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Explosive synchronization in populations of cooperative and competitive oscillators

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Abstract

Synchronization is a subject of interdisciplinary relevance, interpolating between efficiency in transportation and digital data transfers to disease in cardiac and neural tissue. While continuous transitions to synchronization are gradual and easy to control, explosive transitions may occur suddenly and can have catastrophic effects. Here we report that in populations of cooperative and competitive oscillators the transition can be tuned between continuous and explosive simply by adjusting the balance between the two oscillator types. We show that this phenomenon is independent of the network topology, and can be described

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analytically already in the mean-field approximation. Moreover, we provide evidence that the difference between the two transitions is due to a merging process of clusters which is forbidden by adaptation, and that the hysteresis associated to the explosive transition is enhanced when the adaptive mechanisms span larger scales.

Keywords: Explosive synchronization, cooperative, competitive, mean-field

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1. Introduction

One of the fundamental contributions of the 20th century to statistical physics has been the theory of phase transitions and critical phenomena [1, 2]. Although applications were for decades restricted to conventional physical systems, it was around the turn of the century that the full scale and importance of this theory started to come to light. Indeed, the phase transitions observed originally in condensed matter physics were later found of relevance in phenomena as diverse as catastrophic shifts in ecosystems [3], the emergence of public cooperation in social dilemmas [4], percolation and synchronization in complex systems and networks [5, 6].

Synchronization, in particular, has numerous applications across social, technological and natural sciences, wherein the nature of the phase transition from disorder to synchrony frequently plays a key role [7]. For instance, continuous or second-order phase transitions occur in the standard Kuramoto model [8, 9]. Under certain conditions, however, the transition becomes explosive: an abrupt onset of synchronization follows an infinitesimally small change in the coupling strength, and hysteresis loops may be observed as in a thermodynamic first-order phase transition [6]. Explosive synchronization (ES) has been described in different extensions of the Kuramoto model [10], including inertia [11, 12], noise [13], and different frequency distributions [14, 15, 16]. In particular, the finite-size behavior of systems with uniform and other finitely-supported distributions provided important insights into the mechanism underlying ES [17].

The departure from all-to-all to irregular coupling architectures spurred on by the coming of network science [5, 18], and gave rise to a new wave of interest in phase transitions to synchronization. ES in heterogeneous networks was initially linked to natural frequencies being linearly related to the degrees of the oscillators [19], and this feature was indeed observed experimentally in nonlinear electronic circuits exhibiting chaotic dynamics [20]. However, later on it was clarified that linear frequency-degree correlations are a sufficient but not necessary condition for ES: non linear correlation features can actually arise spontaneously from conditioning or weighting the networks' links according to the mismatch in natural frequencies of neighboring oscillators [21, 22]. In fact, other disassortativity rules are also known to give rise to ES transitions [23, 24].

Ultimately, it became clear that not even correlations between natural frequencies and local topological properties are necessary for such a fascinating phenomenon to occur. Indeed, any restrictive condition preventing either the formation or the merging of synchronization clusters during the transition to the coherent state can lead to ES. While the former mechanism was elucidated already in Refs. [21, 22], the prevention of cluster merging was first shown in Ref. [25] by means of a simple adaptive rule involving a dependence of the nodes' interactions on the local order parameter. Mean-field analysis and simulations of very large size networks reveal that any nonzero fraction of oscillators adapting cooperatively their coupling to the local order parameter is already sufficient for ES in the thermodynamic limit [26].

In this paper, we significantly advance such recent studies in two fundamen-tal directions. First, we consider a population fragmented into cooperative and competitive units, i.e. we account for the two adaptation mechanisms describing dynamical competition and cooperation (or interdependence) in networks [27]. Second, we examine the role of different graph's mesoscales in the feedback leading to adaption. Not only we reveal that the type of transition can in fact be controlled simply by means of the balance between the two oscillator types, but we also show that the irreversibility associated to ES is actually enhanced when the adaptation feedback occurs from global to local scales. These findings have important implications for the robustness of ES in different network topologies, for its analytical treatment by means of mean-field approximations, and for elucidating once and for all the fundamental role played by synchronization clusters along the synchronization transition.

2. Model and Results

We start by considering an adaptive Kuramoto model consisting of N phase oscillators, where the instantaneous phase of oscillator i (denoted by θ_i $(i = 1, 2, \cdots, N)$) evolves in time as

$$\dot{\theta}_i = \omega_i + \lambda\alpha_i \sum_{j=1}^{N} A_{ij} \sin(\theta_j - \theta_i), \qquad (1)$$

where ω_i is the natural frequency of the oscillator, λ is the overall coupling strength, and A_{ij} are the elements of the adjacency matrix ($A_{ij} = 1$ if oscillators i and j are coupled and zero otherwise). The degree of a given oscillator i is defined as $k_i = \sum_{j=1}^{N} A_{ij}$.

