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Strong convergence of a vector-BGK model to the incompressible Navier-Stokes equations via the relative entropy method

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ABSTRACT

The aim of this paper is to prove the strong convergence of the solutions to a vector-BGK model under the diffusive scaling to the incompressible Navier-Stokes equations on the two-dimensional torus. This result holds in any interval of time $[0, T]$, with $T > 0$. We also provide the global in time uniform boundedness of the solutions to the approximating system. Our argument is based on the use of local in time H^s-estimates for the model, established in a previous work, combined with the L^2-relative entropy estimate and the interpolation properties of the Sobolev spaces.

RÉSUMÉ

L'objectif de cet article est de présenter un résultat de convergence forte des solutions d'un modèle BGK vectoriel, en régime diffusif, vers les équations de Navier-Stokes incompressibles. Le domaine est le tore bidimensionnel et ce résultat est valide dans l'intervalle de temps $[0, T]$, où $T > 0$. En outre, on démontre que les solutions du modèle BGK restent uniformément bornées. Notre approche est basée sur des estimations dans les espaces de Sobolev H^s, obtenues dans un article précédent, combinées avec la méthode de l'entropie relative et les propriétés d'interpolation des espaces de Sobolev.

1. Introduction

In this paper we deal with the incompressible Navier-Stokes equations in two space dimensions,

$$\begin{cases} \partial_t \mathbf{u}^{NS} + \nabla \cdot (\mathbf{u}^{NS} \otimes \mathbf{u}^{NS}) + \nabla P^{NS} = \nu \Delta \mathbf{u}^{NS}, \\ \nabla \cdot \mathbf{u}^{NS} = 0, \end{cases} \tag{1}$$

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with $(t,x) \in [0,+\infty) \times \mathbb{T}^2$, and initial datum

$$\mathbf{u}^{NS}(0,x) = \mathbf{u}_0(x), \qquad \nabla \cdot \mathbf{u}_0 = 0. \tag{2}$$

In (1), $\mathbf{u}^{NS}$ and ∇P^{NS} are respectively the velocity field and the gradient of the pressure term, and $\nu > 0$ is the viscosity coefficient.

Here we consider a vector-BGK model for the incompressible Navier-Stokes equations, i.e. a discrete velocities BGK system endowed with a vectorial structure, whose general formulation has been introduced in [15], while further developments were presented in [11] from the numerical side and in [8] from the analytical point of view. Precisely, we study the following five velocities (15 equations) vector-BGK approximation to the incompressible Navier-Stokes equations,

$$\begin{cases} \partial_t f_1^\varepsilon + \frac{\lambda}{\varepsilon}\partial_x f_1^\varepsilon = \frac{1}{\tau\varepsilon^2}(M_1(w^\varepsilon) - f_1^\varepsilon), \\ \partial_t f_2^\varepsilon + \frac{\lambda}{\varepsilon}\partial_y f_2^\varepsilon = \frac{1}{\tau\varepsilon^2}(M_2(w^\varepsilon) - f_2^\varepsilon), \\ \partial_t f_3^\varepsilon - \frac{\lambda}{\varepsilon}\partial_x f_3^\varepsilon = \frac{1}{\tau\varepsilon^2}(M_3(w^\varepsilon) - f_3^\varepsilon), \\ \partial_t f_4^\varepsilon - \frac{\lambda}{\varepsilon}\partial_y f_4^\varepsilon = \frac{1}{\tau\varepsilon^2}(M_4(w^\varepsilon) - f_4^\varepsilon), \\ \partial_t f_5^\varepsilon = \frac{1}{\tau\varepsilon^2}(M_5(w^\varepsilon) - f_5^\varepsilon), \end{cases} \tag{3}$$

where

$$w^\varepsilon = (\rho^\varepsilon, \varepsilon\rho^\varepsilon u_1^\varepsilon, \varepsilon\rho^\varepsilon u_2^\varepsilon) = (\rho^\varepsilon, \varepsilon\rho^\varepsilon \mathbf{u}^\varepsilon) = (\rho^\varepsilon, \mathbf{q}^\varepsilon) = \sum_{i=1}^{5} f_i^\varepsilon. \tag{4}$$

Its main properties are as follows:

- f_i^ε, $M_i(w^\varepsilon)$, $i = 1, \cdots 5$, are vector-valued functions taking values in $\mathbb{R}^3$;
- $\rho^\varepsilon(t,x)$ on $\mathbb{R}^+ \times \mathbb{T}^2$ is the approximating density, taking values in $\mathbb{R}^+$;
- $\mathbf{u}^\varepsilon(t,x) = (u_1^\varepsilon(t,x), u_2^\varepsilon(t,x))$ on $\mathbb{R}^+ \times \mathbb{T}^2$ is the approximating velocity field, taking values in $\mathbb{R}^2$.

Precise compatibility conditions to be satisfied by the constant parameters of the model and the Maxwellian functions, together with their explicit expressions, will be provided in details in Section 2.

BGK models were introduced by Bhatnagar, Gross and Krook as a modified version of the Boltzmann equation, characterized by the relaxation of the collision operator. Since they present most of the basic properties of hydrodynamics, they are considered interesting models even though they do not contain all the relevant features of the Boltzmann equation. Essentially, vector-BGK models are inspired by the hydrodynamic limits of the Boltzmann equation [3,4,16,20,24], but later they have been generalized as approximating equations for different kinds of systems. In this regard, one of the main directions has been the approximation of hyperbolic systems with discrete velocities BGK models, as in [13,29,35,10,36,17,18]. Similar results have been obtained for convection-diffusion systems under the diffusive scaling [32,12,30,2,28, 25,9]. Originally, they presented continuous velocities, see [36], but later on discrete velocities BGK models inspired by the relaxation method have been introduced, see [34] for a survey. In the spirit of the relaxation approximations, see for instance [22] and references therein for general diffusive relaxation systems, the main advantage of discrete velocities BGK models is to deal with semilinear systems, see [35,14,26,40,45].

Now, let us present our main result. We prove the strong convergence in the Sobolev spaces, for any interval of time $[0,T]$, $T > 0$, of the vector-BGK model presented in (3) to the incompressible Navier-Stokes equations on the two-dimensional torus. To achieve this result, the novelty relies in using local in time H^s-estimates from a previous work, see [8], combined with the L^2-relative entropy estimate and the standard interpolation theorem. More precisely, part of the results of [8] provides uniform (in ε) estimates of Gronwall type in the Sobolev spaces, which hold in $[0, T^*]$, where $T^* > 0$ is depending on a fixed constant $M > 0$ and on the norm of the initial data. These local bounds guarantee the existence, the minimality and the dissipative property of the kinetic entropy, i.e. a convex entropy for (3), see [10]. Next, the relative entropy allows us to get a precise rate of convergence of the solutions to our model to the Navier-Stokes equations, which holds for $t \in [0, T^*]$, see Theorem 3.3. Thus, the interpolation theorem for Sobolev spaces applied to the relative entropy estimate provides a bound for the solutions to our system which is much more precise than the previous pessimistic Gronwall type estimates. This is the key point in order to close the argument and to prove the strong convergence for all times of the solutions to (3) to system (1), together with the global in time boundedness of the approximating solution itself, in Theorem 3.1. In particular, Lemma 2.8 plays a crucial role in quantifying the dissipation term coming from the entropy inequality. At the best of our understanding, the expansions in Lemma 2.8 are the only way to establish the relative entropy inequality when, as in our case, the explicit dependency of the kinetic entropy on the singular relaxation parameter is not known.

We start from initial data in (16) that are small perturbations of the Maxwellians and, thanks to the uniform bounds, in the end we prove that everything remains in a bounded set of the densities. This local setting perfectly fits the framework described in [10].

The relative entropy method, [19,21], represents an efficient mathematical tool for studying stability and limiting process and it is based on a direct calculation of the relative entropy between a dissipative solution and a conservative smooth solution for the considered system, which provides a remarkable stability estimate. Far from being complete, we collect here a pair of references for hydrodynamic limits [23,38]. In the context of singular hyperbolic scaled systems, we refer to [43,44]. Let us point out that this procedure has been successfully applied to the vector-BGK model considered in this paper and presented below (3) to prove its convergence to the isentropic Euler equations under the hyperbolic scaling, see [39]. Again, the relative entropy in hyperbolic relaxation has been used for one-dimensional discrete velocities Boltzmann schemes, see [5], while in the multidimensional case the question in this context seems to be open. On the other hand, the relative entropy method in diffusive relaxation is of course a more delicate issue, being the diffusive limit the next order approximation of the starting system in the Chapman-Enskog expansion, see [37]. Besides hydrodynamic limits of the Boltzmann equation, our main reference in this framework is [31]. In this paper, the authors apply the relative entropy method to the equations of compressible gas dynamics with friction under the diffusive scaling, so obtaining precise estimates coming from the entropy of the limit hyperbolic system. However, in our case, further complications are due to the fact that the explicit dependency of the kinetic entropy on the singular parameter is not known for our model (3). The BGK framework in [10] only guarantees the existence of such an entropy, whose expression is defined by means of the inverse function theorem. This difficulty requires a better understanding of the dissipative terms provided by the entropy inequality in diffusive relaxation, and new ideas are needed with respect to the existing works, for instance [31,5].

The paper is organized as follows. In Section 2 we introduce the vector-BGK model and provide some preliminary results. Section 3 is devoted to the relative entropy inequality and the strong convergence of the model for all times, in the Sobolev spaces. In the last part of this section we also show the global in time boundedness of the solutions to our model.

2. Presentation of the model, formal limit, and intermediate results

First, we aim at providing a relative entropy inequality for a vector-BGK model approximating the two-dimensional incompressible Navier-Stokes equations. Next, this inequality will allow us to extend for long times the local convergence for smooth solutions achieved in [8]. Let us introduce the setting that will be taken into account hereafter.

Our approximating vector-BGK model has been presented in (3), together with a list of the main properties. We point out that, in order to get consistency with the incompressible Navier-Stokes equations, the Maxwellian functions $M_i(w^\varepsilon),\ i = 1, \cdots, 5$, need to satisfy the following compatibility conditions:

- $\sum_{i=1}^{5} M_i(w^\varepsilon) = w^\varepsilon$;
- $\sum_{i=1}^{5} \lambda_{ij} M_i(w^\varepsilon) = A_j(w^\varepsilon), \quad j = 1, 2$, with A_j in (6),

with discrete velocities $\lambda_i = (\lambda_{i1}, \lambda_{i2})$. More precisely, $\lambda_1 = (\lambda, 0),\ \lambda_2 = (0, \lambda),\ \lambda_3 = (-\lambda, 0),\ \lambda_4 = (0, -\lambda),\ \lambda_5 = (0, 0)$, where λ is a positive constant value.

We provide here the explicit expressions of the Maxwellian functions

$$M_{1,3}(w^\varepsilon) = aw^\varepsilon \pm \frac{A_1(w^\varepsilon)}{2\lambda}, \qquad M_{2,4}(w^\varepsilon) = aw^\varepsilon \pm \frac{A_2(w^\varepsilon)}{2\lambda}, \qquad M_5(w^\varepsilon) = (1-4a)w^\varepsilon, \tag{5}$$

$$A_1(w^\varepsilon) = \begin{pmatrix} q_1^\varepsilon \\ \frac{(q_1^\varepsilon)^2}{\rho^\varepsilon} + P(\rho^\varepsilon) \\ \frac{q_1^\varepsilon q_2^\varepsilon}{\rho^\varepsilon} \end{pmatrix}, \qquad A_2(w^\varepsilon) = \begin{pmatrix} q_2^\varepsilon \\ \frac{q_1^\varepsilon q_2^\varepsilon}{\rho^\varepsilon} \\ \frac{(q_2^\varepsilon)^2}{\rho^\varepsilon} + P(\rho^\varepsilon) \end{pmatrix}, \tag{6}$$

$$P(\rho^\varepsilon) = \frac{(\rho^\varepsilon)^2 - \bar{\rho}^2}{2\bar{\rho}}, \tag{7}$$

where $\bar{\rho} > 0$ is constant value.

The consistency with respect to the incompressible Navier-Stokes equations (1) and the stability of our vector-BGK model hold under the following hypotheses.

Assumptions 2.1. Let us assume

$$a = \frac{\nu}{2\lambda^2\tau}, \qquad 0 < a < \frac{1}{4}, \tag{8}$$

where ν is the viscosity coefficient in (1). Besides, we also take the parameter $\lambda > 0$ "big enough". This is necessary in order to:

- guarantee the positivity of the symmetrizer in [8];
- satisfy the sub-characteristic condition, i.e. the positivity of the spectrum of the Jacobian matrices of the Maxwellians.

Let us discuss more precisely the previous assumptions.

Remark 2.2 *(Consistency and stability under certain assumptions).*
• First of all, equality $\nu = 2a\tau\lambda^2$ is needed to get consistency of the vector-BGK model (3) with respect to the incompressible Navier-Stokes equations, as remarked below.
• From (5), one observes that $M_5(w^\varepsilon) = (1-4a)w^\varepsilon$. This means that $0 < a < \frac{1}{4}$ is a necessary condition for the positivity of the spectrum of the Jacobian of $M_5(w^\varepsilon)$. As widely discussed in the course of the manuscript and in Remark 3.2, the result in [[10], Thm. 2.1] provides the existence of a kinetic entropy for the vector-BGK model (3) under certain hypotheses. We roughly explain the idea behind this result. One considers a convex entropy for the limit system under the hyperbolic scaling (the system that is obtained by fixing ε and sending τ to zero in (3), i.e. the isentropic Euler equations). Then, the discrete version of the Boltzmann H-Theorem (see [10]) allows us to get an (implicit) expression for the kinetic entropy by taking the inverse of the Maxwellian functions. In order to invert the Maxwellians by using the inverse function theorem, one needs the positivity of the spectrum of the Jacobians of the Maxwellians. For $M_5(w^\varepsilon)$, this is

true if $0 < a < \frac{1}{4}$. For the other Maxwellians, as a further discussion in Remark 3.2 and simple computations show, the conditions rely on a lower bound for the discrete velocity λ,

$$\lambda > \frac{1}{2a}\left(\frac{\varepsilon M}{\bar{\rho} - \varepsilon M} + \sqrt{1 + \frac{\varepsilon M}{\bar{\rho}}}\right), \tag{9}$$

where the constant M is introduced in (24).

