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The effect of model misspecification of the bounded transformed gamma process on maintenance optimization

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ABSTRACT

Although the degradation growth of several technological units is naturally bounded, stochastic models used to describe them are typically unbounded. In general, this manifest contradiction does not significantly affect the effectiveness of unbounded degradation models, because degrading units are conventionally considered failed when their degradation level exceeds a threshold that is far below the physical bound. Yet, this is not always the case. Motivated by these arguments, the main aim and novel contribution of this paper is investigating the impact on a condition based maintenance policy and related costs of using a degradation model that neglects the presence of the bound when it exists. In particular, the paper focuses on situations where bound and failure threshold have comparable values. The study is conducted considering as competing models a bounded and an unbounded transformed gamma. The competing models are used to formulate a condition-based maintenance policy for the liners of a marine engine. An example of application based on real data is firstly developed. Hence, the results of a Monte Carlo simulation study are presented and discussed. Obtained results highlight that, neglecting the presence of the upper bound when it really exists, can cause substantial (unnecessary) additional maintenance costs.

1. Introduction

Interest of reliability and maintenance engineers in degradation models is mostly motivated by the fact that many technological units are subject to degradation phenomena that, in the long-run, cause a progressive loss of functionality or performance. In fact, these units are usually considered failed when their degradation level exceeds a predetermined failure threshold.

Describing the degradation process of these units by stochastic models is twofold useful. Indeed, degradation models can be used both to obtain reliability estimates from degradation data, even in the absence of failures, and to perform condition based (i.e., degradation based) estimates of remaining useful life (RUL) and residual reliability, by taking advantage of degradation data collected, in real time, during the operational life of the units. These latter prognostic tools are very useful for formulating condition-based maintenance (CBM) policies, which provide a better trade-offs between corrective and preventive maintenance costs with respect to traditional age-based maintenance (ABM) and time-based maintenance (TBM) strategies (see, e.g., Wang and Pham [1], Ahmad and Kamaruddin [2], and Gertsbakh [3]). Potential benefits of CBM with respect to ABM and TBM strategies are discussed in detail in de Jonge et al. [4].

Obviously, the success of any maintenance policy mainly depends on the ability of the adopted stochastic model to adequately describe the real-world degradation phenomena of interest.

Performances of CBM have been largely investigated in the literature (see, e.g., Alaswad and Xiang [5] and the more recent paper of Ali and Abdelhadi [6] for a comprehensive review of proposed modeling solutions). The large majority of the proposed CBM models describe the growth of the degradation over time by using stochastic processes with independent increments, such as the classical gamma, inverse Gaussian, and Wiener degradation processes (see, e.g., [7–12]), which allow for a good level of flexibility and great convenience from the mathematical point of view. For example, [13–23] construct CBM policies based on the gamma process. Still focusing on gamma degrading units, Esposito et al. [24] suggest a hybrid age-/condition-based maintenance policy that accounts for the presence of unit to unit variability and measurement errors, whereas Yuan e al. [25] use a Bayesian pre-posterior analysis to quantify the economic value of an inspection and preventive replacement program. [26–29] suggest CBM strategies that use the inverse

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Gaussian process, whereas [30–35] propose CBM approaches that adopt the Wiener process. Some (very few) CBM policies refer to degradation processes with increments that depend both on age and state. Two examples are given in [36,37], where the degradation phenomenon is described by using a transformed gamma process [38].

A potential limitation of all these processes is that they assume that the degradation level can increase indeterminately. In the practice, this latter assumption is often not realistic, because many real-world degradation phenomena are intrinsically bounded above (see, e.g. Ling et al. [39], and Deng and Pandey [40]). Taking into account such considerations, recently Fouladirad et al. [41] proposed a new bounded version of the transformed gamma process [38] and used it to describe the degradation process of the cylinder liners of a Diesel engine for marine propulsion. Model parameters were estimated from a set of real wear data. The upper bound was treated as an unknown parameter and was estimated directly from the available dataset. The proposed process was then used to estimate the remaining useful life and residual reliability of the liners. Obtained results were compared to those obtained by using a classical (unbounded) transformed gamma process. The study showed that, in the examined case, the unbounded transformed gamma process provides estimates much more pessimistic than those obtained by using the new bounded model.

Although, due to risk aversion, the use of a more conservative unbounded model could be generally preferred to a bounded one, the high cost of the liners, the strong interest to extend their operating life, and the fact that the considered wear induced soft failure does not produce catastrophic consequences, rise the doubt that neglecting the presence of the bound, when available data give statistical evidence of its existence, could significantly affect the prognostic ability of degradation models and consequently undermine the effectiveness of implemented CBM policy.

Motivated by all these arguments, considered that there are very few papers that focus on modeling of bounded degradation phenomena and that, to the best of our knowledge, there are no studies dealing jointly with CBM and bounded degradation phenomena in the literature, the aim and novel contribution of this paper is to investigate the effect on CBM policy optimization and related maintenance costs, of using an unbounded process in place of a bounded one, when the true process is bounded above. In particular, inspired by a real world example of application, where a CBM policy is applied to the cylinder liners of a Diesel engine for marine propulsion, the paper focuses on experimental situations where bound and failure threshold have comparable values.

The degradation process is described by adopting the bounded transformed gamma process, suggested in [41]. The bounded state function, which characterizes the process, is modeled by using three new functional forms that generalize the ones suggested in [41] and enhance the flexibility of the existing model. The CBM policy used to perform the investigation is similar to the one suggested in [36]. Here, with respect to [36], we use a more general cost function, which includes an additional item that accounts for the effect of downtime. As competing (unbounded) degradation process we consider the transformed gamma process adopted in [36].

The rest of the paper is structured as it follows. Section 2 briefly resumes characteristics and properties of the (asymptotically) bounded transformed gamma process and introduces the new functional forms suggested for its state function. Section 3 is devoted to the formulation of the reliability function and conditional distribution of the RUL. Section 4 addresses the maximum likelihood estimation of model parameters. Sections 5 and 6 focus on the application of the bounded transformed gamma process and of the considered CBM policy to the cylinder liners of a Diesel engine for marine propulsion. The application is developed by using the same set of real wear data analyzed in [36,41]. Purpose of these sections is to motivate the use of a bounded process, to demonstrate the affordability and the effectiveness of the proposed approach, and to show that neglecting the presence of a bound (when there is statistical evidence in favor of the bounded model) could cause (costly) preventive premature replacements of liners. Section 7 presents the results of a small Monte Carlo simulation study aimed to evaluate the actual effect of a model misspecification on maintenance decisions and costs in an experimental situation similar to that of the real wear data. Lastly, Section 8 is devoted to final comments and conclusions.

2. The bounded transformed gamma process

Let $\eta(t)$ be a non-negative, monotone increasing function of time t, with $\eta(0) = 0$, here referred to as age function, and let $w_{\lim}$ denote the asymptotic upper bound of the degradation process. As stated in [41], the increasing bounded degradation process $\{W(t);\ t \geq 0\}$ is a bounded transformed gamma process (BTGP) if:

(i) the degradation increments over disjoint time intervals are not independent;

(ii) the degradation increment $\Delta W(t, t+\Delta t) \equiv W(t+\Delta t) - W(t)$ over the time interval $(t, t+\Delta t)$ depends on the process history up to t only through the current time t and the current degradation level (status) $w_t = W(t)$, being independent on the past (i.e., the process enjoys the Markov property);

(iii) the conditional probability density function (pdf) of the increment $\Delta W(t, t+\Delta t)$, given $W(t) = w_t$, can be expressed as:

$$f_{\Delta W(t,t+\Delta t)|W(t)}(\delta|w_t) = g'(w_t+\delta)\frac{[\Delta g(w_t,\ w_t+\delta)]^{\Delta\eta(t,t+\Delta t)-1}}{\Gamma(\Delta\eta(t,t+\Delta t))} \times \exp(-\Delta g(w_t, w_t+\delta)),\ 0<\delta<w_{\lim}-w_t \tag{1}$$

where $g(w)$ is a non-negative, monotone increasing and differentiable function, with bounded domain, of the degradation level w $(0 \leq w < w_{\lim})$, called the state function, with $g(0) = 0$ and

$$\lim_{w \to w_{\lim}} g(w) = \infty, \tag{2}$$

$g'(w_t+\delta)$ is the first derivative of $g(w)$ evaluated at $w_t+\delta$, $\Delta g(w_t, w_t+\delta) = g(w_t+\delta) - g(w_t)$, $\Delta\eta(t, t+\Delta t) = \eta(t+\Delta t) - \eta(t)$, and $\Gamma(\cdot)$ is the complete gamma function.

Of course, the BTGP is fully defined once the functional forms of the state and age functions are specified. Suitable 2-parameter forms for the state function of a BTGP, referred to as "bounded" state functions, were proposed by Fouladirad et al. [41]:

$$g_1(w) = -\beta \ln\left(1 - \frac{w}{w_{\lim}}\right) \tag{3}$$

$$g_2(w) = \beta \frac{w}{w_{\lim} - w} \tag{4}$$

$$g_3(w) = \beta \tan\left(\frac{\pi}{2}\frac{w}{w_{\lim}}\right), \tag{5}$$

To enhance the flexibility of the BTGP, the following new (more general) 3-parameter "bounded" state functions:

$$g_4(w) = -\beta \ln\left(1 - \left(\frac{w}{w_{\lim}}\right)^{\gamma}\right) \tag{6}$$

$$g_5(w) = \beta \left(\frac{w}{w_{\lim} - w}\right)^{\gamma} \tag{7}$$

$$g_6(w) = \beta \tan\left(\frac{\pi}{2}\left(\frac{w}{w_{\lim}}\right)^{\gamma}\right), \tag{8}$$

are here proposed, which reduce to the (2-parameter) state functions in Eqs. (3)–(5) for $\gamma = 1$. The first derivatives of the functions in Eqs, (6)-

(8) are, respectively:

$$g_4'(w) = \frac{\gamma\,\beta}{w}\,\frac{1}{(w_{\lim}/w)^{\gamma}-1} \tag{9}$$

$$g_5'(w) = \frac{\gamma\,\beta\,w_{\lim}}{w^2}\left(\frac{w}{w_{\lim}-w}\right)^{\gamma+1} \tag{10}$$

$$g_6'(w) = \frac{\gamma\,\beta}{w}\left(\frac{w}{w_{\lim}}\right)^{\gamma}\frac{\pi/2}{\cos^2\left(\frac{\pi}{2}\left(\frac{w}{w_{\lim}}\right)^{\gamma}\right)}\,. \tag{11}$$

In Fig. 1 the "bounded" state functions in Eqs. (6)–(8) are depicted, by assuming $w_{\lim}=10$ and several values of the shape parameter γ, and by setting the parameter β so that $g_l(8)=4$ ($l=4,\ 5,\ 6$). The figure gives evidence that the suggested state functions cover a wide range of behaviors of monotonically increasing functions with bounded domain, showing in some cases the presence of an inflection point.

Suitable (possibly non-linear) forms for the age function are the power-law function

$$\eta(t)=(t/a)^b,\ \ a,b>0, \tag{12}$$

which is convex (concave) when the shape parameter b is larger than 1 (smaller than 1), and the exponential function

$$\eta(t)=(b\,/\,a)[\exp(t\,/\,b)-1],\ \ a>0,\ \ -\infty<b<\infty, \tag{13}$$

which is convex (concave) when the shape parameter b is larger than 0 (smaller than 0). Clearly, the age function in Eq. (12) is linear with the age t when $b=1$, whereas the age function in Eq. (13) tends to be linear with t when $b\to\infty$. Note that when $\eta(t)$ is linear with the age t, so that $\Delta\eta(t,t+\Delta t)\propto\Delta t$, then from Eq. (1) we have that the conditional distribution of the degradation increment $\Delta W(t,t+\Delta t)$, given the current state $W(t)=w_t$, does not depend on the current age t. Otherwise, the conditional distribution of $\Delta W(t,t+\Delta t)$, given the current state $W(t)=w_t$, depends also on the current age t and hence the process is said to be age- and state-dependent.

Similarly, the functional form of the state function $g(w)$ determines how, given the age t, the increment $\Delta W(t,t+\Delta t)$ depends on the current degradation state $W(t)=w_t$.

From Eq. (1), the conditional cumulative distribution function (Cdf) of the degradation increment $\Delta W(t,t+\Delta t)$ is given by:

$$F_{\Delta W(t,t+\Delta t)|W(t)}(\delta|w_t) = \begin{cases}\dfrac{\Gamma_{\mathrm{L}}(\Delta g(w_t,w_t+\delta);\Delta\eta(t,t+\Delta t))}{\Gamma(\Delta\eta(t,t+\Delta t))}, & \text{for } \delta<w_{\lim}-w_t\\ 1, & \text{for } \delta\ge w_{\lim}-w_t\end{cases}, \tag{14}$$

where:

$$\Gamma_{\mathrm{L}}(x;a)=\int_0^x u^{a-1}\exp(-u)\,\mathrm{d}u \tag{15}$$

is the (lower) incomplete gamma function. From Eqs. (1) and (14), by using $g(w)$ and $\eta(t)$ in place of $\Delta g(w_t,w_t+\delta)$ and $\Delta\eta(t,t+\Delta t)$, respectively, the pdf and the Cdf of the degradation level $W(t)$ at the time t of a new (unused) unit (that is, of a unit with age $t=0$ and state $W(0)=0$) are obtained:

$$f_{W(t)}(w)=g'(w)\,\frac{[g(w)]^{\eta(t)-1}}{\Gamma(\eta(t))}\exp(-g(w)),\ \ 0<w<w_{\lim} \tag{16}$$

$$F_{W(t)}(w)=\begin{cases}\dfrac{\Gamma_{\mathrm{L}}(g(w);\eta(t))}{\Gamma(\eta(t))}, & \text{for } w<w_{\lim}\\ 1, & \text{for } w\ge w_{\lim}\end{cases}. \tag{17}$$

The conditional mean and variance of the degradation increment, given the state w_t at the time t, are not in a closed form, and require numerical integration:

$$E\{\Delta W(t,t+\Delta t)|W(t)=w_t\}=\int_0^{w_{\lim}-w_t}\delta\,f_{\Delta W(t,t+\Delta t)|W(t)}(\delta|w_t)\,\mathrm{d}\delta \tag{18}$$

$$V\{\Delta W(t,t+\Delta t)|W(t)=w_t\} = \int_0^{w_{\lim}-w_t}\delta^2\,f_{\Delta W(t,t+\Delta t)|W(t)}(\delta|w_t)\,\mathrm{d}\delta - E^2\{\Delta W(t,t+\Delta t)|W(t)=w_t\}. \tag{19}$$

Likewise, the mean and variance of the degradation level $W(t)$ are given by:

$$E\{W(t)\}=\int_0^{w_{\lim}}w\,f_{W(t)}(w)\mathrm{d}w \tag{20}$$

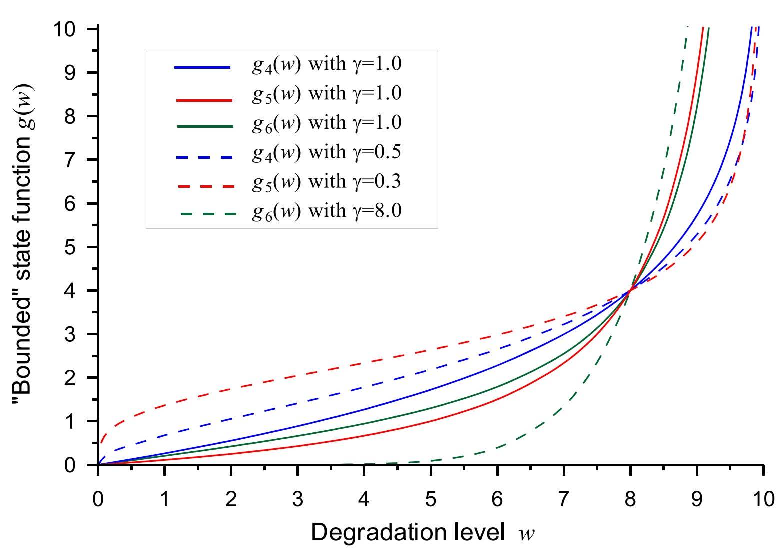

Fig. 1. The "bounded" state functions $g_l(w)$ ($l=4,\ 5,\ 6$), for $w_{\lim}=10$, several values of γ, and β such that $g_l(8)=4$ ($l=4,\ 5,\ 6$).

$$V\{W(t)\} = \int_0^{w_{\lim}} w^2 f_{W(t)}(w)\mathrm{d}w \; - E^2\{W(t)\} \,. \tag{21}$$

Under the considered settings, also these mean and variance functions are not available in closed form.

