ON THE FIXED-POINT OF A SYSTEM OF EQUATIONS ARISING IN A LAYOUT EXPANSION PROBLEM

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ABSTRACT.

In this paper we present some results concerning the fixed-point of a system of equations which regulates the global effect of many local expansions in a VLSI layout, determined by widening wires. Indeed we give tight lower and upper bounds to the width of each wire, we show conditions based upon the layout configuration under which the problem can be remarkably simplified, and we study the convergence properties of an iterative method for the numerical computation of the fixed-point.

1. INTRODUCTION

In this paper we present some results concerning the fixed-point of a system of equations (see [RAM]), which regulates the global effect of many local expansions in a VLSI layout, determined by widening wires.

For a variety of reasons, one may require the width of a wire to be function of its lenght. Since the lenght of wires in a VLSI layout is known only after the circuit is laid out, the problem arises of determining the global effect of widening wires mutally interacting, if one wants to expand wires without distorting the layout configuration.

This problem has been studied in [RAM], where upper bounds to the worst case area increase have been given.

In the next section we give tight lower and upper bounds to the width of each wire; moreover we show conditions based upon the layout configuration, under which the problem can be remarkably simplified.

In section 3, we study the convergence properties of an iterative method for the numerical computation of the fixed-point.

2. THEORETICAL RESULTS

In the following we consider a layout with m wires of initial lenght l_i and width x_i, $i=1,\dots,m$. We need to compute the x_i's, by considering that the lenght of each wire is increased by the interaction with other wires, to a new value

$$l'_i = l_i + \sum_{j=1}^{m} d_{ij} x_j \,, \quad i=1,\dots,m, \text{ where}$$

d_{ij} is a nonnegative integer which expresses the number of tracks associated to wire j, intersecting wire i.

As suggested in [RAM], to give the functional dependence of the width of a wire on its lenght we consider the p-power (0<p<1) and the log functions, namely

$$x_i = f_i(x) = (l_i + \sum_{j=1}^{m} d_{ij} x_j)^p \,, \quad i=1,\dots,m, \qquad (2.1)$$

and $$x_i = g_i(x) = \log (l_i + \sum_{j=1}^{m} d_{ij} x_j), \quad i=1,\dots,m. \qquad (2.2)$$

PROPOSITION 2.1

Let $D = (d_{ij})$. If the sum of the elements in each row of D is constant (=s), and $l_i = 1$, i=1,...,m, then the system

$$x_i = h (1 + \sum_{j=1}^{m} d_{ij} x_j), \quad i=1,\dots,m, \qquad (2.3)$$

where h denotes either the log or the p-power function, is equivalent to the single equation

$$z = h (1 + s z), \qquad (2.4)$$

and the solution $\bar{x}$ of (2.1) verifies $\bar{x} = \bar{z} [1 \dots 1]^T$, with $\bar{z}$ solution of (2.4).

Proof.

If we set

$$x = z\,u, \text{ where } u = [\,1\ 1\ \ldots\ 1\,]^T, \qquad (2.5)$$

then, by substituting (2.5) in (2.3), we obtain (2.4).

Moreover note that (2.4) has always a positive fixed-point $\bar{z}$, since the function h verifies the hypotheses of Theorem 1 in [RAM]. ∎

In Proposition 2.1 we assumed D to have a particular structure, to give a strong result, namely the equivalence of a system of equations to a single equation, with a great reduction of the complexity of the computation of the fixed-point.

Moreover this result will be used in the following to produce upper and lower bounds to the fixed-point, either under the hypotheses of Proposition 2.1, or in general.

