MASONRY-LIKE MATERIALS WITH BOUNDED COMPRESSIVE STRENGTH

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$$\mathcal{R}_3 = \{\mathbf{E} \in \mathrm{Sym};\ 2(1+\alpha)e_2 + \alpha e_3 + \varepsilon^c < 0,\ (2+3\alpha)e_3 - \alpha\varepsilon^c - (1+\alpha)\varepsilon^t \leq 0,$$

$$(2+3\alpha)e_3 + \varepsilon^c \geq 0\},$$

$$\mathcal{R}_4 = \{\mathbf{E} \in \mathrm{Sym};\ (2+3\alpha)e_3 + \varepsilon^c < 0\},$$

$$\mathcal{R}_5 = \{\mathbf{E} \in \mathrm{Sym};\ 2e_3 + \alpha \operatorname{tr}\mathbf{E} - \varepsilon^t > 0,\ 2(1+\alpha)e_2 + \alpha e_1 - \varepsilon^t \leq 0,$$

$$4(1+\alpha)e_1 + 2\alpha e_3 + \alpha\,\varepsilon^t + (2+\alpha)\varepsilon^c \geq 0\},$$

$$\mathcal{R}_6 = \{\mathbf{E} \in \mathrm{Sym};\ 2(1+\alpha)e_2 + \alpha e_1 - \varepsilon^t > 0,\ (2+3\alpha)e_1 - \varepsilon^t \leq 0,$$

$$(2+3\alpha)e_1 + \alpha\varepsilon^t + (1+\alpha)\varepsilon^c \geq 0\},$$

$$\mathcal{R}_7 = \{\mathbf{E} \in \mathrm{Sym};\ (2+3\alpha)e_1 - \varepsilon^t \geq 0\},$$

$$\mathcal{R}_8 = \{\mathbf{E} \in \mathrm{Sym};\ 2(2+3\alpha)e_2 - \alpha\varepsilon^c - (2+\alpha)\varepsilon^t \geq 0,$$

$$(2+3\alpha)e_1 + \alpha\varepsilon^t + (1+\alpha)\varepsilon^c \leq 0\},$$

$$\mathcal{R}_9 = \{\mathbf{E} \in \mathrm{Sym};\ 2(2+3\alpha)e_2 + \alpha\varepsilon^t + (2+\alpha)\varepsilon^c \leq 0,$$

$$(2+3\alpha)e_3 - \alpha\varepsilon^c - (1+\alpha)\varepsilon^t \geq 0\},$$

$$\mathcal{R}_{10} = \{\mathbf{E} \in \mathrm{Sym};\ 4(1+\alpha)e_1 + 2\alpha e_2 + \alpha\varepsilon^t + (2+\alpha)\varepsilon^c < 0,$$

$$4(1+\alpha)e_3 + 2\alpha e_2 - \alpha\varepsilon^c - (2+\alpha)\varepsilon^t > 0,\ 2(2+3\alpha)e_2 - \alpha\varepsilon^c - (2+\alpha)\varepsilon^t \leq 0,$$

$$2(2+3\alpha)e_2 + \alpha\varepsilon^t + (2+\alpha)\varepsilon^c \geq 0\},$$

where we have put $\alpha = \lambda/\mu$ [1], $\varepsilon^c = \sigma^c/\mu$ and $\varepsilon^t = \sigma^t/\mu$. Moreover, we suppose that the eigenvalues e_1, e_2 and e_3 are ordered in such a way that $e_1 \leq e_2 \leq e_3$. It is easy to prove that in the

[1] In the following we assume $\lambda \geq 0$, so that we have $\alpha \geq 0$.

regions $\mathbb{R}_2$, $\mathbb{R}_6$ and $\mathbb{R}_8$ we have $e_1 < e_2 \leq e_3$ and that in $\mathbb{R}_3, \mathbb{R}_5$ and $\mathbb{R}_9$, $e_1 \leq e_2 < e_3$; finally in $\mathbb{R}_{10}$ the eigenvalues of **E** are distinct.

Solving system (2.5), we obtain the principal components of $\mathbf{E}^t$, $\mathbf{E}^c$ and **T**:

(2.6)$_1$ if $\mathbf{E} \in \mathbb{R}_1$ then

$$
\begin{aligned}
a_1 &= 0, \\
a_2 &= 0, \\
a_3 &= 0, \\
\\
b_1 &= 0, \\
b_2 &= 0, \\
b_3 &= 0, \\
\\
t_1 &= \mu[(2+\alpha)e_1 + \alpha(e_2+e_3)], \\
t_2 &= \mu[(2+\alpha)e_2 + \alpha(e_1+e_3)], \\
t_3 &= \mu[(2+\alpha)e_3 + \alpha(e_1+e_2)];
\end{aligned}
$$

(2.6)$_2$ if $\mathbf{E} \in \mathbb{R}_2$ then

$$
\begin{aligned}
a_1 &= 0, \\
a_2 &= 0, \\
a_3 &= 0, \\
\\
b_1 &= e_1 + \frac{\alpha}{2+\alpha}(e_2+e_3) + \frac{\varepsilon^c}{2+\alpha}, \\
b_2 &= 0, \\
b_3 &= 0, \\
\\
t_1 &= -\sigma^c, \\
t_2 &= \mu\left\{2e_2 + \frac{\alpha}{2+\alpha}\left[2(e_2+e_3) - \varepsilon^c\right]\right\}, \\
t_3 &= \mu\left\{2e_3 + \frac{\alpha}{2+\alpha}\left[2(e_2+e_3) - \varepsilon^c\right]\right\};
\end{aligned}
$$

$$a_1 = 0,\quad a_2 = 0,\quad a_3 = 0,$$

(2.6)$_3$ if $\mathbf{E} \in \mathbb{R}_3$ then

$$b_1 = e_1 + \frac{\alpha}{2(1+\alpha)}\, e_3 + \frac{\varepsilon^c}{2(1+\alpha)},$$
$$b_2 = e_2 + \frac{\alpha}{2(1+\alpha)}\, e_3 + \frac{\varepsilon^c}{2(1+\alpha)},$$
$$b_3 = 0,$$

$$t_1 = -\sigma^c,\quad t_2 = -\sigma^c,\quad t_3 = \frac{\mu}{1+\alpha}\left[(2+3\alpha)e_3 - \alpha\varepsilon^c\right];$$

$$a_1 = 0,\quad a_2 = 0,\quad a_3 = 0,$$

(2.6)$_4$ if $\mathbf{E} \in \mathbb{R}_4$ then

$$b_1 = e_1 + \frac{\varepsilon^c}{2+3\alpha},\quad b_2 = e_2 + \frac{\varepsilon^c}{2+3\alpha},\quad b_3 = e_3 + \frac{\varepsilon^c}{2+3\alpha},$$

$$t_1 = -\sigma^c,\quad t_2 = -\sigma^c,\quad t_3 = -\sigma^c;$$

$$a_1 = 0,\quad a_2 = 0,$$
$$a_3 = e_3 + \frac{\alpha}{2+\alpha}\,(e_1 + e_2) - \frac{\varepsilon^t}{2+\alpha},$$

(2.6)$_5$ if $\mathbf{E} \in \mathbb{R}_5$ then

$$b_1 = 0,\quad b_2 = 0,\quad b_3 = 0,$$

$$t_1 = \frac{\mu}{2+\alpha}\left[4(1+\alpha)e_1 + 2\alpha e_2 + \alpha\varepsilon^t\right],$$
$$t_2 = \frac{\mu}{2+\alpha}\left[4(1+\alpha)e_2 + 2\alpha e_1 + \alpha\varepsilon^t\right],$$
$$t_3 = \sigma^t;$$

$$(2.6)_9 \qquad \text{if } \mathbf{E} \in \mathcal{R}_9 \text{ then} \qquad \begin{aligned} a_1 &= 0, \\ a_2 &= 0, \\ a_3 &= e_3 - \frac{\alpha\varepsilon^c}{2+3\alpha} - \frac{(1+\alpha)\varepsilon^t}{2+3\alpha}, \\ \\ b_1 &= e_1 + \frac{(\alpha+2)\varepsilon^c}{2(2+3\alpha)} + \frac{\alpha\varepsilon^t}{2(2+3\alpha)}, \\ b_2 &= e_2 + \frac{(\alpha+2)\varepsilon^c}{2(2+3\alpha)} + \frac{\alpha\varepsilon^t}{2(2+3\alpha)} \\ b_3 &= 0, \\ \\ t_1 &= -\sigma^c, \\ t_2 &= -\sigma^c, \\ t_3 &= \sigma^t; \end{aligned}$$

$$(2.6)_{10} \qquad \text{if } \mathbf{E} \in \mathcal{R}_{10} \text{ then} \qquad \begin{aligned} a_1 &= 0, \\ a_2 &= 0, \\ a_3 &= e_3 + \frac{\alpha}{2(1+\alpha)}\, e_2 - \frac{\alpha+2}{4(1+\alpha)}\, \varepsilon^t - \frac{\alpha}{4(1+\alpha)}\, \varepsilon^c, \\ \\ b_1 &= e_1 + \frac{\alpha}{2(1+\alpha)}\, e_2 + \frac{\alpha+2}{4(1+\alpha)}\, \varepsilon^c + \frac{\alpha}{4(1+\alpha)}\, \varepsilon^t, \\ b_2 &= 0, \\ b_3 &= 0, \\ \\ t_1 &= -\sigma^c, \\ t_2 &= \frac{\mu}{2(1+\alpha)}\left[2(2+3\alpha)e_2 + \alpha(\varepsilon^t - \varepsilon^c)\right], \\ t_3 &= \sigma^t. \end{aligned}$$

Therefore, given a symmetric tensor $\mathbf{E} = \sum_{i=1}^{3} e_i\, \mathbf{q}_i \otimes \mathbf{q}_i$ and having determined the region $\mathcal{R}_k$ to which $\mathbf{E}$ belongs, the solution to the constitutive equation (2.1)-(2.2)-(2.4) is

$$\mathbf{E}^t = \sum_{i=1}^{3} a_i\, \mathbf{q}_i \otimes \mathbf{q}_i, \qquad \mathbf{E}^c = \sum_{i=1}^{3} b_i\, \mathbf{q}_i \otimes \mathbf{q}_i, \qquad \mathbf{T} = \sum_{i=1}^{3} t_i\, \mathbf{q}_i \otimes \mathbf{q}_i,$$

with a_i, b_i and t_i given in $(2.6)_k$.

We shall denote by $\hat{\mathbf{T}}$ the function $\hat{\mathbf{T}}$: Sym $\rightarrow$ Sym which associates to every tensor $\mathbf{E} = \sum_{i=1}^{3} e_i\, \mathbf{q}_i \otimes \mathbf{q}_i$ the stress $\mathbf{T} = \hat{\mathbf{T}}(\mathbf{E}) = \sum_{i=1}^{3} t_i\, \mathbf{q}_i \otimes \mathbf{q}_i$. $\hat{\mathbf{T}}$ is a continuous non-linear, non-injective function, positively homogeneous of degree one [1],

$$\hat{\mathbf{T}}(\alpha\mathbf{E}) = \alpha\, \hat{\mathbf{T}}(\mathbf{E}) \qquad \forall\, \alpha \geq 0, \qquad \forall\, \mathbf{E} \in \mathrm{Sym}$$

and isotropic,

$$\hat{\mathbf{T}}(\mathbf{Q}\mathbf{E}\mathbf{Q}^T) = \mathbf{Q}\hat{\mathbf{T}}(\mathbf{E})\mathbf{Q}^T \qquad \forall\, \mathbf{Q} \in \mathrm{Orth}^{(2)},\ \forall\, \mathbf{E} \in \mathrm{Sym};$$

moreover, we shall prove that $\hat{\mathbf{T}}$ is differentiable in the internal part of every region $\mathcal{R}_i$.
Now we analyse the plane strain and the plane stress separately.
If $\mathbf{E}$ is a plane strain and, in particular, $e_3 = \mathbf{q}_3 \cdot \mathbf{E}\, \mathbf{q}_3 = 0$, then $a_3 = b_3 = 0$ and $t_3 = \frac{\alpha}{2(1+\alpha)}(t_1 + t_2)$. Let us designate $\mathbf{E}$, $\mathbf{E}^t$, $\mathbf{E}^c$ and $\mathbf{T}$ as the restrictions of $\mathbf{E}$, $\mathbf{E}^t$, $\mathbf{E}^c$ and $\mathbf{T}$ to the two-dimensional subspace of $\mathcal{V}$, orthogonal to the vector $\mathbf{q}_3$. Calculation of a_1, a_2, b_1, b_2, t_1 and t_2 which satisfy system (2.5) requires definition of the following sets:

$$\mathcal{S}_1 = \{\mathbf{E} \in \mathrm{Sym};\ \alpha e_1 + (2+\alpha)e_2 - \varepsilon^t \leq 0,\ (2+\alpha)e_1 + \alpha e_2 + \varepsilon^c \geq 0\},$$

$$\mathcal{S}_2 = \{\mathbf{E} \in \mathrm{Sym};\ e_1 > \frac{\varepsilon^t}{2(1+\alpha)}\},$$

$$\mathcal{S}_3 = \{\mathbf{E} \in \mathrm{Sym};\ \alpha e_1 + (2+\alpha)e_2 - \varepsilon^t > 0,\ e_1 \leq \frac{\varepsilon^t}{2(1+\alpha)},\ e_1 \geq -\frac{(2+\alpha)\varepsilon^c + \alpha\varepsilon^t}{4(1+\alpha)}\},$$

$$\mathcal{S}_4 = \{\mathbf{E} \in \mathrm{Sym};\ (2+\alpha)e_1 + \alpha e_2 + \varepsilon^c < 0,\ e_2 \geq -\frac{\varepsilon^c}{2(1+\alpha)},$$
$$e_2 \leq \frac{\alpha\varepsilon^c + (2+\alpha)\varepsilon^t}{4(1+\alpha)}\},$$

$$\mathcal{S}_5 = \{\mathbf{E} \in \mathrm{Sym};\ e_2 < -\frac{\varepsilon^c}{2(1+\alpha)}\},$$

[2] Orth denotes the set of all tensors Q such that $Q^T = Q^{-1}$.

$$\mathcal{S}_6 = \{\mathbf{E} \in \mathrm{Sym};\ e_2 > \frac{\alpha\varepsilon^c + (2+\alpha)\varepsilon^t}{4(1+\alpha)},\ e_1 < -\ \frac{\alpha\varepsilon^t + (2+\alpha)\varepsilon^c}{4(1+\alpha)}\ \}.$$

We still suppose that the eigenvalues e_1 and e_2 of **E** are ordered in such a way that $e_1 \le e_2$. We observe that in $\mathcal{S}_3$, $\mathcal{S}_4$ and $\mathcal{S}_6$ the eigenvalues e_1 and e_2 are distinct.
Regions $\mathcal{S}_1, \ldots, \mathcal{S}_6$ in the e_1-e_2 plane are illustrated in Figure 1.