The novelty of the model lies in the coupling strength being controlled by the global order parameter R through the local variable α_i. R satisfies $Re^{\mathrm{i}\Phi} = \frac{\sum_{j=1}^{N}\sum_{k=1}^{N} A_{jk} e^{\mathrm{i}\theta_k}}{\sum_{j=1}^{N} k_j}$, where Φ is the average phase. On the other hand, R quantifies the synchronization level in the network ($0 \leq R \leq 1$, with the two extreme values corresponding to incoherence and full synchrony, respectively [7]). Our network is furthermore partitioned into two sets of nodes (oscillators): there are competitors ($\alpha_i = 1 - R$) which decouple from their neighbors as global synchrony increases, and cooperators ($\alpha_i = R$) which tend instead to couple more strongly to their neighbors as the network becomes more synchronized. We here consider $N\rho$ competitors and $N(1-\rho)$ cooperators, the populations being controlled by the competition fraction ρ. The case $\rho = 0$ has been studied in Ref. [25], where ES was found. On the other hand, at $\rho = 1$ the ability of synchronous oscillators to form small clusters is enhanced by competition, and therefore one expects to have a continuous synchronization transition.

Our results refer to networks consisting of $N = 1,000$ oscillators, with natural frequencies $\{\omega_i\}$ randomly drawn from a uniform probability distribution in

$[-1, 1]$. These are Erdős-Rényi (ER) networks [28] and Barabási-Albert scale-free (SF) networks [29], which are paradigmatic examples of homogeneous and heterogeneous graphs, respectively, with an average degree $\langle k \rangle \equiv \frac{1}{N}\sum_{i=1}^{N} k_i = 12$. At each value of ρ, λ is gradually increased from 0 to 0.5 (where already one has $R \sim 1$) for the tracking of the forward transition (Fw), and then decreased back again to 0 for the tracking of the backward transition (Bw), with a step $\delta\lambda = 0.02$. The system is allowed to reach its stationary state for each λ value, and the order parameter at stationarity is computed [30].

In Fig. 1 (a,b), R vs. λ curves for Fw and Bw are reported for two values of the competition fraction ρ in ER (a) and BA (b) networks. One can see clearly that when competitors are in majority (i.e., $\rho = 0.8$ for ER), continuous transitions occur as in the all-to-all classical Kuramoto model. By contrast, a large fraction of cooperators (for instance $\rho = 0.2$ for ER) induces abrupt (explosive) transitions with hysteresis effects. The insets in Figs. 1(a,b) give information on the critical points λ_B (for Bw) and λ_F (for Fw). By calculating the same Fw and Bw at different values of ρ, one can reconstruct the phase diagram showing the order parameter R as a function of ρ and the coupling strength λ. This is shown in Fig. 1(c) for an ER network with the same properties discussed above. The (continuous) transition is seen to become steeper as ρ is decreased, but it continues being reversible, which makes it possible to assign a single value to

R for the forward and backwards sweeps. For $\rho \lesssim 0.5$, the transition becomes abrupt and hysteresis effect appear, which means the order parameter R for fixed ρ is no longer a single-valued function of λ. Similar results are obtained on BA network, as shown in Fig. 1(d).

More information on the mechanisms behind ES can be gathered by monitoring each oscillator's effective frequency $\omega_i^{\text{eff}} = \frac{1}{T}\int_{\tau-T}^{\tau} \dot{\theta}_i(t)dt$. The probability distribution function $f(\omega^{\text{eff}})$ at different λ is shown in Fig. 2 for ER networks (τ and T are set as 10^5 and 10^4, respectively). At $\rho = 0.2$, $f(\omega^{\text{eff}})$ turns sharply after the threshold, while a much smoother behavior is observed at $\rho = 0.8$, where the transition is instead continuous. One therefore sees that the main difference between ES and a continuous transition lies in the fact that in the latter case (for large ρ) the giant synchronization cluster coexists with several small clusters, while in the former case it forms abruptly at the transition point without relying on any pre-existing mesoscale correlations.

