Consiglio Nazionale delle Ricerche

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DELLA INFORMAZIONE

PISA

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Nota Interna B79-22

Settembre 1979

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Abstract. Two relevant problems in the diagnostic model of Preparata, Metze and Chien consist in identifying classes of optimal sequentially diagnosable systems and in evaluating the complexity of syndrome decoding for sequential diagnosis. The problem of determining optimal sequentially diagnosable systems is still unsolved except for the cases when the diagnosability equals the smallest or the largest permissible value; and the known bounds to the complexity of optimal systems are supposed to be quite weak. Further, the problem of syndrome decoding is known to be NP-complete in the general case. A class of diagnostic systems is introduced, whose members attain all permissible values of sequential diagnosability. It is shown that the upper bound to complexity of optimal system established by such class is tighter than those previously known, and that complexity of syndrome decoding is $O(|V|)$, where $|V|$ is the number of units.

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*Preprint of paper to be presented at the IFIP Working Conference on "Reliable Computing and Fault-tolerance in the 1980's", London, September 26-29, 1979.

1. Introduction

This paper reconsiders some problems underlying the theory of system diagnosis. In the approach to system diagnosis - first introduced by Praparata, Metze and Chien [1], which ideallyholds for functionally distributed system (e.g., multiprocessor or multicomputer systems), it is assumed that the system under consideration be partitioned into a number of units. Each unit is supposed to possess computational resources sufficient to enable it to test one or more of the remaining units. Each test is supposed to be complete for the class of relevant faults in the tested unit and each test outcome is supposed to be binary. The test outcome is fully significant only if the testing unit is fault-free. Different assumptions have been made to take into account the degree of test invalidation ensuing from the presence of faults in the testing unit. Such assumptions result in different diagnostic models [2,3,4,5]. In this paper the original model due to Preparata, Metze and Chien will be retained, and the test outcome will be considered completely unreliable if the testing unit is faulty.

This model consists of a directed graph $G=(V,A)$,where each vertex $v_i \in V$ corresponds to a unit of the system and there exists one arc (v_i,v_j) from vertex v_i to vertex v_j if and only if the unit represented by v_i tests the unit represented by v_j. Following to application of the test set each arc is labeled with the corresponding test outcome $s(v_i,v_j)$, where $s(v_i,v_j)=0$ if the test passes and $s(v_i,v_j)=1$ if the test fails. The set s of arc labelings resulting from an application of the test set is called a syndrome.

If all units which are faulty at the time of test execution are identified by a process of syndrome decoding, one-step diagnosis is said to occur. If the syndrome decoding enables identification of at least one faulty unit, the diagnosis is called sequential or with repair. For any given system both one-step and sequential diagnosis are possible for all syndromes provided that the number of faulty units does not exceed critical values, called the one-step diagnosability t_0 and sequential diagnosability t_r; such values are integers with $t_r \geq t_0 \geq 0$. The reader is referred to [1,2,3,4,6,7] for more details and results on sequential and one-step diagnosis of digital systems.

One problem of interest in system diagnosis consists in evaluating the complexity of self-diagnosable systems. A reasonable measure of the complexity is given by the number of arcs in the diagnostic model. Such measure needs to be related to the number $|V|$ of units and to the value of diagnosability, either one-step or sequential. Classes of optimal diagnosable systems (that is, diagnosable systems of minimal complexity) have been determined in the model of Barsi, Grandoni and Maestrini for all values of $|V|$ and for any permissible value of diagnosability, both one step and sequential [4]; the same problem has been solved in the model of Preparata Metze and Chien limited to the case of one-step diagnosis [7]. The problem of determining optimal sequentially diagnosable systems in the model of Preparata, Metze and Chien is still unsolved, although optimal diagnosable systems have been found for any $n=|V|$ when $t_r=t_{r\,min}=\lceil 2\sqrt{n}\rceil-3$ [6] or $t_r=t_{r\,max}=\left\lfloor\frac{n-1}{2}\right\rfloor$.(1) An upper bound to the complexity of optimal sequentially diagnosable systems is also known for any permissible t_r [3]. A tighter bound will be established in this paper, by analysing a class of sequentially diagnosable systems whose diagnostic graph is a rosace [4].