Due to the "bounded" nature of the state function $g(w)$, for $t\to\infty$ the pdf in Eq. (16) tends to the Dirac delta distribution with support $w_{\lim}$, and hence the mean $E\{W(t)\}$ of the degradation process tends to $w_{\lim}$ for $t\to\infty$. In turns, the variance $V\{W(t)\}$, which initially grows with time, approaches to zero for $t\to\infty$.

Similarly, for $\Delta t\to\infty$, (given t) the conditional pdf in Eq. (1) tends to the Dirac delta distribution with support $w_{\lim} - w_t$, and hence the conditional mean $E\{\Delta W(t, t+\Delta t)|W(t) = w_t\}$ of the degradation increment tends to $w_{\lim} - w_t$, and the conditional variance $V\{\Delta W(t, t+\Delta t)|W(t) = w_t\}$ approaches to zero for $\Delta t\to\infty$. Of course, for $w_t\to w_{\lim}$, the conditional mean $E\{\Delta W(t, t+\Delta t)|W(t) = w_t\}$ tends to 0.

In addition, from Eq. (16), we have that, under the state functions in Eqs. (6)–(8), the upper limit $w_{\lim}$ acts as a scale parameter for the pdf $f_{W(t)}(w)$ of the degradation level at the time t, which then can be rewritten as:

$$f_{W(t)}(w) = \frac{1}{w_{\lim}} h(w\,/\,w_{\lim}; \eta(t), \beta, \gamma), \tag{22}$$

where the function $h(z; \eta(t), \beta, \gamma)$, with $z = w/w_{\lim}$, depends on $\eta(t)$, on the functional form of $g(w)$, on β, and on γ. Thus, since $\mathrm{d}w\,/\mathrm{d}z = w_{\lim}$, the (dimensionless) random variable $Z(t) = W(t)/w_{\lim}$ has pdf $f_{Z(t)}(z) = h(z; \eta(t), \beta, \gamma)$ that does not depend on $w_{\lim}$. In particular, under the state functions in Eqs. (6)–(8), the function $f_{Z(t)}(z)$ is given by, respectively:

$$f_{Z(t)}(z) = \frac{\gamma\, z^{\gamma-1}\beta^{\eta(t)}\,\left[-\ln(1-z^\gamma)\right]^{\eta(t)-1}}{\Gamma(\eta(t))}(1-z^\gamma)^{\beta-1}, \;\; 0 < z < 1$$

$$f_{Z(t)}(z) = \frac{\gamma\, \beta^{\eta(t)}\; z^{\gamma\eta(t)-1}}{(1-z)^{\gamma\eta(t)+1}\;\Gamma(\eta(t))} \exp\left(-\beta\left(\frac{z}{1-z}\right)^\gamma\right), \;\; 0 < z < 1$$

$$f_{Z(t)}(z) = \frac{\pi}{2}\,\frac{\gamma\,\beta^{\eta(t)}\;z^{\gamma-1}}{\cos^2\left(\frac{\pi}{2}z^\gamma\right)}\,\frac{\left[\tan\left(\frac{\pi}{2}z^\gamma\right)\right]^{\eta(t)-1}}{\Gamma(\eta(t))}\;\exp\left(-\beta\,\tan\left(\frac{\pi}{2}z^\gamma\right)\right),$$

$$0 < z < 1.$$

This implies that the mean and the variance of $W(t)$ depend linearly on $w_{\lim}$ and $w_{\lim}^2$, respectively:

$$E\{W(t)\} = w_{\lim}\int_0^1 z\, f_{Z(t)}(z)\,\mathrm{d}z$$

$$V\{W(t)\} = w_{\lim}^2\left[\int_0^1 z^2 f_{Z(t)}(z)\,\mathrm{d}z - \left(\int_0^1 z\, f_{Z(t)}(z)\,\mathrm{d}z\right)^2\right].$$

By assuming the state function $g_4(w)$ in Eq. (6) and the power-law age function $\eta(t) = (t/a)^b$ in Eq. (12), the curves of the degradation mean $E\{W(t)\}$ are depicted in Fig. 2, for $w_{\lim} = 10$, and different selected values of the process parameters a, b, β, and γ, that is: (1, 1, 1, 1), (1, 0.5, 1, 1), (1, 1, 2, 1), (1, 2, 5, 1), (3, 1, 1, 1), and (1, 2, 5, 0.5), respectively. We have that the mean curve has an inflection point only when the age parameter b is larger than 1 (see the dashed green and black lines of Fig. 2 for which $b = 2.0$), that is, when the age function is convex.

In Fig. 3, the variance $V\{W(t)\}$ of the same degradation model is depicted for $w_{\lim} = 10$ and for the same selected values of the process parameters a, b, β, and γ, used for Fig. 2. We note that, since the age function $\eta(t)$, being non negative, continuous, and unbounded, only affects the "time" scale of the process, the maximum value $V_{\max}\{W(t)\}$ of the variance does not depend on the functional form of the age function and on the values of its parameters a and b (see, at this purpose, the dashed brown line and the continuous blue and red lines in Fig. 3, whose maximum is the same). Thus, the ratio $V_{\max}\{W(t)\}/w_{\lim}^2$ does not depends on a and b, but only on the parameter β of the state function.

In addition, given a, b, β and γ, the time at which the variance reaches its maximum value does not depend on the value of $w_{\lim}$, and given b, β and γ is proportional to a, regardless of the value of $w_{\lim}$. At this purpose, see the continuous blue and red lines of Fig. 3 (indexed by the same values of b, β, and γ), where the time at which the corresponding variances reach their maximum value is equal to 0.81 (blue line, when $a = 1.0$) and 2.43 (red line, when $a = 3.0$).

Similar behavior of the mean and variance curves and the same properties of the mean and variance of $W(t)$ are provided by assuming the state functions in Eqs. (7) and (8) and/or the exponential age function in Eq. (13).

3. The remaining useful life and reliability function

In the context of increasing degradation processes, a unit is

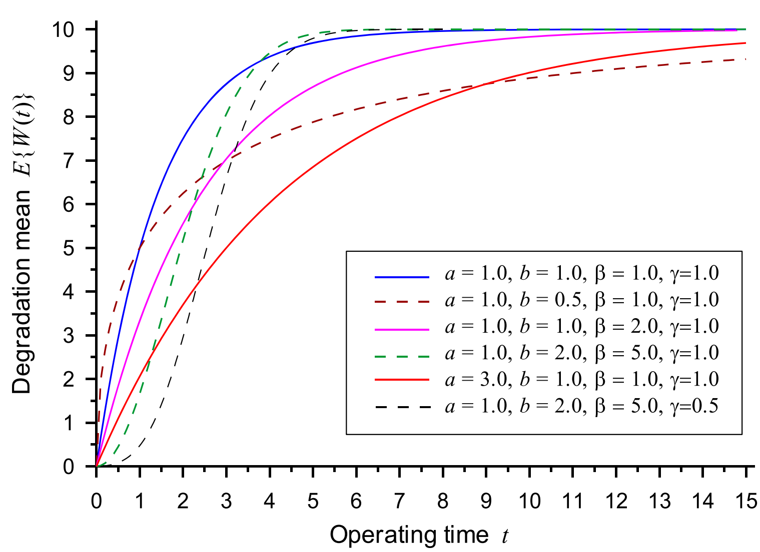

Fig. 2. Behavior of the degradation mean $E\{W(t)\}$, for $w_{\lim} = 10$ and selected values of the process parameters a, b, β, and γ, when $g(w) = g_4(w) = -\beta\,\ln(1-(w/w_{\lim})^\gamma)$ and $\eta(t) = (t/a)^b$.

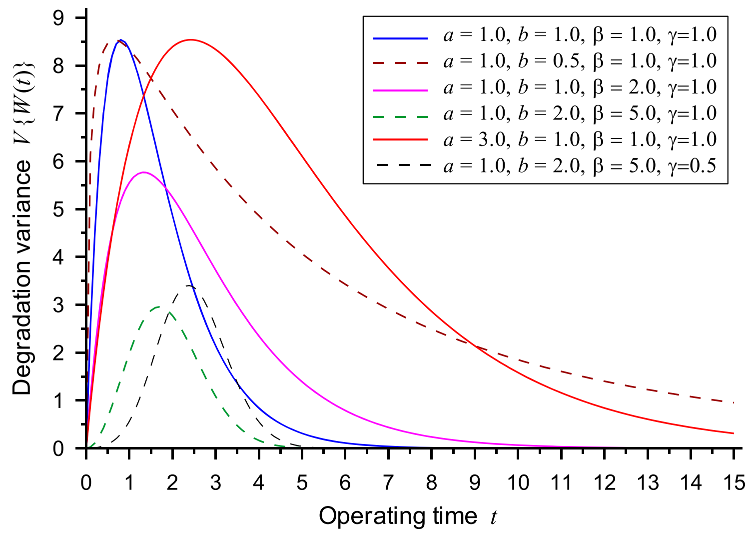

Fig. 3. Behavior of the degradation variance $V\{W(t)\}$, for $w_{\lim} = 10$ and selected values of the process parameters a, b, β, and γ, when $g(w) = g_4(w) = -\beta \ln(1 - (w/w_{\lim})^\gamma)$ and $\eta(t) = (t/a)^b$.

conventionally assumed to fail when its degradation level W exceeds a threshold limit D ($D < w_{\lim}$). Then, the unit lifetime X is defined as the operating time to the first, and sole, passage beyond the limit D.

Likewise, the remaining useful life (RUL) X_t of a unit at time t is defined as $X_t = \max\{0,\ X - t\}$, so that X_t is equal to $X - t$ if the unit at t is unfailed, and is assumed to be 0 otherwise.

Then, by using the conditional Cdf in Eq. (14) of the degradation increment $\Delta W(t, t + \Delta t)$, the residual reliability, that is the conditional probability of the RUL X_t exceeding the time x, given the current state $W(t) = w_t < D$ at the current age t, is given by:

$$\begin{aligned} R_t(x|w_t) &= \Pr\{\Delta W(t, t + x) \le D - w_t | W(t) = w_t\} \\ &= \frac{\Gamma_L(\Delta g(w_t, D); \Delta\eta(t, t + x))}{\Gamma(\Delta\eta(t, t + x))}. \end{aligned} \tag{23}$$

If the age function $\eta(t)$ is differentiable with respect to t, then the conditional pdf of the RUL X_t can be obtained by deriving the residual reliability in Eq. (23) with respect to x:

$$f_{X_t|W(t)}(x|w_t) = -\frac{d}{dx} \int_0^{\Delta g(w_t, D)} \frac{u^{\Delta\eta(t,t+x)-1}}{\Gamma(\Delta\eta(t, t + x))} \exp(-u) du. \tag{24}$$

Both the suggested age functions, say the power-law function $\eta(t) = (t/a)^b$ in Eq. (12) and the exponential function $\eta(t) = (b/a)[\exp(t/b) - 1]$ in Eq. (13), are differentiable with respect to t, and the derivatives of $\Delta\eta(t, t+x)$ with respect to x are, respectively, equal to: $d\Delta\eta(t, t+x)/dx = (b/a)[(t + x)/a]^{b-1}$ and $d\Delta\eta(t, t + x)/dx = (1/a) \exp((t + x)/b)$.

Then, by using arguments provided in [36], instead of resorting to numerical integrations, the conditional pdf of X_t can be expressed in the following analytical form:

$$\begin{aligned} f_{X_t|W(t)}(x|w_t) = \frac{d\Delta\eta(t, t + x)}{dx} \frac{1}{\Gamma(\Delta\eta(t, t + x))} \\ \times \left\{ \Gamma_L(\Delta g(w_t, D); \Delta\eta(t, t + x)) \, [\psi(\Delta\eta(t, t + x)) - \ln(\Delta g(w_t, D))] + \sum_{k=0}^{\infty} \frac{(-1)^k [\Delta g(w_t, D)]^{\Delta\eta(t,t+x)+k}}{[\Delta\eta(t, t + x) + k]^2 \ k!} \right\}, \end{aligned} \tag{25}$$

where $\psi(z)$ denotes the digamma function.