Let us now further extend our study, and consider explicitly that adaptation may depend on specific graph's mesoscales. For this purpose, we define a *penetration depth* l for the adaptation feedback (with $1 \leq l \leq D$, and with D being the diameter of the graph), and introduce a scaled order parameter γ_i^l which measures the level of synchronization between oscillator i and the subgraph formed by all other oscillators whose distances from i are smaller or equal to l. One then has

$$\gamma_i^l = \frac{\sum_{j\in\mathcal{N}_l(i)} \sum_{j=1}^{N} A_{ij} e^{\mathrm{i}\phi_j}}{\sum_{j\in\mathcal{N}_l(i)} k_j}, \tag{2}$$

where $\mathcal{N}_l(i)$ is the set of l-order neighbors of oscillator i. Therefore, $\mathcal{N}_0(i) = \{i\}$ and γ_i^0 recovers the local order parameter used in Ref.[25], while $\mathcal{N}_D(i)$ is the entire graph minus node i (implying that $\gamma_i^D \simeq R$). Then $\alpha_i = \gamma_i^l$ for a fraction $1-\rho$ of oscillators, and $\alpha_i = 1 - \gamma_i^l$ for the remaining ones. Our results show that ES may emerge at all values of the penetration depth. Furthermore, one can see from Fig. 3 that the hysteresis associated to ES is actually enhanced as l increases on both ER and BA networks ($| \langle\lambda_F\rangle - \langle\lambda_B\rangle |$ becomes larger in both cases).

In order to gain analytical insight into the problem at hand, one can recast Eq. (1) as

$$\dot{\theta}_i = \omega_i + \lambda\alpha_i k_i r_i \sin(\Phi_i - \theta_i), \tag{3}$$

where r_i and Φ_i ($i = 1, 2, \cdots, N$) are, respectively, the local order parameter and the local mean phase, which satisfy $r_i e^{i\Phi_i} = \frac{1}{k_i}\sum_{j=1}^{N} A_{ij} e^{\mathrm{i}\phi_j}$. The mean-field approximation consists in replacing $r_i \to R$ and $\Phi_i \to \Phi$, and to referring to the rotating frame in terms of deviations from the mean phase $\Delta\theta_i = \theta_i - \Phi$:

$$\Delta\dot{\theta}_i = \omega_i - \Omega - \lambda\alpha_i k_i R \sin(\Delta\theta_i) \tag{4}$$

where $\dot{\Phi} = \Omega$ is the mean angular velocity.

For vanishing ρ, Eq. (4) becomes $\Delta\dot{\theta}_i = \omega_i - \Omega - \lambda k_i R^2 \sin(\Delta\theta_i)$, and oscillators satisfying $|\omega_i - \Omega| \leq \lambda k_i R^2$ reach a fixed point given by $\Delta\theta_i = \arcsin\left(\frac{\omega_i - \Omega}{\lambda k_i R^2}\right)$ (i.e. they phase-lock to the mean field). The number of oscillators that do satisfy this requirement grows with increasing λ, or k_i, or R (as they are cooperative oscillators). All other oscillators are either too fast or too slow to synchronize, and drift away from the mean field at all times. To provide a mean-field solution, one can define

$$\mathcal{R} = \frac{\sum_{j=1}^{N} k_j r_j}{\sum_{j=1}^{N} k_j} = \frac{\sum_{j=1}^{N} e^{-\mathrm{i}\Phi_j} \sum_{k=1}^{N} A_{jk} e^{\mathrm{i}\theta_k}}{\sum_{j=1}^{N} k_j}. \tag{5}$$

If the average phases Φ_j associated with the local order parameter r_j are similar ($\Phi_1 \approx \Phi_2 \cdots \approx \Phi_N$), they can be simply denoted by Φ, and $e^{-i\Phi_j} \approx e^{-i\Phi}$ can be taken out of the sum as a common factor, yielding

$$\mathcal{R} \approx e^{-\mathrm{i}\Phi} \frac{\sum_{j=1}^{N} k_j e^{\mathrm{i}\theta_j}}{\sum_{j=1}^{N} k_j}. \tag{6}$$

Then

$$\mathcal{R} = \frac{\sum_{j=1}^{N} k_j r_j}{\sum_{j=1}^{N} k_j} \approx \left| \frac{\sum_{j=1}^{N} k_j e^{\mathrm{i}\theta_j}}{\sum_{j=1}^{N} k_j} \right| = R, \tag{7}$$

i.e., one can (for strong phase coherence) approximate R by $\mathcal{R}$ and one has

$$R = \frac{1}{N\langle k \rangle} \sum_{|\omega_i| < \lambda R^2 k_j} k_j \cos(\Delta\theta_j). \tag{8}$$