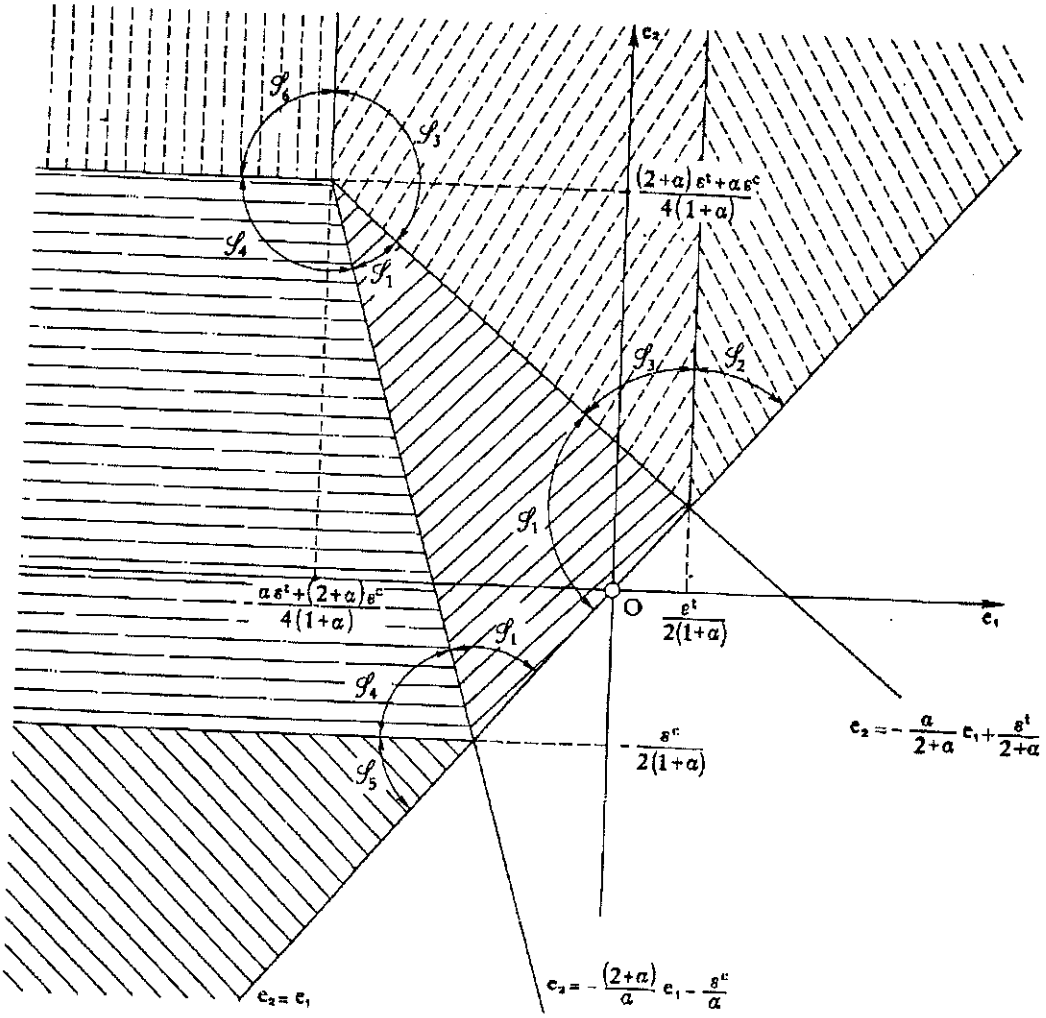

Figure 1. Subdivision of the half-plane $e_1 \le e_2$ into the regions $\mathcal{S}_i$, $i = 1, \ldots, 6$.

The principal components of $\mathbf{E}^t$, $\mathbf{E}^c$ and $\mathbf{T}$ can be calculated from the relations

(2.7)$_1$ if $\mathbf{E}\in \mathcal{S}_1$, then

$$a_1 = 0, \qquad a_2 = 0,$$
$$b_1 = 0, \qquad b_2 = 0,$$
$$t_1 = \mu[(2+\alpha)e_1 + \alpha\, e_2], \qquad t_2 = \mu[(2+\alpha)e_2 + \alpha\, e_1];$$

(2.7)$_2$ if $\mathbf{E}\in \mathcal{S}_2$, then

$$a_1 = e_1 - \frac{\varepsilon^t}{2(1+\alpha)}, \qquad a_2 = e_2 - \frac{\varepsilon^t}{2(1+\alpha)},$$
$$b_1 = 0, \qquad b_2 = 0,$$
$$t_1 = \sigma^t, \qquad t_2 = \sigma^t;$$

(2.7)$_3$ if $\mathbf{E}\in \mathcal{S}_3$, then

$$a_1 = 0, \qquad a_2 = e_2 + \frac{\alpha}{2+\alpha}e_1 - \frac{\varepsilon^t}{2+\alpha},$$
$$b_1 = 0, \qquad b_2 = 0,$$
$$t_1 = \frac{4\mu(1+\alpha)}{2+\alpha}e_1 + \frac{\alpha}{2+\alpha}\sigma^t, \qquad t_2 = \sigma^t;$$

(2.7)$_4$ if $\mathbf{E}\in \mathcal{S}_4$, then

$$a_1 = 0, \qquad a_2 = 0,$$
$$b_1 = e_1 + \frac{\alpha}{2+\alpha}e_2 + \frac{\varepsilon^c}{2+\alpha}, \qquad b_2 = 0,$$
$$t_1 = -\sigma^c, \qquad t_2 = \frac{4\mu(1+\alpha)}{2+\alpha}e_2 - \frac{\alpha}{2+\alpha}\sigma^c;$$

(2.7)$_5$ if $\mathbf{E}\in \mathcal{S}_5$, then

$$a_1 = 0, \qquad a_2 = 0,$$
$$b_1 = e_1 + \frac{\varepsilon^c}{2(1+\alpha)}, \qquad b_2 = e_2 + \frac{\varepsilon^c}{2(1+\alpha)},$$
$$t_1 = -\sigma^c, \qquad t_2 = -\sigma^c;$$

$$a_1 = 0, \qquad a_2 = e_2 - \frac{\alpha\varepsilon^c + (2+\alpha)\varepsilon^t}{4(1+\alpha)},$$

(2.7)$_6$ if $\mathbf{E} \in \mathcal{S}_6$, then $$b_1 = e_1 + \frac{(2+\alpha)\varepsilon^c + \alpha\varepsilon^t}{4(1+\alpha)}, \qquad b_2 = 0,$$

$$t_1 = -\sigma^c, \qquad t_2 = \sigma^t.$$

From the relation $t_3 = \frac{\alpha}{2(1+\alpha)}(t_1 + t_2)$ and from the non-negativeness of α, it follows that the eigenvalue t_3 of $\mathbf{T}$ satisfies the inequalities $-\sigma^c \leq t_3 \leq \sigma^t$ as well.
Now let us consider a plane stress and suppose $t_3 = \mathbf{q}_3 \cdot \mathbf{T}\,\mathbf{q}_3 = 0$. Then a_3 can be set equal to zero and b_3, by virtue of the positiveness of σ^c, must be equal to zero, so that we have $e_3 = \frac{\alpha}{2+\alpha}(a_1 + a_2 + b_1 + b_2 - e_1 - e_2)$. Let us still designate $\mathbf{E}$, $\mathbf{E}^t$, $\mathbf{E}^c$ and $\mathbf{T}$ as the restrictions of $\mathbf{E}$, $\mathbf{E}^t$, $\mathbf{E}^c$ and $\mathbf{T}$ to the two-dimensional subspace of $\mathcal{V}$, orthogonal to the vector $\mathbf{q}_3$. In order to calculate the values of a_1, a_2, b_1, b_2, t_1 and t_2 which satisfy system (2.5) we define the following sets:

$$\mathcal{T}_1 = \{\mathbf{E} \in \mathrm{Sym};\ 2\alpha e_1 + 4(1+\alpha)e_2 - \varepsilon^t(2+\alpha) \leq 0,$$
$$4(1+\alpha)e_1 + 2\alpha e_2 + \varepsilon^c(2+\alpha) \geq 0\},$$

$$\mathcal{T}_2 = \{\mathbf{E} \in \mathrm{Sym};\ e_1 > \frac{(2+\alpha)\varepsilon^t}{2(2+3\alpha)}\},$$

$$\mathcal{T}_3 = \{\mathbf{E} \in \mathrm{Sym};\ 2\alpha e_1 + 4(1+\alpha)e_2 - (2+\alpha)\varepsilon^t > 0,\ e_1 \leq \frac{(2+\alpha)\varepsilon^t}{2(2+3\alpha)},$$
$$e_1 \geq -\frac{2(1+\alpha)\varepsilon^c + \alpha\varepsilon^t}{2(2+3\alpha)}\},$$

$$\mathcal{T}_4 = \{\mathbf{E} \in \mathrm{Sym};\ 4(1+\alpha)e_1 + 2\alpha e_2 + (2+\alpha)\varepsilon^c < 0,\ e_2 \geq -\frac{(2+\alpha)\varepsilon^c}{2(2+3\alpha)},$$
$$e_2 \leq \frac{\alpha\varepsilon^c + 2(1+\alpha)\varepsilon^t}{2(2+3\alpha)}\},$$

$$\mathcal{T}_5 = \{\mathbf{E} \in \mathrm{Sym};\ e_2 < -\ \frac{(2+\alpha)\varepsilon^c}{2(2+3\alpha)}\},$$

$$\mathcal{T}_6 = \{\mathbf{E} \in \mathrm{Sym};\ e_2 > \frac{\alpha\varepsilon^c + 2(1+\alpha)\varepsilon^t}{2(2+3\alpha)},\ e_1 < -\ \frac{\alpha\varepsilon^t + 2(1+\alpha)\varepsilon^c}{2(2+3\alpha)}\}.$$

We observe that in $\mathcal{T}_3$, $\mathcal{T}_4$ and $\mathcal{T}_6$ the eigenvalues e_1 and e_2 are distinct. The principal components of $\mathbf{E}^t$, $\mathbf{E}^c$ and $\mathbf{T}$ can be calculated from the relations

(2.8)$_1$ if $\mathbf{E} \in \mathcal{T}_1$, then

$$a_1 = 0, \qquad a_2 = 0,$$

$$b_1 = 0, \qquad b_2 = 0,$$

$$t_1 = 2\mu\{e_1 + \frac{\alpha}{2+\alpha}(e_1+e_2)\}, \qquad t_2 = 2\mu\{e_2 + \frac{\alpha}{2+\alpha}(e_1+e_2)\};$$