From Eq. (23), by using $g(D)$ and $\eta(x)$ in place of $\Delta g(w_t, D)$ and $\Delta\eta(t, t + x)$, respectively, the reliability function of a new unit is given by:

$$R(x) = \Pr\{W(x) \le D\} = \frac{\Gamma_L(g(D); \eta(x))}{\Gamma(\eta(x))} \ . \tag{26}$$

Likewise, from Eq. (25), the pdf of the lifetime X of a new unit is given by:

$$\begin{aligned} f_X(x) = \frac{d\eta(x)}{dx} \frac{1}{\Gamma(\eta(x))} \\ \times \left\{ \Gamma_L(g(D); \eta(x)) \, [\psi(\eta(x)) - \ln(g(D))] + \sum_{k=0}^{\infty} \frac{(-1)^k [g(D)]^{\eta(x)+k}}{[\eta(x) + k]^2 \ k!} \right\}. \end{aligned} \tag{27}$$

Finally, the mean $E\{X_t | W(t) = w_t\}$ of the RUL and the mean lifetime $E\{X\}$ of a new item are given by, respectively:

$$\begin{aligned} E\{X_t | W(t) = w_t\} &= \int_0^\infty R_t(x|w_t) dx \\ &= \int_0^\infty \frac{\Gamma_L(\Delta g(w_t, D); \Delta\eta(t, t + x))}{\Gamma(\Delta\eta(t, t + x))} dx \end{aligned}$$

and

$$E\{X\} = \int_0^\infty R(x) dx = \int_0^\infty \frac{\Gamma_L(g(D); \eta(x))}{\Gamma(\eta(x))} dx \ .$$

4. The maximum likelihood estimation procedure

Let suppose that m identical units operate under the same conditions over the time intervals $(0,\ T_i)$ $(i = 1, \ldots, m)$, and that n_i degradation measurements are made on the unit i by performing periodic inspections at the ages $t_{i,1},\ldots,t_{i,n_i}$ which are possibly non equispaced and different from unit to unit. Moreover, let $w_{i,j}$ $(i = 1,\ldots,m;\ j = 1,\ldots,n_i)$ denote the degradation level of the unit i at the epoch of its j-th inspection.

From the conditional pdf in Eq. (1), the conditional pdf of the wear increment $\Delta W(t_{i,j-1}, t_{i,j})$ accumulated by the unit i during the inspection interval $(t_{i,j-1},\ t_{i,j})$, given the state $W(t_{i,j-1}) = w_{i,j-1}$ of the unit i at the beginning of the interval, is:

$$f_{\Delta W\left(t_{i,j-1},t_{i,j}\right)|W\left(t_{i,j-1}\right)}\left(\delta_{i,j}|w_{i,j-1}\right) = g'\left(w_{i,j-1}+\delta_{i,j}\right)\frac{\left(\Delta g_{i,j}\right)^{\Delta\eta_{i,j}-1}}{\Gamma\left(\Delta\eta_{i,j}\right)}\exp\left(-\Delta g_{i,j}\right),\ 0<\delta_{i,j}<w_{\lim}-w_{i,j-1}, \tag{28}$$

where $\delta_{i,j} = w_{i,j} - w_{i,j-1}$ is the observed degradation increment, $\Delta g_{i,j} = \Delta g(w_{i,j-1},\ w_{i,j-1} + \delta_{i,j})$, and $\Delta\eta_{i,j} = \Delta\eta(t_{i,j-1}, t_{i,j})$, with $w_{i,0} = t_{i,0} = 0$ for all i. From Eq. (28), the log-likelihood function relative to the whole observed dataset $\boldsymbol{w} = (w_{1,1}, \ldots, w_{1,n_1}, \ldots, w_{m,1}, \ldots, w_{m,n_m})$ is:

$$\ell(\boldsymbol{w};\boldsymbol{\theta},w_{\lim}) = \sum_{i=1}^{m}\sum_{j=1}^{n_i}\ln\left(f_{\Delta W\left(t_{i,j-1},t_{i,j}\right)|W\left(t_{i,j-1}\right)}\left(\delta_{i,j}|w_{i,j-1}\right)\right), \tag{29}$$

where $\boldsymbol{\theta}$ denotes the vector of parameters which, together to $w_{\lim}$, index the age and state functions. For example, if the age function is the power-law function in Eq. (12) or the exponential function in Eq. (13) and the state function is one of those in Eqs. (6)–(8), then $\boldsymbol{\theta} = (a, b, \beta, \gamma)$.

The maximum likelihood (ML) estimates $\widehat{\boldsymbol{\theta}}$ and $\widehat{w}_{\lim}$ of the process parameters can be easily obtained by numerically maximizing the log-likelihood function in Eq. (29) with respect to $\boldsymbol{\theta}$ and $w_{\lim}$. Of course, the maximization procedure of the log-likelihood in Eq. (29) is "naturally" constrained because $w_{\lim}$ must be greater than the maximum observed degradation level $w_M = \max(w_{1,n_1},\ldots,w_{m,n_m})$, that is, it must be $w_{\lim} > w_M$.

Moreover, in several circumstances, previous experiences and/or physical considerations on the degradation phenomenon allow the analyst to fix a lower limit $w_L > w_M$ or an interval $(w_L,\ w_U)$ of plausible values for $w_{\lim}$. In this case, the constrained maximization procedure must satisfy also these additional (possibly more restrictive) constraints.

Once the vector of unknown parameters has been estimated, the ML estimate of any quantity of interest, say $\phi(\boldsymbol{\theta}, w_{\lim})$, can be easily obtained by substituting $\widehat{\boldsymbol{\theta}}$ and $\widehat{w}_{\lim}$ to $\boldsymbol{\theta}$ and $w_{\lim}$ in $\phi(\boldsymbol{\theta}, w_{\lim})$. In particular, by using this approach, one can easily obtain the ML estimates of the conditional distribution $f_{X_t|W(t)}(x|w_t)$ of the RUL, the mean lifetime $E\{X\}$, the mean RUL $E\{X_t|W(t) = w_t\}$, the residual reliability, and the conditional distribution of the degradation growth over a future time interval.

Table 1
Wear $w_{i,j} = W(t_{i,j})$ [mm] accumulated by liner i up to the inspection time $t_{i,j}$ [h].

i	$w_{i,1}$	$t_{i,1}$	$w_{i,2}$	$t_{i,2}$	$w_{i,3}$	$t_{i,3}$	$w_{i,4}$	$t_{i,4}$
1	0.90	11,300	1.30	14,680	2.85	31,270		
2	1.50	11,300	2.00	21,970				
3	1.00	12,300	1.35	16,300				
4	1.90	14,810	2.25	18,700	2.75	28,000		
5	1.20	10,000	2.75	30,450	3.05	37,310		
6	0.50	6860	1.45	17,200	2.15	24,710		
7	0.40	2040	2.00	12,580	2.35	16,620		
8	0.50	7540	1.10	8840	1.15	9770	2.10	16,300

5. Case study

Let now consider the wear measurements of the liners of the 8-cylinder SULZER engine which equips a cargo ship of the Grimaldi Lines, which has operated under homogeneous conditions. The dataset, given in Table 1 and depicted in Fig. 4, consists of a total of 23 wear measurements made via ad hoc inspections carried out between 1994 and 2004. At each inspection, the wear of the inspected liner is measured by positioning a micrometer inside a predetermined hole located at the top dead center of the liner (the point where maximum mechanical and thermal loads are concentrated). Each datum consists of the age (in hours) of the liner at the inspection epoch and of the corresponding wear measurement (in mm).

Hereinafter, for the sake of simplicity, we will use interchangeably the wording "age of the liner at the inspection epoch" and "inspection time".

These wear data were previously analyzed by Giorgio et al. [36] by using a transformed gamma process (TGP) with the "unbounded" power-law state function $g(w) = (w/\alpha)^\beta$, and two different age functions: the power-law function $\eta(t) = (t/a)^b$ in Eq. (12), and the exponential function $\eta(t) = (b/a)[\exp(t/b) - 1]$ in Eq. (13).

The TGP with power-law age function provided the best fit for these data, but although this TGP showed to be able to fit quite well the empirical mean of the wear process (see the dashed line in the following Fig. 5), it was not able to fit adequately the empirical variance (see the dashed line in the following Fig. 6), in particular in the region where the empirical variance decreases quickly.

On the other side, physical considerations related to the wear mechanism suggest that the liner wear cannot growth up to the thickness of the liners (i.e., 100 mm), so that an asymptotical upper bound $w_{\lim}$ for the wearing phenomenon exists, although the same physical considerations alone do not allow to determine it exactly. However, previous experiences suggest that the liner wear can approach the value of 4.3 mm, a wear value which is obviously larger than the maximum value $w_M = 3.05$ mm of the considered dataset. Thus, the liner wear data were analyzed in [41] under the BTGP by treating the upper bound as an unknown parameter subject to the constraint $w_{\lim} \geq w_L = 4.3$ mm.

In particular, the three "bounded" state functions in Eqs. (3)–(5) and the power-law age function $\eta(t) = (t/a)^b$ in Eq. (12) were there considered. According to the Akaike information criterion (see Akaike [42]), the BTGP that provided the best fit to the wear data was the (4-parameters) BTGP with the state function $g_2(w)$ in Eq. (4), here denoted by "PL-BTGP2".

In this paper we have analyzed the same wear data by using the more general state functions in Eqs. (6)–(8), jointly with the power-law age function $\eta(t) = (t/a)^b$ in Eq. (12) or the exponential age function $\eta(t) = (b/a)[\exp(t/b) - 1]$ in Eq. (13). Estimates have been obtained by maximizing the log-likelihood function in Eq. (29). To account for the presence of a constraint on $w_{\lim}$, the maximization has been performed by using the DBCPOL subroutine of the IMSL Math/Library [43], a double-precision Fortran routine that minimizes a function (in this case, the negative log-likelihood) of n variables subject to bounds on the variables using a direct search complex algorithm.

In Table 2 the ML estimates of the parameters of the considered 5-parameter BTGPs, the corresponding estimated log-likelihood $\widehat{\ell}$, and the Akaike information criterion (AIC) value (AIC $= 2\nu - 2\widehat{\ell}$, where ν denotes the number of model parameters) are provided, and compared to the estimates obtained both in [41] under the PL-BTGP2 and in [36] under the TGP with power-law age and state functions.

Note that the 5-parameter BTGP with "bounded" state function $g_l(w)$ $(l = 4,5,6)$ in Eqs. (6)–(8) and power-law age function is here denoted as "PL-BTGPl", whereas the 5-parameter BTGP with "bounded" state function $g_l(w)$ $(l = 1,\ \ldots,\ 3)$ in Eqs. (6)–(8) and exponential age function is denoted as "Ex-BTGPl" $(l = 4, 5, 6)$.

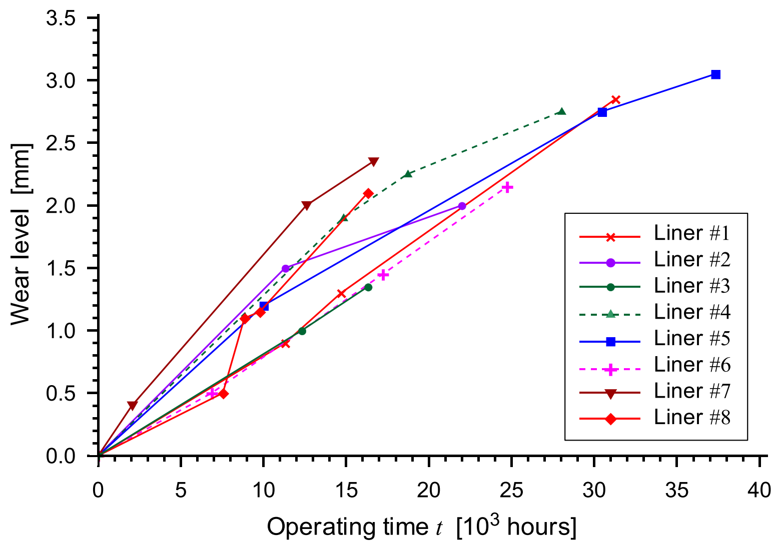

Fig. 4. Observed wear paths of the liners (measurements that pertain to the same liner are linearly connected for graphical convenience).

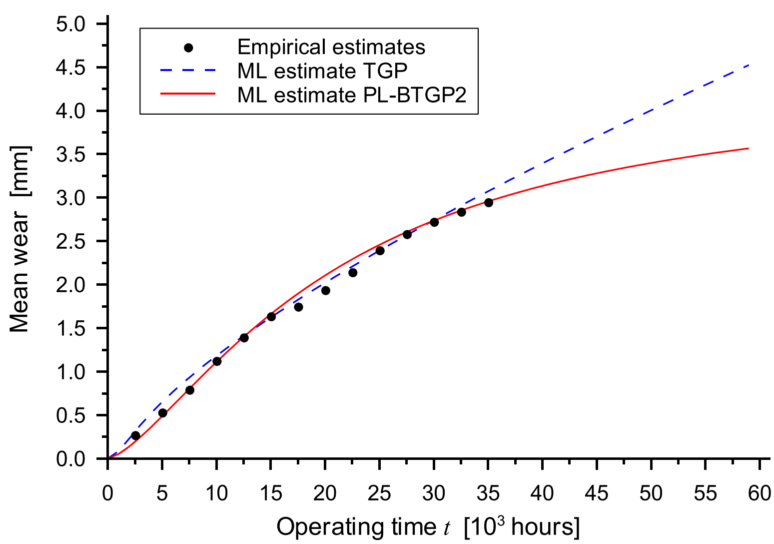

Fig. 5. Empirical and ML estimates of the mean wear $E\{W(t)\}$ under the PL-BTGP2 and the TGP.

The estimation results given in Table 2 show that all the proposed BTGPs fit the liner wear data better than the TGP, because the corresponding AIC values are smaller than the AIC value relative to the TGP. In addition, among the BTGPs, the process that provides the best fit is the PL-BTGP2, already used in [41], with the state function $g_2(w) = \beta\, w/(w_{\text{lim}} - w)$ given in Eq. (4) and the age function $\eta(t) = (t/a)^b$ given in Eq. (12). Indeed, although the estimated log-likelihoods under the PL-BTGP5 and the Ex-BTGP5 are both larger than the estimated log-likelihood under the PL-BTGP2, this latter bounded process is favored in terms of AIC by the fact that it is indexed by only $\nu = 4$ parameters, instead of 5. Thus, in the following, we will mainly focus on PL-BTGP2 and TGP to analyze the wear data of the cylinder liners.

In Figs. 5 and 6 the ML estimates of the mean $E\{W(t)\}$ and variance $V\{W(t)\}$ of the wear process under the PL-BTG2 and the TGP are compared to the empirical estimates.

Since the inspection times generally differ from liner to liner, and hence the wear measurements generally refer to different operating times of the liners, the empirical estimates of mean and variance were obtained by using the interpolation procedure at selected equi-spaced times τ_k ($\tau_k = k{\cdot}2.5{\cdot}10^3$ h) already adopted in [36]. In particular, for each liner i such that $t_{i,n_i} \geq \tau_k$, the linearly interpolated wear value $w_i(\tau_k)$ at the time τ_k is obtained by:

$$w_i(\tau_k) = \frac{w_{i,j} - w_{i,j-1}}{t_{i,j} - t_{i,j-1}} \left(\tau_k - t_{i,j-1}\right) + w_{i,j-1},$$

where $t_{i,j}$ is the smallest inspection time of liner i larger than or equal to τ_k.