• Finally we aim at providing more details on the assumption on the parameter λ "big enough". More precisely, following [8] and Remark 2.3, for fixed $0 < a < \frac{1}{4}$, and $0 < \mu < \frac{1}{4a}$, we choose

$$\lambda > \max\left\{\frac{1}{\sqrt{2a^2(1-1/\mu)}}, \sqrt{\frac{2}{a(1-4a\mu)}}, \sqrt{\frac{1+5a(1-4a)}{4a^2(1-4a)}}\right\} \cap (9). \tag{10}$$

The meaning of the lower bound in (9) has already been explained above. The other one assures both the positivity of the constant symmetrizer and the negativity of the linear part of the source term of system (12), as proved in [8]. In other words, the local in time bounds in [[8], Lemma 4.2 and Proposition 3] hold under that condition.

Now, the change of variables introduced in [8],

$$\begin{aligned} &w^\varepsilon = \sum_{i=1}^{5} f_i^\varepsilon, \quad m^\varepsilon = \frac{\lambda}{\varepsilon}(f_1^\varepsilon - f_3^\varepsilon), \quad \xi^\varepsilon = \frac{\lambda}{\varepsilon}(f_2^\varepsilon - f_4^\varepsilon), \\ &k^\varepsilon = f_1^\varepsilon + f_3^\varepsilon, \qquad h^\varepsilon = f_2^\varepsilon + f_4^\varepsilon, \end{aligned} \tag{11}$$

allows us to recover the consistency with respect to (1) in a simple way at the formal level. Thus, the vector-BGK model (3) reads:

$$\begin{cases} \partial_t w^\varepsilon + \partial_x m^\varepsilon + \partial_y \xi^\varepsilon = 0, \\ \partial_t m^\varepsilon + \frac{\lambda^2}{\varepsilon^2}\partial_x k^\varepsilon = \frac{1}{\tau\varepsilon^2}(\frac{A_1(w^\varepsilon)}{\varepsilon} - m^\varepsilon), \\ \partial_t \xi^\varepsilon + \frac{\lambda^2}{\varepsilon^2}\partial_y h^\varepsilon = \frac{1}{\tau\varepsilon^2}(\frac{A_2(w^\varepsilon)}{\varepsilon} - \xi^\varepsilon), \\ \partial_t k^\varepsilon + \partial_x m^\varepsilon = \frac{1}{\tau\varepsilon^2}(2aw^\varepsilon - k^\varepsilon), \\ \partial_t h^\varepsilon + \partial_y \xi^\varepsilon = \frac{1}{\tau\varepsilon^2}(2aw^\varepsilon - h^\varepsilon). \end{cases} \tag{12}$$

Moreover, we denote by $\mathcal{M}_i(w^\varepsilon) := f^\varepsilon$ the perturbed Maxwellian, i.e. the next order correction, due to the diffusive scaling, to $M_i(w^\varepsilon)$ in (5).

The relaxation formulation (12) of the system gives:

$$\begin{aligned} &m^\varepsilon = \frac{\lambda}{\varepsilon}(f_1^\varepsilon - f_3^\varepsilon) := \frac{\lambda}{\varepsilon}(\mathcal{M}_1(w^\varepsilon) - \mathcal{M}_3(w^\varepsilon)) = \frac{A_1(w^\varepsilon)}{\varepsilon} - \tau\lambda^2\partial_x k^\varepsilon + O(\varepsilon^2), \\ &\xi^\varepsilon = \frac{\lambda}{\varepsilon}(f_2^\varepsilon - f_4^\varepsilon) := \frac{\lambda}{\varepsilon}(\mathcal{M}_2(w^\varepsilon) - \mathcal{M}_4(w^\varepsilon)) = \frac{A_2(w^\varepsilon)}{\varepsilon} - \tau\lambda^2\partial_y h^\varepsilon + O(\varepsilon^2), \\ &k^\varepsilon = f_1^\varepsilon + f_3^\varepsilon = \mathcal{M}_1(w^\varepsilon) + \mathcal{M}_3(w^\varepsilon) = 2aw^\varepsilon + O(\varepsilon^2), \\ &h^\varepsilon = f_2^\varepsilon + f_4^\varepsilon = \mathcal{M}_2(w^\varepsilon) + \mathcal{M}_4(w^\varepsilon) = 2aw^\varepsilon + O(\varepsilon^2). \end{aligned} \tag{13}$$

Substituting the last two expressions of (13) in the other ones, from the first equation of (12), one gets

$$\partial_t w^\varepsilon + \frac{\partial_x A_1(w^\varepsilon)}{\varepsilon} + \frac{\partial_y A_2(w^\varepsilon)}{\varepsilon} = 2a\lambda^2\tau\Delta w^\varepsilon + O(\varepsilon^2).$$

From Assumptions 2.1, $\nu = 2a\tau\lambda^2$, therefore we get

$$\partial_t w^\varepsilon + \frac{\partial_x A_1(w^\varepsilon)}{\varepsilon} + \frac{\partial_y A_2(w^\varepsilon)}{\varepsilon} = \nu\Delta w^\varepsilon + O(\varepsilon^2),$$

which explains the meaning of the consistency condition $\nu = 2a\lambda^2\tau$ in Assumptions 2.1, see also [15]. More explicitly, from the expressions of $w^\varepsilon, A_1(w^\varepsilon), A_2(w^\varepsilon)$ in (4)-(6),

$$\begin{aligned}\partial_t \begin{pmatrix} \rho^\varepsilon - \bar\rho \\ \varepsilon\rho^\varepsilon u_1^\varepsilon \\ \varepsilon\rho^\varepsilon u_2^\varepsilon \end{pmatrix} + \partial_x \begin{pmatrix} \rho^\varepsilon u_1^\varepsilon \\ \varepsilon\rho^\varepsilon (u_1^\varepsilon)^2 + \frac{(\rho^\varepsilon)^2 - \bar\rho^2}{2\bar\rho\varepsilon} \\ \varepsilon\rho^\varepsilon u_1^\varepsilon u_2^\varepsilon \end{pmatrix} + \partial_y \begin{pmatrix} \rho^\varepsilon u_2^\varepsilon \\ \varepsilon\rho^\varepsilon u_1^\varepsilon u_2^\varepsilon \\ \varepsilon\rho^\varepsilon (u_2^\varepsilon)^2 + \frac{(\rho^\varepsilon)^2 - \bar\rho^2}{2\bar\rho\varepsilon} \end{pmatrix} \\ = \nu\Delta \begin{pmatrix} \rho^\varepsilon - \bar\rho \\ \varepsilon\rho^\varepsilon u_1^\varepsilon \\ \varepsilon\rho^\varepsilon u_2^\varepsilon \end{pmatrix} + O(\varepsilon^2).\end{aligned}$$

Dividing the last two lines by ε, this yields

$$\begin{cases} \partial_t(\rho^\varepsilon - \bar\rho) + \nabla\cdot \mathbf{u}^\varepsilon = \nu\Delta(\rho^\varepsilon - \bar\rho) + O(\varepsilon), \\ \partial_t(\rho^\varepsilon \mathbf{u}^\varepsilon) + \nabla\cdot(\rho^\varepsilon \mathbf{u}^\varepsilon \otimes \mathbf{u}^\varepsilon) + \frac{\nabla((\rho^\varepsilon)^2 - \bar\rho^2)}{2\bar\rho\varepsilon^2} = \nu\Delta(\rho^\varepsilon \mathbf{u}^\varepsilon) + O(\varepsilon), \end{cases} \tag{14}$$

which is the compressible approximation to the incompressible Navier-Stokes equations. Let us make a few comments on the last system.

Remark 2.3. [The incompressible limit to the Navier-Stokes equations] The vanishing ε-limit of system (14) to the incompressible Navier-Stokes equations, known as *low Mach number limit* or *acoustic limit*, is quite classical and can be found for instance in [42] (or in [33] for the inviscid case). We briefly recall the idea at the formal level and we refer to the quoted books for a rigorous proof. We assume the asymptotic expansions:

$$\begin{aligned} \rho^\varepsilon &= \bar\rho + \varepsilon^2 P + O(\varepsilon^4), \\ \mathbf{u}^\varepsilon &= \mathbf{u}_{lim} + \varepsilon^2 \mathbf{u}_1 + O(\varepsilon^4). \end{aligned}$$

The incompressible Navier-Stokes equations are obtained at main order by inserting the previous ansatz in system (14). Note that the divergence free condition comes from the mass balance equation in (14), which is linked to the term P in the expansion for ρ^ε. The pressure P is incompressible, it does not depend on the density (which, by the way, is constant) and it can be simply seen as a Lagrange multiplier related to the divergence free constraint.

We point out that a different relaxation approximation to the two dimensional Navier-Stokes equations is presented and analysed in [14]. The main difference with respect to our system is precisely related to compressibility. In the semilinear hyperbolic approximation proposed in [14], the velocity field is divergence free, and the relaxation parameter leads to the limit nonlinear system, i.e. the incompressible Navier-Stokes equations. For the vector-BGK system considered in (3), the diffusive relaxation parameter ε encodes both the approximation to the nonlinear system (in the spirit of relaxation models) and the compressible limit, as it can be seen from (14). This is a very good point for our BGK in (3), since the divergence free condition is only reached in the limit. By contrast, it is a constraint of the model [14], where a projection on the divergence free vector fields (which increases the error from the numerical point of view) is needed.

Now we find an expression of the formal limit in terms of the original kinetic variables (3). The limit solution is obtained by solving the linear system (13) in the unknowns $\mathcal{M}_i(w^\varepsilon),\ i = 1, \cdots, 5$, so providing

$$\begin{aligned}
\mathcal{M}_1(w^\varepsilon) &= M_1(w^\varepsilon) - a\varepsilon\lambda\tau\partial_x w^\varepsilon, \\
\mathcal{M}_2(w^\varepsilon) &= M_2(w^\varepsilon) - a\varepsilon\lambda\tau\partial_y w^\varepsilon, \\
\mathcal{M}_3(w^\varepsilon) &= M_3(w^\varepsilon) + a\varepsilon\lambda\tau\partial_x w^\varepsilon, \\
\mathcal{M}_4(w^\varepsilon) &= M_4(w^\varepsilon) + a\varepsilon\lambda\tau\partial_y w^\varepsilon, \\
\mathcal{M}_5(w^\varepsilon) &= M_5(w^\varepsilon).
\end{aligned} \tag{15}$$

In order to avoid further complications due to initial layers, in our convergence proof the two-dimensional vector-BGK model is endowed with the following initial data:

$$f_i^\varepsilon(0,x) = \overline{\mathcal{M}}_i(\bar{\rho}, \varepsilon\bar{\rho}\bar{\mathbf{u}}_0), \quad i = 1, \cdots, 5, \tag{16}$$

where $\mathbf{u}_0$ is in (2) and $\bar{\rho}$ is a positive constant value.

2.1. Preliminary results

Here we collect some preliminary results, which hold for local times, essentially due to our previous work [8]. Let us start with the following remark.

Remark 2.4. We discuss some differences between [8] and our current setting.

- In [8], the compressible pressure $P(\rho^\varepsilon)$ in (7) is linear. More precisely, from [[8], (10)],

$$\tilde{P}(\rho^\varepsilon) = \rho^\varepsilon - \bar{\rho}.$$

  In this paper, we consider the case of a quadratic pressure $P(\rho^\varepsilon)$ in (7). A simple remark shows that, from (7),

$$\begin{aligned}
P(\rho^\varepsilon) &= \frac{(\rho^\varepsilon)^2 - \bar{\rho}^2}{2\bar{\rho}} \\
&= \frac{2\bar{\rho}(\rho^\varepsilon - \bar{\rho}) + (\rho^\varepsilon - \bar{\rho})^2}{2\bar{\rho}} \\
&= (\rho^\varepsilon - \bar{\rho}) + \frac{(\rho^\varepsilon - \bar{\rho})^2}{2\bar{\rho}}.
\end{aligned}$$

  Thus, the estimates in [8] still hold here: the quadratic pressure only provides an additional quadratic term in the fifth and the ninth line of the nonlinear vector $N(w + \bar{w})$ in [[8], (26)]. These supplementary quadratic terms can be handled exactly as the other ones in the energy estimates in [8]. However, we point out that the same argument holds exactly in the same way for a general compressible pressure

$$P(\rho^\varepsilon) = \begin{cases} \frac{k}{\gamma-1}[(\rho^\varepsilon)^\gamma - \bar{\rho}^\gamma], & \gamma > 1, \\ k[\rho^\varepsilon log(\rho^\varepsilon) - \bar{\rho} log(\bar{\rho})], & \gamma = 1, \end{cases}$$

  where k is a positive constant value.
- In [[8], (18)-(19)], we consider a translated version of the relaxation system (12). Of course this is an equivalent formulation of the approximating model, and since the translation vector $(\bar{\rho}, 0, 0)$ in [[8], (18)] is constant in t and x, most of the energy estimates in [8] can be used here.

- A further change of variables, involving the dissipative constant right symmetrizer Σ in [[8], (28)] is defined in [[8], (30)]. However, here the energy estimates from [8] are expressed in terms of the original relaxation variables (11) to avoid further complications. The explicit change of variables is written in [[8], (78)].

Taking into account Remark 2.4, we state some results that will be applied below. Hereafter, we denote by T^ε the maximum time of existence of the solution to the semilinear vector-BGK approximation (3) with initial data (16), see [33]. Of course T^ε could depend on ε. In the following, we recall and adapt some results from [8], showing that there exist ε_0 and a fixed and positive time T^*, independent of ε and depending on the Sobolev norm of the initial data, such that, for $\varepsilon \le \varepsilon_0$, some local in time H^s-estimates on the solutions to the approximating system hold uniformly with respect to ε. In this context, we consider the constant vector $(\bar\rho, 0, 0)$ and the translated variables:

$$\begin{aligned} w^*(t,x) &= w(t,x) - (\bar\rho, 0, 0), \\ k^*(t,x) &= k(t.x) - 2a(\bar\rho, 0, 0), \\ h^*(t,x) &= h(t.x) - 2a(\bar\rho, 0, 0), \end{aligned} \tag{17}$$

where $w^\varepsilon, k^\varepsilon, h^\varepsilon$ are defined in (11). We also remark that hereafter we drop the apex ε when there is no ambiguity.