Substituting $\Delta\theta_j$ into Eq. (8) leads to

$$R = \frac{1}{N\langle k \rangle} \sum_{|\omega_i| < \lambda R^2 k_j} k_j \sqrt{1 - (\omega_j/\lambda R^2 k_j)^2}, \tag{9}$$

and in the continuum form one has

$$R = \frac{1}{\langle k \rangle} \int_{|\omega| < \lambda R^2 k} h(k,\omega) k \sqrt{1 - \left(\frac{\omega}{\lambda R^2 k}\right)^2} d\omega dk. \tag{10}$$

Here $h(k,\omega)$ is a joint distribution and can be written as $h(k,\omega) = P(k)g(\omega)$, with $P(k)$ being the network's degree distribution (if oscillators' degrees and natural frequencies are independent). In the more general condition where $\rho \in$

$[0,1]$, one has $F(k,\omega,\lambda,\alpha,R) = \int_{|\omega|<\lambda\alpha Rk} h(k,\omega)k\sqrt{1-\left(\frac{\omega}{\lambda\alpha Rk}\right)^2}d\omega dk$, and

$$\begin{aligned} R =& \frac{1}{\langle k\rangle}\left(\rho F(k,\omega,\lambda,R,R)\right. \\ &\left.+(1-\rho)F(k,\omega,\lambda,1-R,R)\right). \end{aligned} \tag{11}$$

By means of Eq. (11) one can calculate the mean-field solution for R on a network with given degree distribution. The case of a ER network with $\langle k\rangle = 12$ and $\rho = 0.3$ is plotted in Fig. 1 (e), in which one clearly sees that the existence of an unstable solution is responsible for the emergence of ES.

3. Conclusion

In summary, we reported on an adaptive Kuramoto model wherein cooperative (or interdependent) oscillators increase the coupling strength with their neighbors in proportion to the degree of synchronization, whilst competitive oscillators do the opposite. The continuity of the synchronization transition in this model can be controlled simply by adjusting the balance between the two oscillator populations. Our observations are independent of the network topol-ogy, and they can be captured analytically already at the level of a mean-field approximation. By focusing on different network mesoscales, we further showed that the scale plays an important role in facilitating ES. Lastly, we have shown that the hysteresis associated to ES is enhanced with a shift from the local to the global order parameter.

These results significantly deepen our understanding of ES transitions, as they reveal an alternative way to control whether the onset of synchronization is gradual or abrupt. This in turn opens up many avenues for the application of such shift for synchronization transitions, in particular on multilayer networks [31], where a duality of oscillatory types is likely on two or more different network layers [27]. Cooperative and competitive duality is also common in several social settings, for example in deciding whether or not to cooperate and vaccinate [32]. As recent research emphasizes the importance and prevalence of using different strategies with different partners over time [33], our research opens up the prospect of moderating the temporal activity in communities and different network layers to achieve an optimal timing in collective action dilemmas. Our results are furthermore of value in brain dynamics: competitors and cooperators in our model resemble indeed inhibitory/excitatory neurons in the brain, and an imbalance ratio between these two types of populations is reported to be at the origin of disorders like Epilepsy and Fibromyalgia, both being significant examples of ES [34, 35, 36]. Moreover, anesthetized brain states undergo frequent loss and recovery of consciousness during which hysteresis loops are observed and demonstrates conditions of ES [37]. There is also little doubt that our research can be verified and extended in more traditional experimental settings, for example using nonlinear electronic circuits, as has been done successfully in the past [20].

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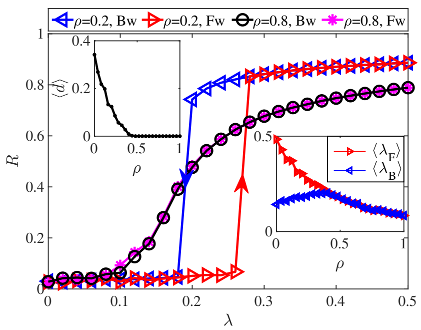

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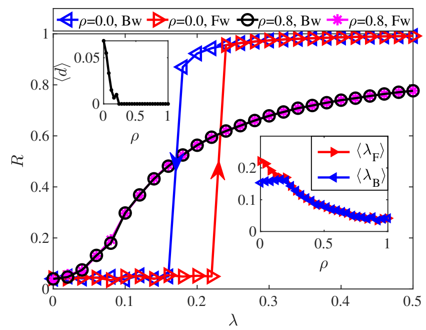