From Fig. 5 we have that the proposed PL-BTGP2 fits the empirical mean a little better that the TGP, in particular for large operating time t where the mean of the TGP, unlike the one of the PL-BTGP2, is not able to bend down. On the other side, Fig. 6 shows that the PL-BTGP2, unlike the TGP, is able to fit adequately both the initial growth of the empirical variance and its subsequent rapid reduction.

In Fig. 7 the ML estimates of the reliability function of a new liner under the PL-BTGP2 and the TGP are plotted. As noted also in [41] the TGP greatly underestimates the unit reliability, and consequently also the mean lifetime $E\{X\}$, with respect to the PL-BTGP2. Indeed, the ML estimates of $E\{X\}$ is equal to 109,994 h under the PL-BTGP2, whereas under the TGP it is equal to only 49,954 h.

Fig. 8 depicts the ML estimates of the conditional pdf $f_{X_t|W(t)}(x|w_t)$ of

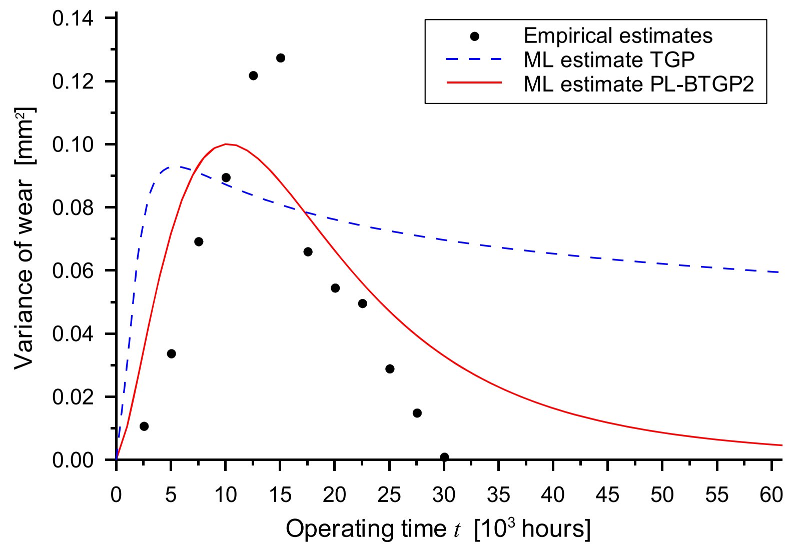

Fig. 6. Empirical and ML estimates of the variance $V\{W(t)\}$ under the PL-BTGP2 and the TGP.

Table 2
Estimation results under the BTGP and the TGP.

Process	$\hat{a}$ [h]	$\hat{b}$	$\hat{w}_{\mathrm{lim}}$ [mm]	$\hat{\alpha}$ [mm]	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\ell}$	AIC
PL-BTGP4	1936	1.239	4.3		33.69	1.204	2.693	4.614
PL-BTGP5	2298	1.359	4.3		19.13	0.931	3.862	2.276
PL-BTGP6	1527	1.187	4.3		21.81	1.010	3.837	2.326
Ex-BTGP4	1250	$1.000{\cdot}10^{11}$ [h]	4.3		29.83	0.854	2.572	4.856
Ex-BTGP5	868	$4.341{\cdot}10^{5}$ [h]	4.3		21.11	0.852	3.981	2.038
Ex-BTGP6	797	$6.323{\cdot}10^{5}$ [h]	4.522		19.78	0.740	3.531	2.938
PL-BTGP2	2682	1.434	4.363		18.62		3.843	0.314
TGP	5107	1.701		0.750	2.312		0.590	6.820

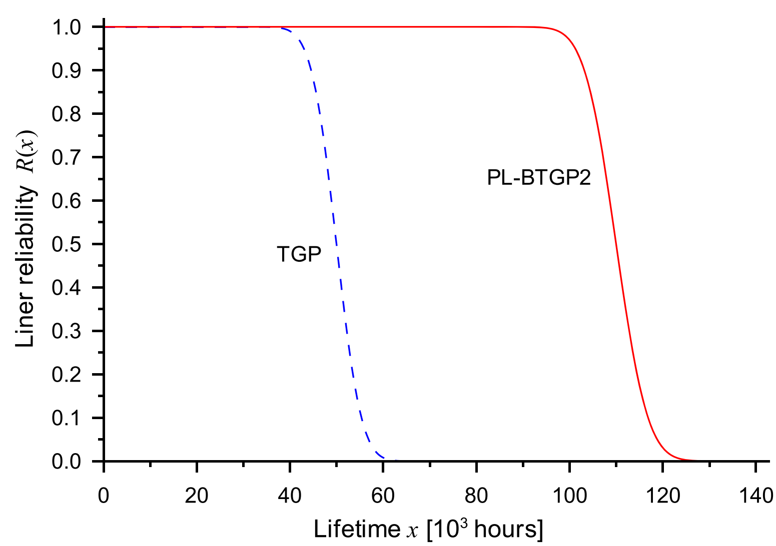

Fig. 7. ML estimates of the reliability $R(x)$ of a new liner under the PL-BTGP2 and the TGP.

the RUL X_t of the liners #3 and 5 under the PL-BTGP2 and the TGP, given their wear level $w_t = 1.35$ and 3.05 mm at the last inspection time $t = 16,300$ and 37,310 h, respectively. Note that liners #3 and 5 were chosen for the comparative analysis because these liners are those that at the last inspection reached the lowest and the highest wear level, respectively. These plots show that the TGP greatly underestimates the RUL of both the liners. It should be noted that similar results have been obtained for the other liners.

Table 3 gives the ML estimate $\widehat{E}\{X_t|W(t) = w_t\}$ of the mean RUL of all the liners, given the age t and the wear level w_t at the last inspection epoch of each liner, under the PL-BTGP2 and the TGP. We can note that the TGP strongly underestimates the mean RUL of the liners with respect to the PL-BTGP2, sometimes being the estimate of $E\{X_t|W(t) = w_t\}$ obtained under the TGP four times smaller than the one obtained under the PL-BTGP2. The greater the current wear level is, the more the TGP underestimates the liner RUL.

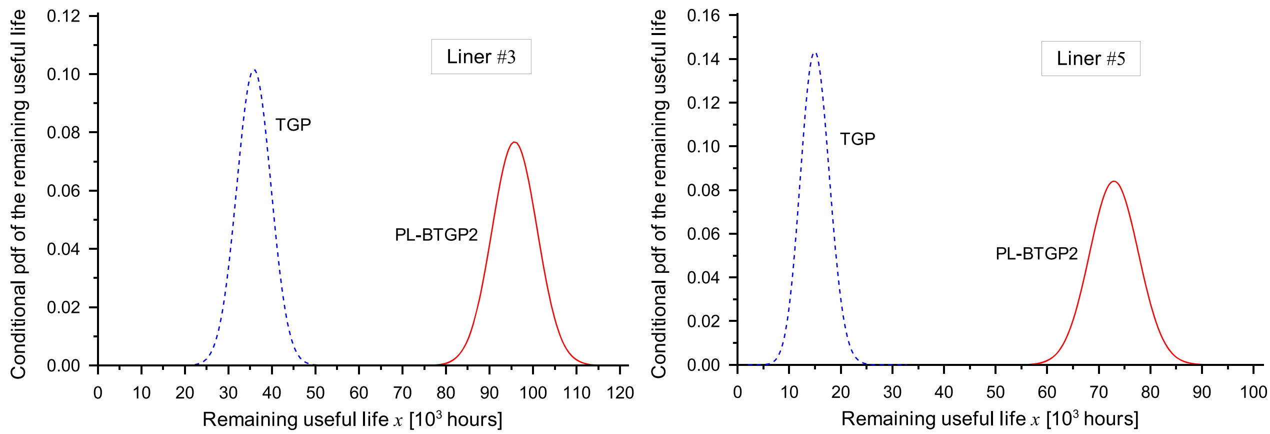

Fig. 8. ML estimates of the conditional pdf $f_{X_t|W(t)}(x|w_t)$ of the remaining useful life X_t of liners #3 and 5, given the age t and the wear level w_t at the last inspection epoch, under the PL-BTGP2 and the TGP.

Table 3
Age t and state w_t of each liner i ($i = 1, \ldots, 8$) at the last inspection epoch and ML estimates of the mean RUL $E\{X_t|W(t) = w_t\}$ under the PL-BTGP2 and the TGP.

	Liner							
	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$	$i = 6$	$i = 7$	$i = 8$
Current age t [h]	31,270	21,970	16,300	28,000	37,310	24,710	16,620	16,300
Current state w_t [mm]	2.85	2.00	1.35	2.75	3.05	2.15	2.35	2.10
$\widehat{E}\{X_t|W(t) = w_t\}$ under PL-BTGP2	78,270	89,750	95,533	80,822	72,822	87,536	90,344	92,201
$\widehat{E}\{X_t|W(t) = w_t\}$ under TGP	18,653	29,389	35,649	20,693	14,965	27,180	29,224	31,434

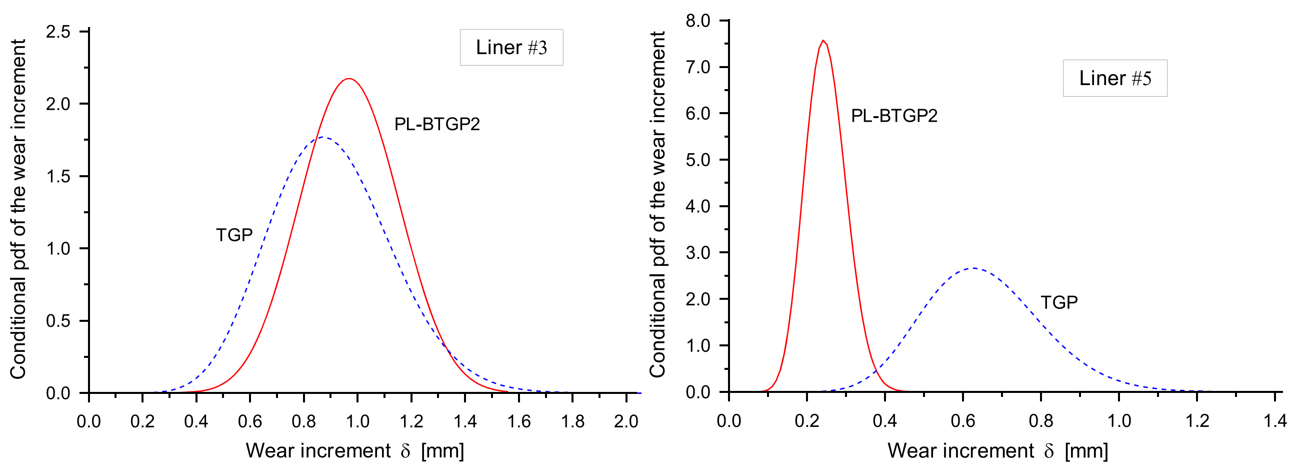

Fig. 9. ML estimates of the conditional pdf $f_{\Delta W(t,t+\Delta t)|W(t)}(\delta|w_t)$ of the wear increment of liners #3 & 5 at the last inspection epoch during the future time interval of width $\Delta t = 10,000$ h, given their current age t and the wear level w_t, under the TGP and the PL-BTGP2.

Table 4
Age t and state w_t of each liner i ($i = 1, \ldots, 8$) at the last inspection epoch and ML estimates of the mean wear growth $E\{\Delta W(t, t+\Delta t)|W(t) = w_t\}$ under the PL-BTGP2 and the TGP for $\Delta t = 10,000$ h.

	Liner							
	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$	$i = 6$	$i = 7$	$i = 8$
Current age t [h]	31,270	21,970	16,300	28,000	37,310	24,710	16,620	16,300
Current state w_t [mm]	2.85	2.00	1.35	2.75	3.05	2.15	2.35	2.10
$\widehat{E}\{\Delta W(t, t+\Delta t)|W(t) = w_t\}$ under PL-BTGP2	0.353	0.693	0.969	0.382	0.289	0.639	0.489	0.597
$\widehat{E}\{\Delta W(t, t+\Delta t)|W(t) = w_t\}$ under TGP	0.637	0.750	0.903	0.623	0.653	0.743	0.563	0.623

It must also be noted that so large differences between the predictions of the RUL X_t under the PL-BTG2 and the TGP, even when the current wear level is far below the estimated upper bound $\widehat{w}_{\lim}$ (as occurs in particular for the liner #3), are due to the fact that the upper bound of the liner wear process is slightly greater than the threshold value D.

Finally, Fig. 9 provides the ML estimates of the conditional pdf $f_{\Delta W(t,t+\Delta t)|W(t)}(\delta|w_t)$ of the wear increment $\Delta W(t,t+\Delta t)$ of liners #3 and 5 during the future time interval of width $\Delta t = 10,000$ h, under the PL-BTGP2 and the TGP, given their wear level $w_t = 1.35$ and 3.05 mm and age $t = 13,300$ and 37,310 h, respectively, at the last inspection epoch.

From the plots in Fig. 9 we note that, compared to the PL-BTGP2, the TGP greatly overestimates the future wear growth of the liner #5, whose current wear level $w_t = 3.05$ is no far from the estimated $\widehat{w}_{\text{lim}} = 4.363$. On the contrary, for the liner #3 whose current wear level $w_t = 1.35$ is far from the estimated $\widehat{w}_{\text{lim}}$, the TGP underestimates a little the wear growth with respect to the PL-BTGP2.

Finally, in Table 4, the ML estimates $\widehat{E}\{\Delta W(t,t+\Delta t)|W(t) = w_t\}$ of the mean wear growth of all the liners under the PL-BTGP2 and the TGP are provided, for $\Delta t = 10,000$ h. We note that, always with respect to the PL-BTGP2, the TGP greatly overestimates the mean wear growth of almost all liners (the overestimation is larger than 40%, being in particular equal to 165% for the liner #5), with the exception of liners #2 and 3, whose current wear levels are the lowest ones, say 2.00 and 1.35 mm, respectively. In particular, the TGP overestimates the mean wear growth of the liner #2 by only 8.3%, whereas the TGP even underestimates the mean wear growth of the liner #3 by 6.8%.

For the sake of comparison, and also to check whether the use of another bounded process in place of the PL-BTGP4 could lead to different conclusions, in Tables 5 and 6 we report the estimates of the RUL and mean wear growth $E\{\Delta W(t,t+\Delta t)|W(t) = w_t\}$, respectively, computed under the other bounded processes with the power-law age function (12) and the 3-parameter "bounded" state functions (6)-(8), already considered in Table 2.