Additional interest resides in evaluating the complexity of syndrome decoding, that is the computational effort required to identify the faulty units, or at least one of them, when the syndrome is given. It has been proved [9] that syndrome decoding belongs to the class of NP-complete problems, both in one-step and sequential diagnosis, although there may exist classes of diagnosable systems (e.g. the simple circuit in the case of sequential diagnosis) for which the syndrome decoding is computationally efficient. It will be shown that the class of sequentially diagnossble systems introduced in this paper is easily diagnosable, since the complexity of syndrome decoding is $O(|V|)$. The relevance of this result resides in the fact that the class under consideration contains sequentially diagnosable systems with arbitrary values of t_r and the complexity of the system itself reaches the best known upper bound to optimality in most cases.

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(1) $\lceil x \rceil$ denotes the smallest integer not less that x; similarly $\lfloor x \rfloor$ is the greatest integer not greater than x.

2. Sequential diagnosability

Let $G=(V,A)$ be the diagnostic graph of a given system and s be a syndrome. Subset $F \subseteq V$ is called a consistent fault pattern of s if $s(v_i,v_j)=1$ for all $(v_i,v_j) \in A$ such that $v_i \in V-F$ and $v_j \in F$, and $s(v_i,v_j)=0$ for all $(v_i,v_j) \in A$ such that $v_i \in V-F$, $v_j \in V-F$. Consider the set $\mathcal{F}_s$ of consistent fault patterns of s, such that $|F| \leq t$ for each $F \in \mathcal{F}_s$. If $\bigcap_{F \in \mathcal{F}_s} F \neq \emptyset$, then any unit in $\bigcap_{F \in \mathcal{F}_s} F$ will be diagnosed as faulty under the hypothesis that the number of units does not exceed t. The greatest integer t_r such that this property holds for any syndrome s is the sequential diagnosability of the given graph [3].

Assuming, without any loss of generality [4] that the diagnostic graph is strongly connected, the following ideas provide a constructive approach to determining the sequential diagnosability, or at least a lower bound to this parameter. Given a syndrome s and any $v \in V$, let $g_0^s(v)$ and $g_1^s(v)$ denote the minimal cardinalities of consistent fault patterns F_0 and F_1 of s such that $v \notin F_0$ and $v \in F_1$, respectively. Under the hypothesis that the number of faults does not exceed $x_s(v)= \max(g_0^s(v), g_1^s(v)) - 1$, v is unambiguously recognized to be faulty if $g_1^s(v) \leq x_s(v)$ and otherwise fault-free. In turn, identifying v as fault-free enables identification of at least one faulty unit under the hypothesis that the diagnostic graph is strongly connected, since any path beginning at v and consisting of h arcs labeled with 0 ($h \geq 0$) followed by an arc (v_i,v_j) labeled with 1, will identify v_j as faulty.

Observe that, whenever $g_0^s(v)=g_1^s(v)$, the cardinality of every consistent fault pattern of syndrome s is greater than $x_s(v)$ and s cannot occur in the hypothesis of at most $x_s(v)$ faults. Also, every $v \in V$ other than v, has $g_0^s(v')=g_0^s(v')$ and $g_1^s(v')=g_1^s(v)$ if $g_0^s(v')=g_1^s(v')$ and $\max(g_0^s(v'),g_1^s(v')) > g_0^s(v)$ otherwise.

Consider $t_r(s)= \max_{v \in V} (x_s(v))$ and let S be the set of all syndromes: the integer $t_r= \min_{s \in S}(t_r(s))$ is the sequential diagnosability. Further, assume that $x_s(v)=\max(g_0^s(v),g_1^s(v))-1$; $v \in V$, is known for all $s \in S$: then $t'_r= \min_{s \in S}(x_s(v))$ is a lower bound to sequential diagnosability t_r. Tighter bounds may be determined by considering the index $x_s(v)$ above defined for each $v \in V'$, where $V' \subset V$ and letting $t'_r(s)= \max_{v \in V'} (x_s(v))$ and $t'_r= \min_{s \in S}(t'_r(s))$.

For example, consider a diagnostic graph $G=(V,A)$ consisting of a simple circuit of n vertices (Fig. 1) and, for any syndrome s, the unique [1] partition of the set V into sequences $(v_{i1},v_{i2},\ldots,v_{ik})$, $k\geq 2$, such that $s(v_{i(k-1)},v_{ik})=1$ and $s(v_{i(q-1)},v_{iq})=0$ for $1<q<k$. Assuming that syndrome s partitions V into ν sequences and λ be the number of