(2.8)$_2$ if $\mathbf{E} \in \mathcal{T}_2$, then

$$a_1 = e_1 - \frac{(2+\alpha)}{2(2+3\alpha)}\varepsilon^t, \qquad a_2 = e_2 - \frac{(2+\alpha)}{2(2+3\alpha)}\varepsilon^t,$$

$$b_1 = 0, \qquad b_2 = 0,$$

$$t_1 = \sigma^t, \qquad t_2 = \sigma^t;$$

(2.8)$_3$ if $\mathbf{E} \in \mathcal{T}_3$, then

$$a_1 = 0, \qquad a_2 = e_2 + \frac{\alpha}{2(1+\alpha)}e_1 - \frac{(2+\alpha)}{2(1+\alpha)}\varepsilon^t,$$

$$b_1 = 0, \qquad b_2 = 0,$$

$$t_1 = \frac{\mu(2+3\alpha)}{1+\alpha}e_1 + \frac{\alpha}{2(1+\alpha)}\sigma^t, \quad t_2 = \sigma^t;$$

$$a_1 = 0, \qquad a_2 = 0,$$

(2.8)$_4$ if $\mathbf{E} \in \mathcal{T}_4$, then $$b_1 = e_1 + \frac{\alpha}{2(1+\alpha)} e_2 + \frac{(2+\alpha)\varepsilon^c}{4(1+\alpha)}, \qquad b_2 = 0,$$

$$t_1 = -\sigma^c, \qquad t_2 = \frac{\mu(2+3\alpha)}{1+\alpha} e_2 - \frac{\alpha}{2(1+\alpha)} \sigma^c;$$

$$a_1 = 0, \qquad a_2 = 0,$$

(2.8)$_5$ if $\mathbf{E} \in \mathcal{T}_5$, then $$b_1 = e_1 + \frac{(2+\alpha)\varepsilon^c}{2(2+3\alpha)}, \qquad b_2 = e_2 + \frac{(2+\alpha)\varepsilon^c}{2(2+3\alpha)},$$

$$t_1 = -\sigma^c, \qquad t_2 = -\sigma^c;$$

$$a_1 = 0, \qquad a_2 = e_2 - \frac{\alpha\varepsilon^c + 2(1+\alpha)\varepsilon^t}{2(2+3\alpha)},$$

(2.8)$_6$ if $\mathbf{E} \in \mathcal{T}_6$, then $$b_1 = e_1 + \frac{2(1+\alpha)\varepsilon^c + \alpha\varepsilon^t}{2(2+3\alpha)}, \qquad b_2 = 0,$$

$$t_1 = -\sigma^c, \qquad t_2 = \sigma^t.$$

III. THE BOUNDARY-VALUE PROBLEM

The equilibrium problem for masonry-like solids (infinitely resistant to compression and with $\sigma^t = 0$) has been studied in recent years and the existence of a solution has been proven solely for a rather restricted class of load conditions [6], [7]. On the other hand, the uniqueness of the solution is guaranteed only in terms of stress, in the sense that different displacement and strain fields can correspond to the same stress field.

Similar considerations can be made for a BCS masonry-like material; in this section we prove that the stress field which satisfies the equilibrium problem for a BCS masonry-like material is unique. To this end, let $\mathcal{B}$ be a solid made up of a BCS material and let $\mathcal{S}_u$ and $\mathcal{S}_f$ be two subsets of the boundary $\partial\mathcal{B}$ of $\mathcal{B}$, such that their union covers $\partial\mathcal{B}$ and their interiors are disjointed.

A load $(\mathbf{b}, \mathbf{s}_0)$ defined in $\mathcal{B} \times \mathcal{S}_f$ with values in $\mathcal{V} \times \mathcal{V}$ is *admissible* if the corresponding boundary-value problem has a solution, *i. e.* if there exists a triple $[\mathbf{u}, \mathbf{E}, \mathbf{T}]$, constituted by a stress field $\mathbf{T}$, a strain field $\mathbf{E}$ and a displacement field $\mathbf{u}$ defined on $\overline{\mathcal{B}}$, piecewise C^2, such that

(3.1)$_1$ $$\mathbf{E} = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right),$$

(3.1)$_2$ $$\mathbf{T} = \hat{\mathbf{T}}(\mathbf{E}) = \mathbb{C}[\mathbf{E} - \mathbf{E}^t - \mathbf{E}^c],$$

(3.1)$_3$ $$\mathbf{u} = \mathbf{0} \text{ on } \partial_u,$$

(3.1)$_4$ $$\mathbf{Tn} = \mathbf{s}_0 \text{ on } \partial_f,$$

(3.1)$_5$ $$\operatorname{div}\mathbf{T} + \mathbf{b} = \mathbf{0} \text{ on } \mathcal{B},$$

where $\mathbf{n}$ is the outward unit normal to ∂_f, $\mathbb{C} = 2\mu\, \mathbb{1} + \lambda\, \mathbf{I} \otimes \mathbf{I}$ is the elasticity tensor and $\mathbf{E}^t$ and $\mathbf{E}^c$ satisfy with $\mathbf{T}$ the constitutive equation (2.1)-(2.4).
It is easy to prove that if $(\mathbf{b}, \mathbf{s}_0)$ is an admissible load and $[\mathbf{u}_1, \mathbf{E}_1, \mathbf{T}_1]$ and $[\mathbf{u}_2, \mathbf{E}_2, \mathbf{T}_2]$ are two solutions to (3.1), then $\mathbf{T}_1(x) = \mathbf{T}_2(x)$ for every $x \in \mathcal{B}$.
In fact, the triple $[\overline{\mathbf{u}}, \overline{\mathbf{E}}, \overline{\mathbf{T}}]$ with $\overline{\mathbf{u}} = \mathbf{u}_1 - \mathbf{u}_2$, $\overline{\mathbf{E}} = \mathbf{E}_1 - \mathbf{E}_2$ and $\overline{\mathbf{T}} = \mathbf{T}_1 - \mathbf{T}_2$ satisfies (3.1)$_1$ and (3.1)$_3$; moreover it satisfies (3.1)$_4$ and (3.1)$_5$ with $\mathbf{s}_0 = \mathbf{0}$ and $\mathbf{b} = \mathbf{0}$. Thus, in agreement with the hypothesis on the smoothness of the solutions, a simple application of the principle of virtual work proves that

(3.2) $$\int_{\mathcal{B}} \overline{\mathbf{T}} \cdot \overline{\mathbf{E}}\, dV = 0.$$

On the other hand,

(3.3) $$\overline{\mathbf{E}} = \overline{\mathbf{E}}^e + \mathbf{E}_1^t + \mathbf{E}_1^c - \mathbf{E}_2^t - \mathbf{E}_2^c,$$

where $\overline{\mathbf{E}}^e = \mathbf{E}_1^e - \mathbf{E}_2^e$, and $\mathbf{E}_1^e$, $\mathbf{E}_1^t$, $\mathbf{E}_1^c$, $\mathbf{E}_2^e$, $\mathbf{E}_2^t$ and $\mathbf{E}_2^c$ are the elastic part, the fracture strain and the crushing strain corresponding to $\mathbf{E}_1$ and $\mathbf{E}_2$, respectively. From (3.2), by using (3.3) we obtain

(3.4) $$\int_{\mathcal{B}} \overline{\mathbf{T}} \cdot \overline{\mathbf{E}}^e\, dV = \int_{\mathcal{B}} (\mathbf{T}_1 - \mathbf{T}_2) \cdot (\mathbf{E}_2^t + \mathbf{E}_2^c - \mathbf{E}_1^t - \mathbf{E}_1^c)\, dV;$$

the first member in (3.4) is equal to $\int_{\mathcal{B}} \overline{\mathbf{T}} \cdot \mathbb{C}^{-1}[\overline{\mathbf{T}}]\, dV$ and then it is non-negative because $\mathbb{C}$ is positive definite. By using (2.4)$_3$, the second member of (3.4) results equal to

$$\int_{\mathcal{B}} [(\mathbf{T}_1 - \sigma^t\mathbf{I}) \cdot \mathbf{E}_2^t + (\mathbf{T}_2 - \sigma^t\mathbf{I}) \cdot \mathbf{E}_1^t + (\mathbf{T}_1 + \sigma^c\mathbf{I}) \cdot \mathbf{E}_2^c + (\mathbf{T}_2 + \sigma^c\mathbf{I}) \cdot \mathbf{E}_1^c]\, dV,$$

which is non-positive by virtue of $(2.1)_2$, $(2.1)_3$, $(2.4)_1$ and $(2.4)_2$. From the equality (3.4) we obtain $\overline{\mathbf{T}} \cdot \mathbb{C}^{-1}[\overline{\mathbf{T}}] = 0$ everywhere in $\mathcal{B}$ and thus $\overline{\mathbf{T}} = 0$, which is the desired result.

In order to solve the equilibrium problems for BCS masonry-like solids by using the finite element method, we are often obliged for numerical reasons, to assign the load incrementally. To this end, although the material being considered is elastic, we must also consider the load processes and incremental equilibrium problem associated with them.

We then intend to prove that the numerical solution obtained by using an incremental procedure is independent of the particular load process chosen; instead, it depends solely on the final assigned load, provided that the load process considered is admissible in the sense specified as follows.

A *load process* $\gamma(\tau)$, $\tau \in [0, \bar{\tau}]$, is a function pair $(\mathbf{b}(x, \tau), \mathbf{s}_0(x, \tau))$ with $\mathbf{b}$ and $\mathbf{s}_0$ defined on $\mathcal{B} \times [0, \bar{\tau}]$ and $\mathcal{S}_t \times [0, \bar{\tau}]$, respectively, differentiable with respect to τ and such that $\gamma(0) = 0$.

Given a process γ, let us suppose that for every τ, $\gamma(\tau) = (\mathbf{b}(x, \tau), \mathbf{s}_0(x, \tau))$ is an admissible load and let $[\mathbf{u}(\tau), \mathbf{E}(\tau), \mathbf{T}(\tau)]$ be a solution to (3.1) with $\mathbf{b} = \mathbf{b}(\tau)$ and $\mathbf{s}_0 = \mathbf{s}_0(\tau)$. A curve $[\mathbf{u}(\tau), \mathbf{E}(\tau), \mathbf{T}(\tau)]$ of solutions to (3.1) is said to be *regular* if it is differentiable with respect to τ.

A load process γ on $[0, \bar{\tau}]$ is *admissible* if, for every $\tau \in [0, \bar{\tau}]$, $\gamma(\tau)$ is an admissible load and if there exists at least one regular curve $[\mathbf{u}(\tau), \mathbf{E}(\tau), \mathbf{T}(\tau)]$ of solutions to (3.1).

Let γ be a load process on $[0, \bar{\tau}]$; a regular curve $[\mathbf{u}(\tau), \mathbf{E}(\tau), \mathbf{T}(\tau)]$ is an *incremental solution* to the boundary-value problem if for each $\tau \in [0, \bar{\tau}]$ we have

$$\dot{\mathbf{E}} = \frac{1}{2}\left(\nabla\dot{\mathbf{u}} + \nabla\dot{\mathbf{u}}^t\right),$$

$$\dot{\mathbf{T}} = D_E\,\hat{\mathbf{T}}(\mathbf{E}(\tau))[\dot{\mathbf{E}}],$$

$$\dot{\mathbf{u}} = \mathbf{0} \ \text{ on } \mathcal{S}_u, \tag{3.5}$$

$$\dot{\mathbf{T}}\,\mathbf{n} = \dot{\mathbf{s}}_0 \ \text{ on } \mathcal{S}_f,$$

$$\operatorname{div}\dot{\mathbf{T}} + \dot{\mathbf{b}} = \mathbf{0} \ \text{ on } \mathcal{B},$$

and

$$\mathbf{u}(x, 0) = \mathbf{0}, \qquad \mathbf{E}(x, 0) = \mathbf{0}, \qquad \mathbf{T}(x, 0) = \mathbf{0} \qquad \text{on } \mathcal{B}, \tag{3.6}$$

where the dot $\cdot$ denotes the derivatives with respect to τ.

It is immediately verifiable that, if γ is an admissible process, then every regular curve of solutions to (3.1) is a solution to (3.5). Moreover, each incremental solution to the boundary-value problem is a regular curve of solutions to (3.1).

In fact, if $[\mathbf{u}(\tau), \mathbf{E}(\tau), \mathbf{T}(\tau)]$ is a regular curve of solutions to (3.1), differentiating (3.1) with respect to τ, we can immediately verify that $[\mathbf{u}, \mathbf{E}, \mathbf{T}]$ satisfies (3.5). On the other hand, if $[\mathbf{u}(\tau), \mathbf{E}(\tau), \mathbf{T}(\tau)]$ is an incremental solution, integrating (3.5) on $[0, \tau]$ and taking into account (3.6), we deduce that $[\mathbf{u}(\tau), \mathbf{E}(\tau), \mathbf{T}(\tau)]$ satisfies (3.1) for each $\tau \in [0, \bar{\tau}]$.

From this result it follows that:

a) if γ is an admissible process, there exists at least one incremental solution to the boundary-value problem;

b) the solution to the incremental problem, if it exists, is unique in terms of stress, *i. e.* if $[\mathbf{u}_1(\tau), \mathbf{E}_1(\tau), \mathbf{T}_1(\tau)]$ and $[\mathbf{u}_2(\tau), \mathrm{E}_2(\tau), \mathbf{T}_2(\tau)]$ are two solutions to (3.5) then

(3.7) $$\mathbf{T}_1(x, \tau) = \mathbf{T}_2(x, \tau), \quad (x, \tau) \in \mathcal{B} \times [0, \bar{\tau}].$$

c) if γ and φ are two admissible processes on $[0, \bar{\tau}]$, such that $\gamma(\bar{\tau}) = \varphi(\bar{\tau})$ and $[\mathbf{u}_1(\tau), \mathrm{E}_1(\tau), \mathbf{T}_1(\tau)]$ and $[\mathbf{u}_2(\tau), \mathrm{E}_2(\tau), \mathbf{T}_2(\tau)]$ are two incremental solutions corresponding to γ and φ respectively, then

(3.8) $$\mathbf{T}_1(x, \bar{\tau}) = \mathbf{T}_2(x, \bar{\tau}) \qquad \text{for each } x \in \mathcal{B}.$$

This last result guarantees that the incremental solution does not depend on the load process at least regarding the stress. In fact, the common value of $\mathbf{T}_1$ and $\mathbf{T}_2$ at the end of the two processes is the solution to the boundary-value problem (3.1) corresponding to the load $\gamma(\bar{\tau}) = \varphi(\bar{\tau})$.