The results reported in Table 5 show that all the bounded processes provide estimates of the mean RUL that are much greater than the corresponding estimates obtained under the TGP (see Table 3), clearly due to the effect of the finite bound w_{lim}. We also note that the estimates of the mean RUL provided by the PL-BTGP5 and PL-BTGP6 are always rather close to the ones obtained under the PL-BTGP2, whereas the mean RUL estimates provided by the PL-BTGP4 model, although larger than the estimates under the unbounded TGP, are quite pessimistic when compared to the PL-BTGP2, PL-BTGP5, and PL-BTGP6 models.

Differences among the mentioned estimates are more evident in the case of liners whose degradation level is closer to the estimated bound $\widehat{w}_{\text{lim}}$. For example, from Fig. 10 it is evident that, in the case of the liner #1, whose current degradation level is 2.85 mm, the pdfs of the RUL at $t = 31,270$ h obtained under all the bounded processes are much more optimistic than the one estimated by using the unbounded TGP. In fact, also the conditional pdf of X_t obtained under the PL-BTGP4 (that, according to the Akaike information criterion, see Table 2, is the bounded process that provides the worst fit for the considered liner data is quite shifted to the right with respect to the conditional pdf obtained under the unbounded TGP.

From Table 6, we see that the estimates of mean wear growth provided by the PL-BTGP4, PL-BTGP5, and PL-BTGP6 are very similar to those obtained under the PL-BTGP2 (provided in Table 4), and that the estimates of the mean wear growth provided by the TPG are similar to those provided by the bounded processes only for the liners # 2, 3, 6, 7, and 8, whose current wear level is quite smaller than the estimated bound $\widehat{w}_{\text{lim}}$, due to relatively short time horizon of $10,000$ h. In the other cases, all the bounded processes provide estimates of the mean wear growth significantly smaller than those provided by the unbounded TGP.

This circumstance is clearly shown by the Figs. 11 and 12, which evidence how the difference between the estimates of the conditional pdf of the increment $\Delta W(t,t+\Delta t)$ given the current state $W(t) = w_t$ provided by PL-BTGPs and unbounded TGP for the liner #1, which current degradation level and age are $w_t = 2.85$ mm and $t = 31,270$ h, is much larger than the difference existing between the corresponding estimated pdfs obtained for the liner #2, which current degradation level and age are $w_t = 2.00$ mm and $t = 21,970$ h.

Based on the results of Tables 3–6, and on Figs 10–12, we can conclude that the unbounded TGP provides estimates of the mean RUL and of the mean wear growth that are more pessimistic than the ones provided by the bounded processes, regardless of the adopted bounded state function (i.e., either $g_2(w)$, $g_4(w)$, $g_5(w)$, or $g_6(w)$). Moreover, obtained results indicate that differences between the estimates provided by unbounded and bounded TGPs are primarily influenced by the time horizon, rather than by the specific adopted bounded state function.

In fact, for example, the unbounded TGP provides estimates of the mean wear growth that are similar to those provided by the bounded processes only when the time horizon is small and the current degradation level of the liner is far below the estimated bound $\widehat{w}_{\text{lim}}$.

Based on these preliminary results, in Section 6, where we present a condition based maintenance policy, we focus our attention only on the PL-BTGP2 (i.e., on the model that according to the Akaike information criterion is the best one, among the considered alternatives, for the liner data in Table 1) and on the TGP which (among the unbounded TGP processes considered in [36]) provides the best fit for the same data. Then, in Section 7, where we investigate the consequences caused by a misspecification of a PL-BTGP2 with a TGP in terms of maintenance decision and maintenance costs by carrying out a small Monte Carlo simulation study, we also consider different values of the width of the time horizon which the decision relates to, for evaluating the sensitivity of results to this feature of the maintenance policy.

6. The condition-based maintenance policy

A condition-based maintenance policy very similar to the one proposed in Giorgio et al. [36] is apply in this paper. The only (slight) difference concerns the cost function, which includes a new (additional) term that account for the effect of downtime.

The policy is conceived taking into account the maintenance activities that are carried out in real settings on the considered cylinder liners of marine engines and the nature of the failure these units are subject to.

Due to a contract clause, whose failure to comply determines the forfeiture of the guarantee, the liners should be replaced before their degradation level exceeds a threshold limit D that is defined by the manufacturer of the engine. In compliance with this contractual agreement, a liner whose degradation level has passed this threshold value is

Table 5
Age t and state w_t of each liner i ($i = 1, \ldots, 8$) at the last inspection epoch and ML estimates of the mean RUL $E\{X_t|W(t) = w_t\}$ under the PL-BTGP4, PL-BTGP5, and PL-BTGP6.

	Liner							
	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$	$i=7$	$i=8$
Current age t [h]	31,270	21,970	16,300	28,000	37,310	24,710	16,620	16,300
Current state w_t [mm]	2.85	2.00	1.35	2.75	3.05	2.15	2.35	2.10
$\widehat{E}\{X_t|W(t)=w_t\}$ under PL-BTGP4	47,285	57,646	62,796	48,840	43,760	56,192	54,047	56,690
$\widehat{E}\{X_t|W(t)=w_t\}$ under PL-BTGP5	87,305	98,997	104,778	89,926	81,850	96,823	99,108	101,103
$\widehat{E}\{X_t|W(t)=w_t\}$ under PL-BTGP6	98,967	111,191	117,089	101,389	93,666	109,174	109,617	112,079

Table 6
Age t and state w_t of each liner i ($i = 1, \ldots, 8$) at the last inspection epoch and ML estimates of the mean wear growth $E\{\Delta W(t, t+\Delta t)|W(t) = w_t\}$ under the PL-BTGP4, PL-BTGP5, and PL-BTGP6 for $\Delta t = 10,000$ h.

	Liner							
	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$	$i = 6$	$i = 7$	$i = 8$
Current age t [h]	31,270	21,970	16,300	28,000	37,310	24,710	16,620	16,300
Current state w_t [mm]	2.85	2.00	1.35	2.75	3.05	2.15	2.35	2.10
$\widehat{E}\{\Delta W(t, t+\Delta t)|W(t) = w_t\}$ under PL-BTGP4	0,468	0,719	0,904	0,493	0,414	0,681	0,577	0,654
$\widehat{E}\{\Delta W(t, t+\Delta t)|W(t) = w_t\}$ under PL-BTGP5	0,348	0,689	0,964	0,379	0,282	0,633	0,495	0,601
$\widehat{E}\{\Delta W(t, t+\Delta t)|W(t) = w_t\}$ under PL-BTGP6	0,348	0,688	0,944	0,383	0,277	0,631	0,520	0,623

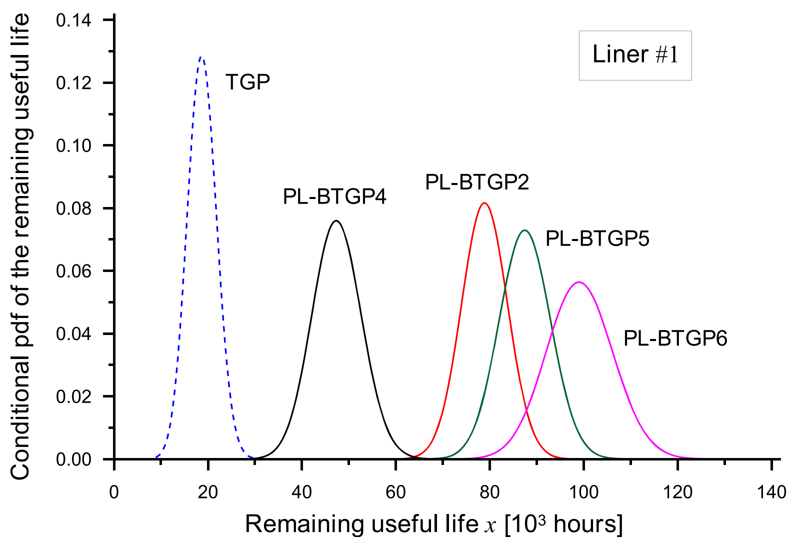

Fig. 10. ML estimates of the conditional pdf $f_{X_t|W(t)}(x|w_t)$ of the remaining useful life X_t of liner #1, given the age $t = 31,270$ h and the wear level $w_t = 2.85$ mm at the last inspection epoch, under the TGP, PL-BTGP2, PL-BTGP4, PL-BTGP5, and PL-BTGP6.

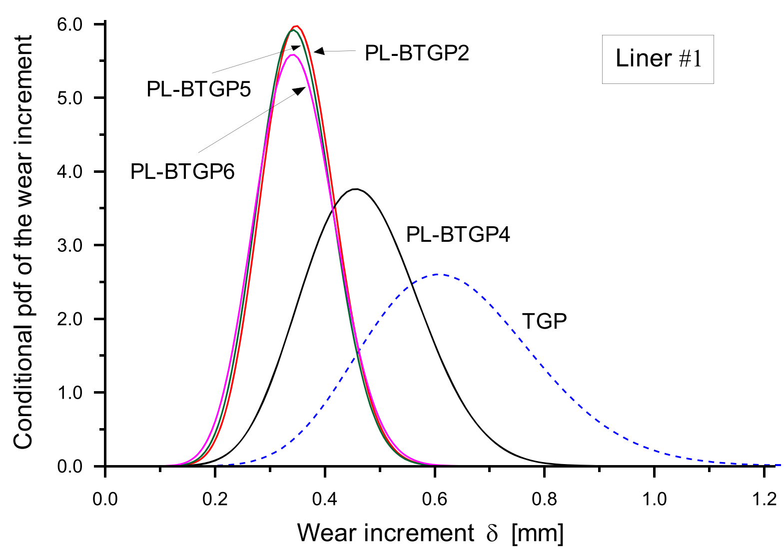

Fig. 11. ML estimates of the conditional pdf $f_{\Delta W(t,t+\Delta t)|W(t)}(\delta|w_t)$ of the wear increment of liner #1 at the last inspection epoch during the future time interval of width $\Delta t = 10,000$ h, given its current age $t = 31,270$ h and wear level $w_t = 2.85$ mm, under the TGP, PL-BTGP2, PL-BTGP4, PL-BTGP5, and PL-BTGP6.

said "failed", although it is not necessarily unable to operate. In fact, in the most of the cases, a liner (soft) failure induced by excessive wear only causes a loss of power.

During their operating life, the liners are subject to sequential non-periodic inspections that are planned to perform several checks or maintenance actions on the whole engine, and then cannot be skipped for the sole purpose of optimizing the liner maintenance. During each inspection, the wear accumulated by each liner is measured. If the wear level w_t of a liner at the inspection time t (measured from the time at which the liner is placed in service) exceeds the threshold value D, then the liner is immediately replaced with a new one. Otherwise, given the current state (i.e., the wear level) w_t of the liner and the scheduled time $t + \tau$ for the next inspection, one has to decide what action to take between two alternatives:

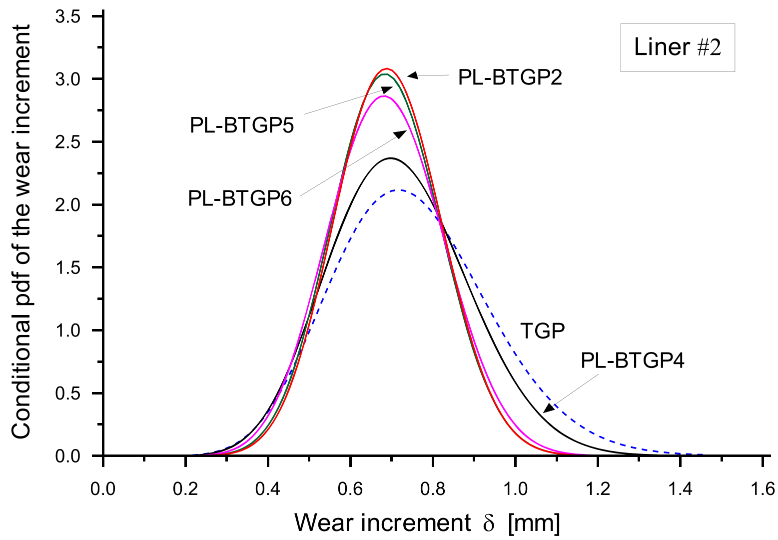

Fig. 12. ML estimates of the conditional pdf $f_{\Delta W(t,t+\Delta t)|W(t)}(\delta|w_t)$ of the wear increment of liner #2 at the last inspection epoch during the future time interval of width $\Delta t = 10,000$ h, given its current age $t = 21,970$ h and the wear level $w_t = 2.00$ mm, under the TGP, PL-BTGP2, PL-BTGP4, PL-BTGP5, and PL-BTGP6.

(a) deferring the replacement of the liner to a later time (the next inspection time $t+\tau$ or even later), or
(b) replacing immediately the liner with a new one.

Of course, the decision must be taken on the basis of the expected utility loss associated with each possible action.

Let c_L denote the total cost of a new liner (including the cost of mounting) and let $E\{X\}$ be its mean lifetime. The utility loss $UL_a(\tau|w_{t+\tau}, t_D)$ associated to the action *(a)* depends on the (unknown) wear level $w_{t+\tau}$ of the liner of age $t+\tau$ at the next planned inspection and on the (unknown) downtime t_D of the unit in the future time interval $(t,\ t+\tau)$ (that is, the unknown amount of time, in the interval in $(t,\ t+\tau)$, during which the wear level exceeds the threshold D), and is given by:

$$UL_a(\tau|W(t+\tau) = w_{t+\tau}, T_D = t_D) = U_c(w_t) - U(w_{t+\tau}, t_D), \tag{30}$$

where: *i)* $U_c(w_t) = c_L\ E\{X_t|W(t) = w_t\}/E\{X\}$ is the current utility (or residual economic value) of the liner, assumed to be proportional to its current (conditional) mean RUL $E\{X_t|W(t) = w_t\}$, and *ii)* $U(w_{t+\tau}, t_D)$ is the utility of same liner at the time $t+\tau$ of the next planned inspection. The utility $U(w_{t+\tau}, t_D)$ depends on the (unknown) liner wear level $w_{t+\tau}$ at the time of the next inspection, as well as on the (unknown) downtime t_D of the unit in $(t,\ t+\tau)$, and is given by:

$$U(w_{t+\tau}, t_D) = \begin{cases} c_L\ E\{X_{t+\tau}|W(t+\tau) = w_{t+\tau}\}/E\{X\}; & w_{t+\tau} < D \\ 0; & w_{t+\tau} = D \\ -[(w_{t+\tau} - D)/c]^d - q\,t_D; & w_{t+\tau} > D \end{cases}. \tag{31}$$

From Eq. (31), we have that the utility $U(w_{t+\tau}, t_D)$ is:

(1) positive, if the liner will not fail within $t+\tau$ (that is, if $w_{t+\tau} < D$), and proportional to the (conditional) mean RUL $E\{X_{t+\tau}|W(t+\tau) = w_{t+\tau}\}$ at the next inspection time $t+\tau$;
(2) null, if the liner will reach the threshold limit D exactly at $t+\tau$, and
(3) negative, if the liner will exceed the threshold limit before the next inspection. In this case, by adopting a more general formulation with respect to Giorgio et al. [36], the (negative) utility is assumed to be the sum of a power-law function of the (possible) excess of wear $w_{t+\tau} - D$ (already considered in [36]) and a (new) term consisting in a linear function of the downtime t_D.