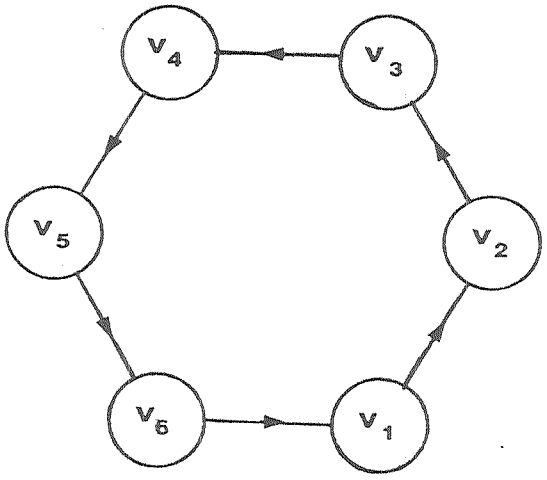

Fig. 1

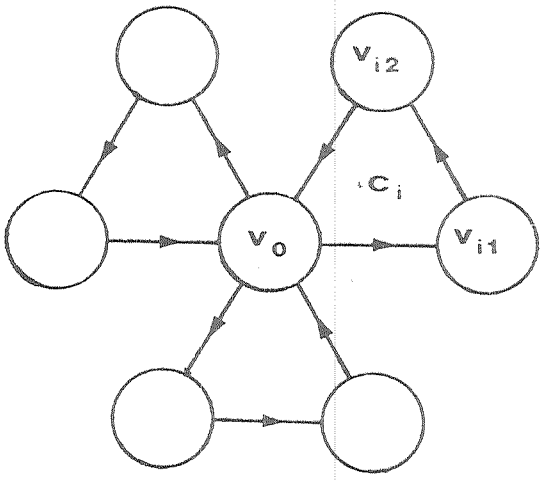

Fig. 2

vertices in a longest sequence, let v_f be the last vertex in a sequence of length λ. It is easily seen that $g_0^s(v_f)=\nu+\lambda-2$ and $g_1^s(v_f)=\nu$. The integer $t_r'=\lceil 2\sqrt{n}\rceil-3$, obtained by minimizing $x_s(v_f)=\max(g_0^s(v_f), g_1^s(v_f))-1$ over the set of permissible pairs (ν,λ) (that is, over the set S of syndromes), is a lower bound to sequential diagnosability of the simple circuit of n vertices. However, since $\lceil 2\sqrt{n}\rceil-3$ is the upper bound to sequential diagnosability of the simple circuit [6], t_r' actually coincides with the sequential diagnosability t_r.

3. A class of sequentially diagnosable systems

Consider a rosace consisting of k circuits, where each circuit has 3 vertices (Fig. 2). In order to determine the sequential diagnosability of this graph, the indices $g_0(v_0)$ and $g_1(v_0)$ will be evaluated. For each circuit C_i, $1\leq i\leq k$ in the rosace consider the set of all possible arc

labelings, listed in Table 1. If an arc labeling denoted type ξ occurs, the vertex whose incoming arc is labeled with 1 is necessarily faulty and sequential diagnosis is trivial. Thus, consideration will be limited to the non-trivial case of syndromes s resulting in arc labelings of type α, β, γ or δ in all circuits. For each such arc labeling, the minimum number of faults in the node subset $C_i-\{v_0\}$ is listed in Table 1 for both hypotheses of v_0 faulty and v_0 non-faulty. Denoting by j the number of arc-labelings of type j in the rosace, the indices $g_0^s(v_0)$ and $g_1^s(v_0)$ are

Type	Arc labels v_{i1} → v_{i2}			Faults in $C_i-\{v_0\}$	
				$v_0 \in V-F$	$v_0 \in F$
ξ	0	0	1	--	--
ξ	0	1	0	--	--
ξ	1	0	0	--	--
α	1	1	0	1	1
α	0	1	1	1	1
β	1	0	1	2	0
γ	0	0	0	0	2
δ	1	1	1	2	1

Table 1

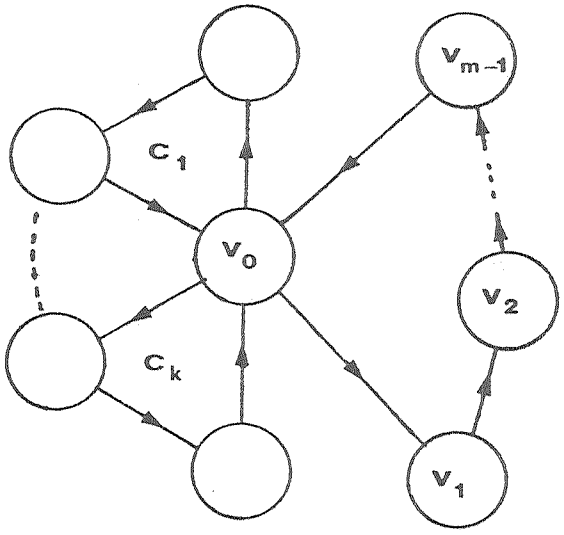

Fig. 3

immediately computed as follows:

$$g_0^s(v_0) = \alpha + 2\beta + 2\delta$$
$$g_1^s(v_0) = \alpha + 2\gamma + \delta + 1 \qquad (1)$$

