} \left\{ \frac{\hbar}{4} \sigma^{ab} \omega_{ab\nu} \right\} - e A_\nu \right) \right) \\
& = - \left(m^2 c^2 + m V_{qu} \right) \end{aligned} \tag{A103}$$

where

$$\begin{aligned}
 V_{qu} = & -\frac{\hbar^2}{m} \left(\begin{aligned} & \left(\frac{\partial_\mu (g^{\mu\nu} \partial_\nu |\psi_{\pm 1}|)}{|\psi_{\pm 1}|} \right) + \frac{\hbar^2}{2} \text{Im}\{\sigma^{ab} \omega_{ab\mu}\} g^{\mu\nu} \begin{pmatrix} \frac{\partial_\nu |\psi_{\pm 1}|}{|\psi_{\pm 1}|} \\ \frac{\partial_\nu |\psi_{\pm 2}|}{|\psi_{\pm 2}|} \end{pmatrix} \\ & + \partial_\mu g^{\mu\nu} \left(\frac{\text{Im}\{\sigma^{ab} \omega_{ab\nu}\}}{4} \right) + g^{\mu\nu} \text{Im}\{\frac{1}{4} \sigma^{ab} \omega_{ab\mu}\} \text{Im}\{\frac{1}{4} \sigma^{ab} \omega_{ab\nu}\} \end{aligned} \right) \\
 & + \text{Re} \left(g^{\mu\beta} g^{\nu\alpha} \begin{pmatrix} \frac{ie}{2\hbar} \begin{pmatrix} \alpha_{\pm\beta\alpha 1j} \frac{|\psi_{\pm 1}|}{|\psi_{\pm 1}|} \\ \alpha_{\pm\beta\alpha 2j} \frac{|\psi_{\pm 1}|}{|\psi_{\pm 2}|} \end{pmatrix} F_{\mu\nu} \\ -\frac{i}{4} \alpha_{\pm\beta\alpha} \sigma^{ab} \begin{pmatrix} \omega_{ab\mu} \begin{pmatrix} \frac{\partial_\nu |\psi_{\pm 1}|}{|\psi_{\pm 1}|} \\ \frac{\partial_\nu |\psi_{\pm 2}|}{|\psi_{\pm 2}|} \end{pmatrix} + \begin{pmatrix} \frac{\partial_\mu \omega_{ab\nu} |\psi_{\pm 1}|}{|\psi_{\pm 1}|} \\ \frac{\partial_\mu \omega_{ab\nu} |\psi_{\pm 2}|}{|\psi_{\pm 2}|} \end{pmatrix} \end{pmatrix} \right) \end{pmatrix} \right) \tag{A104}
 \end{aligned}$$