Lemma 2.5. *Consider the vector-BGK system (3) with initial data (16), and $\mathbf{u}_0$ in (2) belonging to $H^s(\mathbb{T}^2)$, for $s > 3$. Then, the following estimates hold true.*

$$\begin{aligned} &\|w^*(t)\|_s^2 + \varepsilon^2(\|m(t)\|_s^2 + \|\xi(t)\|_s^2) + \|k^*(t)\|_s^2 + \|h^*(t)\|_s^2 \\ &\quad + \int_0^T \frac{1}{\varepsilon^2}\|w^*(\theta)\|_s^2 + \|m(\theta)\|_s^2 + \|\xi(\theta)\|_s^2 + \frac{1}{\varepsilon^2}(\|k^*(\theta)\|_s^2 + \|h^*(\theta)\|_s^2)\, d\theta \\ &\quad \le c\varepsilon^2(\|\mathbf{u}_0\|_s^2 + \|\nabla \mathbf{u}_0\|_s^2) \\ &\quad + c(|\rho - \bar\rho|_{L_t^\infty L_x^\infty}/\varepsilon,\ |\mathbf{u}|_{L_t^\infty L_x^\infty}) \int_0^T \|w^*(\theta)\|_s^2 + \varepsilon^2(\|m(\theta)\|_s^2 + \|\xi(\theta)\|_s^2)\, d\theta \\ &\quad + c(|\rho - \bar\rho|_{L_t^\infty L_x^\infty}/\varepsilon,\ |\mathbf{u}|_{L_t^\infty L_x^\infty}) \int_0^T \|k^*(\theta)\|_s^2 + \|h^*(\theta)\|_s^2\, d\theta, \quad t < T^\varepsilon. \end{aligned} \tag{18}$$

$$\begin{aligned} &\|w^*(t)\|_s^2 + \varepsilon^2(\|m(t)\|_s^2 + \|\xi(t)\|_s^2) + \|k^*(t)\|_s^2 + \|h^*(t)\|_s^2 \\ &\quad \le c\varepsilon^2(\|\mathbf{u}_0\|_s^2 + \|\nabla \mathbf{u}_0\|_s^2) e^{c(|\rho-\bar\rho|_{L_t^\infty L_x^\infty}/\varepsilon,\ |\mathbf{u}|_{L_t^\infty L_x^\infty})t}, \quad t < T^\varepsilon. \end{aligned} \tag{19}$$

$$\begin{aligned} &\|\partial_t w^*(t)\|_{s-1}^2 + \varepsilon^2(\|\partial_t m(t)\|_{s-1}^2 + \|\partial_t \xi(t)\|_{s-1}^2) + \|\partial_t k^*(t)\|_{s-1}^2 + \|\partial_t h^*(t)\|_{s-1}^2 \\ &\quad \le c\varepsilon^2(\|\mathbf{u}_0\|_{s-1}^2 + \|\nabla \mathbf{u}_0\|_{s-1}^2 + \|\nabla^2 \mathbf{u}_0\|_{s-1}^2) e^{c(|\rho-\bar\rho|_{L_t^\infty L_x^\infty}/\varepsilon,\ |\mathbf{u}|_{L_t^\infty L_x^\infty})t},\ t < T^\varepsilon. \end{aligned} \tag{20}$$

Moreover, there exist ε_0, M and $T^* < T^\varepsilon$ fixed such that, for $\varepsilon \le \varepsilon_0$,

$$|\rho \mathbf{u}(t)|_\infty \le M, \quad |\rho(t) - \bar\rho|_\infty \le \varepsilon M, \quad t \in [0, T^*], \tag{21}$$

$$|\rho(t)|_\infty \le \bar\rho + \varepsilon M, \quad |\mathbf{u}(t)|_\infty \le \frac{M}{\bar\rho - \varepsilon M}, \quad t \in [0, T^*]. \tag{22}$$

$$\int_0^T |\rho(t)-\bar{\rho}|_\infty \, dt \leq c(M)\varepsilon^2, \quad T \in [0,T^*]. \tag{23}$$

Proof. We discuss each result separately.

- Estimate (18) follows from [[8], Lemma 4.2], the change of variables [[8], (30)] and the Sobolev embedding theorem. The only difference is the dependency of $c(|\rho-\bar{\rho}|_{L^\infty_t L^\infty_x}/\varepsilon,\cdot)$ on $|\rho-\bar{\rho}|_{L^\infty_t L^\infty_x}/\varepsilon$, which is a consequence of the quadratic extra term in the compressible pressure (7), see Remark 2.4. We briefly sketch here the argument that allows us to handle that quadratic extra term, essentially by following the proof in [[8], Lemma 4.2]. Let us recall that, by using the change of variables (11), our vector-BGK system (12) can be written in compact form as follows,

$$\partial_t W^\varepsilon + B_1^\varepsilon \partial_x W^\varepsilon + B_2^\varepsilon \partial_y W^\varepsilon = -L^\varepsilon W^\varepsilon + N(w^\varepsilon),$$

where $W^\varepsilon := (w^\varepsilon, \varepsilon^2 m^\varepsilon, \varepsilon^2 \xi^\varepsilon, \varepsilon^2 k^\varepsilon, \varepsilon^2 h^\varepsilon)$,

$$-L^\varepsilon = \frac{1}{\tau}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\varepsilon}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & -\frac{1}{\varepsilon^2} Id & 0 & 0 & 0 \\ \frac{1}{\varepsilon}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} & 0 & -\frac{1}{\varepsilon^2} Id & 0 & 0 \\ 2aId & 0 & 0 & -\frac{1}{\varepsilon^2} Id & 0 \\ 2aId & 0 & 0 & 0 & -\frac{1}{\varepsilon^2} Id \end{pmatrix}$$

and

$$N(w^\varepsilon) = \frac{1}{\tau}\begin{pmatrix} \mathbf{0} \\ \begin{pmatrix} 0 \\ u_1^\varepsilon w_2^\varepsilon + \frac{(\rho^\varepsilon-\bar{\rho})^2}{2\varepsilon\bar{\rho}} \\ u_2^\varepsilon w_2^\varepsilon \end{pmatrix} \\ \begin{pmatrix} 0 \\ u_1^\varepsilon w_3^\varepsilon \\ u_2^\varepsilon w_3^\varepsilon + \frac{(\rho^\varepsilon-\bar{\rho})^2}{2\varepsilon\bar{\rho}} \end{pmatrix} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}.$$

Under Assumptions 2.1, one can find a positive constant right symmetrizer Σ^ε for the system, such that $-L^\varepsilon\Sigma^\varepsilon$ is semi-negative definite (cf.. *conservative form* in [8] and *conservative-dissipative form* in [7]). Therefore, we define the new variable $\tilde{W}^\varepsilon$ such that $W^\varepsilon := \Sigma^\varepsilon \tilde{W}^\varepsilon$, and the symmetrized system reads:

$$\Sigma\partial_t \tilde{W} + B_1\Sigma\partial_x\tilde{W} + B_2\Sigma\partial_y\tilde{W} = -L\Sigma\tilde{W} + N((\Sigma\tilde{W})_1 + \bar{w}).$$

The energy estimate and the higher order estimates on the symmetric system above are explained in details in [8]. Starting from that, one recovers the bounds on the original variable W^ε by using the diffeomorphism $W^\varepsilon = \Sigma^\varepsilon\tilde{W}^\varepsilon$. Here we limit ourselves at showing how to control the extra quadratic term in the compressible pressure, in the estimate for the nonlinear term.

Again, from [8],

$$
\begin{aligned}
&(N(w+\bar{w}), \tilde{W})_0 \\
&= \frac{1}{\tau}\{(u_1 w_2 + \frac{(\rho-\bar{\rho})^2}{2\varepsilon\bar{\rho}}, \varepsilon^2 \tilde{m}_2)_0 + (u_1 w_3, \varepsilon^2 \tilde{m}_3)_0 \\
&+ (u_2 w_2, \varepsilon^2 \tilde{\xi}_2)_0 + (u_2 w_3 + \frac{(\rho-\bar{\rho})^2}{2\varepsilon\bar{\rho}}, \varepsilon^2 \tilde{\xi}_3)_0\} \\
&\le \frac{1}{2\tau}\{\|u_1 w_2\|_0^2 + \varepsilon^4 \|\tilde{m}_2\|_0^2 + \|u_1 w_3\|_0^2 + \frac{1}{4\varepsilon^2\bar{\rho}^2}\|(\rho-\bar{\rho})^2\|_0^2 \\
&+ \varepsilon^4\|\tilde{m}_3\|_0^2 + \|u_2 w_2\|_0^2 + \varepsilon^4\|\tilde{\xi}_2\|_0^2 + \|u_2 w_3\|_0^2 + \varepsilon^4\|\tilde{\xi}_3\|_0^2 + \frac{1}{4\varepsilon^2\bar{\rho}^2}\|(\rho-\bar{\rho})^2\|_0^2\} \\
&\le c(|\mathbf{u}|_{L_t^\infty L_x^\infty},\ |\rho-\bar{\rho}|_{L_t^\infty L_x^\infty}/\varepsilon)\|w\|_0^2 + \frac{\varepsilon^4}{2\tau}(\|\tilde{m}\|_0^2 + \|\tilde{\xi}\|_0^2).
\end{aligned}
$$

The higher order estimates follow in the same way from [[8], Lemma 4.2].

- By applying Gronwall's inequality to (18), one gets (19).
- Estimate (20) follows from [[8], Proposition 3 and (30)].
- For a fixed constant $M > M_0 := c_S\bar{\rho}\|\mathbf{u}_0\|_{s+1}$, where c_S is the Sobolev embedding constant, let us define

$$
T^* := \sup_{t\in[0,T^\varepsilon)} \left\{ \frac{|\rho(t)-\bar{\rho}|_\infty}{\varepsilon} + |\rho\mathbf{u}(t)|_\infty \le M \right\}. \tag{24}
$$

The Sobolev embedding theorem applied to (19) yields, thanks to (17),

$$
\frac{|\rho(t)-\bar{\rho}|_\infty}{\varepsilon} + |\rho\mathbf{u}(t)|_\infty \le M_0 e^{c(\frac{|\rho(t)-\bar{\rho}|_\infty}{\varepsilon},\ |\mathbf{u}|_{L_t^\infty L_x^\infty})t}, \qquad t \le T^*. \tag{25}
$$

The uniform bounds (21)-(22) are due to the Sobolev embedding theorem applied to (19) and the definition of T^*, which depends on M_0, M.

- The last uniform bound is a consequence of the Sobolev embedding theorem applied to (18), the previous bounds in (21)-(22), and the definition of w^* in (17). □

2.2. Kinetic entropies and the relative entropy

Here we recall the definition and the conditions that assure the existence of a kinetic entropy for a discrete velocities BGK model, see [10] for a detailed discussion.

Let $\mathcal{E}$ be a non-empty set of convex entropies for a given limit system. Assume also that $\mathcal{E}$ is separable. A general BGK model under the diffusive scaling reads as follows

$$
\partial_t f_i + \frac{\lambda_i}{\varepsilon}\cdot\nabla_x f_i = \frac{1}{\varepsilon^2}(M_i(\mathbf{u}^\varepsilon) - f_i), \qquad i = 1, \cdots, L, \tag{26}
$$

where $L \ge d$, for $i = 1, \cdots, L$,

$$
\begin{aligned}
&f_i(t,x) = (f_i^1, \cdots, f_i^N) : \mathbb{R}\times\mathbb{R}^d \to \mathbb{R}^N, \\
&\lambda_i = (\lambda_i^1, \cdots, \lambda_i^d), \\
&M_i(\mathbf{u}^\varepsilon) = (M_i^1, \cdots, M_i^N) : \mathbb{R}^N \to \mathbb{R}^N,
\end{aligned}
$$

and, under precise consistency conditions, see [10,15,1,2,12] for a detailed discussion, $\mathbf{u}^\varepsilon = \sum_{i=1}^L f_i \in \mathbb{R}^N$ is the approximating vector field to the solution to the limit system. An important feature of these approximations is the existence, under some reasonable assumptions, of a kinetic entropy. Set $\mathcal{D}_i := \{M_i(\mathbf{u}) : \mathbf{u} \in \mathcal{U}\}$.

Definition 2.6. A kinetic entropy for system (26) is a convex function $\mathcal{H}(\mathbf{f}) = \sum_{i=1}^L \mathcal{H}_i(f_i)$, with $\mathcal{H}_i : \mathcal{D}_i \to \mathbb{R}$, such that, for $\eta(\mathbf{u}) \in \mathcal{E}$,

- (E1) $\mathcal{H}(M(\mathbf{u})) = \eta(\mathbf{u})$ for every $\mathbf{u} \in \mathcal{U}$,
- (E2) $\mathcal{H}(M(\mathbf{u}_f)) \leq \mathcal{H}(\mathbf{f})$, where $\mathbf{u}_f := \sum_{i=1}^L f_i \in \mathcal{U},\ f_i \in \mathcal{D}_i$.