THEOREM 2.1 [HOUS]

The real positive roots of the algebraic equation

$$x^n - a\,x - b = 0, \quad a,b>0, \quad n \geq 2,$$

satisfy the relations:

$$\max\left\{ b^{1/n}, a^{1/(n-1)} \right\} \leq x \leq \max\left\{ (2b)^{1/n}, (2a)^{1/(n-1)} \right\}. \quad ∎$$

COROLLARY 2.1

Under the hypotheses of Proposition 2.1, if $h(x) = x^p$, and $1/p$ integer, the solution of (2.1) verifies:

$$\bar{x} = \bar{z}\,[\,1 \ldots 1\,]^T, \text{ with}$$

$$\max\left\{ l^p, s^{p/(1-p)} \right\} \leq \bar{z} \leq \max\left\{ (2l)^p, (2s)^{p/(1-p)} \right\}. \quad ∎$$

COROLLARY 2.2

Under the hypotheses of Proposition 2.1, and if $h(x) = \log x$, the solution of (2.1) verifies:

$$\bar{x} = \bar{z}\,[\,1 \ldots 1\,]^T, \text{ with}$$

$$\log\,(l + s \log l) \leq \bar{z} \leq 2 \log\,(l + s \log l).$$

Proof.

Note that the function

$g(x) = \log(l+s\log l) - (\log l)s/(l+s\log l) + xs/(l+s\log l)$

satisfies: $g(x) > \log(1 + s x)$.

Therefore the fixed-point verifies the inequality:

$\bar{x} < \log(l+s\log l) - (\log l)s/(l+s\log l) + \bar{x}s/(l+s\log l)$, i.e.

$$\bar{x} < 2 \log\,(l + s \log l),\ l \geq 2.$$

The lower bound proof is trivial. ∎

DEFINITION 2.1

Let $D = (d_{ij})$ and $B = (b_{ij})$. Then we say that $D \leq B$ if and only if $d_{ij} \leq b_{ij}$ for any i and j.

LEMMA 2.2

Let D be the matrix associated to a fixed-point problem, with function h. Let $D'' \leq D \leq D'$. Then each component of the fixed-point of the original problem is lower and upper bounded by each component of the fixed-point of the problems associated with D'' and D', respectively. ∎

PROPOSITION 2.2

Given the system

$$x_i = (l_i + \sum_{j=1}^{m} d_{ij} x_j)^p, \quad i=1,\dots,m, \qquad \text{with } 1/p \text{ integer,}$$

the solution verifies:

$$\max\left\{(l_m)^p, (s_m)^{p/(1-p)}\right\} \le \bar{x}_i \le \max\left\{(2l_M)^p, (2s_M)^{p/(1-p)}\right\}, \quad i=1,\dots,m,$$

where l_m and l_M are the minimum and the maximum wire lenght, and s_m and s_M are the minimum and the maximum sum of the elements of each row of $D = (d_{ij})$.

Proof.

Follows from LEMMA 2.2, THEOREM 2.1, and COROLLARY 2.1. ∎

PROPOSITION 2.3

Given the system

$$x_i = \log (l_i + \sum_{j=1}^{m} d_{ij} x_j), \quad i=1,\dots,m,$$

the solution verifies:

$$\log(l_m + s_m \log l_m) \le \bar{x}_i \le 2 \log (l_M + s_M \log l_M),$$

for any i, where l_m, l_M, s_m and s_M are defined as in Proposition 2.2. ∎

Note that the lower and upper bounds shown above are close together, up to a small constant, under the hypotheses of Proposition 2.1. Moreover, in the worst case, they are of the same order of the ones shown in [RAM], with a smaller multiplicative constant, since $s_M \leq l_M$.

In the following we present some observations which allow decreasing the computational efforts, in the detection of the fixed-point, for particular layouts, by reducing the problem to an equivalent simpler one.

REMARK 2.1

Let $D = (d_{ij})$, with $d_{ii} = 0$, $i=1,\ldots,m$. If D is lower triangular, then the fixed-point can be computed by the following direct algorithm:

$$x_1 = h(l_1)$$

$$x_i = h(l_i + \sum_{j=1}^{i-1} d_{ij} x_j), \quad i=2,\ldots,m. \quad \blacksquare$$

An analogous algorithm can be applied if D is upper triangular.

REMARK 2.2

If the i-th and the j-th rows of the mxm matrix D are equal, and if $l_i = l_j$, then $x_i = x_j$. Moreover it is possible to reduce the matrix D into an $(m-1)\times(m-1)$ matrix $\tilde{D}$, where $\tilde{D}$ is obtained by deleting the j-th row and column of D, and by substituting the i-th column of D with the sum of the i-th and the j-th column.