Moreover, by grouping the imaginary terms of Equation (A98) as follows

$$\begin{aligned}
 & \left(\begin{aligned} & \begin{pmatrix} \partial_\mu (\frac{i}{\hbar} g^{\mu\nu} \partial_\nu S_{\pm 1}) \\ \partial_\mu (\frac{i}{\hbar} g^{\mu\nu} \partial_\nu S_{\pm 2}) \end{pmatrix} + \begin{pmatrix} \frac{i}{\hbar} \partial_\mu S_{\pm 1} (g^{\mu\nu} \partial_\nu |\psi_{\pm 1}|) \\ \frac{i}{\hbar} \partial_\mu S_{\pm 2} (g^{\mu\nu} \partial_\nu |\psi_{\pm 2}|) \end{pmatrix} \\ & + \begin{pmatrix} 2(-\frac{i}{4} \text{Re}\{\sigma^{ab}\} \omega_{ab\mu} + \frac{ie}{\hbar} A_\mu) g^{\mu\nu} \begin{pmatrix} (\partial_\nu |\psi_{\pm 1}|) \\ (\partial_\nu |\psi_{\pm 2}|) \end{pmatrix} \\ + 2g^{\mu\nu} (\frac{1}{4} \text{Im}\{\sigma^{ab}\} \omega_{ab\nu}) \begin{pmatrix} (\frac{i}{\hbar} |\psi_{\pm 1}| \partial_\mu S_{\pm 1}) \\ (\frac{i}{\hbar} |\psi_{\pm 2}| \partial_\mu S_{\pm 2}) \end{pmatrix} \\ + \begin{pmatrix} |\psi_{\pm 1}| \\ |\psi_{\pm 2}| \end{pmatrix} \partial_\mu g^{\mu\nu} (-\frac{i}{4} \sigma^{ab} \omega_{ab\nu} + \frac{ie}{\hbar} A_\nu) \end{pmatrix} \\ -g^{\mu\nu} (-\frac{1}{4} \text{Re}\{\sigma^{ab}\} \omega_{ab\mu} + \frac{e}{\hbar} A_\mu) (-\frac{1}{4} \text{Im}\{\sigma^{ab}\} \omega_{ab\nu}) \begin{pmatrix} |\psi_{\pm 1}| \\ |\psi_{\pm 2}| \end{pmatrix} \\ -g^{\mu\nu} (-\frac{1}{4} \text{Im}\{\sigma^{ab}\} \omega_{ab\mu}) (-\frac{1}{4} \text{Re}\{\sigma^{ab}\} \omega_{ab\nu} + \frac{e}{\hbar} A_\nu) \begin{pmatrix} |\psi_{\pm 1}| \\ |\psi_{\pm 2}| \end{pmatrix} \end{pmatrix} \\ = -\left(\frac{ie}{2\hbar} \text{Re} \left\{ g^{\mu\beta} g^{\nu\alpha} \alpha_{\pm\beta\alpha} \left(F_{\mu\nu} - \frac{\hbar}{2e} \sigma^{ab} \begin{pmatrix} \omega_{ab\mu} \begin{pmatrix} \frac{\partial_\nu |\psi_{\pm 1}|}{|\psi_{\pm 1}|} \\ \frac{\partial_\nu |\psi_{\pm 2}|}{|\psi_{\pm 2}|} \end{pmatrix} + \begin{pmatrix} \frac{\partial_\mu \omega_{ab\nu} |\psi_{\pm 1}|}{|\psi_{\pm 1}|} \\ \frac{\partial_\mu \omega_{ab\nu} |\psi_{\pm 2}|}{|\psi_{\pm 2}|} \end{pmatrix} \right) \right\} \right) \begin{pmatrix} |\psi_{\pm 1}| \\ |\psi_{\pm 2}| \end{pmatrix} \end{aligned} \right) \tag{A105}
 \end{aligned}$$

the current conservation equation can finally be obtained (the derivation is not given here).

Appendix D

It is useful to note that in the quantum case, due to the force generated by the quantum potential (that is a function of the co-ordinates q_μ), generally speaking, we have that

$$\left(\sum_{\pm i} T_{\pm i \mu}{}^\nu \right)_{, \nu} + T_{\mu}{}^\nu{}_{; \nu em} = \sum_{\pm i} (|\psi_{\pm i}|^2 T_{\pm i \mu}{}^\nu)_{, \nu} + T_{\mu}{}^\nu{}_{; \nu em} \neq 0 \tag{A106}$$

It is easy to check this point by the inspection of the quantum hydrodynamic motion Equation (A47) [19,31] for a scalar uncharged particle obeying to the KGE (A69) that reads

$$\begin{aligned}
 \frac{dp_\mu}{ds} &= \frac{d}{ds} \left(mc u_\mu \sqrt{1 - \frac{V_{qu}}{mc^2}} \right) = -\frac{\gamma}{c} \frac{\partial L}{\partial q^\mu} \\
 &= mc \frac{\partial}{\partial q^\mu} \sqrt{1 - \frac{V_{qu}}{mc^2}} \tag{A107}
 \end{aligned}$$