Such a property provides an energy inequality which gives robustness for the scheme. Indeed, it is easy to see that, multiplying the BGK system (26) by $\nabla_f \mathcal{H}(\mathbf{f})$, the minimality (E2) together with the convexity property, provide the following entropy inequality

$$\partial_t \mathcal{H}(\mathbf{f}) + \frac{\Lambda}{\varepsilon} \cdot \nabla_x \mathcal{H}(\mathbf{f}) = \frac{1}{\varepsilon^2} \nabla_{\mathbf{f}} \mathcal{H}(\mathbf{f}) \cdot (M(\mathbf{u}) - \mathbf{f}) \leq 0, \tag{27}$$

which means that, according to the definition given in [27], the kinetic entropy $\mathcal{H}(\mathbf{f})$ is dissipative. More precisely, properties (E1)-(E2) under the hypotheses of [[10], Thm. 2.1] assure that, for any $\eta(\mathbf{u}) \in \mathcal{E}$, defining the projector $\mathcal{P}$ such that

$$\mathcal{P}\mathbf{f} = \sum_{i=1}^L f_i = \mathbf{u}, \tag{28}$$

then

$$\eta(\mathbf{u}) = \min_{\mathcal{P}\mathbf{f} = \mathbf{u}} \mathcal{H}(\mathbf{f}) = \mathcal{H}(M(\mathbf{u})). \tag{29}$$

In this context, the Gibbs principle for relaxation and, in particular, [[43], Prop. 2.1], imply that

$$\nabla_{\mathbf{f}} \mathcal{H}(M(\mathbf{u})) \perp Ker(\mathcal{P}). \tag{30}$$

Since $\mathbf{f} - M(\mathbf{u}) \in Ker(\mathcal{P})$, the convexity property of $\mathcal{H}(\mathbf{f})$ together with condition (30) allow us to get the following inequality:

$$\nabla_{\mathbf{f}} \mathcal{H}(\mathbf{f}) \cdot (\mathbf{f} - M(\mathbf{u})) \leq -c|\mathbf{f} - M(\mathbf{u})|^2, \qquad c = c(|\mathbf{f}|_\infty), \tag{31}$$

meaning that the kinetic entropy $\mathcal{H}(M(\mathbf{u}))$ is strictly dissipative, as in [27]. According to the theory developed by Bouchut [10], the existence of a kinetic entropy for system (3) is subjected to the existence of a convex entropy for the limit of system (3) under the hyperbolic scaling. The hyperbolic parameter of the vector-BGK approximation (3) is represented by τ and the limit equations approximated by (3) in the vanishing parameter of the hyperbolic scaling τ are the isentropic Euler equations. The convergence of the hyperbolic-scaled system is guaranteed by the structural properties of our vector-BGK model listed before, see [15], while a rigorous proof is provided in [39]. A convex entropy for the limit equations under the hyperbolic scaling, i.e., the isentropic Euler equations, is given by

$$\eta(w^\varepsilon) = \frac{1}{2} \frac{|\mathbf{q}^\varepsilon|^2}{\rho^\varepsilon} + k(\rho^\varepsilon)^2. \tag{32}$$

We can immediately state the following result.

Proposition 2.7. *Consider the vector-BGK approximation (3) under Assumptions 2.1 and emanating from smooth initial data (16). Then, in the time interval* $[0, T^*]$*, there exists a kinetic entropy* $\mathcal{H}(\mathbf{f}^\varepsilon) = \sum_{i=1}^5 \mathcal{H}_i({f_i}^\varepsilon)$ *for system (3), satisfying the properties listed in Definition 2.6 in a neighbourhood of the Maxwellians* $M(w^\varepsilon)$ *(5), with* $\eta(w^\varepsilon)$ *in (32).*

Proof. First of all, the local in time estimates in Lemma 2.5 provide the boundedness of the densities $\mathbf{f}^\varepsilon$. This result and Assumptions 2.1 allow us to prove the positivity of the spectrum of the Jacobian matrices of the Maxwellians (5), see Remark 3.2. The statement follows from [[10], Theorem 2.1]. □

The relative entropy can be seen as a perturbation of the kinetic entropy near to the equilibrium represented by the solution to the limit system. A precise definition in the context of hyperbolic relaxation is provided in [43]. For diffusive relaxation, we will use the following:

$$\begin{aligned}\tilde{\mathcal{H}}(\mathbf{f}|\bar{\mathbf{f}}) &= \mathcal{H}(\mathbf{f}) - \mathcal{H}(\overline{\mathcal{M}}(\bar{w})) - \nabla_{\mathbf{f}}\mathcal{H}(\overline{\mathcal{M}}(\bar{w})) \cdot (\mathbf{f} - \overline{\mathcal{M}}(\bar{w})) \\ &= \sum_i \mathcal{H}_i(f_i) - \mathcal{H}_i(\overline{\mathcal{M}}_i(\bar{w})) - \nabla_{f_i}\mathcal{H}_i(\overline{\mathcal{M}}_i(\bar{w})) \cdot (f_i - \overline{\mathcal{M}}_i(\bar{w})),\end{aligned} \tag{33}$$

where $\mathcal{H}(\mathbf{f})$ is in Definition 2.6, and $\overline{\mathcal{M}}(\bar{w}) = (\overline{\mathcal{M}}_i(\bar{w}))_{i=1,\cdots,5}$ are the perturbed Maxwellians in (15), evaluated in the solution $\bar{w} = (\bar{\rho}, \varepsilon\bar{\rho}\bar{\mathbf{u}})$ to the incompressible Navier-Stokes equations (1).

2.3. Quantifying the dissipation

The aim of this part is to characterize and to quantify the dissipative terms resulting from the relative entropy estimate.

Lemma 2.8. *Let* $\eta(w)$ *be defined in (32). Let* $\mathcal{H}(\mathbf{f}) = \sum_{i=1}^5 \mathcal{H}_i(f_i)$ *be a kinetic entropy associated with the vector-BGK model in (3), such that* $\mathcal{H}(M(w)) = \eta(w)$*. Then the following entropy expansion is satisfied:*

$$\begin{aligned}&\frac{1}{\varepsilon^2}\int_0^T\iint\sum_{i=1}^5 \nabla_{f_i}\mathcal{H}_i(f_i)\cdot(M_i - f_i)\,dt\,dx\,dy \\ &\quad= -\int_0^T\iint\left[\frac{\nabla_w^2\eta(w)}{2a\lambda^2\tau}\cdot\left(m - \frac{A_1(w)}{\varepsilon}\right)\right]\cdot\left(m - \frac{A_1(w)}{\varepsilon}\right)dt\,dx\,dy \\ &\quad\quad -\int_0^T\iint\left[\frac{\nabla_w^2\eta(w)}{2a\lambda^2\tau}\cdot\left(\xi - \frac{A_2(w)}{\varepsilon}\right)\right]\cdot\left(\xi - \frac{A_2(w)}{\varepsilon}\right)dt\,dx\,dy \\ &\quad\quad -\int_0^T\int\int\left[\frac{\nabla_w^2\eta(w)}{2a\varepsilon^2\tau}\cdot(k - 2aw)\right]\cdot(k - 2aw)\,dt\,dx\,dy \\ &\quad\quad -\int_0^T\iint\left[\frac{\nabla_w^2\eta(w)}{2a\varepsilon^2\tau}\cdot(h - 2aw)\right]\cdot(h - 2aw)\,dt\,dx\,dy \\ &\quad\quad -\int_0^T\iint\left[\frac{\nabla_w^2\eta(w)}{(1-4a)\tau\varepsilon^2}\cdot(4aw - (k+h))\right]\cdot(4aw - (k+h))\,dt\,dx\,dy \\ &\quad\quad + O(\varepsilon^3).\end{aligned}$$

Proof. First of all, Proposition 2.7 guarantees the existence of a kinetic entropy for (3), such that

$$\mathcal{H}(M(w)) = \eta(w) \text{ in (32)}, \quad \nabla_{f_i}\mathcal{H}_i(M_i(w)) = \nabla_w \eta(w), \quad i = 1, \cdots, 5.$$

We point out that the spectrum of the Jacobian matrices of the Maxwellians in (5) is positive provided that the parameter a in the expressions (5) is positive and $\lambda > 0$ is big enough (Assumptions 2.1 and Remark 2.2). This remark, together with the bounds in (22), assure the existence of a kinetic convex and dissipative entropy for our system, thanks to [[10], Theorem 2.1], as stated in Proposition 2.7. Notice that in the course of our computations, the densities $\mathbf{f}^\varepsilon$ remain in a bounded set, close enough to the hyperbolic equilibrium.

Now we consider the following expansion

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\sum_{i=1}^{5}\nabla_{f_i}\mathcal{H}_i(f_i)\cdot(M_i - f_i)\\
&= \frac{1}{\varepsilon^2}\sum_{i=1}^{5}\nabla_{f_i}\mathcal{H}_i(M_i)\cdot(M_i - f_i) + \frac{1}{\varepsilon^2}\sum_{i=1}^{5}\nabla^2_{f_i}\mathcal{H}_i(M_i)\cdot(f_i - M_i)\cdot(M_i - f_i)\\
&+ O(\frac{|f_i - M_i|^3}{\varepsilon^2})\\
&= -\frac{1}{\varepsilon^2}\sum_{i=1}^{5}\nabla^2_{f_i}\mathcal{H}_i(M_i)\cdot(f_i - M_i)\cdot(f_i - M_i)\\
&+ O(\frac{|f_i - M_i|^3}{\varepsilon^2}),
\end{aligned} \tag{34}$$

where the first term vanishes thanks to the orthogonality property [[43], Proposition 2.1]. For $i = 1, \cdots, 4$, the first term of the last equality reads

$$\begin{aligned}
&-\frac{1}{\varepsilon^2}\int_0^T\iint \nabla^2_{f_i}\mathcal{H}_i(M_i)\cdot(f_i - M_i)\cdot(f_i - M_i)\,dt\,dx\,dy\\
&= -\frac{1}{\varepsilon^2}\int_0^T\iint \nabla^2\mathcal{H}_i(aw \pm \frac{A_i(w)}{2\lambda})\cdot(f_i - M_i)\cdot(f_i - M_i)\,dt\,dx\,dy.
\end{aligned}$$

Note that, from (4)-(6) and Lemma 2.5,

$$w = \begin{pmatrix}\rho\\ \varepsilon\rho u_1\\ \varepsilon\rho u_2\end{pmatrix} = \begin{pmatrix}O(1)\\ O(\varepsilon)\\ O(\varepsilon)\end{pmatrix},$$

$$A_1(w) = \begin{pmatrix}\varepsilon\rho u_1\\ \varepsilon^2\rho u_1^2 + \frac{\rho^2-\bar\rho^2}{2\bar\rho}\\ \varepsilon^2\rho u_1 u_2\end{pmatrix} = \begin{pmatrix}O(\varepsilon)\\ O(\varepsilon^2)\\ O(\varepsilon^2)\end{pmatrix},$$

$$A_2(w) = \begin{pmatrix}\varepsilon\rho u_1\\ \varepsilon^2\rho u_1 u_2\\ \varepsilon^2\rho u_2^2 + \frac{\rho^2-\bar\rho^2}{2\bar\rho}\end{pmatrix} = \begin{pmatrix}O(\varepsilon)\\ O(\varepsilon^2)\\ O(\varepsilon^2)\end{pmatrix}.$$

This way,

$$\nabla_{f_i}\mathcal{H}(M_i(w)) = \nabla_{f_i}\mathcal{H}\left(aw \pm \frac{A_i(w)}{2\lambda}\right) = \nabla_{f_i}\mathcal{H}(aw) + O(\varepsilon).$$

Moreover, from [10], it is also known that

$$\nabla_{f_i}\mathcal{H}(M_i(w)) = \nabla_w \eta(w).$$

Differentiating again the previous equivalent expressions,

$$\nabla^2_{f_i}\mathcal{H}_i(aw) = \frac{1}{a}\nabla^2_w \eta(w) + O(\varepsilon). \tag{35}$$

Thus, the last equality yields

$$\begin{aligned}
&-\frac{1}{\varepsilon^2}\int_0^T \iint \nabla^2_{f_i}\mathcal{H}_i(aw \pm \frac{A_i(w)}{2\lambda}) \cdot (f_i - M_i)\cdot(f_i - M_i)\, dt\, dx\, dy \\
&\le -\frac{1}{\varepsilon^2}\int_0^T \iint \frac{1}{a}\nabla^2_w \eta(w)\cdot(f_i - M_i)\cdot(f_i - M_i)\, dt\, dx\, dy \\
&+ \frac{c(|w|_{L^\infty_t L^\infty_x})}{\varepsilon}|f_i - M_i|^2_{L^\infty_t L^\infty_x}.
\end{aligned}$$

Now, from (3)-(11),

$$\begin{aligned}
\frac{M_1 - f_1}{\varepsilon^2} &= \partial_t f_1 + \frac{\lambda}{\varepsilon}\partial_x f_1 = \frac{1}{2}(\partial_t k + \partial_x m) + \frac{1}{2\lambda\varepsilon}(\varepsilon^2\partial_t m + \lambda^2\partial_x k),\\
\frac{M_3 - f_3}{\varepsilon^2} &= \partial_t f_3 - \frac{\lambda}{\varepsilon}\partial_x f_3 = \frac{1}{2}(\partial_t k + \partial_x m) - \frac{1}{2\lambda\varepsilon}(\varepsilon^2\partial_t m + \lambda^2\partial_x k),\\
\frac{M_2 - f_2}{\varepsilon^2} &= \partial_t f_2 + \frac{\lambda}{\varepsilon}\partial_y f_2 = \frac{1}{2}(\partial_t h + \partial_y \xi) + \frac{1}{2\lambda\varepsilon}(\varepsilon^2\partial_t \xi + \lambda^2\partial_y h),\\
\frac{M_4 - f_4}{\varepsilon^2} &= \partial_t f_4 - \frac{\lambda}{\varepsilon}\partial_y f_4 = \frac{1}{2}(\partial_t h + \partial_y \xi) - \frac{1}{2\lambda\varepsilon}(\varepsilon^2\partial_t \xi + \lambda^2\partial_y h).
\end{aligned} \tag{36}$$

Lemma 2.5 and the previous equalities imply that

$$\frac{c(|w|_{L^\infty_t L^\infty_x})}{\varepsilon}|f_i - M_i|^2_{L^\infty_t L^\infty_x} = O(\varepsilon^3),$$

and so, by using the change of variables (11),

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\int_0^T \iint \sum_{i=1}^4 \nabla_{f_i}\mathcal{H}_i(f_i)\cdot(M_i - f_i)\, dt\, dx\, dy \\
&= -\frac{1}{\varepsilon^2}\int_0^T \iint \sum_{i=1}^4 \frac{1}{a}\nabla^2_w \eta(w)\cdot(M_i - f_i)\cdot(M_i - f_i)\, dt\, dx\, dy + O(\varepsilon^3) \\
&= \int_0^T \iint \left[\frac{\nabla^2_w \eta(w)}{2a\lambda^2\tau}\cdot(m - \frac{A_1(w)}{\varepsilon})\right]\cdot(m - \frac{A_1(w)}{\varepsilon})\, dt\, dx\, dy
\end{aligned}$$

$$- \int_0^T \iint \left[\frac{\nabla_w^2 \eta(w)}{2a\lambda^2\tau} \cdot (\xi - \frac{A_2(w)}{\varepsilon}) \right] \cdot (\xi - \frac{A_2(w)}{\varepsilon})\, dt\, dx\, dy$$
$$- \int_0^T \int\int \left[\frac{\nabla_w^2 \eta(w)}{2a\varepsilon^2\tau} \cdot (k - 2aw) \right] \cdot (k - 2aw)\, dt\, dx\, dy$$
$$- \int_0^T \iint \left[\frac{\nabla_w^2 \eta(w)}{2a\varepsilon^2\tau} \cdot (h - 2aw) \right] \cdot (h - 2aw)\, dt\, dx\, dy + O(\varepsilon^3).$$