From Eqs. (30) and (31), the expected utility loss associated to the action *(a)* is then given by:

$$\begin{aligned} E\{UL_a(\tau)\} &= \int_{w_t}^{A}\int_{0}^{\tau} [U_c(w_t) - U(w_{t+\tau}, t_D)]\, f_{W(t+\tau)|W(t)}(w_{t+\tau}|w_t)\, f_{T_D|W(t),W(t+\tau),}(t_D|w_t, w_{t+\tau})\ \mathrm{d}t_D\ \mathrm{d}w_{t+\tau} \\ &= c_L\,\frac{E\{X_t|W(t) = w_t\}}{E\{X\}} - \int_{w_t}^{D} c_L \frac{E\{X_{t+\tau}|W(t+\tau) = w_{t+\tau}\}}{E\{X\}} f_{W(t+\tau)|W(t)}(w_{t+\tau}|w_t)\ \mathrm{d}w_{t+\tau} \\ &\quad + \int_{D}^{A}\left(\frac{w_{t+\tau} - D}{c}\right)^d f_{W(t+\tau)|W(t)}(w_{t+\tau}|w_t)\ \mathrm{d}w_{t+\tau} + \int_{0}^{\tau} q\ t_D f_{T_D|W(t)}(t_D|w_t)\ \mathrm{d}t_D, \end{aligned} \tag{32}$$

where:

$$A = \begin{cases} \infty; & \text{if the degradation process is not bounded} \\ w_{\lim}; & \text{if the degradation process is bounded} \end{cases},$$

$f_{W(t+\tau)|W(t)}(w_{t+\tau}|w_t)$ is the (conditional) pdf of the wear level $W(t+\tau)$ at the next inspection time $t+\tau$, given the current wear level w_t:

$$f_{W(t+\tau)|W(t)}(w_{t+\tau}|w_t) = g'(w_{t+\tau})\frac{[\Delta g(w_t,\ w_{t+\tau})]^{\Delta\eta(t,t+\tau)-1}}{\Gamma(\Delta\eta(t,t+\tau))} \times \exp(-\Delta g(w_t, w_{t+\tau})),\ w_t < w_{t+\tau} < A, \tag{33}$$

$f_{T_D|W(\tau),W(t+\tau)}(t_D|w_\tau, w_{t+\tau})$ is the conditional pdf of the downtime T_D of the used unit, given $W(\tau) = w_\tau$ and $W(t+\tau) = w_{t+\tau}$, $f_{T_D|W(t)}(t_D|w_t)$ is the "marginal" pdf of T_D, given $W(t) = w_t$:

$$f_{T_D|W(t)}(t_D|w_t) = \int_{w_t}^{A} f_{T_D|W(\tau),W(t+\tau)}(t_D|w_\tau, w_{t+\tau}) f_{W(t+\tau)|W(t)}(w_{t+\tau}|w_t)\ \mathrm{d}w_{t+\tau}\ .$$

and the last integral in Eq. (32) arises from the equality:

$$\int_0^\tau q\ t_D f_{T_D|W(t)}(t_D|w_t)\ \mathrm{d}t_D = \int_{w_t}^{A}\int_0^\tau q\ t_D f_{T_D|W(t),W(t+\tau)}(t_D|w_t, w_{t+\tau}) \times f_{W(t+\tau)|W(t)}(w_{t+\tau}|w_t)\ \mathrm{d}t_D\ \mathrm{d}w_{t+\tau}\ .$$

Moreover, since in this case $T_D = \max(0, \tau - X_t)$, where X_t is the remaining useful life (RUL), the last integral in Eq. (32) can be rewritten as:

$$\begin{aligned}&\int_0^\tau q\ t_D f_{T_D|W(t)}(t_D|w_t)\ \mathrm{d}t_D \\ &= q\ \tau\int_0^\tau f_{X_t|W(t)}(x_t|w_t)\ \mathrm{d}x_t - q\int_0^\tau x_t\ f_{X_t|W(t)}(x_t|w_t)\ \mathrm{d}x_t \\ &= q\ \tau\ [1 - R_t(\tau|w_t)] - q\int_0^\tau x_t\ f_{X_t|W(t)}(x_t|w_t)\ \mathrm{d}x_t\ .\end{aligned}$$

Likewise, the utility loss $UL_b(\tau|W(\tau) = w_\tau, T_D = t_D)$ associated to the action *(b)*: replacing immediately the liner with a new one, given the (unknown) value of the wear level w_τ of the new liner of age τ at the next planned inspection and the (unknown) downtime t_D of the new liner, is given by:

$$UL_b(\tau|W(\tau) = w_\tau, T_D = t_D) = U_c(w_t) + U_c(0) - U(w_\tau, t_D), \tag{34}$$

where: *i)* $U_c(w_t) = c_L\ E\{X_t|W(t) = w_t\}/E\{X\}$ is the current residual economic value of the (used) liner of age t that is replaced even if not yet failed, *ii)* $U_c(0) = c_L$ is the current utility (economic value) of the new liner that is mounted in place of the current one, and *iii)* $U(w_\tau, t_D)$ is the utility of the new liner at the time τ of the next planned inspection.

The utility $U(w_\tau, t_D)$ of the new liner depends on the (unknown) liner wear level w_τ at the time of the next inspection and on the (unknown) downtime t_D of the new liner, and is given by:

$$U(w_\tau, t_D) = \begin{cases} c_L\ E\{X_\tau|W(\tau) = w_\tau\}/E\{X\}; & w_\tau < D \\ 0; & w_\tau = D \\ -[(w_\tau - D)/c]^d - q\,t_D; & w_\tau > D \end{cases}, \tag{35}$$

where, similarly to Eq. (31), $U(w_\tau)$ is: *i)* positive, if the new liner will not exhaust his life within τ ($w_\tau < D$), *ii)* null, if the new liner will reach the threshold limit D exactly at τ, and *iii)* negative, if the new liner will exceed the threshold limit before the next inspection, being now $w_\tau - D$ the (possible) excess wear.

Thus, from Eqs. (34) and (35), the expected utility loss associated to the action *(b)* is given by:

$$\begin{aligned}&E\{UL_b(\tau)\} \\ &= U(w_t) + \int_0^A\int_0^\tau [U_c(0) - U(w_\tau, t_D)]\ f_{W(\tau)}(w_\tau)\ f_{T_D|W(\tau)}(t_D|w_\tau)\ \mathrm{d}t_D\ \mathrm{d}w_\tau \\ &= c_L\ \frac{E\{X_t|W(t) = w_t\}}{E\{X\}} + c_L - \int_0^D c_L\frac{E\{X_\tau|W(\tau) = w_\tau\}}{E\{X\}} f_{W(\tau)}(w_\tau)\ \mathrm{d}w_\tau \\ &+ \int_D^A \left(\frac{w_\tau - D}{c}\right)^d f_{W(\tau)}(w_\tau)\ \mathrm{d}w_\tau + \int_0^\tau q\ t_D f_{T_D}(t_D)\ \mathrm{d}t_D,\end{aligned} \tag{36}$$

where $f_{W(\tau)}(w_\tau)$ is the pdf in Eq. (16) of the wear level $W(\tau)$ of the new liner at the next inspection time τ, $f_{T_D|W(\tau)}(t_D|w_\tau)$ is the conditional pdf of the downtime T_D of a new unit, given $W(\tau) = w_\tau$, and $f_{T_D}(t_D)$ is the marginal pdf of T_D, given by:

$$f_{T_D}(t_D) = \int_0^A f_{T_D|W(\tau)}(t_D|w_\tau) f_{W(\tau)}(w_\tau)\ \mathrm{d}w_\tau,$$

and the last integral in Eq. (36) arises from the equality:

$$\int_0^\tau q\ t_D f_{T_D}(t_D)\mathrm{d}t_D = \int_0^A\int_0^\tau q\ t_D\ f_{W(\tau)}(w_\tau)\ f_{T_D|W(\tau)}(t_D|w_\tau)\ \mathrm{d}t_D\ \mathrm{d}w_\tau\ .$$

Moreover, since in this case $T_D = \max(0,\ \tau - X)$, where X is the lifetime, the last integral in Eq. (36) can be rewritten as:

$$\begin{aligned}\int_0^\tau q\ t_D f_{T_D}(t_D)\mathrm{d}t_D &= q\ \tau\int_0^\tau f_X(x)\ \mathrm{d}x - q\int_0^\tau x\ f_X(x)\mathrm{d}x \\ &= q\ \tau\ [1 - R(\tau)] - q\int_0^\tau x\ f_X(x)\mathrm{d}x\ .\end{aligned}$$

Let now assume that $c_L = 10,000$ €, and that the cost due to an excess wear of 0.2 mm is equal to 100,000 €. By setting $d = 1.5$, we have that the failure cost parameter c is $c = 9.283{\cdot}10^{-5}$ mm. In addition, we assume that the cost due to a downtime of 10,000 hours is equal to 20,000 €, from which $q = 2$ €/h.

By using the MLE of the model parameters in Table 2 under the PL-BTGP2, the expected utility losses in Eqs. (32) and (36), associated, respectively, to the actions *(a)* and *(b)*, are estimated for each liner, by assuming that the next inspection of the liners is planned after $\tau = 15,000$ h. As shown in Table 7, the estimated $E\{UL_a(\tau)\}$ under the PL-BTGP2 is always smaller than the estimated $E\{UL_b(\tau)\}$, and then we decide to defer the replacement of all the liners.

For a comparative purpose, we have also estimated the expected utility losses $E\{UL_a(\tau)\}$ and $E\{UL_b(\tau)\}$ under the (unbounded) TGP with power-law age and state functions. As shown in Table 7, under the (unbounded) TGP the estimated $E\{UL_a(\tau)\}$ is larger than $E\{UL_b(\tau)\}$ for liners #1 and 5. This implies that, if the (unbounded) TGP is used to describe the liner wear process, then the liners #1 and 5 would have been immediately replaced with new liners at their last inspection times, say 31,270 and 37,310 h

A deeper insight into the observed difference is provided by the Fig. 13, where the expected utility losses $E\{UL_a(\tau)\}$ and $E\{UL_b(\tau)\}$ of the liner #5 at its last inspection time $t = 37,310$ h, under the PL-BTGP2 and the TGP are depicted as the time τ to the next inspection varies. From the plotted expected losses, we have that the "indifferent time" τ^* for the future inspection of the liner #5, that is the time at which the expected utility losses associated to the actions *(a)* and *(b)* the same value, is equal to 73,860 h under the PL-BTGP2 process, and is equal to 11,090 h under the TGP.

As depicted in Fig. 13, from Eqs. (32) and (36) we have that:

Table 7
Age t and state w_t of each liner at the last inspection time and estimated expected losses (in €) $E\{UL_a(\tau)\}$ and $E\{UL_b(\tau)\}$, associated to actions *(a)* and *(b)*, respectively, under the PL-BTGP2 and the TGP, when $\tau = 15,000$ h.

	Liner							
	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$	$i=7$	$i=8$
Current age t [h]	31,270	21,970	16,300	28,000	37,310	24,710	16,620	16,300
Current state w_t [mm]	2.85	2.00	1.35	2.75	3.05	2.15	2.35	2.10
$E\{UL_a(\tau)\}$ under PL-BTGP2	1364	1364	1364	1364	1364	1364	1364	1364
$E\{UL_b(\tau)\}$ under PL-BTGP2	8480	9524	10,050	8722	7985	9323	9578	9747
$E\{UL_a(\tau)\}$ under TGP	8362	3004	3003	4409	43,290	3009	3004	3003
$E\{UL_b(\tau)\}$ under TGP	6737	8886	10,140	7145	5998	8444	8853	9997

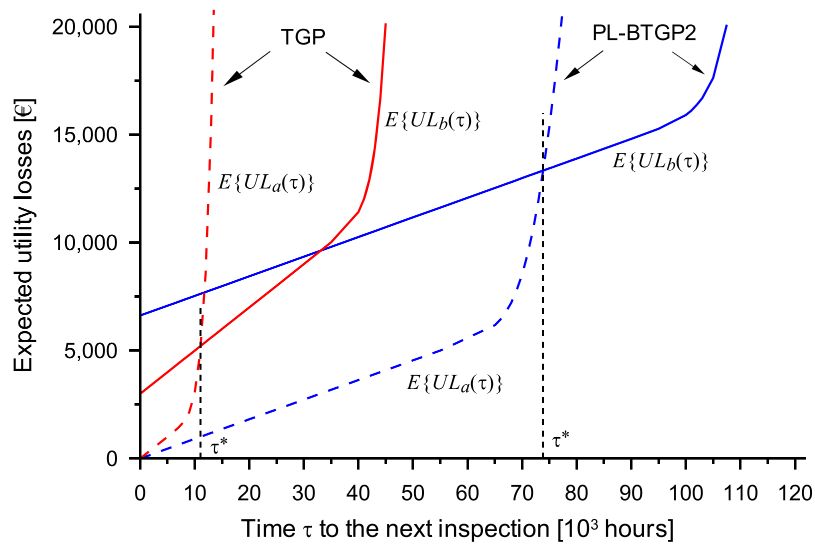

Fig. 13. Expected utility losses $E\{UL_a(\tau)\}$ and $E\{UL_b(\tau)\}$ of the liner #5 under the PL-BTGP2 and the TGP.