IV. SOME EXPLICIT SOLUTIONS

In this Section we analyse a circular ring and a spherical container made up of a BCS material subjected to uniform radial pressures p_e and p_i acting, respectively, on the outer and inner boundary and we explicitly calculate the stress field at equilibrium with these loads and the corresponding strain and displacement fields that, in this case, are unique. The explicit solutions thus obtained will be compared in section VI with the corresponding numerical results.
In the following, ν and E are respectively the Poisson ratio and the Young modulus of the material. Moreover, we suppose $\sigma^t = 0$, $\sigma^c > 0$ to be fixed, and that p_e and p_i satisfy the compatibility conditions $p_e \leq \sigma^c$ and $p_i \leq \sigma^c$.
A stress field in equilibrium with loads p_e and p_i, satisfying $(2.4)_1$ and $(2.4)_2$ will be said to be *statically admissible.*

The circular ring.
The circular ring Ω shown in Figure 2, having inner radius a and outer radius b, is subjected to a plane strain as a consequence of the action of two uniform radial pressures p_e and p_i acting, respectively, on the outer and inner boundary. Let us choose a cylindrical reference system $\{O, \rho, \theta, z\}$ in which the origin coincides with the centre of the ring and the z axis is orthogonal to its plane.

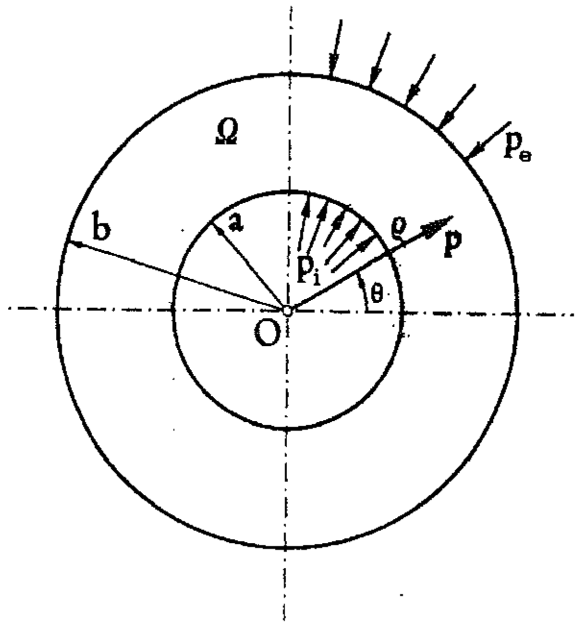

Figure 2. The circular ring.

It is known [8] that if $\frac{p_e}{p_i} \geq \frac{a^2+b^2}{2\,b^2}$, then the stress field $\mathbf{T}^{(e)}$ corresponding to a linear elastic material, having principal components

$$\sigma_\rho^{(e)}(\rho) = \frac{a^2b^2(p_e - p_i)}{b^2 - a^2}\,\frac{1}{\rho^2} + \frac{p_i a^2 - p_e b^2}{b^2 - a^2},$$

$$\sigma_\theta^{(e)}(\rho) = -\,\frac{a^2b^2(p_e - p_i)}{b^2 - a^2}\,\frac{1}{\rho^2} + \frac{p_i a^2 - p_e b^2}{b^2 - a^2}, \tag{4.1}$$

$$\sigma_z^{(e)}(\rho) = \nu\,[\sigma_\rho^{(e)}(\rho) + \sigma_\theta^{(e)}(\rho)] = \frac{2\nu\,(p_i a^2 - p_e b^2)}{b^2 - a^2},$$

is negative semi-definite.

Let us begin by supposing

$$\frac{p_e}{p_i} \geq 1,$$

then, for the circumferential stress, which is a monotonic function of ρ, the inequalities $\sigma_\theta^{(e)}(a) \leq \sigma_\theta^{(e)}(b) \leq -\,p_e$ hold.

If, in particular p_e and p_i are such that the condition

$$\frac{\sigma^c - p_e}{\sigma^c - p_i} \geq \frac{a^2 + b^2}{2\,b^2}$$

is also satisfied, or, equivalently, if $p_e \leq \frac{a^2 + b^2}{2\,b^2} p_i + \frac{b^2 - a^2}{2\,b^2} \sigma^c$, then the stress field $\mathbf{T}^{(e)}$ is statically admissible. On the other hand, if p_e and p_i are such that the inequality

$$\frac{\sigma^c - p_e}{\sigma^c - p_i} \leq \frac{a^2 + b^2}{2\,b^2}$$

holds, then $\sigma_\theta^{(e)}(a) < - \sigma^c$, and $\mathbf{T}^{(e)}$ does not satisfy condition $(2.4)_2$. A statically admissible stress field $\mathbf{T}$ can be obtained by starting from $\mathbf{T}^{(e)}$ and using a procedure similar to that in [8] for a circular ring made up of an elastic non-linear material with bounded tensile strength.

In the attempt to find the solution, we may suppose that $\sigma_\theta(\rho)$ is equal to $- \sigma^c$ in a circular ring $\Omega_1 = \{(\rho, \theta);\ \rho \in [a, \rho_c]\}$, where $\rho_c \in [a, b]$ is unknown. In this region, for equilibrium reasons, σ_ρ has the expression

$$\sigma_\rho(\rho) = \frac{a}{\rho} (\sigma^c - p_i) - \sigma^c.$$

Consequently, the circular ring $\Omega_2 = \{(\rho, \theta);\ \rho \in [\rho_c, b]\}$ is subjected to both external pressure p_e and an internal pressure whose value is $p_c = \sigma^c - \frac{a}{\rho_c} (\sigma^c - p_i)$. Moreover, for continuity reasons, $\sigma_\theta(\rho_c^+) = - \sigma^c$. On the other hand, in Ω_2 the solution coincides with the linear elastic one; thus,

$$\sigma_\theta(\rho_c^+) = \frac{p_c(b^2 + \rho_c^2) - 2p_e b^2}{b^2 - \rho_c^2}$$

and ρ_c is a solution to the algebraic equation

$$a(\sigma^c - p_i)\rho_c^2 - 2b^2(\sigma^c - p_e)\rho_c + ab^2(\sigma^c - p_i) = 0,$$

which, if $\frac{\sigma^c - p_e}{\sigma^c - p_i} \geq \frac{a}{b}$, that is if $p_e \leq \frac{a}{b} p_i + \frac{b - a}{b} \sigma^c$, has in [a, b] the sole root

$$(4.2) \qquad \rho_c = \frac{b}{a} \, \frac{b(\sigma^c - p_e) - \sqrt{b^2(\sigma^c - p_e)^2 - a^2(\sigma^c - p_i)^2}}{\sigma^c - p_i}.$$

It can be seen that when the ratio $\frac{\sigma^c - p_e}{\sigma^c - p_i}$ decreases from $\frac{a^2 + b^2}{2\,b^2}$ to a/b, ρ_c correspondingly varies from a to b. Finally, the stress $\mathbf{T}$ having principal components

$$(4.3)_1 \qquad \sigma_\rho(\rho) = \begin{cases} \dfrac{a}{\rho}\,(\sigma^c - p_i) - \sigma^c, & \rho \in [a, \rho_c], \\[2ex] \dfrac{a}{2}\,(\sigma^c - p_i)\left(\dfrac{\rho_c}{\rho^2} + \dfrac{1}{\rho_c}\right) - \sigma^c, & \rho \in [\rho_c, b]; \end{cases}$$

$$(4.3)_2 \qquad \sigma_\theta(\rho) = \begin{cases} -\,\sigma^c, & \rho \in [a, \rho_c], \\[2ex] \dfrac{a}{2}\,(\sigma^c - p_i)\left(-\dfrac{\rho_c}{\rho^2} + \dfrac{1}{\rho_c}\right) - \sigma^c, & \rho \in [\rho_c, b]. \end{cases}$$

is statically admissible.
In agreement with the constitutive equation (2.1)-(2.2)-(2.4), in Ω_2 the fracture strain and the crushing strain are nil; the total deformation has components

$$(4.4)_1 \qquad \varepsilon_\rho(\rho) = \frac{1+\nu}{2E}\left\{(1-2\nu)\left[(\sigma^c - p_i)\,\frac{a}{\rho_c} - 2\sigma^c\right] + (\sigma^c - p_i)\,\frac{a\rho_c}{\rho^2}\right\}, \qquad \rho \in [\rho_c, b],$$

$$(4.4)_2 \qquad \varepsilon_\theta(\rho) = \frac{1+\nu}{2E}\left\{(1-2\nu)\left[(\sigma^c - p_i)\,\frac{a}{\rho_c} - 2\sigma^c\right] - (\sigma^c - p_i)\,\frac{a\rho_c}{\rho^2}\right\}, \qquad \rho \in [\rho_c, b];$$

the radial displacement is

$$u(\rho) = \frac{1+\nu}{2E}\left\{(1-2\nu)\left[(\sigma^c - p_i)\,\frac{a}{\rho_c} - 2\sigma^c\right]\rho - (\sigma^c - p_i)\,\frac{a\rho_c}{\rho}\right\}, \qquad \rho \in [\rho_c, b].$$

In Ω_1 the fracture strain is nil and the total deformation has components

$$(4.5)_1 \qquad \varepsilon_\rho(\rho) = \varepsilon^e_\rho(\rho) = \frac{1+\nu}{E}\left\{(1-\nu)(\sigma^c - p_i)\,\frac{a}{\rho} - (1-2\nu)\sigma^c\right\}, \qquad \rho \in [a, \rho_c],$$

$$(4.5)_2 \qquad \varepsilon_\theta(\rho) = \varepsilon^e_\theta(\rho) + \varepsilon^c_\theta(\rho) = \frac{1+\nu}{E}\left\{-\,\nu(\sigma^c - p_i)\,\frac{a}{\rho} - (1-2\nu)\sigma^c\right\} + \varepsilon^c_\theta(\rho), \qquad \rho \in [a, \rho_c],$$

where the circumferential crushing strain ε^c_θ is a non-positive function of ρ which needs to be determined. The radial displacement, obtained by integrating ε_ρ is

$$u(\rho) = \frac{1+\nu}{E}\left\{(1-\nu)(\sigma^c - p_i)\,a\ln\rho - (1-2\nu)\sigma^c\rho\right\} + k, \qquad \rho \in [a, \rho_c],$$

where $k = -\frac{1+\nu}{E}(\sigma^c - p_i)\, a\, [\nu + (1-\nu) \ln \rho_c]$ is a constant whose value is determined by imposing the continuity of the radial displacement at $\rho = \rho_c$. By virtue of $(4.5)_2$ we have

$$(4.6) \qquad \varepsilon_\theta^c(\rho) = \frac{1-\nu^2}{E}\frac{a}{\rho}(\sigma^c - p_i) \ln\left(\frac{\rho}{\rho_c}\right), \qquad \rho \in [a, \rho_c],$$

therefore the crushing strain is negative in Ω_1 and zero when $\rho = \rho_c$. It is interesting to remark that if $\frac{a}{b} < \frac{\sigma^c - p_e}{\sigma^c - p_i} \le \frac{a^2+b^2}{2b^2}$, besides the stress field, the strain and displacement fields are also unique, whereas if $\frac{\sigma^c - p_e}{\sigma^c - p_i} = \frac{a}{b}$, the displacement and thus the circumferential crushing strain are not unique and depend upon the constant k. If the ratio $\frac{\sigma^c - p_e}{\sigma^c - p_i}$ is less than the critical value a/b, there are no statically admissible stress fields.

Now, let us suppose

$$\frac{a^2+b^2}{2b^2} \le \frac{p_e}{p_i} \le 1.$$

In this case we have $\sigma_\theta^{(e)}(b) \le \sigma_\theta^{(e)}(a) \le -p_i$, and moreover, by virtue of the inequalities

$$\frac{\sigma^c - p_e}{\sigma^c - p_i} \ge \frac{p_e}{p_i} \ge \frac{a^2+b^2}{2b^2} \ge \frac{2a^2}{a^2+b^2},$$

the condition $\sigma_\theta^{(e)}(b) \ge -\sigma^c$ is always satisfied, so $\mathbf{T}^{(e)}$ is a statically admissible stress field.