where

$$u_\mu = \frac{\gamma}{c} \dot{q}_\mu \quad (\text{A108})$$

that leads to

$$mc \sqrt{1 - \frac{V_{qu}}{mc^2}} \frac{du_\mu}{ds} = -mcu_\mu \frac{d}{ds} \left(\sqrt{1 - \frac{V_{qu}}{mc^2}} \right) + mc \frac{\partial}{\partial q^\mu} \left(\sqrt{1 - \frac{V_{qu}}{mc^2}} \right) \quad (\text{A109})$$

where by using the energy-impulse tensor $T_{(k)\mu}{}^\nu$ expression [19]

$$T_{(k)\mu}{}^\nu = \left(\dot{q}_\mu \frac{\partial L_{(k)n}}{\partial \dot{q}_\nu} - L_{(k)n} \delta_\mu{}^\nu \right) = \frac{mc^2}{\gamma} \sqrt{1 - \frac{V_{qu(k)}}{mc^2}} (u_\mu u^\nu - \delta_\mu{}^\nu) \quad (\text{A110})$$

Equation (A99) more synthetically reads

$$mc \sqrt{1 - \frac{V_{qu(k)}}{mc^2}} \frac{du_\mu}{ds} = -\frac{\gamma}{c} \frac{\partial T_{(k)\mu}{}^\nu}{\partial q^\nu} \quad (\text{A111})$$

Since, in absence of an external Hamiltonian potential, $T_{(k)\mu}{}^\nu$ depends by the co-ordinates, through the quantum potential, there exists a local acceleration of the mass distribution being $\frac{du_\mu}{ds} \neq 0$.

This characteristics of the hydrodynamic quantum evolution is quite general and it is confirmed also for the motion Equation (A48) for the generic superposition of states.

To explain this point, it is sufficient to observe that the force due to the quantum potential is self-generated by the shape of the mass distribution and, hence, there may exist a force acting on the mass density, at each point, letting it accelerating, even in absence of an external force, (e.g., the ballistic enlargement of a free Gaussian packet).

In the case of the eigenstates (that are stationary states (i.e., $\frac{du_\mu}{ds} = 0$)), from (A110) it follows that

$$\frac{\partial T_{(k)\mu}{}^\nu}{\partial q^\nu} = T_{(k)\mu}{}^\nu{}_{;\nu} = 0 \quad (\text{A112})$$

The same results holds in presence of an external potential $\mathbf{U}_{(q_i)}$ considering the overall EITD

$$T_{(k)\mu}{}^\nu{}_{tot} = T_{(k)\mu}{}^\nu + T_{ext\mu}{}^\nu \quad (\text{A113})$$

where

$$T_{ext\mu}{}^\nu = \mathbf{U}_{(q_i)} \delta_\mu^\nu \quad (\text{A114})$$

leading to

$$mc \sqrt{1 - \frac{V_{qu(k)}}{mc^2}} \frac{du_\mu}{ds} = T_{(k)\mu}{}^\nu{}_{;\nu} + T_{ext\mu}{}^\nu{}_{;\nu} = 0 \quad (\text{A115})$$

The above property comes from the stationarity of the eigenstates generated by the fact that the quantum potential force equals the external forces point by point [23].

Moreover, since the eigenstates are the only states that survives to the decoherence, thence, it follows that

$$\begin{aligned} & \left(\lim_{dec} \sum_{\pm i} (|\psi_{\pm i}|^2 T_{\pm i\mu}{}^\nu)_{;\nu F} \right) + T_{\mu}{}^\nu{}_{;\nu em} \\ & = \sum_{\pm i} (|\psi_{\pm i(k)}|^2 T_{\pm i(k)\mu}{}^\nu)_{;\nu F} + T_{\mu}{}^\nu{}_{;\nu em} = 0 \end{aligned} \quad (\text{A116})$$