By introducing $k=\alpha+\beta+\delta+\gamma$, expressions (1) become:

$$g_0^s(v_0) = k + \beta - \gamma + \delta = k + x + \delta$$
$$g_1^s(v_0) = k + 1 - (\beta - \gamma) = k - x + 1 \qquad (2)$$

where $x=\beta-\gamma$ is an integer depending on the syndrome s.

Since $g_0^s(v_0) + g_1^s(v_0) = 2k + 1 + \delta$, assuming $g_i^s(v_0) \leq k$, $i \in \{0,1\}$, implies $g_{1-i}^s(v_0) \geq k+1$ for all syndromes. It follows $t'_r = \max(g_0(v_0), g_1(v_0)) - 1 \geq k$ and the rosace is at least k-sequentially diagnosable. On the other hand, the sequential diagnosability of the rosace shown in Fig. 2 cannot exceed k since $k = \left\lfloor \frac{n-1}{2} \right\rfloor$ coincides with the upper bound to sequential diagnosability [1], where $n=2k+1=|V|$.

The preceding result shows that the connection assignment of n nodes corresponding to the rosace of Fig. 2 yields sequentially diagnosable systems whose diagnosability t_r reaches the upper bound of $\left\lfloor \frac{n-1}{2} \right\rfloor$, while the number of arcs, equal to $a = 3 \cdot \frac{n-1}{2} = 3\left\lceil \frac{n}{2} \right\rceil - 3$, is slightly above the lower bound of $3\left\lceil \frac{n}{2} \right\rceil - 5$ established by Ciompi and Simoncini [8] for sequentially diagnosable systems with $t_r = \left\lfloor \frac{n-1}{2} \right\rfloor$. The class of sequentially diagnosable systems to be analyzed in the following is a generalization of the graph shown in Fig. 2, and consists of rosaces formed by k circuits of 3 vertices and one circuit of m vertices ($m \geq 3$).

Given a rosace in this class (Fig. 3), consider a non-trivial syndrome s resulting in arc labelings of type α, β, γ, δ in the circuits of length 3 and define x as above. Further, assume that the circuit of length m has ν sequences and λ is the maximum length of such sequences. Using the same techniques described in the preceding examples, the indices $g_0^s(v_0)$ and $g_1^s(v_0)$ are evaluated as follows:

1) v_0 is the last vertex in a sequence of length λ: $g_0^s(v_0) \geq k+x+\nu+\lambda-2$ $g_1^s(v_0) = k-x+\nu$

2) v_0 is the last vertex in a sequence of length $\ell < \lambda$: $g_0^s(v_0) \geq k+x+\nu+\lambda-2-(\lambda-\ell)$; $g_1^s(v_0) = k-x+\nu$

3) v_0 is not the last vertex in a sequence: $g_0^s(v_0) \geq k+x+\nu$; $g_1^s(v_0) \geq k-x+\nu$,

where the inequality in expressions for $g_0^s(v)$ results from neglecting δ. Let $t_r'(s)$ be an estimate of the maximum number of faults under which the state of at least one unit can be diagnosed when syndrome s occurs. If $g_0^s(v_0) > t_r'(s)\{g_1^s(v_0) > t_r'(s)\}$, unit v_0 is diagnosed as faulty {fault-free} in the hypothesis of $t_r'(s)$ or less faults, the case where $g_0^s(v_0) > t_r'(s)$ and $g_1^s(v_0) > t_r'(s)$ corresponding to the occurrence of more than $t_r'(s)$ faults. If $g_0^s(v_0) \leq t_r'(s)$, $g_1^s(v_0) \leq t_r'(s)$ the state of unit v_0 cannot be diagnosed; however denoting by v_f the last unit in a sequence of length λ it is easily seen that in this hypothesis $g_0^s(v_f) = k+2\nu-t_r'(s)+\lambda-2$ in case 1 and 3 and $g_0^s(v_f) = k+2\nu-t_r'(s)+\lambda-2+\ell-2 \geq k+2\nu-t_r'(s)+\lambda-2$ in case 2, where the last inequality derives from $\ell \geq 2$ [1]. In order v_f to be diagnosed as faulty in the assumption of $t_r'(s)$ or less faults, must be $t_r'(s) \leq g_0(v_f)-1$ and $t_r'(s)$ is determined from the preceding expressions as $t_r'(s) = k+\nu+\left\lfloor \frac{\lambda-3}{2} \right\rfloor$. A lower bound t_r' to sequential diagnosability is thus determined by minimizing $t_r'(s)$ over the set of permissible pairs (ν,λ), resulting from consideration of all syndromes, as stated by the following theorem whose proof is omitted for the sake of brevity.