The expansion

$$\frac{1}{\varepsilon^2} \int_0^T \iint \nabla_{f_5} \mathcal{H}_5(f_5) \cdot (M_5 - f_5)\, dt\, dx\, dy$$
$$= - \int_0^T \iint \left[\frac{\nabla_w^2 \eta(w)}{(1-4a)\tau\varepsilon^2} \cdot (4aw - (k+h)) \right] \cdot (4aw - (k+h))\, dt\, dx\, dy + O(\varepsilon^3)$$

is obtained in analogous way. □

Lemma 2.9. *Consider $\overline{\mathcal{M}}_i$, for $i = 1, \cdots, 4$, in (15). Then*

$$\begin{aligned} \nabla_{f_i} \overline{\mathcal{H}}_i(\overline{\mathcal{M}}_i) &= \nabla_{f_i} \mathcal{H}_i(\overline{M}_i) \mp a\varepsilon\lambda\tau \nabla^2_{f_i} \mathcal{H}_i(\overline{M}_i) \partial_{x_j} \bar{w} + O(\varepsilon^3) \\ &= \nabla_w \eta(\bar{w}) \mp \lambda\varepsilon\tau \nabla_w^2 \eta(\bar{w}) \partial_{x_j} \bar{w} + O(\varepsilon^3), \quad j = 1, 2. \end{aligned} \tag{37}$$

Proof. The proof follows by Taylor expansions and (35), in the spirit of Lemma 2.8. □

3. Relative entropy estimate for the vector-BGK model

Our main result is stated here.

Theorem 3.1. *Consider the vector-BGK model in (3) for the two-dimensional incompressible Navier-Stokes equations in (1) on $[0, +\infty) \times \mathbb{T}^2$, endowed with a kinetic entropy $\mathcal{H}(\boldsymbol{f}^\varepsilon)$, whose existence and properties are given by Proposition 2.7. Let $\bar{\mathbf{u}} = (\bar{u}_1, \bar{u}_2)$, $\nabla \bar{P}$ be a smooth velocity field and pressure satisfying the incompressible Navier-Stokes equations (1) on $[0, +\infty) \times \mathbb{T}^2$ and $\{\boldsymbol{f}^\varepsilon\}$ be a family of smooth solutions to (3) and emanating from smooth initial data $\mathbf{u}_0$ in (2) and $\mathbf{f}_0 = (f_i(0, x))_{i=1,\cdots,5}$ in (16). Then, defining $w^\varepsilon = \sum_i f_i^\varepsilon = (\rho^\varepsilon, \varepsilon\rho^\varepsilon \mathbf{u}^\varepsilon)$, the following estimate holds for any $T > 0$ and for $\varepsilon \leq \varepsilon_0$, where ε_0 is fixed and depends on $M_0 = \bar{\rho}\|\mathbf{u}_0\|_{s+1}$,*

$$\sup_{t \in [0,T]} \frac{\|\rho^\varepsilon(t) - \bar{\rho}\|_{s'}}{\varepsilon} + \|\mathbf{u}^\varepsilon(t) - \bar{\mathbf{u}}(t)\|_{s'} \leq c\varepsilon^{\frac{1}{2} - \delta},$$

with $s > 3$, $0 < s' < s$ and $\delta := \frac{s-s'}{2s}$. Moreover, for $\varepsilon \leq \varepsilon_0$, the solutions $(\rho^\varepsilon, \mathbf{u}^\varepsilon)$ to the approximating system (3) are globally bounded in time, and for $\varepsilon \to 0$,

$$\frac{\nabla((\rho^\varepsilon)^2 - \bar{\rho}^2)}{\varepsilon^2} \rightharpoonup^{\star} \nabla \bar{P} \quad \text{in } L_t^\infty H_x^{s-3}.$$

Remark 3.2 *(Looking at the macroscopic variables).* Note that the existence of a kinetic entropy $\mathcal{H}(\mathbf{f}^\varepsilon)$ is due to the boundedness of $\{\mathbf{f}^\varepsilon\}$, as stated in Proposition 2.7. The reason lies in the assumptions of [[10], Theorem 2.1], where the positivity of the Jacobians of the Maxwellian functions is the crucial property. More precisely, the spectrum of these Jacobian matrices is the following:

$$\sigma_{\nabla_w M_{1,3}(w)} = a \pm \frac{\varepsilon u_1}{2\lambda}, \quad \sigma_{\nabla_w M_{2,4}(w)} = a \pm \frac{\varepsilon u_2}{2\lambda},$$
$$\mu^{\pm}_{\nabla_w M_{1,3}(w)} = a \pm \frac{\varepsilon u_1}{2\lambda} \pm \frac{1}{2\lambda}\sqrt{\frac{\rho}{\bar\rho}}, \quad \mu^{\pm}_{\nabla_w M_{2,4}(w)} = a \pm \frac{\varepsilon u_2}{2\lambda} \pm \frac{1}{2\lambda}\sqrt{\frac{\rho}{\bar\rho}}.$$

This implies that the eigenvalues are strictly positive provided that $w^\varepsilon = (\rho, \varepsilon u_1, \varepsilon u_2)$ is bounded (Lemma 2.5) and $\lambda > 0$ is big enough (Assumptions 2.1 and Remark 2.2). This way, the setting exactly fits the hypothesis of [[10], Theorem 2.1], which guarantees the existence of a *dissipative entropy*,

$$\mathcal{H}(\mathbf{f}^\varepsilon) = H(w^\varepsilon, \varepsilon^2 m^\varepsilon, \varepsilon^2\xi^\varepsilon, \varepsilon^2 k^\varepsilon, \varepsilon^2 h^\varepsilon),$$

at least for local times, where Lemma 2.5 holds. Actually, the dissipative entropy can be interpreted as an energy for the whole system, in terms of the macroscopic variables (12). The following estimate of Theorem 3.3 will be provided in L^2, since a little information on the entropy, which is defined by means of the inverse function theorem, is not enough to get more. However, the precise local in time estimates of Lemma 2.5 will be used in the final argument to obtain the strong convergence in $H^{s'}(\mathbb{T}^2)$, $0 < s' < s$, of $(\mathbf{u}^\varepsilon, \nabla P^\varepsilon = \frac{\nabla((\rho^\varepsilon)^2 - \bar\rho^2)}{\varepsilon^2})$ to the smooth solutions to the incompressible Navier-Stokes equations $(\bar{\mathbf{u}}, \nabla \bar P)$ for any positive time $T > 0$.

The global in time convergence proof is based on the use of the relative entropy inequality, which is stated and proved here.

Theorem 3.3. *Under the hypothesis of Theorem 3.1, let T^* be defined in (24). Then the relative entropy method provides the following estimate:*

$$\sup_{t\in[0,T^*]} \frac{\|\rho^\varepsilon(t) - \bar\rho\|_0}{\varepsilon} + \|\rho^\varepsilon \mathbf{u}^\varepsilon(t) - \bar\rho\bar{\mathbf{u}}(t)\|_0 \le c\sqrt{\varepsilon}.$$

Proof. We start by recalling the definition of the relative entropy in (33),

$$\begin{aligned}\tilde{\mathcal{H}}(\mathbf{f}|\bar{\mathbf{f}}) &= \mathcal{H}(\mathbf{f}) - \mathcal{H}(\overline{\mathcal{M}}) - \nabla_{\mathbf{f}}\mathcal{H}(\overline{\mathcal{M}}) \cdot (\mathbf{f} - \overline{\mathcal{M}}) \\ &= \sum_i \mathcal{H}_i(f_i) - \mathcal{H}_i(\overline{\mathcal{M}}_i) - \nabla_{f_i}\mathcal{H}_i(\overline{\mathcal{M}}_i) \cdot (f_i - \overline{\mathcal{M}}_i),\end{aligned}$$

where $\overline{\mathcal{M}}_i = \overline{\mathcal{M}}_i(\bar\rho, \varepsilon\bar\rho\bar{\mathbf{u}})$, $i = 1, \cdots, 5$, are in (15), $\bar\rho$ is a constant density, $\bar{\mathbf{u}}$ is the smooth solution to (1), and the associated entropy-flux is given by

$$\begin{aligned}\tilde{Q}(\mathbf{f}|\bar{\mathbf{f}}) = &\frac{\lambda}{\varepsilon}\begin{pmatrix} \mathcal{H}_1(f_1) - \mathcal{H}_3(f_3) - (\mathcal{H}_1(\overline{\mathcal{M}}_1) - \mathcal{H}_3(\overline{\mathcal{M}}_3)) \\ \mathcal{H}_2(f_2) - \mathcal{H}_4(f_4) - (\mathcal{H}_2(\overline{\mathcal{M}}_2) - \mathcal{H}_4(\overline{\mathcal{M}}_4)) \end{pmatrix} \\ &- \frac{\lambda}{\varepsilon}\begin{pmatrix} \nabla_{f_1}\mathcal{H}_1(\overline{\mathcal{M}}_1)(f_1 - \overline{\mathcal{M}}_1) - \nabla_{f_3}\mathcal{H}_3(\overline{\mathcal{M}}_3)(f_3 - \overline{\mathcal{M}}_3) \\ \nabla_{f_2}\mathcal{H}_2(\overline{\mathcal{M}}_2)(f_2 - \overline{\mathcal{M}}_2) - \nabla_{f_4}\mathcal{H}_4(\overline{\mathcal{M}}_4)(f_4 - \overline{\mathcal{M}}_4) \end{pmatrix}.\end{aligned}$$

Hereafter, we adopt the following notation, $\overline{\mathcal{H}}_i := \mathcal{H}_i(\overline{\mathcal{M}}_i)$.

Now we proceed to get the desired inequality.

$$\int_0^T \iint \partial_t \tilde{\mathcal{H}}(\mathbf{f}|\bar{\mathbf{f}}) + \nabla_x \cdot \tilde{Q}(\mathbf{f}|\bar{\mathbf{f}})\, dt\, dx\, dy$$
$$= \int_0^T \iint \partial_t \mathcal{H}(\mathbf{f}) + \frac{\lambda}{\varepsilon}\partial_x(\mathcal{H}_1(f_1) - \mathcal{H}_3(f_3)) + \frac{\lambda}{\varepsilon}\partial_y(\mathcal{H}_2(f_2) - \mathcal{H}_4(f_4))\, dt\, dx\, dy$$

$$- \int_0^T \iint \partial_t \mathcal{H}(\overline{\mathcal{M}}) + \frac{\lambda}{\varepsilon}\partial_x(\mathcal{H}_1(\overline{\mathcal{M}}_1) - \mathcal{H}_3(\overline{\mathcal{M}}_3))\, dt\, dx\, dy$$
$$- \int_0^T \iint \frac{\lambda}{\varepsilon}\partial_y(\mathcal{H}_2(\overline{\mathcal{M}}_2) - \mathcal{H}_4(\overline{\mathcal{M}}_4))\, dt\, dx\, dy$$

$$- \int_0^T \iint \partial_t(\nabla_{f_1}\mathcal{H}_1(\overline{\mathcal{M}}_1)(f_1 - \overline{\mathcal{M}}_1) + \nabla_{f_2}\mathcal{H}_2(\overline{\mathcal{M}}_2)(f_2 - \overline{\mathcal{M}}_2))\, dt\, dx\, dy$$
$$- \int_0^T \iint \partial_t(\nabla_{f_3}\mathcal{H}_3(\overline{\mathcal{M}}_3)(f_3 - \overline{\mathcal{M}}_3) + \nabla_{f_4}\mathcal{H}_4(\overline{\mathcal{M}}_4)(f_4 - \overline{\mathcal{M}}_4))\, dt\, dx\, dy$$
$$- \int_0^T \iint \partial_t(\nabla_{f_5}\mathcal{H}_5(\overline{\mathcal{M}}_5)(f_5 - \overline{\mathcal{M}}_5))\, dt\, dx\, dy$$

$$- \int_0^T \iint \frac{\lambda}{\varepsilon}\partial_x(\nabla_{f_1}\mathcal{H}_1(\overline{\mathcal{M}}_1)(f_1 - \overline{\mathcal{M}}_1) - \nabla_{f_3}\mathcal{H}_3(\overline{\mathcal{M}}_3)(f_3 - \overline{\mathcal{M}}_3))\, dt\, dx\, dy$$
$$- \int_0^T \iint \frac{\lambda}{\varepsilon}\partial_y(\nabla_{f_2}\mathcal{H}_2(\overline{\mathcal{M}}_2)(f_2 - \overline{\mathcal{M}}_2) - \nabla_{f_4}\mathcal{H}_4(\overline{\mathcal{M}}_4)(f_4 - \overline{\mathcal{M}}_4))\, dt\, dx\, dy$$
$$= I_1 + I_2 + I_3 + I_4.$$

First of all, I_1 is already estimated in Lemma 2.8. Now, let us consider I_2.