Appendix E

In quasi-Minkowskian curved space-time, with particles very far from the Planckian mass density ($\frac{m_p}{l_p^3} = \frac{c^5}{\hbar G^2}$), the vielbein can be approximated as following

$$e^a{}_{\mu} = \delta^a{}_{\mu} + \frac{\varepsilon^a{}_{\mu}}{2} \quad (\text{A117})$$

with

$$\varepsilon_{\mu\nu}\varepsilon^{\mu\nu} \ll 1 \quad (\text{A118})$$

and, being

$$g_{\nu\mu} = e^a{}_{\mu}\eta_{ab}e^b{}_{\nu} \quad (\text{A119})$$

where

$$\eta_{\nu\mu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad (\text{A120})$$

by posing

$$\begin{aligned} \varepsilon_{sim}{}^{\mu}{}_{\nu} &= \frac{\eta_{\mu a}\varepsilon^a{}_{\nu} + \eta_{\nu a}\varepsilon^a{}_{\mu}}{2} \\ &= \begin{bmatrix} \varepsilon^0{}_0 & \varepsilon^0{}_1 - \varepsilon^1{}_0 & \varepsilon^0{}_2 - \varepsilon^2{}_0 & \varepsilon^0{}_3 - \varepsilon^3{}_0 \\ \varepsilon^0{}_1 - \varepsilon^1{}_0 & -\varepsilon^1{}_1 & -(\varepsilon^1{}_2 + \varepsilon^2{}_1) & -(\varepsilon^1{}_3 + \varepsilon^3{}_1) \\ \varepsilon^0{}_2 - \varepsilon^2{}_0 & -(\varepsilon^1{}_2 + \varepsilon^2{}_1) & -\varepsilon^2{}_2 & -(\varepsilon^2{}_3 + \varepsilon^3{}_2) \\ \varepsilon^0{}_3 - \varepsilon^3{}_0 & -(\varepsilon^1{}_3 + \varepsilon^3{}_1) & -(\varepsilon^2{}_3 + \varepsilon^3{}_2) & -\varepsilon^3{}_3 \end{bmatrix} \end{aligned} \quad (\text{A121})$$

it follows that

$$g_{\mu\nu} \cong \eta_{\mu\nu} + \frac{\eta_{\mu a}\varepsilon^a{}_{\nu} + \eta_{\nu a}\varepsilon^a{}_{\mu}}{2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} + \varepsilon_{sim\mu\nu} \quad (\text{A122})$$

$$\Gamma_{\mu\alpha}^{\beta} = \frac{1}{2}\eta^{\beta\gamma}(\partial_{\alpha}\varepsilon_{sim\gamma\mu} + \partial_{\mu}\varepsilon_{sim\gamma\alpha} - \partial_{\gamma}\varepsilon_{sim\mu\alpha}) \quad (\text{A123})$$

$$\begin{aligned} \omega_{ab\mu} &= f_b^{\alpha}e_{a\beta}\Gamma_{\mu\alpha}^{\beta} - f_b^{\alpha}\partial_{\mu}e_{a\alpha} = f_b^{\alpha}e^b{}_{\beta}g_{ba}\Gamma_{\mu\alpha}^{\beta} - f_b^{\alpha}\partial_{\mu}e_{a\alpha} \\ &= \delta_{\beta}^{\alpha}(\eta_{ba} + \varepsilon_{sim}{}^b{}_a)\Gamma_{\mu\alpha}^{\beta} - f_b^{\alpha}\partial_{\mu}e_{a\alpha} \\ &= (\eta_{ba} + \varepsilon_{sim}{}^b{}_a)\Gamma_{\mu\alpha}^{\alpha} - \delta_b^{\alpha}\partial_{\mu}\frac{\varepsilon_{a\alpha}}{2} \\ &= (\eta_{ba} + \varepsilon_{sim}{}^b{}_a)\partial_{\mu}\ln\sqrt{-g} - \partial_{\mu}\frac{\varepsilon_{ab}}{2} \\ &= (\eta_{ba} + \varepsilon_{sim}{}^b{}_a)\partial_{\mu}\ln\sqrt{-\frac{1}{|\eta^{\beta\chi} + \varepsilon_{sim}{}^{\beta\chi}|}} - \partial_{\mu}\frac{\varepsilon_{ab}}{2} \\ &= (\eta_{ba} + \varepsilon_{sim}{}^b{}_a)\partial_{\mu}\ln\sqrt{1 + |\varepsilon_{sim}{}^{\beta\chi}|} - \partial_{\mu}\frac{\varepsilon_{ab}}{2} \\ &= (\eta_{ba} + \varepsilon_{sim}{}^b{}_a)\partial_{\mu}\ln\left(1 + \frac{|\varepsilon_{sim}{}^{\beta\chi}|}{2}\right) - \partial_{\mu}\frac{\varepsilon_{ab}}{2} \\ &= (\eta_{ba} + \varepsilon_{sim}{}^b{}_a)\partial_{\mu}\frac{|\varepsilon_{sim}{}^{\beta\chi}|}{2} - \partial_{\mu}\frac{\varepsilon_{ab}}{2} \\ &= \frac{1}{2}\partial_{\mu}(\eta_{ba}|\varepsilon_{sim}{}^{\beta\chi}| - \varepsilon_{ab}) \end{aligned} \quad (\text{A124})$$