Finally, we need to consider the case

$$\frac{p_e}{p_i} \le \frac{a^2+b^2}{2b^2}.$$

If p_e and p_i also satisfy the inequality $\frac{p_e}{p_i} \ge \frac{a}{b}$, then the semi-definite negative stress field $\mathbf{T}$ calculated in [8], having principal components:

$$(4.7)_1 \qquad \sigma_\rho(\rho) = \begin{cases} -\dfrac{a\, p_i}{\rho}, & \rho \in [a, \rho_t], \\[2ex] -\dfrac{a\, p_i}{2}\left(\dfrac{\rho_t}{\rho^2} + \dfrac{1}{\rho_t}\right), & \rho \in [\rho_t, b]; \end{cases}$$

$$(4.7)_2 \qquad \sigma_\theta(\rho) = \begin{cases} 0, & \rho \in [a, \rho_t], \\ \dfrac{a\,p_i}{2}\left(\dfrac{\rho_t}{\rho^2} - \dfrac{1}{\rho_t}\right), & \rho \in [\rho_t, b]; \end{cases}$$

is statically admissible, since $\sigma_\theta(a) \geq \sigma_\theta(b) \geq - p_e$. The transition radius from the region in which $\mathbf{E}^t \neq \mathbf{0}$ to the one in which $\mathbf{E}^t = \mathbf{0}$. is

$$(4.8) \qquad \rho_t = \frac{b}{a}\,\frac{b p_e - \sqrt{b^2 p_e^2 - a^2 p_i^2}}{p_i},$$

in particular, if $\frac{p_e}{p_i} = \frac{a}{b}$, $\rho_t = b$ and if $\frac{p_e}{p_i} = \frac{a^2 + b^2}{2\,b^2}$, then $\rho_t = a$. The crushing and radial fracture strains are both nil and the circumferential fracture strain is

$$(4.9) \qquad \varepsilon^t_\theta(\rho) = \begin{cases} \dfrac{1 - \nu^2}{E}\,\dfrac{a p_i}{\rho}\,\ln\left(\dfrac{\rho_t}{\rho}\right), & \rho \in [a, \rho_t], \\ 0, & \rho \in [\rho_t, b]. \end{cases}$$

Finally, for values of $\frac{p_e}{p_i}$ less than $\frac{a}{b}$ no statically admissible stress field exists.

Now we increase the external pressure p_e from $\frac{a}{b}\,p_i$ to $\frac{a}{b}\,p_i + \frac{b - a}{b}\,\sigma^c$, while maintaining the internal pressure p_i constant. Figure 3 shows the evolution of the inelastic strain for different values of p_e. When $p_e = \frac{a}{b}\,p_i$ (Figure 3 a), the crushing strain is nil and the fracture strain is non-zero throughout the circular ring; for $p_e \in \left[\frac{a}{b}\,p_i,\ \frac{a^2 + b^2}{2b^2}\,p_i\right]$ (Figure 3 b), the crushing strain is still nil and the region in which the fracture strain is non-zero diminishes progressively and disappears when p_e falls within the interval $\left[\frac{a^2 + b^2}{2b^2}\,p_i,\ \frac{a^2 + b^2}{2b^2}\,p_i + \frac{b^2 - a^2}{2b^2}\,\sigma^c\right]$. In fact, for these values of p_e (Figure 3 c) the crushing and the fracture strain are zero. For p_e increasing from $\frac{a^2 + b^2}{2b^2}\,p_i + \frac{b^2 - a^2}{2b^2}\,\sigma^c$ to $\frac{a}{b}\,p_i + \frac{b - a}{b}\,\sigma^c$ (Figure 3 d), the fracture strain remains equal to zero and the region in which the crushing strain is non-zero progressively extends and covers the whole of the circular ring when p_e reaches the value $\frac{a}{b}\,p_i + \frac{b - a}{b}\,\sigma^c$ (Figure 3 e). For values of p_e less than $\frac{a}{b}\,p_i$ and greater than $\frac{a}{b}\,p_i + \frac{b - a}{b}\,\sigma^c$ there are no statically admissible stress fields.

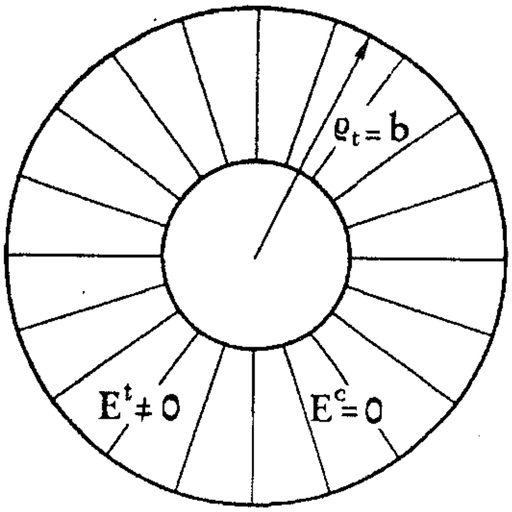

Figure 3 a

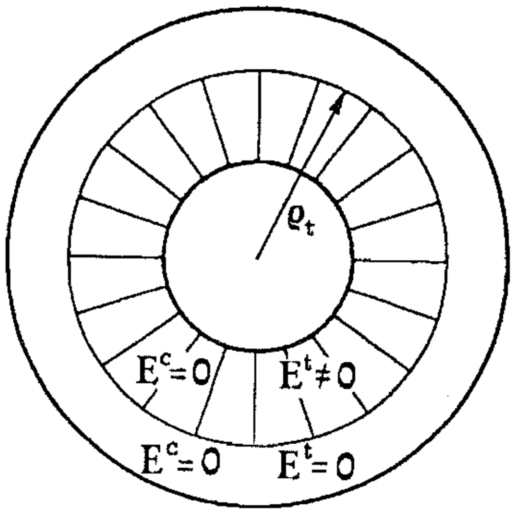

Figure 3 b.

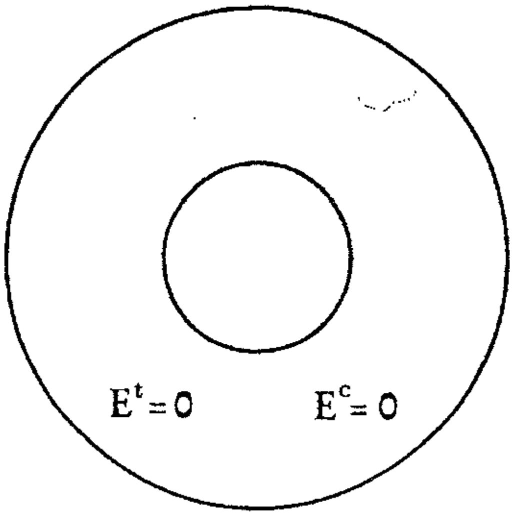

Figure 3 c.

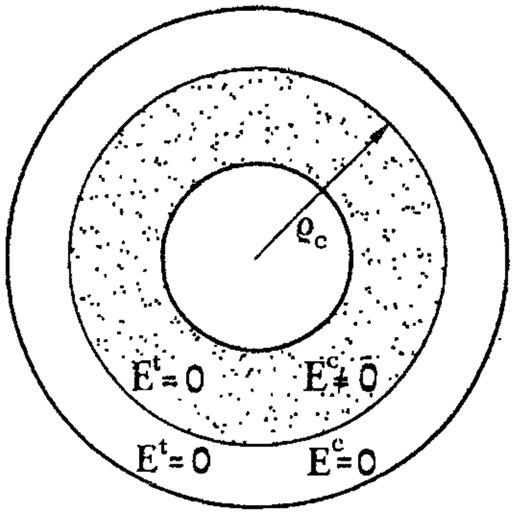

Figure 3 d.

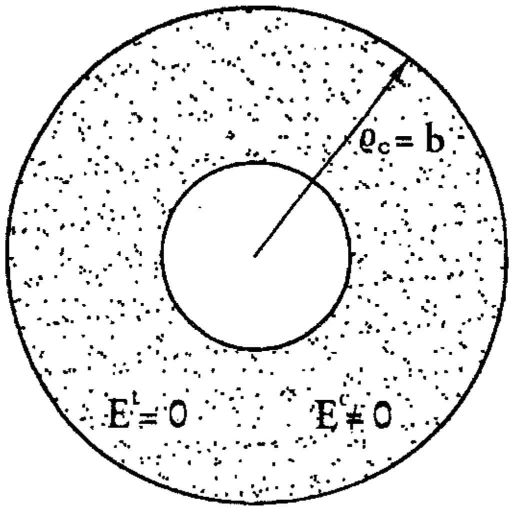

Figure 3 e.

The spherical container.

Let us consider a spherical container Ω_s made up of a BCS material with inner radius a and outer radius b, subjected to two uniform radial pressures: a pressure p_e acting on the external boundary and a pressure p_i acting on the internal boundary. Let $\{O, \rho, \theta, \varphi\}$ be a spherical reference system, with origin O coinciding with the centre of the container.

Bennati *et al.* [8] have shown that if $\frac{p_e}{p_i} \geq \frac{2a^3 + b^3}{3b^3}$, the stress field $\mathbf{T}^{(e)}$ corresponding to a linear elastic material and having the principal components

(4.10)
$$\sigma_\rho^{(e)} = \frac{a^3b^3(p_e - p_i)}{b^3 - a^3}\frac{1}{\rho^3} + \frac{p_i a^3 - p_e b^3}{b^3 - a^3},$$

$$\sigma_\theta^{(e)} = \sigma_\varphi^{(e)} = -\frac{a^3b^3(p_e - p_i)}{2(b^3 - a^3)}\frac{1}{\rho^3} + \frac{p_i a^3 - p_e b^3}{b^3 - a^3},$$

is negative semi-definite. First of all, let us suppose

$$\frac{p_e}{p_i} \geq 1,$$

so the circumferential stress satisfies the boundary inequalities $\sigma_\theta^{(e)}(a) \leq \sigma_\theta^{(e)}(b) \leq -p_e$. Furthermore, if

$$\frac{\sigma^c - p_e}{\sigma^c - p_i} \geq \frac{2a^3 + b^3}{3b^3},$$

then the elastic solution (4.10) satisfies condition $(2.4)_2$ and $T^{(e)}$ is statically admissible. On the contrary, if

$$\frac{\sigma^c - p_e}{\sigma^c - p_i} < \frac{2a^3 + b^3}{3b^3},$$

then $\sigma_\theta^{(e)}(a) < -\sigma^c$ and $T^{(e)}$ is not statically admissible. Using a procedure similar to that used for the circular ring, we may suppose that the spherical region $\Omega_{s1} = \{\rho;\ \rho \in [a, \rho_c]\}$, where ρ_c has to be determined, is subjected to the equilibrated stress field

$$\sigma_\rho(\rho) = \frac{a^2}{\rho^2}(\sigma^c - p_i) - \sigma^c,$$

$$\sigma_\theta(\rho) = \sigma_\varphi(\rho) = -\sigma^c.$$

Consequently, the remaining spherical region $\Omega_{s2} = \{\rho;\ \rho \in [\rho_c, b]\}$ is subjected to the external pressure p_e and to the internal pressure $p_c = \sigma^c - \frac{a^2}{\rho_c^2}(\sigma^c - p_i)$. On the other hand, for continuity reasons, equalities $\sigma_\theta(\rho_c^+) = \sigma_\varphi(\rho_c^+) = -\sigma^c$ must hold. Finally, by virtue of (4.6), we determine that if the ratio $\frac{\sigma^c - p_e}{\sigma^c - p_i}$ satisfies the inequalities

$$\frac{a^2}{b^2} \le \frac{\sigma^c - p_e}{\sigma^c - p_i} \le \frac{2a^3 + b^3}{3b^3},$$

a statically admissible stress field will have components:

$$(4.11)_1 \qquad \sigma_\rho(\rho) = \begin{cases} \dfrac{a^2}{\rho^2}(\sigma^c - p_i) - \sigma^c, & \rho \in [a, \rho_c], \\[2ex] \dfrac{a^2}{3}(\sigma^c - p_i)\left(\dfrac{2\rho_c}{\rho^3} + \dfrac{1}{\rho_c^2}\right) - \sigma^c, & \rho \in [\rho_c, b]; \end{cases}$$

$$(4.11)_2 \qquad \sigma_\theta(\rho) = \sigma_\varphi(\rho) = \begin{cases} -\sigma^c, & \rho \in [a, \rho_c], \\[2ex] \dfrac{a^2}{3}(\sigma^c - p_i)\left(-\dfrac{\rho_c}{\rho^3} + \dfrac{1}{\rho_c^2}\right) - \sigma^c, & \rho \in [\rho_c, b]. \end{cases}$$

The radius ρ_c, which separates the zone where $\mathbf{E}^c \neq \mathbf{0}$ from the zone in which $\mathbf{E}^c = \mathbf{0}$, is the sole real root belonging to [a, b] of the third degree polynomial

$$(4.14)_1 \qquad \sigma_\rho(\rho) = \begin{cases} -\dfrac{a^2 p_i}{\rho^2}, & \rho \in [a, \rho_t], \\[2ex] -\dfrac{a^2 p_i}{3}\left(\dfrac{2\rho_t}{\rho^3} + \dfrac{1}{\rho_t^2}\right), & \rho \in [\rho_t, b]; \end{cases}$$

$$(4.14)_2 \qquad \sigma_\theta(\rho) = \begin{cases} 0, & \rho \in [a, \rho_t], \\[2ex] \dfrac{a^2 p_i}{3}\left(\dfrac{\rho_t}{\rho^3} - \dfrac{1}{\rho_t^2}\right), & \rho \in [\rho_t, b]; \end{cases}$$

where ρ_t is the sole real root belonging to [a, b] of the cubic equation

$$2a^2 p_i \rho^3 - 3b^3 p_e \rho^2 + a^2 b^3 p_i = 0$$

and separates the region in which the circumferential traction strain is non-zero from the region in which it is zero. When $\frac{p_e}{p_i}$ varies from $\frac{2a^3 + b^3}{3b^3}$ to $\frac{a^2}{b^2}$, radius ρ_t correspondingly varies from a to b. The crushing strain is nil, the radial displacement is

$$(4.15) \qquad u(\rho) = \begin{cases} \dfrac{a^2 p_i}{E}\left[\dfrac{1}{\rho} - (1-\nu)\dfrac{1}{\rho_t}\right], & \rho \in [a, \rho_t], \\[2ex] \dfrac{p_i}{3E}\dfrac{a^2}{\rho_t^2}\left[(2\nu - 1)\rho + (1+\nu)\dfrac{\rho_t^3}{\rho^2}\right], & \rho \in [\rho_t, b], \end{cases}$$

and the circumferential fracture strain is

$$(4.16) \qquad \varepsilon_\theta^t(\rho) = \begin{cases} \dfrac{1-\nu}{E}\dfrac{a^2}{\rho}p_i\left(\dfrac{1}{\rho} - \dfrac{1}{\rho_t}\right), & \rho \in [a, \rho_t], \\[2ex] 0, & \rho \in [\rho_t, b]. \end{cases}$$

For values of $\frac{p_e}{p_i}$ less than $\frac{a^2}{b^2}$, there are no statically admissible stress fields.