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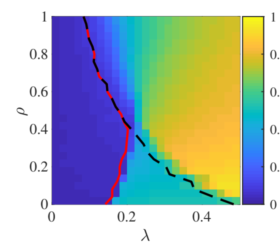

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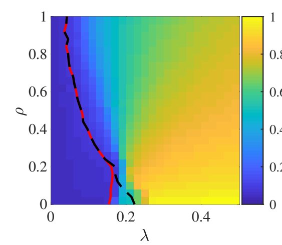

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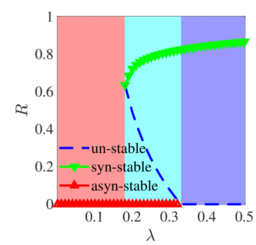

(e)

Figure 1: Forward (Fw) and backward (Bw) synchronization transitions on (a) ER and (b) BA networks with $N = 1,000$ and $\langle k \rangle = 12$. The two insets report the distance d between the critical points λ_B (for Bw) and λ_F (for Fw), as well as λ_B and λ_F themselves. The contour plot of R (color codes in the right bars) in the parameter space (λ, ρ) are displayed in (c) for the ER network, and in (d) for the BA network. In panels (c,d) the cyan area is the parameter region where ES emerges. The red continuous (black dashed) lines represent λ_B (λ_F). Panel (e) reports the mean-field solution of Eq. (11) for an ER network and for $\rho = 0.3$. The blue-dashed-line represents the un-stable solution, while the green-inverted-triangle-line (the red-triangle-line) is the coherent (incoherent) solution.

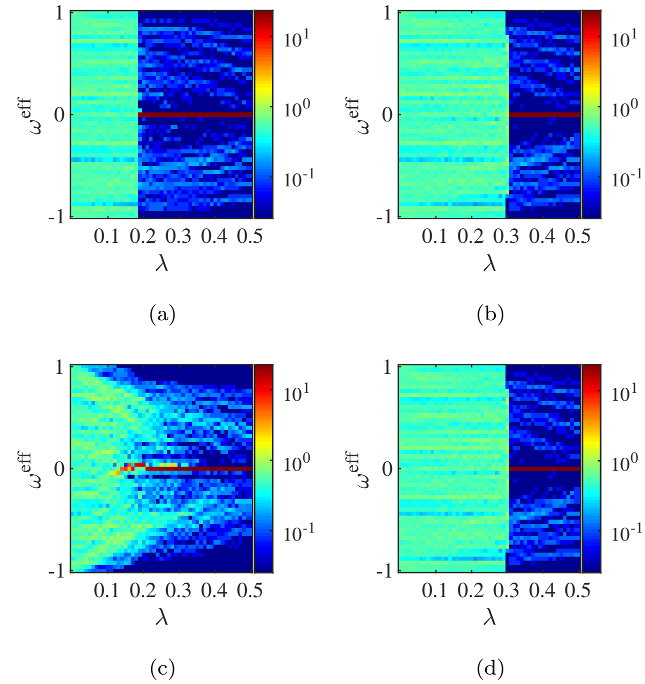

Figure 2: The contour plots (see color codes on the right bars) report the probability distribution function of the effective frequency, $f(\omega^{\rm eff})$ (see text for definition), versus λ for an ER network. Parameters are: (a) Bw for $\rho = 0.2$, (b) Fw for $\rho = 0.2$, (c) Bw for $\rho = 0.8$, and (d) Fw for $\rho = 0.8$.

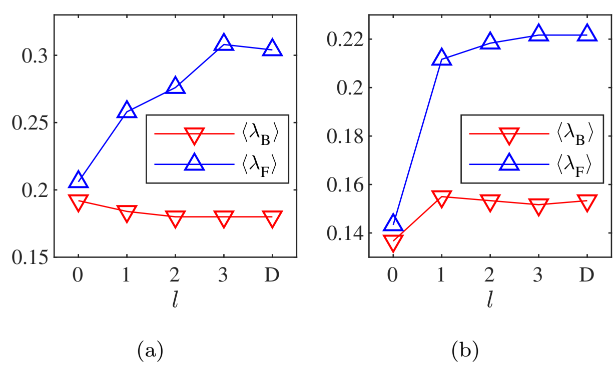

Figure 3: Critical points for Bw ($\langle\lambda_B\rangle$) and Fw ($\langle\lambda_F\rangle$) for (a) an ER network with $\rho = 0.2$ and (b) a BA network with $\rho = 0.0$. Plotted points refer to ensemble averages over 10 distinct network realizations.