where f_b^α is the inverse vielbein.

Thence, given that

$$\sigma^{ab} = -\sigma^{ba} \quad (\text{A125})$$

and that

$$\sigma^{ab}\eta_{ab} = 0 \quad (\text{A126})$$

(6) reads

$$\left(i\hbar\gamma^\mu\left(\partial_\mu + \frac{ie}{\hbar}A_\mu\right) - mc\right)\Psi = \frac{\hbar}{8}\gamma^\mu\sigma^{ab}(\partial_\mu\varepsilon_{ab})\Psi \quad (\text{A127})$$

Appendix F Covariant Anti-Commutation Rules for Fermions in Curved Space-Time

By following the postulate of physical covariance, the anti-commuting rules for the fermion field

$$\begin{aligned} i\{\psi^a_{\nu(q',t)}, \psi^{b\dagger}_{\mu(q,t)}\} &= \left\{\psi^a_{\nu(q',t)}, \pi^b_{\mu(q,t)}\right\} \\ &= i\hbar\delta^{(3)}(q-q')\delta_{ab}\delta_{\nu\mu} = i\hbar\delta^{(3)}(q-q')\frac{\partial\psi^a_{\nu(q',t)}}{\partial\psi^{b\dagger}_{\mu(q,t)}} \end{aligned} \quad (\text{A128})$$

$$\left\{\pi^a_{\nu(q',t)}, \pi^b_{\mu(q,t)}\right\} = i\hbar\delta^{(3)}(q-q')\delta_{ab}\delta_{\nu\mu} = i\hbar\delta^{(3)}(q-q')\frac{\partial\pi^a_{\nu(q',t)}}{\partial\psi^{b\dagger}_{\mu(q,t)}} = 0 \quad (\text{A129})$$

where $\pi^a_{\mu(q,t)}$ is the momentum of the field [QSFT], in curved space-time reads

$$i\{\psi^a_{\nu(q',t)}, \psi^{b\dagger}_{\mu(q,t)}\} = \left\{\psi^a_{\nu(q',t)}, \pi^a_{\mu(q,t)}\right\} = i\hbar\delta^{(3)}(q-q')D_{\psi^b_{\mu}}\psi^a_{\nu(q',t)} \quad (\text{A130})$$

$$\left\{\pi^a_{\nu(q',t)}, \pi^b_{\mu(q,t)}\right\} = D_{\psi^b_{\mu}}\pi^a_{\nu(q',t)} \quad (\text{A131})$$

where

$$D_{(x^b)_\mu}Y^a_{\nu(q',t)} = \frac{\partial Y^a_{\nu(q',t)}}{\partial(x^b)_\mu(q,t)} - \Gamma^m_{\nu\mu(q,t)}Y^a_{m(q',t)} + \omega_{\mu^a c}Y^c_{\nu(q',t)} \quad (\text{A132})$$

is the covariant derivative for affine and spinor connections (where $\Gamma^m_{\nu\mu}$ is the Christoffel symbol) that, for vector bi-spinor, leads to the expression