Theorem 1. A lower bound to sequential diagnosability of the rosace consisting of k circuits of length 3 and one circuit of length m is

$$t'_r = k + \left\lceil \frac{\sqrt{2m+1} - 1}{2} \right\rceil + \left\lfloor \frac{1}{2} \left(\left\lceil \frac{m}{\left\lceil \frac{\sqrt{2m+1} - 1}{2} \right\rceil} \right\rceil - 3 \right) \right\rfloor$$

The class of design corresponding to rosaces of Fig. 3 includes sequentially diagnosable systems whose diagnosability spans the values between the lower and the upper bound. It is interesting to observe that while the rosaces of Fig. 3 correspond to optimal sequentially diagnosable systems in the model of Barsi, Grandoni and Maestrini [4], this property does not hold in the model of Preparata, Metze and Chien. This can be verified by observing that the rosace with m=5, where the number of vertices is n=2k+5 and the number of arcs is a=3k+5, has $t_r = k+2 = \left\lfloor \frac{n-1}{2} \right\rfloor$ and thus reaches the upper bound to sequential diagnosability, while the number of arcs exceeds by one the lower bound of $3\left\lceil \frac{n}{2} \right\rceil - 5$ holding for optimal designs with the same diagnosability [8].

The problem of determining an upper bound to the complexity of optimal sequentially diagnosable systems with arbitrary $|V|$ and any permissible value of t_r has been considered by Maheshwari and Hakimi [3]. They have established a bound of $n+t_r-1$ for t_r-diagnosable systems whose diagnostic graph has n vertices. Since in the rosace of Fig. 3 the number of arcs is a=3 k +m and n=2 k +m, $k = t'_r - \left\lceil \frac{\sqrt{2m+1} - 1}{2} \right\rceil - \left\lfloor \frac{1}{2} \left(\left\lceil \frac{m}{\left\lceil \frac{\sqrt{2m+1} - 1}{2} \right\rceil} \right\rceil - 3 \right) \right\rfloor$, it is easily seen that the complexity of a system whose diagnosability is no less than t'_r is equal to $a = n + t'_r - \left\lceil \frac{\sqrt{2m+1} - 1}{2} \right\rceil - \left\lfloor \frac{1}{2} \left(\left\lceil \frac{m}{\left\lceil \frac{\sqrt{2m+1} - 1}{2} \right\rceil} \right\rceil - 3 \right) \right\rfloor$, which constitutes a tighter bound to optimality.

4. Syndrome decoding

Another problem of interest consists in evaluating the computational effort required to actually diagnose at least one unit, given the syndrome. It has been proved that syndrome decoding in sequential diagnosis is a NP-complete problem [9] and this implies that sequential diagnosis becomes computationally untractable as the size of the problem increases. However efficient decoding algorithms may exist for special classes of diagnostic graphs: this has been actually proved for the simple circuit

of Fig. 1 [9] . A similar result holds for the k-rosace shown in Fig. 3, since it is clear that the following decoding algorithm, which derives from the preceding analysis of the diagnostic properties of the k-rosace, is $O(|V|)$ in time. With the notations defined above, the decoding algorithm is as follows:

A1) If any one of the circuits of length 3 has the arc labeling of type ξ and arc (v_{ij},v_{ik}) is labeled with 1, vertex v_{ik} is recognized to be faulty. Else:

A2) Determine if v_0 is the last element of a sequence in the circuit of length m and if such sequence has length λ, and compute $g_0^s(v_0)$, $g_1^s(v_0)$ using the corresponding formulae. If $g_0^s(v_0) > t'_r$, v_0 is recognized to be faulty in the hypothesis of at most t'_r faults; if $g_1^s(v_0) > t'_r$, v_0 is diagnosed as fault-free in the same hypothesis. Else:

A3) Denoting by v_f the last vertex in a sequence of length λ in the circuit of length m, v_f is recognized to be faulty in the hypothesis of at most t'_r faults.

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