- for $\tau = 0$, it results $E\{UL_a(0)\} = 0$ and $E\{UL_b(0)\} = c_L\, E\{X_t|W(t) = w_t\}/E\{X\}$;
- for τ sufficiently smaller than the mean RUL $E\{X_t|W(t) = w_t\}$, so that the failure probability at $t+\tau$, say $1 - F_{\Delta W(t,t+\tau)|W(t)}(D - w_t|w_t)$, is negligible, the difference between the conditional mean RUL at t and mean RUL at $t+\tau$, given w_t, is approximately:

$$E\{X_t|W(t) = w_t\} - \int_{w_t}^{D} E\{X_{t+\tau}|W(t+\tau) = w_{t+\tau}\} \times f_{W(t+\tau)|W(t)}(w_{t+\tau}|w_t)\, \mathrm{d}w_{t+\tau} \cong \tau,$$

and hence, from Eq. (32), it results that $E\{UL_a(\tau)\} \cong c_L\tau/E\{X\}$, so that $E\{UL_a(\tau)\}$ initially increases linearly with τ, with rate $c_L\,/E\{X\}$;
- for τ sufficiently smaller than the mean lifetime $E\{X\}$, so that the failure probability $1 - F_{W(\tau)}(D)$ of the new liner before the time τ of the next planned inspection is negligible, the difference between the mean lifetime of the new liner and its marginal mean RUL at τ is approximately:

$$E\{X\} - \int_0^D E\{X_\tau|W(\tau) = w_\tau\}\, f_{W(\tau)}(w_\tau)\, \mathrm{d}w_\tau \cong \tau,$$

and hence, from Eq. (36), we have that $E\{UL_b(\tau)\} \cong c_L\, E\{X_t|W(t) = w_t\}/E\{X\} + c_L\tau/E\{X\}$, so that $E\{UL_b(\tau)\}$ initially increases linearly with τ with the same rate of $E\{UL_a(\tau)\}$.

In particular, as depicted in Fig. 13 for the liner #5, the linear approximation of $E\{UL_a(\tau)\}$ and $E\{UL_b(\tau)\}$ under the PL-BTGP2, when the mean RUL $E\{X_t|W(t) = w_t\} = 72,822$ h and the mean lifetime $E\{X\} = 109,994$ h, works very well up to $\tau = 60,000$ h and $\tau = 95,000$ h, respectively. Likewise, under the TGP, when the mean RUL $E\{X_t|W(t) = w_t\} = 14,965$ h and the mean lifetime $E\{X\} = 49,954$ h, the linear approximation of $E\{UL_a(\tau)\}$ and $E\{UL_b(\tau)\}$ works very well up to $\tau = 7000$ h and $\tau = 37,000$ h, respectively.

In a nutshell, based on the results of this case study, it is possible to affirm that the (incorrect) use of a TGP when the true process is the PL-BTGP2 would bring to the (possibly immature) immediate replacement of the liners #5 and 1. From the results obtained under the PL-BTGP2, reported in the first and second rows of Table 7, the (potential) expected utility losses (extra costs) caused by these decisions are estimated to be equal to 8480 - 1364 = 7116 € and 7,985 - 1364 = 6621 €, respectively. On the other hand, in the case of the considered real data it is not possible to be certain that the PL-BTGP2 and the TGP are the true and wrong model, respectively. Indeed, we can only say that, according to Akaike information criterion, the PL-BTGP2 is to prefer to the TGP. Hence, in order to better investigate the effect of a misspecification of a PL-BTGP2 with a TGP we have carried out the small simulation study illustrated in Section 7.

7. Simulation study

In order to investigate the consequences caused by a misspecification of a bounded PL-BTGP2 with an unbounded TGP, we have carried out a small Monte Carlo simulation study. The investigation is conducted by generating 100 pseudo-random samples of degradation data from the PL-BTGP2 whose parameters are equal to the ML estimates reported in Table 2 (i.e., $\hat{a} = 2682$ h, $\hat{b} = 1.434$, $\hat{w}_{\lim} = 4.363$ mm, and $\hat{\beta} = 18.62$). Hereinafter, this PL-BTGP2 will be referred to as “true wear model”.

The 100 datasets have the same structure (i.e., the same number of

Table 8
Fraction of times the true wear model, the PL-BTGP2, and the TGP lead to identify the preventive immediate replacement of the liners as optimal action when $\tau = 15,000$ h.

Process	Liner							
	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$	$i = 6$	$i = 7$	$i = 8$
True wear model	0	0	0	0	0	0	0	0
PL-BTGP2	0	0	0	0	0	0	0	0
TGP	0.25	0.04	0.01	0.18	0.73	0.07	0	0

units, the same number of measurements, and the same measurement times) of the real wear data reported in Table 1. The considered TGP is the one with power-law state function $g(w) = (w/\alpha)^{\beta}$ and power-law age function $\eta(t) = (t/a)^{b}$, which provided the best fit for the real wear data within the (unbounded) TGP.

For each simulated dataset, we have estimated the parameters of both the PL-BTGP2 and TGP models. As in the case of the real data, the MLE of the parameters of the PL-BTGP2 have been obtained under the constraint $w_{\text{lim}} \geq w_{\text{L}} = 4.3$ mm. Hence, for each of the two processes, based on the estimated parameters and the maintenance policy presented in Section 6, we have identified the liners that should be immediately replaced at the last inspection time. In the rest of this section, the estimated PL-BTGP2 and TGP models will be indicated as PL-BTGP2 and TGP, by omitting the term “estimated”.

As in Section 6, we have considered the case where the decision is made under the assumption that the next inspection is planned after $\tau = 15,000$ h. The cost model is also calibrated as in Section 6.

The results of this simulation study are summarized in Table 8. As indicated in the first row of the table, for all the liners the correct decision (i.e., the one defined by using the true wear model) is always to defer the replacement. The other two rows of the table give the fraction of times the PL-BTGP2 (second raw) and the TGP (third raw) have led to identify the preventive immediate replacement of the liner as optimal action (that is, the fraction of times the PL-BTGP2 and the (unbounded) TGP leads to a wrong maintenance decision).

Obtained results show that *i*) the PL-BTGP2 always brings to make the correct decision, and *ii*) the misspecification of the PL-BTGP2 with the (unbounded) TGP leads to a total of 128 incorrect decisions (i.e., premature replacement of the liner) over a total of 800 decisions, especially in the case of the liners that are observed for a longer time, such as the liner #5 (observed up to 37,310 h), #1 (observed up to 31,270 h), and #4 (observed up to 28,000 h).

The expected value of the total extra cost caused (for each liner) by the wrong decisions (i.e., the premature replacement of the liner) made under the TGP when $\tau = 15,000$ h are reported in Table 9. These values are obtained by adding the expected extra costs caused by all the (single) wrong decisions (e.g., in the case of the liner #5, the value 479,913 € is computed by summing the expected extra costs caused by 73 out of 100 wrong decisions). The expected extra cost caused by a single wrong decision is determined by computing the difference between the true expected loss of utility caused by the actions *(b)* and *(a)* (i.e., as $E\{UL_b(\tau)\} - E\{UL_a(\tau)\}$), The true values of the expected utility losses are calculated by using the true model.

From the extra costs in Table 9 under the TGP, whose sum is equal to 878,722 €, it results that the (estimated) mean expected cost of a wrong decision is equal 6865 € and the (estimated) mean expected cost for sample due to wrong decisions is equal to 8787 €. The expected utility loss caused by a single wrong decision ranges between 5852 € and 8468 €.

In order to investigate the sensitivity of the results on the width of the time horizon (that, as highlighted in Section 5, affects significantly the estimate of the mean degradation growth) we have repeated our analyses considering the cases where the decision is made under the assumptions that the next inspection is planned after $\tau = 10,000$ h and $\tau = 25,000$. In Tables 10 and 11 the fraction of times the PL-BTGP2, and the TGP lead to identify the preventive immediate replacement of the liners as optimal action when $\tau = 10,000$ h and $\tau = 25,000$ h, respectively, are provided.

Obtained results show that, as the time interval τ increases, the misspecification of the PL-BTGP2 with the TGP significantly increases the risk of deciding to replace immediately the liner when the correct decision would be to postpone its replacement:

i when $\tau = 10,000$ h, the TGP brings to make the incorrect decision 26 times over 800. In particular, the liner # 5, whose current wear level is the largest one, is subject to the incorrect decision 23 times over 100. On the contrary, the PL-BTGP2 model always brings to the correct decision;
ii when $\tau = 25,000$ h, the PL-BTGP2 brings to make the incorrect decision 413 times out of 800. In particular, the liner # 5 is always

Table 10
Fraction of times the true wear model, the PL-BTGP2, and the TGP lead to identify the preventive immediate replacement of the liners as optimal action when $\tau = 10,000$ h.

Process	Liner							
	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$	$i = 6$	$i = 7$	$i = 8$
True wear model	0	0	0	0	0	0	0	0
PL-BTGP2	0	0	0	0	0	0	0	0
TGP	0.02	0.01	0	0	0.23	0.01	0	0

Table 11
Fraction of times the true wear model, the PL-BTGP2, and the TGP lead to identify the preventive immediate replacement of the liners as optimal action when $\tau = 25,000$ h.

Process	Liner							
	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$	$i = 6$	$i = 7$	$i = 8$
True wear model	0	0	0	0	0	0	0	0
PL-BTGP2	0.02	0	0	0.01	0.02	0.01	0	0
TGP	0.94	0.42	0.08	0.81	1.00	0.66	0.13	0.09

Table 12
Simulated dataset. As in Table 2, $w_{i,j} = W(t_{i,j})$ [mm] indicates the wear level accumulated by the liner i up to the inspection time $t_{i,j}$ [h].

i	$w_{i,1}$	$t_{i,1}$	$w_{i,2}$	$t_{i,2}$	$w_{i,3}$	$t_{i,3}$	$w_{i,4}$	$t_{i,4}$
1	1.37	11,300	1.60	14,680	2.82	31,270		
2	1.81	11,300	2.65	21,970				
3	1.76	12,300	1.98	16,300				
4	1.43	14,810	1.96	18,700	2.33	28,000		
5	1.26	10,000	2.78	30,450	3.04	37,310		
6	0.83	6860	2.38	17,200	2.83	24,710		
7	0.01	2040	0.96	12,580	1.49	16,620		
8	0.31	7540	0.39	8840	0.50	9770	0.99	16,300

Table 9
Total expected value of the extra costs (in €) caused by wrong decisions (out of 100) made under the PL-BTGP2 and the TGP when $\tau = 15,000$ h.

Process	Liner							
	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$	$i = 6$	$i = 7$	$i = 8$
PL-BTGP2	0	0	0	0	0	0	0	0
TGP	176,289	29,011	8,468	133,147	479,913	51,894	0	0

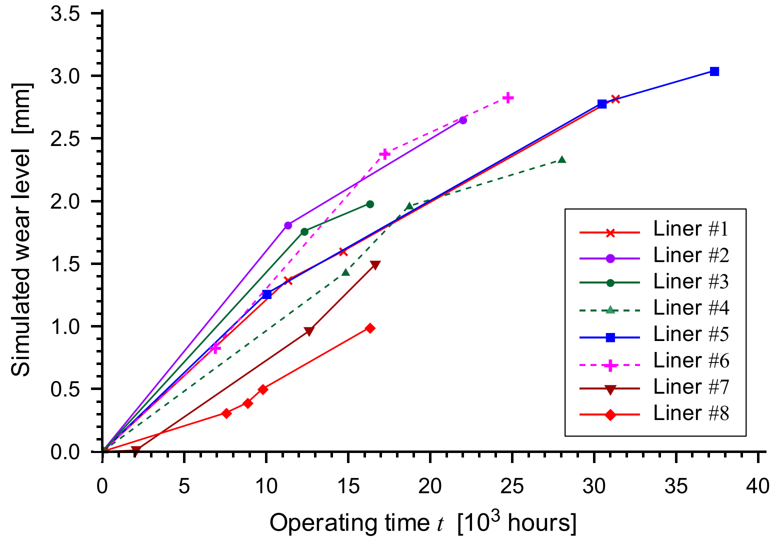

Fig. 14. Simulated wear paths of the liners (measurements that pertain to the same path are linearly connected for graphical convenience).

Table 13
Estimates obtained under the PL-BTGP2 and TGP from the simulated data.

Process	$\hat{a}$ [h]	$\hat{b}$	$\hat{w}_{\lim}$ [mm]	$\hat{\alpha}$ [mm]	$\hat{\beta}$	$\hat{\gamma}$	AIC
PL-BTGP2	3079	1.432	4.3		14.582	6.832	−5.664
TGP	5594	1.617		0.804	2.211	2.016	3.968

Table 14
Age t and state w_t of each liner i ($i = 1, \ldots, 8$) at the last inspection epoch, true mean RUL $E\{X_t|W(t) = w_t\}$ and ML estimates of the mean RUL under the PL-BTGP2 and the TGP.

	Liner							
	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$	$i = 6$	$i = 7$	$i = 8$
Current age t [h]	31,270	21,970	16,300	28,000	37,310	24,710	16,620	16,300
Current state w_t [mm]	2.85	2.00	1.35	2.75	3.05	2.15	2.35	2.10
$E\{X_t\|W(t) = w_t\}$ true wear model	78,660	84,869	92,879	84,798	72,996	81,440	94,886	96,597
$\hat{E}\{X_t\|W(t) = w_t\}$ under PL-BTGP2	90,928	97,314	105,295	97,029	85,135	93,867	107,142	108,794
$\hat{E}\{X_t\|W(t) = w_t\}$ under TGP	19,380	24,109	32,656	25,149	15,504	21,123	35,539	37,710

immediately replaced even if the correct decision is to postpone its replacement. Moreover, even the other liners that are observed for a longer time, such as the liner #1 (observed up to 31,270 h), #4 (observed up to 28,000 h), and #6 (observed up to 24,710 h), are subject to the incorrect decision more times than to the correct one. On the contrary, the PL-BTGP2 model brings to an incorrect decision only 6 times over 800.

Finally, to provide further insight into the effect caused by a misspecification, we report in detail the results obtained from one of the 100 simulated datasets. For comparative purposes, we have selected a dataset for which, as in case of the real data, the use of the TGP leads to immediately replace the liners #1 and 5. Again, as in the case of the real data, we have assumed that $\tau = 15,000$ h. Obviously, the main difference with respect to the application discussed in Sections 5 and 6 is that here the true wear model is known. These simulated data are reported in Table 12 and depicted in Fig. 14.

Table 13 provides the maximum likelihood estimates of the

Table 15
Age t and wear level w_t of all the linear at the last inspection epoch, true expected utility losses (in €) associated to the actions *(a)* and *(b)* evaluated under the true wear model, and the corresponding expected utility losses (in €) estimated under the PL-BTGP2 and TGP.