REMARK 2.3

If it exists a permutation matrix P such that $P D P^T$ is block diagonal, then the variables associated with each block are independent of all the others, and the original problem can be splitted into k disjoint subproblems, where k is the number of blocks of $P D P^T$.

REMARK 2.4

If wires does not bend, then it exists a scheduling of wires such that D has the following structure:

$$\left(\begin{array}{c|c} \underline{0} & \\ \hline & \underline{0} \end{array}\right).$$

REMARK 2.5

If $k \leq 2$, then $d_{ii} = 0$, $i=1,\ldots,m$.

3. THE NUMERICAL SOLUTION OF THE FIXED-POINT.

The theoretical results presented in [RAM] assure that the functions h of the types (2.1) and (2.2) have a fixed-point. Anyway this result does not give informations about the convergence of an iterative process of the type

$$x^{(k)} = h(x^{(k-1)}), \quad k \geq 1.$$

It is well known [CRT] that a sufficient condition for the global convergence is that h is a contraction on its domain.

The log function, defined in (2.2), satisfies this condition, while the p-power function is a contraction only for any x belonging to the set $S_r = \left\{ x \in R^m \mid x_i > r_i ,\ i=1,\dots,m \right\}$.

Therefore the problem arises of determining a point $z \in S_r$ such that $z_i \le \bar{x}_i$, $i=1,\dots m$. This problem can not be easily solved, in general. In the following we present a result which garantees convergence for the p-power function, too, starting from a lower bound to the fixed-point (see the previous section).

DEFINITION 3.1

R^m_+ denotes the set $\left\{ x \in R^m \mid x_i \ge 0 ,\ i=1,\dots,m \right\}$.

PROPOSITION 3.1

Let $f(x_1, x_2, \dots, x_m)$ be a continuous mapping from R^m_+ to R^m_+, where $f(x_1,\dots,x_m) = (f_1(x_1,\dots,x_m),\dots,f_m(x_1,\dots,x_m))$, and each $f_i(x_1,\dots,x_m)$ is a mapping from R^m_+ to R_+.

Assume that:

1. f has a fixed point $\bar{x} \in R^m_+$,

2. It exists $\check{x} \in R^m_+$ such that $\check{x}_i < f_i(\check{x})$, for any i,

3. For each couple of vectors x, y, each belonging to R_+^m, such that $x_i > y_i$, for any i, it holds $f_i(x) > f_i(y)$, for any i.

Then the following properties hold:

1. The sequence $x^{(0)}, x^{(1)}, \ldots$ defined by

$$\begin{cases} x^{(0)} = \tilde{x}, \\ x^{(k)} = f(x^{(k-1)}), \end{cases}$$

satisfies $x_i^{(k-1)} < x_i^{(k)} < \bar{x}_i$, $i=1,\ldots,m$;

2. $\lim_{k \to \infty} x_i^{(k)} = \bar{x}_i$, $i=1,\ldots,m$.

Proof.

Hypothesis 2 implies that

$$x_i^{(1)} = f_i(x^{(0)}) > x_i^{(0)}, \quad i=1,\ldots,m,$$

from which we have

$$f_i(x^{(1)}) > f_i(x^{(0)}), \quad i=1,\ldots,m, \text{ (hypothesis 3)},$$

and thesis 1 readily follows by induction on k.

The rest of the proof is trivial. ∎

COROLLARY 3.1

The functions f_i and g_i, $i=1,\ldots,m$, defined in (2.1) and (2.2), satisfy the hypotheses of Proposition 3.1, provided that $l_i > 0$, $d_{ij} \geq 0$, for any i and j, and $0 < p < 1$. ∎

Corollary 3.1 assures that for any starting point $x^{(0)} \in R_+^m$, and $x_i^{(0)} < \bar{x}_i$, $i=1,\ldots,m$, the iterative methods

$$x_i^{(k)} = f_i(x^{(k-1)}), \quad k>1, \quad i=1,\ldots,m,$$

$$x_i^{(k)} = g_i(x^{(k-1)}), \quad k>1, \quad i=1,\ldots,m,$$

produce sequences convergent to the fixed-point of f and g, respectively.

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