$$\begin{aligned} i\{\psi^a_{\nu(q',t)}, \psi^{b\dagger}_{\mu(q,t)}\} &= \left\{\psi^a_{\nu(q',t)}, \pi^b_{\mu(q,t)}\right\} \\ &= i\hbar\delta^{(3)}(q-q')\left(\frac{\partial\psi^a_{\nu(q',t)}}{\partial\psi^{b\dagger}_{\mu(q,t)}} - \Gamma^m_{\nu\mu(q',t)}\psi^a_{m(q',t)} + \omega_{\nu^a c}\psi^c_{\nu}\right) \quad \text{in the classical limit (i.e.,} \quad (\text{A133}) \\ &= i\hbar\delta^{(3)}(q-q')\left(g_{\nu\mu} - \psi^{a\alpha}_{(q,t)}\frac{\partial g_{a\nu(q',t)}}{\partial\psi^{b\dagger}_{\mu(q,t)}} - \Gamma^m_{\nu\mu(q',t)}\psi^a_{m(q',t)} + \omega_{\nu^a c}\psi^c_{\nu}\right) \end{aligned}$$

where $\psi^a_{\nu} = \begin{pmatrix} \psi^1_{\nu} \\ \psi^2_{\nu} \\ \psi^3_{\nu} \\ \psi^4_{\nu} \end{pmatrix} = \Psi_{\nu}$.

In the case of the Dirac spinor $\Psi = \psi^a = \begin{pmatrix} \psi^1 \\ \psi^2 \\ \psi^3 \\ \psi^4 \end{pmatrix}$ (A130–A133) read

$$i\{\psi^a_{(q',t)}, \psi^{b\dagger}_{(q,t)}\} = \left\{\psi^a_{(q',t)}, \pi^b_{(q,t)}\right\} = i\hbar\delta^{(3)}(q-q')\frac{\partial\psi^a_{(q',t)}}{\partial\psi^b_{(q,t)}} \quad (\text{A134})$$

$$\left\{ \pi^a_{\nu(q',t)}, \pi^b_{\mu(q,t)} \right\} = i\hbar\delta^{(3)}(q-q') \frac{\partial \pi^a(q',t)}{\partial \psi^b(q,t)} \quad (\text{A135})$$

where, in weak gravity, the momentum of the field π^a reads

$$\pi^a = \quad (\text{A136})$$

Appendix G

As shown by Landau and Lifits [29] in the limit of Newtonian gravity

$$g_{\mu\nu} \cong \eta_{\mu\nu} + \varepsilon_{\mu\nu} \quad (\text{A137})$$

where

$$\varepsilon_{\mu\nu}\varepsilon^{\mu\nu} \ll 1 \quad (\text{A138})$$

it holds

$$g_{00} = 1 + \frac{2\varphi}{c^2} \quad (\text{A139})$$

where φ is the Newtonian potential that reads

$$\frac{\partial}{\partial q^\alpha} \frac{\partial \varphi}{\partial q^\alpha} = c^2 R_0^0 \quad (\text{A140})$$

where R_0^0 can be easily calculated by the GE (83) [29].

Moreover, since from (68) it follows that

$$\varepsilon_{sim\mu\nu} \cong \begin{bmatrix} \frac{2\varphi}{c^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (\text{A141})$$

we obtain

$$\mathcal{L}_I = \frac{\hbar}{8c^2} (\bar{\Psi})^{\prime\mu} \sigma^{00} [\partial_\mu \varphi] \Psi = \sigma^{00} J^\mu \partial_\mu \varphi = 0 \quad (\text{A142})$$

It is noteworthy that the non-zero contribution to the gravitational perturbation

$$\mathcal{L}_I = \frac{\hbar}{8} \sigma^{ab} J^\mu [\partial_\mu \varepsilon_{ab}] \quad (\text{A143})$$

due to the antisymmetry of σ^{ab} , comes from the out-diagonal terms of the metric tensor [29] that become relevant in high (non-Newtonian) gravity fields.

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