	Liner							
	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$	$i = 6$	$i = 7$	$i = 8$
Current age t [h]	31,270	21,970	16,300	28,000	37,310	24,710	16,620	16,300
Current state w_t [mm]	2.82	2.65	1.98	2.33	3.04	2.83	1.49	0.99
$E\{UL_a(\tau)\}$ true wear model	1356	1356	1356	1356	1356	1356	1356	1356
$E\{UL_b(\tau)\}$ true wear model	8522	9084	9809	9077	8009	8774	9988	10,140
$E\{UL_a(\tau)\}$ under PL-BTGP2	1227	1227	1227	1227	1227	1227	1227	1227
$E\{UL_b(\tau)\}$ under PL-BTGP2	8665	9187	9838	9164	8191	8905	9991	10,130
$E\{UL_a(\tau)\}$ under TGP	10,861	3640	2974	3276	46,762	6348	2972	2972
$E\{UL_b(\tau)\}$ under TGP	6814	7751	9445	7957	6045	7159	10,020	11,660

parameters of both the PL-BTGP2 and TGP, together to the estimate log-likelihood $\widehat{\ell}$ and the AIC value.

Table 14 provides the ML estimate $\widehat{E}\{X_t|W(t) = w_t\}$ of the mean RUL of all the liners, given their age t and wear level w_t at the last inspection time, under the PL-BTGP2 and the TGP. The true values $E\{X_t|W(t) = w_t\}$ of the mean RUL are given in the third row. By comparing the estimated mean RUL to the true mean RUL values it is apparent that the TGP strongly underestimates the mean RUL of the liners, whereas the PL-BTGP2, provides estimates that are much closer to the true values. We can also note that, for this dataset, the PL-BTGP2 slightly overestimates the mean RUL of all the liners.

Table 15 provides, for the all the liners, the true expected utility losses (in €) associated to the actions *(a)*: deferring the liner replacement, and *(b)*: replacing immediately the liner (i.e., the expected utility losses evaluated under the "true" PL-BTGP2 used to generate the simulated dataset) together with the corresponding expected utility losses evaluated by using the estimated PL-BTGP2 and TGP.

Obtained results show that the expected utility losses evaluated by using the PLBTGP2 are satisfactorily close to the true ones. On the contrary, the comparison between the true expected utility losses and those evaluated under the (unbounded) TGP gives clear evidence that this latter model hugely overestimates the expected utility loss associated to the action *(a)*: deferring the liner replacement, and almost always underestimates the expected utility loss associated to action *(b)*: replacing immediately the liner.

These circumstances, in the case of the examined dataset, lead the TGP to produce incorrect decisions for the liners #1 and 5. In fact, while the true values of expected utility losses associated to the actions (*a*) and (*b*) indicate that, according to the adopted decision rule, none of the liners should be replaced, the results obtained under the TGP would lead to immediately replace the liners #1 and 5. From the true values of the expected utility losses associated to the actions (*a*) and (*b*) given in Table 15, it is easy to assess that the wrong decision of immediately replacing the liners #1 and 5 would cause an unnecessary additional cost of 8522 € – 1356 € = 7166 € and 8009 € – 1356 € = 6653 €, respectively.

Finally, it is also worth to remark that the expected utility losses associated to actions *(a)* and *(b)* evaluated under the PLBTGP2 would lead to make the correct decision for all the liners.

8. Conclusions

This paper investigated the effect on the condition based maintenance decision process and on related maintenance costs of the use of an unbounded degradation model when the degradation phenomena of interest is bounded above. The use of a bounded model is motivated by a case study based on a set of real wear measurements of the liners of an 8-cylinder engine equipping a Diesel engine for marine propulsion.

The degradation phenomenon is modeled by using a bounded transformed gamma process. Three new characterizations are suggested of this process, which extend the model recently proposed in the literature. The condition based maintenance policy used to perform the study is a generalized version of an existing policy, which has been already applied to the considered set of liner data, under an unbounded transformed gamma process that, in this paper, is used as unbounded (competing) alternative to the bounded one.

The main characteristics of the proposed bounded process have been briefly illustrated and discussed. The set of real liner data has been analyzed under the proposed bounded process, and the corresponding estimation results have been compared to those provided by the considered competing unbounded transformed gamma process. The estimated models have been also used to perform a comparative analysis involving the unit reliability, the wear increment during a future time interval, and a condition-based maintenance policy. This comparative analysis shows that modeling the degradation growth by an unbounded transformed gamma process when the model that provides the best fit for the available data is a bounded transformed gamma process can lead to very different (and potentially wrong) reliability estimations, life predictions and maintenance decisions. Finally, a small Monte Carlo Study has been carried out to investigate the practical effect of the wrong assumption of an unbounded transformed gamma process in place of the correct bounded one. Obtained results show that the mentioned misspecification, in the considered experimental scenario, leads very often to wrong maintenance decisions and significant unnecessary additional costs.

The possibility of using a Bayesian inference approach, able to introduce in the loss function estimation prior information on the degradation phenomenon possessed by the analyst, will be investigate in future studies.

CRediT authorship contribution statement

Massimiliano Giorgio: Conceptualization, Formal analysis, Methodology, Software, Writing – original draft, Writing – review & editing. **Gianpaolo Pulcini:** Conceptualization, Formal analysis, Methodology, Software, Writing – original draft, Writing – review & editing.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

The data that supports the findings of this study are available within the article.

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References

[1] Wang H, Pham H. Reliability and optimal maintenance. London: Springer; 2006.
[2] Ahmad R, Kamaruddin S. An overview of time-based and condition-based maintenance in industrial application. Comput Ind Eng 2012;63(1):135149. https://doi.org/10.1016/j.cie.2012.02.002.
[3] Gertsbakh I. Reliability theory: with applications to preventive maintenance. London: Springe; 2013.
[4] de Jonge B, Teunter R, Tinga T. The influence of practical factors on the benefits of condition-based maintenance over time-based maintenance. Reliab Eng Syst Saf 2017;158:21–30.
[5] Alaswad S, Xiang Y. A review on condition-based maintenance optimization models for stochastically deteriorating system. Reliab Eng Syst Saf 2017;157:5463. https://doi.org/10.1016/j.ress.2016.08.009.
[6] Ali A, Abdelhadi A. Condition-based monitoring and maintenance: state of the art review. Appl Sci 2022:12668. https://doi.org/10.3390/app12020688.
[7] MA Abdel-Hameed. A gamma wear process. IEEE Trans Reliab 1975;24(2):152–4. https://doi.org/10.1109/TR.1975.5215123.
[8] van Noortwijk JM. A survey of the application of gamma processes in maintenance. Reliab Eng Syst Saf 2009;94(1):2–21. https://doi.org/10.1016/j.ress.2007.03.019.
[9] Ye ZS, Chen N. The inverse Gaussian process as a degradation model. Technometrics 2014;56(3):302–11. https://doi.org/10.1080/00401706.2013.830074.
[10] Doksum KA, Normand SLT. Gaussian models for degradation processes — Part I: methods for the analysis of biomarker data. Lifetime Data Anal 1995;1(2):131–44. https://doi.org/10.1007/bf00985763.
[11] Whitmore GA, Schenkelberg F. Modelling accelerated degradation data using Wiener diffusion with a time scale transformation. Lifetime Data Anal 1997;3(1):27–45. https://doi.org/10.1023/A:100966410.
[12] Kahle W, Mercier S, Paroissin C. Degradation processes in reliability. John Wiley & Sons Inc; 2016. iste ltd and.

[13] Grall A, Bérenguer C, Dieulle L. A condition-based maintenance policy for stochastically deteriorating systems. Reliab Eng Syst Saf 2002;76(2):167–80. https://doi.org/10.1016/S0951-8320(01)00148-X.
[14] Castanier B, Bérenguer C, Grall A. A sequential condition-based repair/replacement policy with non-periodic inspections for a system subject to continuous wear. Appl Stoch Models Bus Ind 2003;19(4):327–47. https://doi.org/10.1002/asmb.493.
[15] Castanier B, Bérenguer C, Grall A. A condition-based maintenance policy with non-periodic inspections for a two-unit series system. Reliab Eng Syst Saf 2005;87(1):109–20. https://doi.org/10.1016/j.ress.2004.04.013.
[16] Kallen MJ, Van Noortwijk JM. Optimal maintenance decisions under imperfect inspection. Reliab Eng Syst Saf 2005;90(2–3):177–85. https://doi.org/10.1016/j.ress.2004.10.004.
[17] Fouladirad M, Grall A. Condition-based maintenance for a system subject to a non-homogeneous wear process with a wear rate transition. Reliab Eng Syst Saf 2011;96(6):611–8. https://doi.org/10.1016/j.ress.2010.12.008.
[18] Huynh KT, Barros A, Bérenguer C, Castro IT. A periodic inspection and replacement policy for systems subject to competing failure modes due to degradation and traumatic events. Reliab Eng Syst Saf 2011;96(4):497–508. https://doi.org/10.1016/j.ress.2010.12.018.
[19] Cholette ME, Yu H, Borghesani P, Ma L, Kent G. Degradation modeling and condition-based maintenance of boiler heat exchangers using gamma processes. Reliab Eng Syst Saf 2019;183:184–96. https://doi.org/10.1016/j.ress.2018.11.023.
[20] Nguyen KTP, Do P, Huynh KT, Bérenguer C, Grall A. Joint optimization of monitoring quality and replacement decisions in condition-based maintenance. Reliab Eng Syst Saf 2019;189:177–95. https://doi.org/10.1016/j.ress.2019.04.034.
[21] Hao S, Yang J, Bérenguer C. Condition-based maintenance with imperfect inspections for continuous degradation processes. Appl Math Modell 2020;86:311–34. https://doi.org/10.1016/j.apm.2020.05.013.
[22] Bismut E, Pandey MD, Straub D. Reliability-based inspection and maintenance planning of a nuclear feeder piping system. Reliab Eng Syst Saf 2022;224:108521. https://doi.org/10.1016/j.ress.2022.108521.
[23] Corset F, Fouladirad M, Paroissin C. Imperfect condition-based maintenance for a gamma degradation process in presence of unknown parameters. Proc Inst Mech Eng Part O J Risk Reliab 2023;237(3):546–61. https://doi.org/10.1177/1748006X221134132.
[24] Esposito N, Mele A, Castanier B, Giorgio M. A hybrid maintenance policy for a deteriorating unit in the presence of three forms of variability. Reliab Eng Syst Saf 2023;237:109320. https://doi.org/10.1016/j.ress.2023.109320.
[25] Yuan XX, Higo E, Pandey MD. Estimation of the value of an inspection and maintenance program: a Bayesian gamma process model. Reliab Eng Syst Saf 2021;216:107912. https://doi.org/10.1016/j.ress.2021.107912.
[26] 21 Chen N, Ye ZS, Xiang Y, Zhang L. Condition-based maintenance using the inverse Gaussian degradation model. Eur J Oper Res 2015;243(1):190–9. https://doi.org/10.1016/j.ejor.2014.11.029.
[27] 22 Wu ZY, Guo B, Axita T, Xiao T. A dynamic condition-based maintenance model using inverse Gaussian process. IEEE Access 2019;8:104–17. https://doi.org/10.1109/ACCESS.2019.2958137.
[28] 23 Omshi EM, Grall A, Shemehsavar S. A dynamic auto-adaptive predictive maintenance policy for degradation with unknown parameters. Eur J Oper Res 2020;282(1):81–92. https://doi.org/10.1016/j.ejor.2019.08.050.
[29] 24 Huynh KT. An adaptive predictive maintenance model for repairable deteriorating systems using inverse Gaussian degradation process. Reliab Eng Syst Saf 2021;213:107695. https://doi.org/10.1016/j.ress.2021.107695. 2021.
[30] Hontelez JAM, Burger HH, Wijnmalen DJD. Optimum condition-based maintenance policies for deteriorating systems with partial information. Reliab Eng Syst Saf 1996;51:267–74. https://doi.org/10.1016/0951-8320(95)00087-9.
[31] Elwany AH, Gebraeel NZ, Maillart LM. Structured replacement policies for components with complex degradation processes and dedicated sensors. Oper Res 2011;59:684–95. https://doi.org/10.1287/opre.1110.0912.
[32] Guo C, Wang W, Guo B, Si X. A maintenance optimization model for mission-oriented systems based on wiener degradation. Reliab Eng Syst Saf 2013;111:183–94. https://doi.org/10.1016/j.ress.2012.10.015.
[33] Zhang M, Gaudoin O, Xie M. Degradation-based maintenance decision using stochastic filtering for systems under imperfect maintenance. Eur J Oper Res 2015;245(2):531–41. https://doi.org/10.1016/j.ejor.2015.02.050.
[34] Ma J, Cai L, Liao G, Yin H, Si X, Zhang P. A multi-phase Wiener process-based degradation model with imperfect maintenance activities. Reliab Eng Syst Saf 2023;232:109075. https://doi.org/10.1016/j.ress.2022.109075.
[35] Pedersen TI, Vatn J. Optimizing a condition-based maintenance policy by taking the preferences of a risk-averse decision maker into account. Reliab Eng Syst Saf 2022;228:108775. https://doi.org/10.1016/j.ress.2022.108775. 2022.
[36] Giorgio M, Guida M, Pulcini G. A condition-based maintenance policy for deteriorating units. An application to the cylinder liners of marine engine. Appl Stoch Models in Bus Ind 2015;31:339–48. https://doi.org/10.1002/asmb.2029.
[37] Cha JH, Finkelstein M, Levitin G. Age replacement policy for items described by stochastic degradation with dependent increments. J. Manag Math 2021;33(2):273–87. https://doi.org/10.1093/imaman/dpab014.
[38] Giorgio M, Guida M, Pulcini G. A new class of Markovian processes for deteriorating units with state dependent increments and covariates. IEEE Trans Reliab 2015;64(2):562–78. https://doi.org/10.1109/TR.2015.2415891.
[39] Ling M, Tsui HKL, Balakrishnan N. Accelerated degradation analysis for the quality of a system based on the gamma process. IEEE Trans Reliab 2015;64(1):463–72. https://doi.org/10.1109/TR.2014.2337071.
[40] Deng Y, Pandey MG. Modelling of a bounded degradation process based on a transformed gamma process. L. Walls, M. Revie and T. Bedford (Eds.). Risk, reliability and safety: innovating theory and practice. London: CRC Press, Taylor & Francis Group; 2017. p. 1417–24. https://doi.org/10.1201/9781315374987. *Proceedings of ESREL 2016, Glasgow, Scotland, 25-29 September 2016*.
[41] Fouladirad M, Giorgio M, Pulcini G. A transformed gamma process for bounded degradation phenomena. Qual Reliab Eng Int 2022:1–19. https://doi.org/10.1002/qre.3167.
[42] Akaike H. A new look at the statistical model identification. IEEE Trans Autom Control 1974;19(6):716–23. https://doi.org/10.1109/TAC.1974.1100705.
[43] IMSL® Fortran Math Library, Version 2022.1, Rogue wave software, inc. https://help.imsl.com/fortran/current/pdf/Fortran_Math_Library.pdf.