The following expansions are based on Lemma 2.9.

$$- \int_0^T \iint \partial_t(\overline{\mathcal{H}}_1 + \overline{\mathcal{H}}_2 + \overline{\mathcal{H}}_3 + \overline{\mathcal{H}}_4 + \overline{\mathcal{H}}_5)\, dt\, dx\, dy$$
$$- \int_0^T \iint \frac{\lambda}{\varepsilon}\partial_x(\overline{\mathcal{H}}_1 - \overline{\mathcal{H}}_3) + \frac{\lambda}{\varepsilon}\partial_y(\overline{\mathcal{H}}_2 - \overline{\mathcal{H}}_4)\, dt\, dx\, dy$$
$$= - \int_0^T \iint (\nabla_w \eta(\bar{w}) - a\varepsilon\lambda\tau\nabla^2_{f_1}\overline{\mathcal{H}}_1\partial_x\bar{w})(\partial_t\overline{\mathcal{M}}_1 + \frac{\lambda}{\varepsilon}\partial_x\overline{\mathcal{M}}_1)\, dt\, dx\, dy$$

$$-\int_0^T\iint(\nabla_w\eta(\bar w)+a\varepsilon\lambda\tau\nabla^2_{f_3}\overline{\mathcal{H}}_3\partial_x\bar w)(\partial_t\overline{\mathcal{M}}_3-\frac{\lambda}{\varepsilon}\partial_x\overline{\mathcal{M}}_3)\,dt\,dx\,dy$$

$$-\int_0^T\iint(\nabla_w\eta(\bar w)-a\varepsilon\lambda\tau\nabla^2_{f_2}\overline{\mathcal{H}}_2\partial_y\bar w)(\partial_t\overline{\mathcal{M}}_2+\frac{\lambda}{\varepsilon}\partial_y\overline{\mathcal{M}}_2)\,dt\,dx\,dy$$

$$-\int_0^T\iint(\nabla_w\eta(\bar w)+a\varepsilon\lambda\tau\nabla^2_{f_4}\overline{\mathcal{H}}_4\partial_y\bar w)(\partial_t\overline{\mathcal{M}}_4-\frac{\lambda}{\varepsilon}\partial_y\overline{\mathcal{M}}_4)\,dt\,dx\,dy$$

$$-\int_0^T\iint\nabla_w\eta(\bar w)\cdot\partial_t\overline{\mathcal{M}}_5\,dt\,dx\,dy$$

$$=-\int_0^T\iint\nabla_w\eta(\bar w)\cdot\partial_t(\overline{\mathcal{M}}_1+\overline{\mathcal{M}}_2+\overline{\mathcal{M}}_3+\overline{\mathcal{M}}_4+\overline{\mathcal{M}}_5)\,dt\,dx\,dy$$

$$-\int_0^T\iint\nabla_w\eta(\bar w)[\frac{\lambda}{\varepsilon}\partial_x(\overline{\mathcal{M}}_1-\overline{\mathcal{M}}_3)+\frac{\lambda}{\varepsilon}\partial_y(\overline{\mathcal{M}}_2-\overline{\mathcal{M}}_4)]\,dt\,dx\,dy$$

$$+2a\tau\lambda^2\int_0^T\iint(\nabla^2_w\eta(\bar w)\cdot\partial_x\bar w)\partial_x\bar w+(\nabla^2_w\eta(\bar w)\cdot\partial_y\bar w)\partial_y\bar w\,dt\,dx\,dy+O(\varepsilon^3)$$

$$=-\int_0^T\iint\nabla_w\eta(\bar w)[\partial_t\bar w+\partial_x\frac{A_1(\bar w)}{\varepsilon}+\partial_y\frac{A_2(\bar w)}{\varepsilon}-\nu(\partial_{xx}\bar w+\partial_{yy}\bar w)]\,dt\,dx\,dy$$

$$+\int_0^T\iint\tau\frac{\nabla^2_w\eta(\bar w)}{2a\lambda^2}\cdot(\lambda^2\partial_x\bar k)\cdot(\lambda^2\partial_x\bar k)\,dt\,dx\,dy$$

$$+\int_0^T\iint\tau\frac{\nabla^2_w\eta(\bar w)}{2a\lambda^2}\cdot(\lambda^2\partial_y\bar h)\cdot(\lambda^2\partial_y\bar h)\,dt\,dx\,dy$$

$$=-\int_0^T\iint\nabla_w\eta(\bar w)[\partial_t\bar w+\partial_x\frac{A_1(\bar w)}{\varepsilon}+\partial_y\frac{A_2(\bar w)}{\varepsilon}-\nu(\partial_{xx}\bar w+\partial_{yy}\bar w)]\,dt\,dx\,dy$$

$$+\int_0^T\iint\tau\left[\frac{\nabla^2_w\eta(\bar w)}{2a\lambda^2}\cdot(\varepsilon^2\partial_t\bar m+\lambda^2\partial_x\bar k)\right]\cdot(\varepsilon^2\partial_t\bar m+\lambda^2\partial_x\bar k)\,dt\,dx\,dy$$

$$+\int_0^T\iint\tau\left[\frac{\nabla^2_w\eta(\bar w)}{2a\lambda^2}\cdot(\varepsilon^2\partial_t\bar\xi+\lambda^2\partial_y\bar h)\right]\cdot(\varepsilon^2\partial_t\bar\xi+\lambda^2\partial_y\bar h)\,dt\,dx\,dy+O(\varepsilon^3)$$

$$-\tau\int_0^T\iint\left[\frac{\nabla^2_w\eta(\bar w)}{2a\lambda^2}\cdot(\varepsilon^2\partial_t\bar m)\right]\cdot(\varepsilon^2\partial_t\bar m)+\left[\frac{\nabla^2_w\eta(\bar w)}{a\lambda^2}\cdot(\varepsilon^2\partial_t\bar m)\right]\cdot(\lambda^2\partial_x\bar k)\,dt\,dx\,dy$$

$$-\tau\int_0^T\iint\left[\frac{\nabla^2_w\eta(\bar w)}{2a\lambda^2}\cdot(\varepsilon^2\partial_t\bar\xi)\right]\cdot(\varepsilon^2\partial_t\bar\xi)+\left[\frac{\nabla^2_w\eta(\bar w)}{a\lambda^2}\cdot(\varepsilon^2\partial_t\bar\xi)\right]\cdot(\lambda^2\partial_y\bar h)\,dt\,dx\,dy$$

Notice that the order (with respect to ε) of the last two expressions is due to the terms $(\varepsilon^2\partial_t\bar{m})\cdot\partial_x\bar{k}$, $(\varepsilon^2\partial_t\bar{\xi})\cdot\partial_y\bar{h}$, $|\varepsilon^2\partial_t\bar{m}|^2$, $|\varepsilon^2\partial_t\bar{\xi}|^2$, where

$$\begin{aligned}
\partial_x\bar{k} &= 2a\partial_x\bar{w} = 2a\partial_x\begin{pmatrix}\bar{\rho}\\ \varepsilon\bar{\rho}\bar{u}_1\\ \varepsilon\bar{\rho}\bar{u}_2\end{pmatrix} = O(\varepsilon),\\
\partial_y\bar{h} &= 2a\partial_y\bar{w} = O(\varepsilon),
\end{aligned}$$

and

$$\begin{aligned}
\varepsilon^2\partial_t\bar{m} &= \varepsilon^2\partial_t[\frac{A_1(\bar{w})}{\varepsilon} - \nu\partial_x\bar{w}] = \varepsilon^2\partial_t\left[\begin{pmatrix}\bar{\rho}\bar{u}_1\\ \varepsilon\bar{\rho}\bar{u}_1^2+\varepsilon\bar{P}\\ \varepsilon\bar{\rho}\bar{u}_1\bar{u}_2\end{pmatrix} - \nu\partial_x\bar{w}\right] = O(\varepsilon^2),\\
\varepsilon^2\partial_t\bar{\xi} &= \varepsilon^2\partial_t[\frac{A_2(\bar{w})}{\varepsilon} - \nu\partial_y\bar{w}] = \varepsilon^2\partial_t\left[\begin{pmatrix}\bar{\rho}\bar{u}_2\\ \varepsilon\bar{\rho}\bar{u}_1\bar{u}_2\\ \varepsilon\bar{\rho}\bar{u}_2^2+\varepsilon\bar{P}\end{pmatrix} - \nu\partial_y\bar{w}\right] = O(\varepsilon^2).
\end{aligned}$$

This way, every remainder term (the last two expressions above) is $O(\varepsilon^3)$. Next, we consider I_3.

$$\begin{aligned}
I_3 = &-\int_0^T\iint \partial_t[\nabla_{f_1}\mathcal{H}_1(\overline{\mathcal{M}}_1)(f_1-\overline{\mathcal{M}}_1) + \nabla_{f_2}\mathcal{H}_2(\overline{\mathcal{M}}_2)(f_2-\overline{\mathcal{M}}_2)]\,dt\,dx\,dy\\
&-\int_0^T\iint \partial_t[\nabla_{f_3}\mathcal{H}_3(\overline{\mathcal{M}}_3)(f_3-\overline{\mathcal{M}}_3) + \nabla_{f_4}\mathcal{H}_4(\overline{\mathcal{M}}_4)(f_4-\overline{\mathcal{M}}_4)]\,dt\,dx\,dy\\
&-\int_0^T\iint \partial_t[\nabla_{f_5}\mathcal{H}_5(\overline{\mathcal{M}}_5)(f_5-\overline{\mathcal{M}}_5)]\,dt\,dx\,dy\\
= &-\int_0^T\iint \partial_t[\nabla_w\eta(\bar{w})\cdot(w-\bar{w})]\,dt\,dx\,dy\\
&+\varepsilon\lambda\tau\int_0^T\iint \partial_t[\nabla_w^2\eta(\bar{w})\cdot\partial_x\bar{w}\cdot(f_1-f_3-(\overline{\mathcal{M}}_1-\overline{\mathcal{M}}_3))]\,dt\,dx\,dy\\
&+\varepsilon\lambda\tau\int_0^T\iint \partial_t[\nabla_w^2\eta(\bar{w})\cdot\partial_y\bar{w}\cdot(f_2-f_4-(\overline{\mathcal{M}}_2-\overline{\mathcal{M}}_4))]\,dt\,dx\,dy + O(\varepsilon^3)\\
= &-\int_0^T\iint \partial_t[\nabla_w\eta(\bar{w})\cdot(w-\bar{w})]\,dt\,dx\,dy\\
&+\varepsilon^2\tau\int_0^T\iint \partial_t[\nabla_w^2\eta(\bar{w})\cdot\partial_x\bar{w}\cdot(m-\frac{A_1(\bar{w})}{\varepsilon}+2a\lambda^2\tau\partial_x\bar{w})]\,dt\,dx\,dy\\
&+\varepsilon^2\tau\int_0^T\iint \partial_t[\nabla_w^2\eta(\bar{w})\cdot\partial_y\bar{w}\cdot(\xi-\frac{A_2(\bar{w})}{\varepsilon}+2a\lambda^2\tau\partial_y\bar{w})]\,dt\,dx\,dy + O(\varepsilon^3)
\end{aligned}$$

$$= \int_0^T \iint \Big[\nabla_w^2 \eta(\bar{w}) \cdot [\partial_x \frac{A_1(\bar{w})}{\varepsilon} + \partial_y \frac{A_2(\bar{w})}{\varepsilon} - 2a\lambda^2 \tau \partial_{xx} \bar{w} - 2a\lambda^2 \tau \partial_{yy} \bar{w}] \cdot (w - \bar{w})$$

$$+ \nabla_w \eta(\bar{w}) \cdot [\partial_x m + \partial_y \xi - \partial_x \frac{A_1(\bar{w})}{\varepsilon} - \partial_y \frac{A_2(\bar{w})}{\varepsilon} + 2a\lambda^2 \tau \partial_{xx} \bar{w} + 2a\lambda^2 \partial_{yy} \bar{w}]$$

$$+ \tau \nabla_w^2 \eta(\bar{w}) \cdot \partial_x \bar{w} \cdot (\varepsilon^2 \partial_t m + \lambda^2 \partial_x k) - \tau \lambda^2 \nabla_w^2 \eta(\bar{w}) \cdot \partial_x \bar{w} \cdot \partial_x k$$

$$+ \tau \nabla_w^2 \eta(\bar{w}) \cdot \partial_y \bar{w} \cdot (\varepsilon^2 \partial_t \xi + \lambda^2 \partial_y h) - \tau \lambda^2 \nabla_w^2 \eta(\bar{w}) \cdot \partial_y \bar{w} \cdot \partial_y h \Big] \, dt \, dx \, dy$$

$$+ O(\varepsilon^3).$$

It remains to deal with the last term.