V. THE NUMERICAL METHOD

In this section we calculate the derivative $D_E\widehat{\mathbf{T}}$ of $\mathbf{T} = \widehat{\mathbf{T}}(\mathbf{E})$ with respect to $\mathbf{E}$. Knowing this derivative allows calculation of the tangent matrix and determination of the displacements by solving a non-linear system obtained by discretisation into finite elements via the Newton-Raphson method.

The algorithm used for the numerical solution of the equilibrium problem in the presence of incremental loads has already been described in [4] and is thus omitted here.

Differentiating $\widehat{\mathbf{T}}$ with respect to $\mathbf{E}$ requires some preliminary results.

Let Sym* stand for the subset of Sym of all symmetric tensors having distinct eigenvalues. Given $\mathbf{A} \in$ Sym*, let $\lambda_1, \lambda_2, \lambda_3$ with $\lambda_1 < \lambda_2 < \lambda_3$ and $\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3$ be the eigenvalues and the eigenvectors of $\mathbf{A}$, respectively.

Putting, for convenience,

$$\mathbf{G}_1 = \frac{1}{\sqrt{2}}(\mathbf{g}_1\otimes\mathbf{g}_2 + \mathbf{g}_2\otimes\mathbf{g}_1) \quad , \qquad \mathbf{G}_2 = \frac{1}{\sqrt{2}}(\mathbf{g}_1\otimes\mathbf{g}_3 + \mathbf{g}_3\otimes\mathbf{g}_1) \quad ,$$

$$\mathbf{G}_3 = \frac{1}{\sqrt{2}}(\mathbf{g}_2\otimes\mathbf{g}_3 + \mathbf{g}_3\otimes\mathbf{g}_2) \quad ,$$

we propose to prove the following [3]:

$$(5.1)_1 \qquad D_A\,\lambda_1 = \mathbf{g}_1\otimes\mathbf{g}_1 \,,$$

$$(5.1)_2 \qquad D_A\,\lambda_2 = \mathbf{g}_2\otimes\mathbf{g}_2 \,,$$

$$(5.1)_3 \qquad D_A\,\lambda_3 = \mathbf{g}_3\otimes\mathbf{g}_3 \,;$$

$$(5.1)_4 \qquad D_A\,\mathbf{g}_1\otimes\mathbf{g}_1 = \frac{1}{\lambda_1 - \lambda_2}\,\mathbf{G}_1\otimes\mathbf{G}_1 + \frac{1}{\lambda_1 - \lambda_3}\,\mathbf{G}_2\otimes\mathbf{G}_2 \,,$$

$$(5.1)_5 \qquad D_A\,\mathbf{g}_2\otimes\mathbf{g}_2 = \frac{1}{\lambda_2 - \lambda_1}\,\mathbf{G}_1\otimes\mathbf{G}_1 + \frac{1}{\lambda_2 - \lambda_3}\,\mathbf{G}_3\otimes\mathbf{G}_3 \,,$$

$$(5.1)_6 \qquad D_A\,\mathbf{g}_3\otimes\mathbf{g}_3 = \frac{1}{\lambda_3 - \lambda_1}\,\mathbf{G}_2\otimes\mathbf{G}_2 + \frac{1}{\lambda_3 - \lambda_2}\,\mathbf{G}_3\otimes\mathbf{G}_3 \,.$$

It is sufficient to prove $(5.1)_1$ and $(5.1)_4$, because the other relations can be proven in a similar way. Let us consider $\mathbf{A} \in$ Sym*, $\mathbf{H} \in$ Sym to be fixed and $\alpha \in \mathbb{R}$; let $\lambda_1(\alpha)$ and $\mathbf{g}_1(\alpha)$ be the smallest eigenvalue and the corresponding eigenvector of $\mathbf{A} + \alpha\mathbf{H}$, respectively:

---

[3] Here $D_A\lambda_i$ is the derivative with respect to $\mathbf{A}$ of the function λ_i : Sym* $\rightarrow \mathbb{R}$, $\mathbf{A} \mapsto \lambda_i(\mathbf{A})$; analogously $D_A\mathbf{g}_i \otimes \mathbf{g}_i$ is the derivative with respect to $\mathbf{A}$ of the function $\mathbf{g}_i \otimes \mathbf{g}_i$. This last function is well defined since, by virtue of the fact that the eigenvalues of $\mathbf{A}$ are distinct, the eigenvectors $\mathbf{g}_i$ are uniquely determined from the relations $\mathbf{A}\,\mathbf{g}_i = \lambda_i\,\mathbf{g}_i$, i = 1, 2, 3.

$$(5.2) \qquad (\mathbf{A} + \alpha\mathbf{H})\, \mathbf{g}_1(\alpha) = \lambda_1(\alpha)\, \mathbf{g}_1(\alpha) \; .$$

Since we are interested in the behaviour of $\lambda_1(\alpha)$ and $\mathbf{g}_1(\alpha)$ for α near zero, within an error of order $o(\alpha)$ we can put

$$(5.3) \qquad \lambda_1(\alpha) = \lambda_1 + \dot{\lambda}_1(0), \text{ and } \mathbf{g}_1(\alpha) = \mathbf{g}_1 + \dot{\mathbf{g}}_1(0),$$

where $\lambda_1 = \lambda_1(0)$, $\mathbf{g}_1 = \mathbf{g}_1(0)$ and the superimposed dot $\cdot$ denotes differentiation with respect to α. By substituting (5.3) in (5.2) we obtain

$$(5.4) \qquad \mathbf{A}\, \dot{\mathbf{g}}_1(0) + \mathbf{H}\, \mathbf{g}_1 = \dot{\lambda}_1(0)\mathbf{g}_1 + \lambda_1\, \dot{\mathbf{g}}_1(0).$$

Since $\mathbf{g}_1 \cdot \mathbf{g}_1 = 1$, then $\dot{\mathbf{g}}_1(0) \cdot \mathbf{g}_1 = 0$; thus if we multiply (5.4) by $\mathbf{g}_1$ we have

$$(5.5) \qquad \dot{\lambda}_1(0) = \mathbf{g}_1 \cdot \mathbf{H}\mathbf{g}_1 = \mathbf{g}_1 \otimes \mathbf{g}_1 \cdot \mathbf{H} \; .$$

Because, for every $\mathbf{H}$ in Sym we can write

$$\dot{\lambda}_1(0) = \frac{d}{d\alpha} \lambda_1(\mathbf{A} + \alpha\mathbf{H}) \,|_{\alpha=0} = D_{\mathbf{A}}\lambda_1 \cdot \mathbf{H} \, ,$$

by virtue of (5.5) we obtain $(5.1)_1$.

In order to calculate the derivative of $\mathbf{g}_1 \otimes \mathbf{g}_1$, we have to calculate the derivative of $\mathbf{g}_1$. To this end, by substituting (5.5) into (5.4), we obtain

$$(5.6) \qquad \mathbf{A}\, \dot{\mathbf{g}}_1(0) + \mathbf{H}\, \mathbf{g}_1 = (\mathbf{g}_1 \otimes \mathbf{g}_1 \cdot \mathbf{H})\; \mathbf{g}_1 + \lambda_1\, \dot{\mathbf{g}}_1(0).$$

Since $\mathbf{g}_1$ and $\dot{\mathbf{g}}_1(0)$ are orthogonal, we can write

$$(5.7) \qquad \dot{\mathbf{g}}_1(0) = \chi\, \mathbf{g}_2 + \xi\, \mathbf{g}_3 \; ,$$

where χ and ξ are scalars which depend on $\mathbf{A}$. By substituting (5.7) into (5.6) the relation

$$(5.8) \qquad \chi\,(\lambda_2 - \lambda_1)\mathbf{g}_2 + \xi\,(\lambda_3 - \lambda_1)\mathbf{g}_3 = (\mathbf{g}_1 \otimes \mathbf{g}_1 \cdot \mathbf{H})\mathbf{g}_1 - \mathbf{H}\mathbf{g}_1$$

follows. Multiplying (5.8) by $\mathbf{g}_2$ and by $\mathbf{g}_3$, we obtain respectively,

$$(5.9) \qquad \begin{aligned} \chi &= \frac{1}{\lambda_1 - \lambda_2}\, \mathbf{g}_1 \otimes \mathbf{g}_2 \cdot \mathbf{H} \, , \\ \xi &= \frac{1}{\lambda_1 - \lambda_3}\, \mathbf{g}_1 \otimes \mathbf{g}_3 \cdot \mathbf{H} \, . \end{aligned}$$

Thus, from (5.7) and (5.9), by virtue of the symmetry of **H**, we have[4]

(5.10) $$\dot{\mathbf{g}}_1(0) = \frac{d}{d\alpha}\,\mathbf{g}_1(\mathbf{A}+\alpha\mathbf{H})\,|_{\alpha=0} = D_{\mathbf{A}}\mathbf{g}_1[\mathbf{H}] =$$

$$= \frac{1}{2(\lambda_1-\lambda_2)}\,(\mathbf{g}_2\otimes\mathbf{g}_1\otimes\mathbf{g}_2+\mathbf{g}_2\otimes\mathbf{g}_2\otimes\mathbf{g}_1)[\mathbf{H}]\,+$$

$$+\,\frac{1}{2(\lambda_1-\lambda_3)}\,(\mathbf{g}_3\otimes\mathbf{g}_1\otimes\mathbf{g}_3+\mathbf{g}_3\otimes\mathbf{g}_3\otimes\mathbf{g}_1)[\mathbf{H}]\quad .$$

The desired result follows from the relation

$$D_{\mathbf{A}}\mathbf{g}_1\otimes\mathbf{g}_1[\mathbf{H}] = D_{\mathbf{A}}\mathbf{g}_1[\mathbf{H}]\otimes\mathbf{g}_1+\mathbf{g}_1\otimes D_{\mathbf{A}}\mathbf{g}_1[\mathbf{H}]\quad .$$

Now we are in a position to calculate the derivative of the stress with respect to the total deformation in the ten regions $\mathcal{R}_i$. Let us consider the orthonormal basis of Sym

(5.11)
$$\mathbf{O}_1 = \mathbf{q}_1\otimes\mathbf{q}_1\,,$$
$$\mathbf{O}_2 = \mathbf{q}_2\otimes\mathbf{q}_2\,,$$
$$\mathbf{O}_3 = \mathbf{q}_3\otimes\mathbf{q}_3\,,$$
$$\mathbf{O}_4 = \frac{1}{\sqrt{2}}\,(\mathbf{q}_1\otimes\mathbf{q}_2+\mathbf{q}_2\otimes\mathbf{q}_1)\ ,$$
$$\mathbf{O}_5 = \frac{1}{\sqrt{2}}\,(\mathbf{q}_1\otimes\mathbf{q}_3+\mathbf{q}_3\otimes\mathbf{q}_1)\ ,$$
$$\mathbf{O}_6 = \frac{1}{\sqrt{2}}\,(\mathbf{q}_2\otimes\mathbf{q}_3+\mathbf{q}_3\otimes\mathbf{q}_2)\ ,$$

and the spectral representation of **T**

(5.12) $$\mathbf{T} = \sum_{i=1}^{3} t_i\,\mathbf{O}_i$$

where t_1, t_2 and t_3 are given in (2.6). From $(2.6)_1$, $(2.6)_4$ and $(2.6)_7$ it follows that the expression of $D_{\mathbf{E}}\hat{\mathbf{T}}(\mathbf{E})$ for **E** belonging to $\mathcal{R}_1$, $\mathcal{R}_4$ and $\mathcal{R}_7$ can be easily calculated; the calculation of $D_{\mathbf{E}}\hat{\mathbf{T}}(\mathbf{E})$ when **E** belongs to the seven other regions is slightly more complex and requires differentiating (5.12). In order to differentiate (5.12) with respect to **E** we must use the previously calculated derivatives of the eigenvalues of **E** and the tensors $\mathbf{O}_1$, $\mathbf{O}_2$ and $\mathbf{O}_3$ with respect to **E**.