$$I_4 = -\int_0^T \iint \frac{\lambda}{\varepsilon} \partial_x [\nabla_{f_1} \mathcal{H}_1(\overline{\mathcal{M}}_1)(f_1 - \overline{\mathcal{M}}_1) - \nabla_{f_3} \mathcal{H}_3(\overline{\mathcal{M}}_3)(f_3 - \overline{\mathcal{M}}_3)] \, dt \, dx \, dy$$

$$- \int_0^T \iint \frac{\lambda}{\varepsilon} \partial_y [\nabla_{f_2} \mathcal{H}_2(\overline{\mathcal{M}}_2)(f_2 - \overline{\mathcal{M}}_2) - \nabla_{f_4} \mathcal{H}_4(\overline{\mathcal{M}}_4)(f_4 - \overline{\mathcal{M}}_4)] \, dt \, dx \, dy$$

$$= -\int_0^T \iint \frac{\lambda}{\varepsilon} \partial_x \Big[(\nabla_w \eta(\bar{w}) - \varepsilon \lambda \tau \nabla_w^2 \eta(\bar{w}) \partial_x \bar{w})(f_1 - \overline{\mathcal{M}}_1)$$

$$- (\nabla_w \eta(\bar{w}) + \varepsilon \lambda \tau \nabla_w^2 \eta(\bar{w}) \partial_x \bar{w})(f_3 - \overline{\mathcal{M}}_3) \Big] \, dt \, dx \, dy$$

$$- \int_0^T \iint \frac{\lambda}{\varepsilon} \partial_y \Big[(\nabla_w \eta(\bar{w}) - \varepsilon \lambda \tau \nabla_w^2 \eta(\bar{w}) \partial_y \bar{w})(f_2 - \overline{\mathcal{M}}_2)$$

$$- (\nabla_w \eta(\bar{w}) + \varepsilon \lambda \tau \nabla_w^2 \eta(\bar{w}) \partial_y \bar{w})(f_4 - \overline{\mathcal{M}}_4) \Big] \, dt \, dx \, dy + O(\varepsilon^3)$$

$$= -\int_0^T \iint \frac{\lambda}{\varepsilon} \partial_x [\nabla_w \eta(\bar{w}) \cdot ((f_1 - f_3) - (\overline{\mathcal{M}}_1 - \overline{\mathcal{M}}_3))] \, dt \, dx \, dy$$

$$- \int_0^T \iint \frac{\lambda}{\varepsilon} \partial_y [\nabla_w \eta(\bar{w}) \cdot ((f_2 - f_4) - (\overline{\mathcal{M}}_2 - \overline{\mathcal{M}}_4))] \, dt \, dx \, dy$$

$$+ \lambda^2 \tau \int_0^T \iint \partial_x [\nabla_w^2 \eta(\bar{w}) \cdot \partial_x \bar{w} \cdot (f_1 + f_3 - (\overline{\mathcal{M}}_1 + \overline{\mathcal{M}}_3))] \, dt \, dx \, dy$$

$$+ \lambda^2 \tau \int_0^T \iint \partial_y [\nabla_w^2 \eta(\bar{w}) \cdot \partial_y \bar{w} \cdot (f_2 + f_4 - (\overline{\mathcal{M}}_2 + \overline{\mathcal{M}}_4))] \, dt \, dx \, dy + O(\varepsilon^3)$$

$$= -\int_0^T \iint \partial_x [\nabla_w \eta(\bar{w}) \cdot (m - \frac{A_1(\bar{w})}{\varepsilon} + 2a\tau \lambda^2 \partial_x \bar{w})] \, dt \, dx \, dy$$

$$- \int_0^T \iint \partial_y [\nabla_w \eta(\bar{w}) \cdot (\xi - \frac{A_2(\bar{w})}{\varepsilon} + 2a\tau\lambda^2 \partial_y \bar{w})] \, dt \, dx \, dy$$

$$+ \lambda^2 \tau \int_0^T \iint \partial_x [\nabla_w^2 \eta(\bar{w}) \cdot \partial_x \bar{w} \cdot (k - 2a\bar{w})] \, dt \, dx \, dy$$

$$+ \lambda^2 \tau \int_0^T \iint \partial_y [\nabla_w^2 \eta(\bar{w}) \cdot \partial_y \bar{w} \cdot (h - 2a\bar{w})] \, dt \, dx \, dy + O(\varepsilon^3)$$

$$= - \iint \int_0^T \nabla_w \eta(\bar{w}) \cdot [\partial_x m + \partial_y \xi - \frac{A_1(\bar{w})}{\varepsilon} - \partial_y \frac{A_2(\bar{w})}{\varepsilon} + 2a\tau\lambda^2 \partial_{xx} \bar{w} + 2a\tau\lambda^2 \partial_{yy} \bar{w}] \, dt \, dx \, dy$$

$$- \int_0^T \iint \nabla_w^2 \eta(\bar{w}) \cdot \partial_x \bar{w} \cdot (m - \frac{A_1(w)}{\varepsilon}) \, dt \, dx \, dy$$

$$- \int_0^T \iint \nabla_w^2 \eta(\bar{w}) \cdot \partial_x \bar{w} \cdot (\frac{A_1(w)}{\varepsilon} - \frac{A_1(\bar{w})}{\varepsilon}) \, dt \, dx \, dy$$

$$- 2a\tau\lambda^2 \int_0^T \iint \nabla_w^2 \eta(\bar{w}) \cdot \partial_x \bar{w} \cdot \partial_x \bar{w} \, dt \, dx \, dy$$

$$- \int_0^T \iint \nabla_w^2 \eta(\bar{w}) \cdot \partial_y \bar{w} \cdot (\xi - \frac{A_2(w)}{\varepsilon}) \, dt \, dx \, dy$$

$$- \int_0^T \iint \nabla_w^2 \eta(\bar{w}) \cdot \partial_y \bar{w} \cdot (\frac{A_2(w)}{\varepsilon} - \frac{A_2(\bar{w})}{\varepsilon}) \, dt \, dx \, dy$$

$$- 2a\tau\lambda^2 \int_0^T \iint \nabla_w^2 \eta(\bar{w}) \cdot \partial_y \bar{w} \cdot \partial_y \bar{w} \, dt \, dx \, dy$$

$$+ \lambda^2 \tau \int_0^T \iint \nabla_w^2 \eta(\bar{w}) \cdot \partial_x \bar{w} \cdot (\partial_x k - 2a\partial_x \bar{w}) + \nabla_w^2 \eta(\bar{w}) \cdot \partial_{xx} \bar{w} \cdot (k - 2a\bar{w}) \, dt \, dx \, dy$$

$$+ \lambda^2 \tau \int_0^T \iint \nabla_w^2 \eta(\bar{w}) \cdot \partial_y \bar{w} \cdot (\partial_y h - 2a\partial_y \bar{w}) + \nabla_w^2 \eta(\bar{w}) \cdot \partial_{yy} \bar{w} \cdot (h - 2a\bar{w}) \, dt \, dx \, dy$$

$$+ \lambda^2 \tau \int_0^T \iint \nabla_w^3 \eta(\bar{w}) (\partial_x \bar{w})^2 (k - 2a\bar{w}) + \nabla_w^3 \eta(\bar{w}) (\partial_y \bar{w})^2 (h - 2a\bar{w}) \, dt \, dx \, dy + O(\varepsilon^3)$$

$$
\begin{aligned}
&= -\int_0^T \iint \nabla_w \eta(\bar{w}) \cdot [\partial_x m + \partial_y \xi - \frac{A_1(\bar{w})}{\varepsilon} - \partial_y \frac{A_2(\bar{w})}{\varepsilon}]\, dt\, dx\, dy \\
&- 2a\tau\lambda^2 \int_0^T \iint \partial_{xx}\bar{w} + \partial_{yy}\bar{w}\, dt\, dx\, dy \\
&+ \int_0^T \iint \nabla_w^2 \eta(\bar{w}) \cdot \partial_x \bar{w} \cdot \tau(\varepsilon^2 \partial_t m + \lambda^2 \partial_x k)\, dt\, dx\, dy \\
&- \int_0^T \iint \nabla_w^2 \eta(\bar{w}) \cdot \partial_x \bar{w} \cdot (\frac{A_1(w)}{\varepsilon} - \frac{A_1(\bar{w})}{\varepsilon})\, dt\, dx\, dy \\
&- 4a\tau\lambda^2 \int_0^T \iint \nabla_w^2 \eta(\bar{w}) \cdot \partial_x \bar{w} \cdot \partial_x \bar{w}\, dt\, dx\, dy \\
&+ \int_0^T \iint \nabla_w^2 \eta(\bar{w}) \cdot \partial_y \bar{w} \cdot \tau(\varepsilon^2 \partial_t \xi + \lambda^2 \partial_y h)\, dt\, dx\, dy \\
&- \int_0^T \iint \nabla_w^2 \eta(\bar{w}) \cdot \partial_y \bar{w} \cdot (\frac{A_2(w)}{\varepsilon} - \frac{A_2(\bar{w})}{\varepsilon})\, dt\, dx\, dy \\
&- 4a\tau\lambda^2 \int_0^T \iint \nabla_w^2 \eta(\bar{w}) \cdot \partial_y \bar{w} \cdot \partial_y \bar{w}\, dt\, dx\, dy \\
&+ \lambda^2\tau \int_0^T \iint \nabla_w^2 \eta(\bar{w}) \cdot \partial_x \bar{w} \cdot \partial_x k + \nabla_w^2 \eta(\bar{w}) \cdot \partial_y \bar{w} \cdot \partial_y h\, dt\, dx\, dy \\
&+ \lambda^2\tau \int_0^T \iint \nabla_w^2 \eta(\bar{w}) \cdot \partial_{xx} \bar{w} \cdot (k - 2aw) + 2a\nabla_w^2 \eta(\bar{w}) \cdot \partial_{xx} \bar{w} \cdot (w - \bar{w})\, dt\, dx\, dy \\
&+ \lambda^2\tau \int_0^T \iint \nabla_w^2 \eta(\bar{w}) \cdot \partial_{yy} \bar{w} \cdot (h - 2aw) + 2a\nabla_w^2 \eta(\bar{w}) \cdot \partial_{yy} \bar{w} \cdot (w - \bar{w})\, dt\, dx\, dy \\
&+ \lambda^2\tau \int_0^T \iint \nabla_w^3 \eta(\bar{w})[(\partial_x \bar{w})^2(k - 2a\bar{w}) + (\partial_y \bar{w})^2(h - 2a\bar{w})]\, dt\, dx\, dy + O(\varepsilon^3).
\end{aligned}
$$

As an intermediate step, let us look at the sum

$$
\begin{aligned}
I_3 + I_4 = 2\tau \int_0^T \iint \nabla_w^2 \eta(\bar{w}) \cdot \partial_x \bar{w} \cdot (\varepsilon^2 \partial_t m + \lambda^2 \partial_x k)\, dt\, dx\, dy \\
+ 2\tau \int_0^T \iint \nabla_w^2 \eta(\bar{w}) \cdot \partial_y \bar{w} \cdot (\varepsilon^2 \partial_t \xi + \lambda^2 \partial_y h)\, dt\, dx\, dy
\end{aligned}
$$

$$
\begin{aligned}
&-\int_0^T \iint \nabla_w^2\eta(\bar{w})\cdot\partial_x\bar{w}\cdot\Big(\frac{A_1(w)}{\varepsilon}-\frac{A_1(\bar{w})}{\varepsilon}-\frac{A_1'(\bar{w})}{\varepsilon}(w-\bar{w})\Big)\,dt\,dx\,dy\\
&-\int_0^T \iint \nabla_w^2\eta(\bar{w})\cdot\partial_y\bar{w}\cdot\Big(\frac{A_2(w)}{\varepsilon}-\frac{A_2(\bar{w})}{\varepsilon}-\frac{A_2'(\bar{w})}{\varepsilon}(w-\bar{w})\Big)\,dt\,dx\,dy\\
&+\lambda^2\tau\int_0^T \iint \nabla_w^2\eta(\bar{w})\cdot[\partial_{xx}\bar{w}\cdot(k-2aw)+\partial_{yy}\bar{w}\cdot(h-2aw)]\,dt\,dx\,dy\\
&-4a\tau\lambda^2\int_0^T \iint \nabla_w^2\eta(\bar{w})\cdot[\partial_x\bar{w}\cdot\partial_x\bar{w}+\partial_y\bar{w}\cdot\partial_y\bar{w}]\,dt\,dx\,dy\\
&+\lambda^2\tau\int_0^T \iint \nabla_w^3\eta(\bar{w})\cdot[(\partial_x\bar{w})^2(k-2a\bar{w})+(\partial_y\bar{w})^2(h-2a\bar{w})]\,dt\,dx\,dy\\
&+O(\varepsilon^3).
\end{aligned}
$$

We analyse each line separately.

- The first one can be written as

$$
\frac{\tau}{2a\lambda^2}\int_0^T \iint \nabla_w^2\eta(\bar{w})\cdot(\varepsilon^2\partial_t\bar{m}+\lambda^2\partial_x\bar{k})\cdot(\varepsilon^2\partial_t m+\lambda^2\partial_x k)\,dt\,dx\,dy+O(\varepsilon^3).
$$

- Similarly for the second line.
- The third/fourth lines are equivalent to

$$
|\nabla_w^2\eta(\bar{w})|_{L_t^\infty L_x^\infty}\int_0^T \iint |w-\bar{w}|^2\,dt\,dx\,dy+O(\varepsilon^3).
$$

- The fifth line can estimated by

$$
\begin{aligned}
&c_1(|\nabla_w^2\eta(\bar{w})|_{L_t^\infty L_x^\infty})\int_0^T \iint \varepsilon^2|\partial_{xx}\bar{w}|^2+\varepsilon^2|\partial_{yy}\bar{w}|^2\,dt\,dx\,dy\\
&+c_2(|\nabla_w^2\eta(\bar{w})|_{L_t^\infty L_x^\infty})\int_0^T \iint \frac{|k-2aw|^2}{\varepsilon^2}+\frac{|h-2aw|^2}{\varepsilon^2}\,dt\,dx\,dy,
\end{aligned}
$$

  where the first term is $O(\varepsilon^4)$, while the second one is absorbed by the dissipation in I_1.
- The sixth term can be written as

$$
\begin{aligned}
&-\frac{\tau}{a\lambda^2}\int_0^T \iint \nabla_w^2\eta(\bar{w})\cdot(\varepsilon^2\partial_t\bar{m}+\lambda^2\partial_x\bar{k})\cdot(\varepsilon^2\partial_t\bar{m}+\lambda^2\partial_x\bar{k})\,dt\,dx\,dy\\
&-\frac{\tau}{a\lambda^2}\int_0^T \iint \nabla_w^2\eta(\bar{w})\cdot(\varepsilon^2\partial_t\bar{\xi}+\lambda^2\partial_y\bar{h})\cdot(\varepsilon^2\partial_t\bar{\xi}+\lambda^2\partial_y\bar{h})\,dt\,dx\,dy+O(\varepsilon^3).
\end{aligned}
$$

- The last term presents the following form

$$\lambda^2\tau\int_0^T\iint \nabla^3_w\eta(\bar w)(\partial_x\bar w)^2(k-2a\bar w)+\nabla^3_w\eta(\bar w)(\partial_y\bar w)^2(h-2a\bar w)\,dt\,dx\,dy$$

$$=\lambda^2\tau\int_0^T\iint \nabla^3_w\eta(\bar w)(\partial_x\bar w)^2(k-2aw)-2a\nabla^3_w\eta(\bar w)(\partial_x\bar w)^2(\bar w-w)\,dt\,dx\,dy$$

$$+\lambda^2\tau\int_0^T\iint \nabla^3_w\eta(\bar w)(\partial_y\bar w)^2(h-2aw)-2a\nabla^3_w\eta(\bar w)(\partial_y\bar w)^2(\bar w-w)\,dt\,dx\,dy$$

$$\le c(|\nabla^2_w\eta(\bar w)|_{L^\infty_t L^\infty_x})\int_0^T\iint |w-\bar w|^2+\frac{|k-2aw|^2}{\varepsilon^2}+\frac{|h-2aw|^2}{\varepsilon^2}\,dt\,dx\,dy$$

$$+O(\varepsilon^3),$$

where the right-hand side is controlled by using the dissipation coming from I_1.

Remark 3.4. Denoting by $\mu_i(\nabla^2_w\eta(w))$, $\mu_i(\nabla^2_w\eta(\bar w))$ the eigenvalues of $\nabla^2_w\eta(w)$, $\nabla^2_w\eta(\bar w)$ respectively, by simple computations one gets that

$$\int_0^T |\mu_i(\nabla^2_w\eta(w(t)))-\mu_i(\nabla^2_w\eta(\bar w(t)))|_\infty\,dt\le c\int_0^T|\rho(t)-\bar\rho|_\infty\,dt=O(\varepsilon^2),$$

where the last equality follows from Lemma 2.5. Thus, we can write

$$\frac{1}{2a\lambda^2}\int_0^T\iint(\nabla^2_w\eta(w)\cdot(\varepsilon^2\partial_t m+\lambda^2\partial_x k))\cdot(\varepsilon^2\partial_t m+\lambda^2\partial_x k)\,dt\,dx\,dy$$

$$+\frac{1}{2a\lambda^2}\int_0^T\iint(\nabla^2_w\eta(w)\cdot(\varepsilon^2\partial_t\xi+\lambda^2\partial_y h))\cdot(\varepsilon^2\partial_t\xi+\lambda^2\partial_y h)\,dt\,dx\,dy$$

$$=\frac{1}{2a\lambda^2}\int_0^T\iint(\nabla^2_w\eta(\bar w)\cdot(\varepsilon^2\partial_t m+\lambda^2\partial_x k))\cdot(\varepsilon^2\partial_t m+\lambda^2\partial_x k)\,dt\,dx\,dy$$

$$+\frac{1}{2a\lambda^2}\int_0^T\iint(\nabla^2_w\eta(\bar w)\cdot(\varepsilon^2\partial_t\xi+\lambda^2\partial_y h))\cdot(\varepsilon^2\partial_t\xi+\lambda^2\partial_y h)\,dt\,dx\,dy+O(\varepsilon^3).$$

Now we consider the total sum, given by

$$I_1+I_2+I_3+I_4\le|\nabla^2_w\eta(\bar w)|_{L^\infty_t L^\infty_x}\int_0^T\iint|w-\bar w|^2\,dt\,dx\,dy$$

$$-\frac{\tau}{2a\lambda^2}\int_0^T\iint\nabla^2_w\eta(\bar w)\cdot(\varepsilon^2\partial_t m+\lambda^2\partial_x k)\cdot(\varepsilon^2\partial_t m+\lambda^2\partial_x k)\,dt\,dx\,dy$$

$$
\begin{aligned}
&- \frac{\tau}{2a\lambda^2}\int_0^T\iint \nabla_w^2\eta(\bar w)\cdot(\varepsilon^2\partial_t\xi+\lambda^2\partial_y h)\cdot(\varepsilon^2\partial_t\xi+\lambda^2\partial_y h)\,dt\,dx\,dy\\
&+ \frac{\tau}{a\lambda^2}\int_0^T\iint \nabla_w^2\eta(\bar w)\cdot(\varepsilon^2\partial_t m+\lambda^2\partial_x k)\cdot(\varepsilon^2\partial_t \bar m+\lambda^2\partial_x \bar k)\,dt\,dx\,dy\\
&+ \frac{\tau}{a\lambda^2}\int_0^T\iint \nabla_w^2\eta(\bar w)\cdot(\varepsilon^2\partial_t\xi+\lambda^2\partial_y h)\cdot(\varepsilon^2\partial_t\bar\xi+\lambda^2\partial_y \bar h)\,dt\,dx\,dy\\
&- \frac{c(|\nabla_w^2\eta(w)|_{L_t^\infty L_x^\infty}(1-\frac{1}{\delta}))}{2a\tau\varepsilon^2}\int_0^T\iint |k-2aw|^2+|h-2aw|^2\,dt\,dx\,dy\\
&- \frac{c(|\nabla_w^2\eta(w)|_{L_t^\infty L_x^\infty})}{(1-4a)\tau\varepsilon^2}\int_0^T\iint |4aw-(k+h)|^2\,dt\,dx\,dy+O(\varepsilon^3).
\end{aligned}
$$

The Gronwall inequality, together with the definition of w in (4), yields the following estimate

$$
\sup_{t\in[0,T^*]}\frac{\|\rho(t)-\bar\rho\|_0}{\varepsilon}+\|\rho\mathbf{u}(t)-\bar\rho\bar{\mathbf{u}}(t)\|_0\le c\sqrt{\varepsilon}, \tag{38}
$$

where the local time T^* is defined in (24). □

Proof of Theorem 3.1. We start by using the interpolation property of the Sobolev spaces, see [41], for $0<s'<s$ and $t\in[0,T^*]$, which gives

$$
\begin{aligned}
\|\rho\mathbf{u}(t)-\bar\rho\bar{\mathbf{u}}(t)\|_{s'} &\le \|\rho\mathbf{u}(t)-\bar\rho\bar{\mathbf{u}}(t)\|_0^{1-s'/s}\|\rho\mathbf{u}(t)-\bar\rho\bar{\mathbf{u}}(t)\|_s^{s'/s}\\
&\le c\varepsilon^{\frac{s-s'}{2s}}(M_0+cM_0e^{c(|\rho-\bar\rho|_{L_t^\infty L_x^\infty}/\varepsilon,\,|\mathbf{u}|_{L_t^\infty L_x^\infty})t})^{s'/s},
\end{aligned}
\tag{39}
$$

where the last inequality follows by

- the H^s-bound of the solution to the incompressible Navier-Stokes equations on the two-dimensional torus, i.e.

$$
\|\bar\rho\bar{\mathbf{u}}(t)\|_s\le\|\bar\rho\mathbf{u}_0\|_s\le cM_0;
$$

- the Gronwall inequality applied to estimate (18),

$$
\|\rho\mathbf{u}(t)\|_s\le cM_0e^{c(|\rho-\bar\rho|_{L_t^\infty L_x^\infty}/\varepsilon,\,|\mathbf{u}|_{L_t^\infty L_x^\infty})t}.
$$

Taking s' big enough, the Sobolev embedding theorem yields

$$
\begin{aligned}
|\rho\mathbf{u}(t)-\bar\rho\bar{\mathbf{u}}(t)|_\infty &\le c_S\|\rho\mathbf{u}(t)-\bar\rho\bar{\mathbf{u}}(t)\|_{s'}\\
&\le c\varepsilon^{\frac{s-s'}{2s}}(M_0+cM_0e^{c(|\rho-\bar\rho|_{L_t^\infty L_x^\infty}/\varepsilon,\,|\mathbf{u}|_{L_t^\infty L_x^\infty})t})^{s'/s},
\end{aligned}
\tag{40}
$$

and so

$$
|\rho\mathbf{u}(t)|_\infty\le M_0+c\varepsilon^{\frac{s-s'}{2s}}(M_0+cM_0e^{c(|\rho-\bar\rho|_{L_t^\infty L_x^\infty}/\varepsilon,\,|\mathbf{u}|_{L_t^\infty L_x^\infty})t})^{s'/s}.
$$

Similarly,

$$\begin{aligned} |\rho(t)-\bar\rho|_\infty &\le c_S\|\rho(t)-\bar\rho\|_{s'} \\ &\le c_S\|\rho(t)-\bar\rho\|_0^{1-s'/s}\|\rho(t)-\bar\rho\|_s^{s'/s} \\ &\le c\varepsilon^{\frac{3(s-s')}{2s}}(c\varepsilon M_0 e^{c(|\rho-\bar\rho|_{L^\infty_t L^\infty_x}/\varepsilon,\, |\mathbf{u}|_{L^\infty_t L^\infty_x})t})^{s'/s}, \end{aligned} \tag{41}$$

i.e.

$$\begin{cases} \rho(t,x) \ge \bar\rho - c\varepsilon^{\frac{3(s-s')}{2s}}(c\varepsilon M_0 e^{c(|\rho-\bar\rho|_{L^\infty_t L^\infty_x}/\varepsilon,\, |\mathbf{u}|_{L^\infty_t L^\infty_x})t})^{s'/s}, \\ \rho(t,x) \le \bar\rho + c\varepsilon^{\frac{3(s-s')}{2s}}(c\varepsilon M_0 e^{c(|\rho-\bar\rho|_{L^\infty_t L^\infty_x}/\varepsilon,\, |\mathbf{u}|_{L^\infty_t L^\infty_x})t})^{s'/s}, \end{cases}$$

and

$$|\rho(t)|_\infty \le \bar\rho + c\varepsilon^{\frac{3(s-s')}{2s}}(c\varepsilon M_0 e^{c(|\rho-\bar\rho|_{L^\infty_t L^\infty_x}/\varepsilon,\, |\mathbf{u}|_{L^\infty_t L^\infty_x})t})^{s'/s}. \tag{42}$$

Now, since

$$\mathbf{u}-\bar{\mathbf{u}} = \frac{1}{\rho}(\rho\mathbf{u}-\bar\rho\bar{\mathbf{u}}) + \frac{\bar{\mathbf{u}}}{\rho}(\bar\rho-\rho),$$

then from (41)-(40)-(41),

$$\begin{aligned} |\mathbf{u}(t)-\bar{\mathbf{u}}(t)|_\infty \le \frac{1}{\rho}\Bigg(& c\varepsilon^{\frac{s-s'}{2s}}(M_0 + cM_0 e^{c(|\rho-\bar\rho|_{L^\infty_t L^\infty_x}/\varepsilon,\, |\mathbf{u}|_{L^\infty_t L^\infty_x})t})^{s'/s} \\ & + c\varepsilon^{\frac{3(s-s')}{2s}}(c\varepsilon M_0 e^{c(|\rho-\bar\rho|_{L^\infty_t L^\infty_x}/\varepsilon,\, |\mathbf{u}|_{L^\infty_t L^\infty_x})t})^{s'/s}\Bigg), \end{aligned} \tag{43}$$

i.e.,

$$\begin{aligned} |\mathbf{u}(t)|_\infty \le \frac{1}{\bar\rho}\Bigg(& M_0 + c\varepsilon^{\frac{s-s'}{2s}}(M_0 + cM_0 e^{c(|\rho-\bar\rho|_{L^\infty_t L^\infty_x}/\varepsilon,\, |\mathbf{u}|_{L^\infty_t L^\infty_x})t})^{s'/s} \\ & + c\varepsilon^{\frac{3(s-s')}{2s}}(c\varepsilon M_0 e^{c(|\rho-\bar\rho|_{L^\infty_t L^\infty_x}/\varepsilon,\, |\mathbf{u}|_{L^\infty_t L^\infty_x})t})^{s'/s}\Bigg). \end{aligned} \tag{44}$$

Recalling the definition of T^* in (24) and taking $M = 4M_0$, estimate (44) implies that there exists ε_0 fixed such that, for $\varepsilon \le \varepsilon_0$ and $t \le T^*$,

$$|\mathbf{u}(t)|_\infty \le \frac{1}{\bar\rho}(M_0 + c\varepsilon^{\frac12-\delta}) < 2M_0/\bar\rho = \frac{M}{2\bar\rho},$$

for $0 < \delta = \frac{s'}{2s} < \frac12$. Similarly, for $t \le T^*$,

$$\frac{|\rho(t)-\bar\rho|_\infty}{\varepsilon} + |\rho\mathbf{u}(t)|_\infty \le M_0 + c\varepsilon^{\frac12-\delta} < 2M_0 = \frac{M}{2}. \tag{45}$$

Now let us assume $T^* < T^\varepsilon$. Then, by definition (24),

$$\frac{|\rho(T^*)-\bar\rho|_\infty}{\varepsilon} + |\rho\mathbf{u}(T^*)|_\infty = 4M_0 = M.$$

On the other hand, estimate (45) implies that there exists a fixed ε_0, depending on M_0 and small enough such that, for $\varepsilon \le \varepsilon_0$,

$$\frac{|\rho(T^*) - \bar{\rho}|_\infty}{\varepsilon} + |\rho \mathbf{u}(T^*)|_\infty \le M_0 + c\varepsilon^{\frac{1}{2}-\delta} < 2M_0.$$

Now, by contradiction one gets that $T^* \ge T^\varepsilon$ for $\varepsilon \le \varepsilon_0$, where T^* is independent of ε. As a consequence, for $\varepsilon \le \varepsilon_0$ the solutions $(\rho^\varepsilon, \mathbf{u}^\varepsilon)$ to the approximating system evaluated in T^ε are bounded. This way, the Continuation Principle, see [33], implies that they are globally bounded in time. Moreover, since the uniform bounds in Lemma 2.5 are based on the $L_t^\infty L_x^\infty$ boundedness of $(\rho^\varepsilon, \mathbf{u}^\varepsilon)$, it turns out that they hold globally in time for $\varepsilon \le \varepsilon_0$. In the end, we proved:

- the global in time existence and uniform boundedness of $(\rho^\varepsilon, \mathbf{u}^\varepsilon)$ in $H^s(\mathbb{T}^2)$ for a fixed $\varepsilon \le \varepsilon_0$ depending on M_0;
- the strong convergence in $[0, T]$, for any $T > 0$, of the solutions $(\rho^\varepsilon, \mathbf{u}^\varepsilon)$ to the approximating system (3) to the solutions $(\bar{\rho}, \bar{\mathbf{u}})$ to the incompressible Navier-Stokes equations in $H^{s'}(\mathbb{T}^2)$, for $0 < s' < s$ and $s > 3$;
- the rate of this strong convergence.

Finally, the convergence to the gradient of the limit incompressible pressure ∇P^{NS} in (1) is discussed in details in [8]. □

Remark 3.5 *(A comment on the 3D case).* About a possible application of the strategy to the three dimensional case in space, except for the explicit construction of the right symmetrizer that provides the conservative-dissipative form in [8] (i.e. a dissipative entropy for the linearized system), which can be rather different in the three dimensional case, the method should be applicable to a 3D vector-BGK system approximating the 3D incompressible Navier-Stokes equations (with the obvious limit on the smoothness of the solution, see [6]) as presented in [15].

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