---

[4] Here, given **u**, v, w $\in \mathcal{V}$, **u** $\otimes$v$\otimes$w denotes the third-order tensor defined by $\mathbf{u}\otimes\mathbf{v}\otimes\mathbf{w}\,\mathbf{H} = (\mathbf{v}\otimes\mathbf{w}\cdot\mathbf{H})\,\mathbf{u}$, $\mathbf{H}\in$ Lin.

As a single example, we shall calculate $D_E\widehat{\mathbf{T}}(\mathbf{E})$ when $\mathbf{E} \in \mathcal{R}_2$, where $e_1 < e_2 \le e_3$. Let us begin by supposing $e_1 < e_2 < e_3$; from (5.12), $(2.6)_2$ and (5.1), using the relation

$$D_E\widehat{\mathbf{T}}(\mathbf{E}) = D_E t_1 \otimes \mathbf{O}_1 + t_1 \otimes D_E\mathbf{O}_1 + D_E t_2 \otimes \mathbf{O}_2 + t_2 \otimes D_E\mathbf{O}_2 + D_E t_3 \otimes \mathbf{O}_3 + t_3 \otimes D_E\mathbf{O}_3,$$

we obtain

$$(5.13) \qquad D_E\widehat{\mathbf{T}}(\mathbf{E}) = \frac{2\mu}{2+\alpha}\,\frac{\varepsilon^c + 2(1+\alpha)e_2 + \alpha e_3}{e_2 - e_1}\;\mathbf{O}_4 \otimes \mathbf{O}_4 + \\ + \frac{2\mu}{2+\alpha}\,\frac{\varepsilon^c + 2(1+\alpha)e_3 + \alpha e_3}{e_3 - e_1}\;\mathbf{O}_5 \otimes \mathbf{O}_5 + f(e_2, e_3)\;\mathbf{O}_6 \otimes \mathbf{O}_6 + \\ + \frac{\mu(2+3\alpha)}{2+\alpha}\;(\mathbf{O}_2 + \mathbf{O}_3) \otimes (\mathbf{O}_2 + \mathbf{O}_3) + \mu\;(\mathbf{O}_2 - \mathbf{O}_3) \otimes (\mathbf{O}_2 - \mathbf{O}_3)\,,$$

where $f(e_2, e_3) = 2\mu\,\frac{e_3 - e_2}{e_3 - e_2}$. When $e_3 - e_2$ goes to zero, $f(e_2, e_3)$ converges on 2μ and then (5.13), with $f(e_2, e_3) = 2\mu$, holds also when $e_3 = e_2$.

Finally, we summarise the expression of $D_E\widehat{\mathbf{T}}(\mathbf{E})$ in the ten regions $\mathcal{R}_i$:

$$(5.14)_1 \qquad D_E\widehat{\mathbf{T}}(\mathbf{E}) = 2\mu\,\mathbb{1} + \lambda\,\mathbf{I} \otimes \mathbf{I}\,, \qquad \mathbf{E} \in \mathcal{R}_1,$$

$$(5.14)_2 \qquad D_E\widehat{\mathbf{T}}(\mathbf{E}) = \frac{2\mu}{2+\alpha}\,\frac{\varepsilon^c + 2(1+\alpha)e_2 + \alpha e_3}{e_2 - e_1}\;\mathbf{O}_4 \otimes \mathbf{O}_4 + \\ + \frac{2\mu}{2+\alpha}\,\frac{\varepsilon^c + 2(1+\alpha)e_3 + \alpha e_3}{e_3 - e_1}\;\mathbf{O}_5 \otimes \mathbf{O}_5 + 2\mu\;\mathbf{O}_6 \otimes \mathbf{O}_6 + \\ + \frac{\mu(2+3\alpha)}{2+\alpha}\;(\mathbf{O}_2 + \mathbf{O}_3) \otimes (\mathbf{O}_2 + \mathbf{O}_3) + \mu\;(\mathbf{O}_2 - \mathbf{O}_3) \otimes (\mathbf{O}_2 - \mathbf{O}_3)\,, \qquad \mathbf{E} \in \mathcal{R}_2,$$

$$(5.14)_3 \qquad D_E\widehat{\mathbf{T}}(\mathbf{E}) = \frac{\mu}{1+\alpha}\,\frac{\varepsilon^c + (2+3\alpha)e_3}{e_3 - e_1}\;\mathbf{O}_5 \otimes \mathbf{O}_5 + \frac{\mu}{1+\alpha}\,\frac{\varepsilon^c + (2+3\alpha)e_3}{e_3 - e_2}\;\mathbf{O}_6 \otimes \mathbf{O}_6 + \\ + E\;\;\mathbf{O}_3 \otimes \mathbf{O}_3\,, \qquad \mathbf{E} \in \mathcal{R}_3,$$

$$(5.14)_4 \qquad D_E\widehat{\mathbf{T}}(\mathbf{E}) = \mathbb{O}\ , \qquad \mathbf{E}\in \mathcal{R}_4,$$

$$(5.14)_5 \qquad D_E\widehat{\mathbf{T}}(\mathbf{E}) = 2\mu\ \mathbf{O}_4 \otimes \mathbf{O}_4 + \frac{2\mu}{2+\alpha}\ \frac{\varepsilon^t - 2(1+\alpha)e_1 - \alpha e_2}{e_3 - e_1}\ \mathbf{O}_5 \otimes \mathbf{O}_5 +$$

$$+ \frac{2\mu}{2+\alpha}\ \frac{\varepsilon^t - 2(1+\alpha)e_2 - \alpha e_1}{e_3 - e_2}\ \mathbf{O}_6 \otimes \mathbf{O}_6 +$$

$$+ \frac{\mu(2+3\alpha)}{2+\alpha}\ (\mathbf{O}_1 + \mathbf{O}_2) \otimes (\mathbf{O}_1 + \mathbf{O}_2) + \mu\ (\mathbf{O}_1 - \mathbf{O}_2) \otimes (\mathbf{O}_1 - \mathbf{O}_2)\ , \qquad \mathbf{E}\in \mathcal{R}_5,$$

$$(5.14)_6 \qquad D_E\widehat{\mathbf{T}}(\mathbf{E}) = \frac{\mu}{1+\alpha}\ \frac{\varepsilon^t - (2+3\alpha)e_1}{e_2 - e_1}\ \mathbf{O}_4 \otimes \mathbf{O}_4 +$$

$$+ \frac{\mu}{1+\alpha}\ \frac{\varepsilon^t - (2+3\alpha)e_1}{e_3 - e_1}\ \mathbf{O}_5 \otimes \mathbf{O}_5 + E\ \ \mathbf{O}_1 \otimes \mathbf{O}_1\ , \qquad \mathbf{E}\in \mathcal{R}_6,$$

$$(5.14)_7 \qquad D_E\widehat{\mathbf{T}}(\mathbf{E}) = \mathbb{O}\ , \qquad \mathbf{E}\in \mathcal{R}_7,$$

$$(5.14)_8 \qquad D_E\widehat{\mathbf{T}}(\mathbf{E}) = \frac{\sigma^t + \sigma^c}{e_2 - e_1}\ \mathbf{O}_4 \otimes \mathbf{O}_4 + \frac{\sigma^t + \sigma^c}{e_3 - e_1}\ \mathbf{O}_5 \otimes \mathbf{O}_5\ , \qquad \mathbf{E}\in \mathcal{R}_8,$$

$$(5.14)_9 \qquad D_E\widehat{\mathbf{T}}(\mathbf{E}) = \frac{\sigma^t + \sigma^c}{e_3 - e_1}\ \mathbf{O}_5 \otimes \mathbf{O}_5 + \frac{\sigma^t + \sigma^c}{e_3 - e_2}\ \mathbf{O}_6 \otimes \mathbf{O}_6\ , \qquad \mathbf{E}\in \mathcal{R}_9,$$

$$(5.14)_{10} \qquad D_E\widehat{\mathbf{T}}(\mathbf{E}) = \frac{\mu}{2(1+\alpha)}\ \frac{\alpha\varepsilon^t + (2+\alpha)\varepsilon^c + 2(2+3\alpha)e_2}{e_2 - e_1}\ \mathbf{O}_4 \otimes \mathbf{O}_4 +$$

$$+ \frac{\sigma^t + \sigma^c}{e_3 - e_1}\ \mathbf{O}_5 \otimes \mathbf{O}_5 + \frac{\mu}{2(1+\alpha)}\ \frac{\alpha\varepsilon^c + (2+\alpha)\varepsilon^t - 2(2+3\alpha)e_2}{e_3 - e_2}\ \mathbf{O}_6 \otimes \mathbf{O}_6 +$$

$$+ \frac{\mu(2+3\alpha)}{1+\alpha}\ \ \mathbf{O}_2 \otimes \mathbf{O}_2, \qquad \mathbf{E}\in \mathcal{R}_{10},$$

where $\mathbb{1}$ and $\mathbb{O}$ are the fourth-order identity tensor and the fourth-order null tensor, respectively. It is note-worthy that the expressions given in $(5.14)_2$-$(5.14)_{10}$ are the spectral representations of $D_E\widehat{\mathbf{T}}(\mathbf{E})$ in the nine regions $\mathbb{R}_2$- $\mathbb{R}_{10}$. Moreover, it can be easily verified that the eigenvalues of $D_E\widehat{\mathbf{T}}(\mathbf{E})$ are non-negative and so the strain-energy density $\psi(\mathbf{E}) = \frac{1}{2}\widehat{\mathbf{T}}(\mathbf{E})\cdot\mathbf{E}$ is a convex function. The same result has been proven in [1] for materials not supporting tension and infinitely resistant to compression.

We conclude this section by listing the expression for the derivative of the stress for plane strain and plane stress.

For the plane case, let $e_1 < e_2$ be the eigenvalues of $\mathbf{E}$ and $\mathbf{q}_1$ and $\mathbf{q}_2$ be the corresponding eigenvectors, putting

$$\mathbf{O}_1 = \mathbf{q}_1\otimes\mathbf{q}_1\,,$$

$$\mathbf{O}_2 = \mathbf{q}_2\otimes\mathbf{q}_2\,,$$

$$\mathbf{O}_3 = \frac{1}{\sqrt{2}}\,(\mathbf{q}_1\otimes\mathbf{q}_2 + \mathbf{q}_2\otimes\mathbf{q}_1)\;,$$

we have

$$D_E\, e_1 = \mathbf{O}_1, \qquad D_E\, e_2 = \mathbf{O}_2,$$

$$D_E\, \mathbf{O}_1 = \frac{1}{e_1 - e_2}\;\mathbf{O}_3\otimes\mathbf{O}_3, \qquad D_E\, \mathbf{O}_2 = \frac{1}{e_2 - e_1}\;\mathbf{O}_3\otimes\mathbf{O}_3.$$

For plane strain, the derivatives of $\mathbf{T}$ in the six regions $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$, $\mathcal{S}_4$, $\mathcal{S}_5$ and $\mathcal{S}_6$ are

$$(5.15)_1 \qquad D_E\widehat{\mathbf{T}}(\mathbf{E}) = 2\mu\,\mathbb{1} + \lambda\,\mathbf{I}\otimes\mathbf{I}, \qquad \mathbf{E}\in\mathcal{S}_1,$$

$$(5.15)_2 \qquad D_E\widehat{\mathbf{T}}(\mathbf{E}) = \mathbb{O}, \qquad \mathbf{E}\in\mathcal{S}_2,$$

$$(5.15)_3 \qquad D_E\widehat{\mathbf{T}}(\mathbf{E}) = \frac{2\mu}{2+\alpha}\;\frac{\varepsilon^t - 2(1+\alpha)e_1}{e_2 - e_1}\;\mathbf{O}_3\otimes\mathbf{O}_3 + $$

$$+\;\frac{4\mu(1+\alpha)}{2+\alpha}\;\mathbf{O}_1\otimes\mathbf{O}_1, \qquad \mathbf{E}\in\mathcal{S}_3,$$

$$(5.15)_4 \qquad D_E\hat{\mathbf{T}}(\mathbf{E}) = \frac{2\mu}{2+\alpha}\ \frac{\varepsilon^c + 2(1+\alpha)e_2}{e_2 - e_1}\ \mathbf{O}_3 \otimes \mathbf{O}_3 + $$

$$+ \frac{4\mu(1+\alpha)}{2+\alpha}\ \mathbf{O}_2 \otimes \mathbf{O}_2, \qquad \mathbf{E} \in \mathcal{S}_4,$$

$$(5.15)_5 \qquad D_E\hat{\mathbf{T}}(\mathbf{E}) = \mathbb{O}, \qquad \mathbf{E} \in \mathcal{S}_5,$$

$$(5.15)_6 \qquad D_E\hat{\mathbf{T}}(\mathbf{E}) = \frac{\sigma^c + \sigma^t}{e_2 - e_1}\ \mathbf{O}_3 \otimes \mathbf{O}_3, \qquad \mathbf{E} \in \mathcal{S}_6.$$

For plane stress we have

$$(5.16)_1 \qquad D_E\hat{\mathbf{T}}(\mathbf{E}) = 2\mu\ \mathbb{1} + \frac{2\mu\alpha}{2+\alpha}\ \mathbf{I} \otimes \mathbf{I}, \qquad \mathbf{E} \in \mathcal{T}_1,$$

$$(5.16)_2 \qquad D_E\hat{\mathbf{T}}(\mathbf{E}) = \mathbb{O}, \qquad \mathbf{E} \in \mathcal{T}_2,$$

$$(5.16)_3 \qquad D_E\hat{\mathbf{T}}(\mathbf{E}) = \frac{\mu}{2(1+\alpha)}\ \frac{(2+\alpha)\varepsilon^t - 2(2+3\alpha)e_1}{e_2 - e_1}\ \mathbf{O}_3 \otimes \mathbf{O}_3 + $$

$$+ E\ \mathbf{O}_1 \otimes \mathbf{O}_1, \qquad \mathbf{E} \in \mathcal{T}_3,$$

$$(5.16)_4 \qquad D_E\hat{\mathbf{T}}(\mathbf{E}) = \frac{\mu}{2(1+\alpha)}\ \frac{(2+\alpha)\varepsilon^c + 2(2+3\alpha)e_2}{e_2 - e_1}\ \mathbf{O}_3 \otimes \mathbf{O}_3 + $$

$$+ E\ \mathbf{O}_2 \otimes \mathbf{O}_2, \qquad \mathbf{E} \in \mathcal{T}_4,$$

$$(5.16)_5 \qquad D_E\hat{\mathbf{T}}(\mathbf{E}) = \mathbb{O}, \qquad \mathbf{E} \in \mathcal{T}_5,$$

$$(5.16)_6 \qquad D_E\hat{\mathbf{T}}(\mathbf{E}) = \frac{\sigma^c + \sigma^t}{e_2 - e_1}\ \mathbf{O}_3 \otimes \mathbf{O}_3, \qquad \mathbf{E} \in \mathcal{T}_6.$$

VI. NUMERICAL EXAMPLES

The circular ring

In this section we numerically solve the problem of the circular ring considered in Section IV. The finite element analysis is performed using the calculus scheme described in [3], by means of the tangent stiffness matrix calculated with the help of the fourth-order tensor $D_E\hat{T}(E)$ deduced in the previous section. For the numerical calculation of the solution, the following values of the constants have been used

$$\begin{aligned}
a &= 1 \text{ m},\\
b &= 1.5 \text{ m},\\
p_i &= 0.1 \text{ MPa},\\
p_e &= 0.23 \text{ MPa},\\
\sigma^c &= 0.5 \text{ MPa},\\
\nu &= 0.1\\
E &= 5000 \text{ MPa}.
\end{aligned}$$

In this case the ratio $\frac{\sigma^c - p_e}{\sigma^c - p_i} = 0.675$ lies within the interval $\left[\frac{a}{b}, \frac{a^2 + b^2}{2\, b^2}\right] = [0.667,\ 0.722]$ and the transition radius is approximately $\rho_c = 1.28$ m. For symmetry reasons, only a quarter of the circular ring was studied, and this was discretised into four hundred eight-node elements; convergence was reached in three iterations. Figures 4, 5, and 6 show the behaviour of the radial stress, circumferential stress and circumferential crushing strain. The continuous line represents the exact solution, the bold points, the numerical solution.

The circular ring was successively subjected to a load process with $p_i = 0.1$ MPa and p_e increasing from $p_{e0} = 0.0667$ MPa to $p_{ef} = 0.2333$ MPa. In Figure 7 the behaviour of radius $\rho*$, which separates the region in which the inelastic deformation $\mathbf{E}^t + \mathbf{E}^c$ is non-zero from the region in which $\mathbf{E}^t + \mathbf{E}^c = \mathbf{0}$, is shown. In accordance with (4.8) and (4.2), the expression of $\rho*$ is

$$\rho*(p_e/p_i) = \begin{cases}
1.5\left(1.5\,\frac{p_e}{p_i} - \sqrt{2.25\,\frac{p_e^2}{p_i^2} - 1}\right), & p_e/p_i \in [0.667, 0.722\,],\\[2ex]
1, & p_e/p_i \in [0.722, 2.111\,],\\[2ex]
0.375\left[1.5\left(5 - \frac{p_e}{p_i}\right) - \sqrt{2.25\left(5 - \frac{p_e}{p_i}\right)^2 - 16}\right], & p_e/p_i \in [2.111, 2.333\,].
\end{cases}$$

the extrados in the springing and has the value of 7.6 MPa. The region characterized by the openings is illustrated in Figure 10, where the isostatic lines are also drawn.

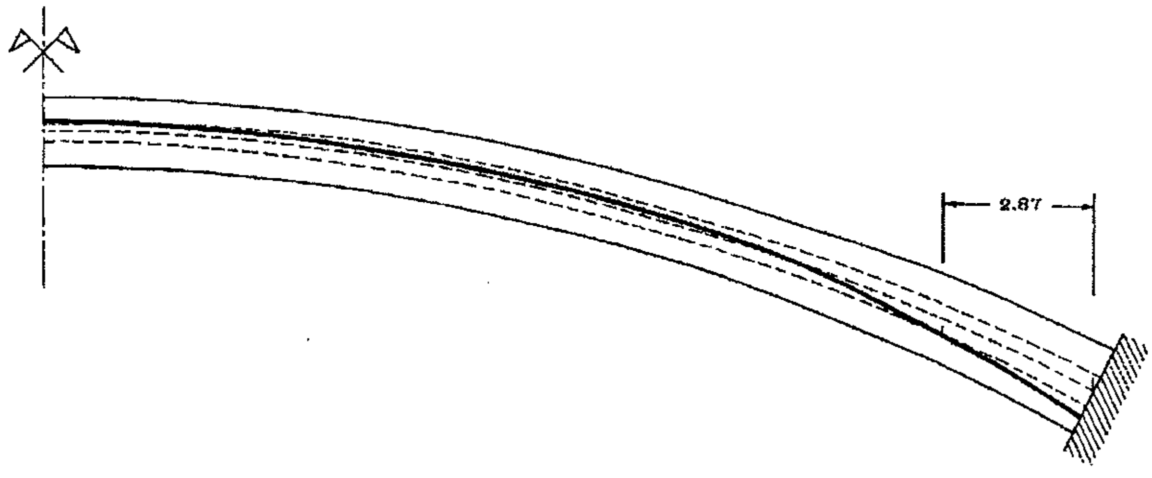

Figure 9. The line of thrust for Mosca's bridge.

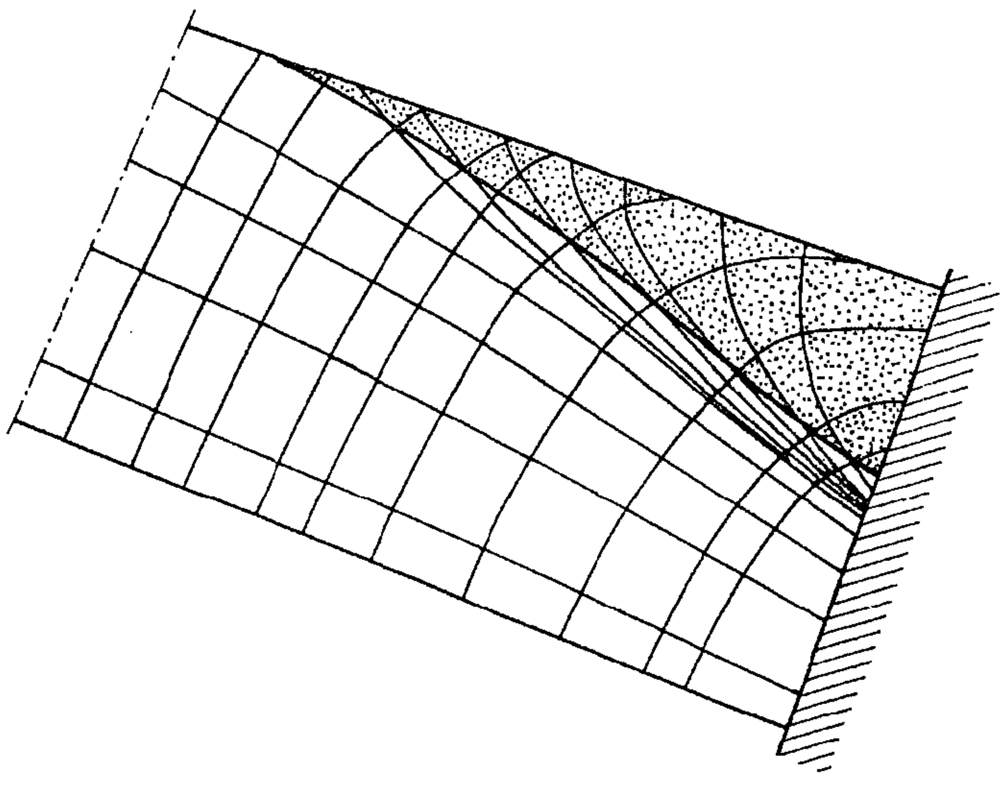

Figure 10. The isostatic lines near the springing.

The three-dimensional arch

Let us consider the reduced circular arch whose springings are fixed, shown in Figure 11 arch is subjected to its own weight and a load p, constant per unit span, distributed alon extrados. For symmetry reasons, only a quarter of the structure was studied and this discretized into 300 isoparametric three-dimensional elements with 20 nodes and 27 G points. We suppose that the material constituting the arch is not resistant to traction ($\sigma^t = 0$) has a maximum compressive strength $\sigma^c = 8.82$ MPa. The distributed load is progressi increased until the value p_c, beyond which the convergence cannot be reached; p_c, interpr here as collapse load, resulted equal to 0.405 MPa.

Collapse occurs because of the formation of a number of hinges sufficient to render the struc labile. The constitutive characteristics of the material suggest supposing that at the instan collapse in the normal sections of the arch where there are the hinges, the normal stress constant and equal to σ^c in an interval having an extremum coinciding with the intrados or extrados and nil elsewhere. Figure 12, where $\sigma = \sigma^c$ for $d \le y \le h/2$ and $\sigma = 0$ for $-h/2 \le y \le d$, shows one of these situations. The straight line parallel to the x axis and a distance equal to d from it is called the *neutral axis*.

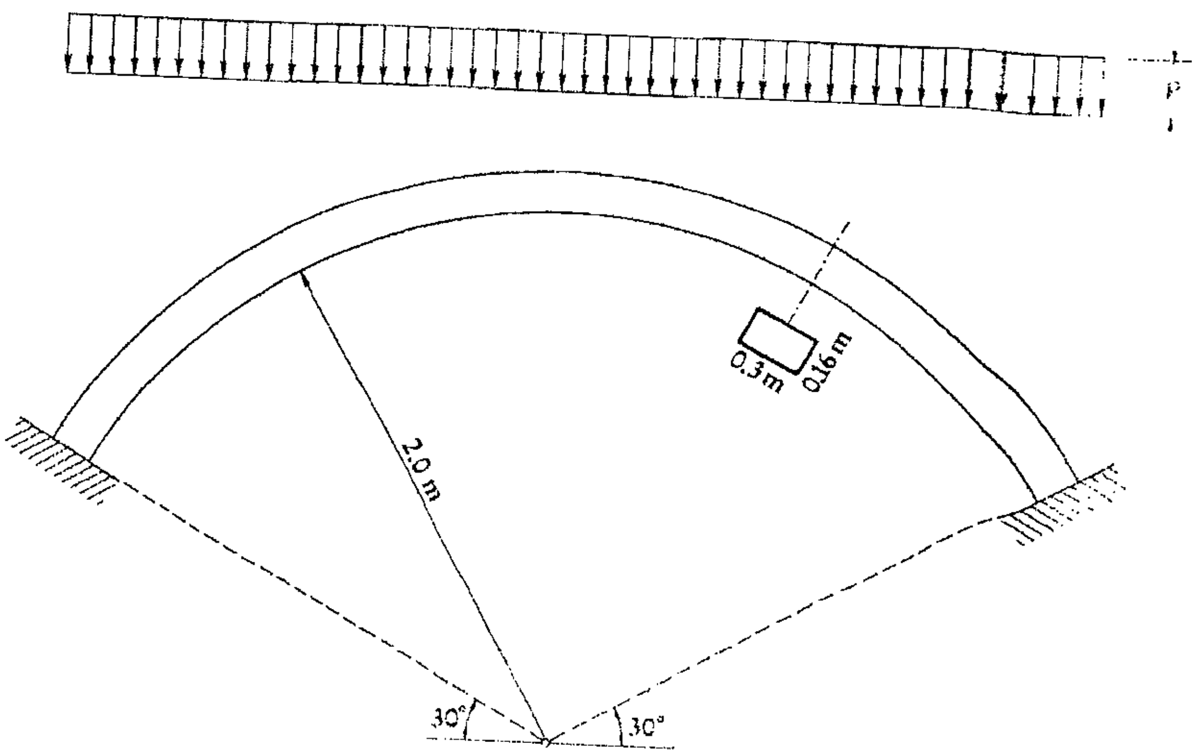

Figure 11. The reduced circular arch.

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